Lecture 10 Activity: Chain Rule

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math.dartmouth.edu/~blogsdon/activity10.pdf

1. Use the chain rule (and other rules) to calculate derivatives of the following functions.
   1.1 $e^{\cos x}$
   1.2 e^{cx}
   1.3 $\sin(cx)$
   1.4 $\sin(e^x)$
   1.5 $(\sin(x))^{1000}$
   1.6 $\frac{\sin(\cos x)}{x^2}$
2. Take the derivatives of the functions below. Which ones require the chain rule?
   2.1 $\frac{x^2-3x+1}{e^{(x^2)}}$
   2.2 $\sin\left(\frac{1}{x}\right)$
   2.3 $\frac{\sin^2 x-\cos^2 x}{x^2}$
3. What is the derivative of $\sin(\cos(e^x))$? (**Hint:** This will require using the chain rule twice. First, use the chain rule to differentiate $\cos(e^x)$; then look at the whole thing.)

Solution 1.1

Let $u = \cos x$. Then

$$\frac{df}{du} = \frac{d}{du}(e^u) = e^u = e^{\cos x}$$

and

$$\frac{du}{dx} = \frac{d}{dx}(\cos x) = -\sin x.$$

So

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = e^{\cos x}(-\sin x).$$

Solution 1.2

Let $u = cx$. Then

$$\frac{df}{du} = \frac{d}{du}(e^u) = e^u = e^{cx}$$

and

$$\frac{du}{dx} = \frac{d}{dx}(cx) = c.$$

So

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = e^{cx} \cdot c = ce^{cx}.$$

Solution 1.3

Let $u = cx$. Then

$$\frac{df}{du} = \frac{d}{du}(\sin u) = \cos u = \cos(cx)$$

and

$$\frac{du}{dx} = \frac{d}{dx}(cx) = c.$$

So

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \cos(cx) \cdot c = c\cos(cx)$$

Solution 1.4

Let $u = e^x$. Then

$$\frac{df}{du} = \frac{d}{du}(\sin u) = \cos u = \cos(e^x)$$

and

$$\frac{du}{dx} = \frac{d}{dx}(e^x) = e^x.$$

So

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \cos(e^x) \cdot e^x = e^x \cos(e^x).$$

Solution 1.5

Let $u = e^x$. Then

$$\frac{df}{du} = \frac{d}{du}(u^{1000}) = 1000u^{999} = 1000(\sin x)^{999}$$

and

$$\frac{du}{dx} = \frac{d}{dx}(\sin x) = \cos x$$

So

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = 1000(\sin x)^{999} \cos x.$$

Solution 1.6

Since this is a quotient, we're going to use the quotient rule. But first look at the numerator on its own: $\sin(\cos x)$. Since the cos is inside the sin, we'll need the chain rule for this part.

Let $u = \cos x$. Then

$$\frac{df}{du} = \frac{d}{du}(\sin u) = \cos u = \cos(\cos x)$$

and

$$\frac{du}{dx} = \frac{d}{dx}(\cos x) = -\sin x$$

So

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \cos(\cos x) \cdot (-\sin x) = -\sin x \cos(\cos x).$$

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Solution 1.6 cont.

Now we are ready to differentiate the whole function.

$$\frac{d}{dx}\left(\frac{\sin(\cos x)}{x^2}\right) = \frac{x^2 \frac{d}{dx}(\sin(\cos x)) - \sin(\cos x)\frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2(-\sin x \cos(\cos x)) - \sin(\cos x) \cdot 2x}{x^4}.$$

It would be okay to leave it like this, but we can simplify it by cancelling out one x from the numerator and denominator:

$$\frac{-x \sin x \cos(\cos x) - 2\sin(\cos x)}{x^3}.$$

Solution 1.7

Since this is a fraction, we'll need the quotient rule. Let's look at the denominator first. Let $u = x^2$. Then

$$\frac{df}{du} = \frac{d}{du}(e^u) = e^u = e^{(x^2)}$$

and

$$\frac{du}{dx} = \frac{d}{dx}(x^2) = 2x.$$

So

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = e^{(x^2)} \cdot 2x = 2xe^{(x^2)}.$$

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Solution 1.7 cont.

Now we're ready to look at the whole thing.

$$\frac{d}{dx}\left(\frac{x^2-3x+1}{e^{(x^2)}}\right) = \frac{e^{(x^2)}\cdot\frac{d}{dx}(x^2-3x+1)-(x^2-3x+1)\frac{d}{dx}\left(e^{(x^2)}\right)}{\left(e^{(x^2)}\right)^2}$$

$$= \frac{e^{(x^2)}(2x-3)-(x^2-3x+1)\left(2xe^{(x^2)}\right)}{\left(e^{(x^2)}\right)^2}.$$

Solution 2.2

Let $u = \frac{1}{x}$. Then

$$\frac{df}{du} = \frac{d}{du}(\sin u) = \cos u = \cos \frac{1}{x}$$

and

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

So

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \left(\cos \frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) = -\frac{\cos \frac{1}{x}}{x^2}.$$

Solution 2.3

Either the chain rule or the product rule can be used for the numerator. In either case, its derivative is $4\sin x\cos x$. We finish up with the quotient rule:

$$\frac{d}{dx}\left(\frac{\sin^2 x - \cos^2 x}{x^2}\right) = \frac{x^2 \cdot (4\sin x\cos x) - (\sin^2 x - \cos^2 x)(2x)}{(x^2)^2}.$$

Solution 3

Let's look at the inside first: $\cos(e^x)$. Using the chain rule, we'll wind up with $-e^x \sin(e^x)$. Now look at the whole thing. Let $u = \cos(e^x)$. Then

$$\frac{df}{du} = \frac{d}{du}(\sin u) = \cos u = \cos(\cos(e^x))$$

and

$$\frac{du}{dx} = \frac{d}{dx}\left(\cos(e^x)\right) = -e^x \sin(e^x).$$

So

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \cos(\cos(e^x)) \cdot \left(-e^x \sin(e^x)\right).$$