Lecture 12 Activity: Logarithmic Differentiation and Inverse Trig Derivatives

Ben Logsdon
Math 3, Fall 2023

October 6, 2023

math.dartmouth.edu/~blogsdon/activity12.pdf

1. Differentiate the following functions.
   1.1 $x \cdot (\ln x)$
   1.2 $\ln(\tan x)$
   1.3 $\ln(c \tan x)$
2. Use logarithmic differentiation to differentiate the following functions.
   2.1 $\frac{x^2+1}{x-2}$
   2.2 $\frac{(x+1)(x-1)\sqrt{x-2}}{x^2+3}$
   2.3 $(x^2+5)^{10}\sqrt{\sin x}$
   2.4 $\frac{\sin^2 x \cdot \sqrt[3]{e^x}}{x^3-5x^2+4}$
3. Use implicit differentiation to derive a formula for $\frac{d}{dx} \arctan x$. (You can practice by finding derivatives of the other inverse trig functions.)

Solution 1.1

This will use the product rule.

$$\frac{d}{dx}(x \cdot \ln x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1.$$

Solution 1.2

There are multiple ways to do this. We'll start by using the log identities to rewrite the function:

$$\ln(\tan x) = \ln\left(\frac{\sin x}{\cos x}\right) = \ln(\sin x) - \ln(\cos x).$$

To take the derivative of this, we'll use the formula $\frac{d}{dx}(\ln u) = \frac{u'}{u}$. The derivative is

$$\frac{\cos x}{\sin x} - \frac{-\sin x}{\cos x} = \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = \cot x + \tan x.$$

Solution 1.3

From the log rules, we know that $\ln(c \tan x) = \ln c + \ln(\tan x)$. Since c is a constant, $\ln c$ is also a constant. Therefore, the derivative of this function will be the same as the derivative from the previous problem: $\cot x + \tan x$.

Solution 2.1

First, take the natural log of both sides and simplify:

$$
\begin{aligned}
y &= \frac{x^2+1}{x-2} \\
\ln y &= \ln\left(\frac{x^2+1}{x-2}\right) \\
\ln y &= \ln(x^2+1) - \ln(x-2).
\end{aligned}
$$

Differentiate both sides:

$$
\begin{aligned}
\frac{d}{dx}(\ln y) &= \frac{d}{dx}(\ln(x^2+1) - \ln(x-2)) \\
\frac{y'}{y} &= \frac{2x}{x^2+1} - \frac{1}{x-2}.
\end{aligned}
$$

Solve for y' and plug in y:

$$
\begin{aligned}
y' &= y\left(\frac{2x}{x^2+1} - \frac{1}{x-2}\right) \\
y' &= \frac{x^2+1}{x-2}\left(\frac{2x}{x^2+1} - \frac{1}{x-2}\right).
\end{aligned}
$$

Solution 2.2

First, take the natural log of both sides and simplify:

$$y = \frac{(x+1)(x-1)\sqrt{x-2}}{x^2+3}$$
$$\ln y = \ln(x+1) + \ln(x-1) + \ln\left(\sqrt{x-2}\right) - \ln(x^2+3)$$
$$\ln y = \ln(x+1) + \ln(x-1) + \frac{1}{2}\ln(x-2) - \ln(x^2+3).$$

Differentiate both sides:

$$\frac{y'}{y} = \frac{1}{x+1} + \frac{1}{x-1} + \frac{1}{2}\frac{1}{x-2} - \frac{2x}{x^2+3}.$$

Solve for y' and plug in y:

$$y' = y\left(\frac{1}{x+1} + \frac{1}{x-1} + \frac{1}{2}\frac{1}{x-2} - \frac{2x}{x^2+3}\right)$$
$$y' = \frac{(x+1)(x-1)\sqrt{x-2}}{x^2+3}\left(\frac{1}{x+1} + \frac{1}{x-1} + \frac{1}{2}\frac{1}{x-2} - \frac{2x}{x^2+3}\right).$$

Solution 2.3

First, take the natural log of both sides and simplify:

$$y = (x^2 + 5)^{10}\sqrt{\sin x}$$
$$\ln y = 10 \ln(x^2 + 5) + \frac{1}{2} \ln(\sin x).$$

Differentiate both sides:

$$\frac{y'}{y} = 10\frac{2x}{x^2 + 5} + \frac{1}{2} \cdot \frac{\cos x}{\sin x}$$
$$\frac{y'}{y} = \frac{20x}{x^2 + 5} + \cdot\frac{\cos x}{2\sin x}.$$

Solve for y' and plug in y:

$$y' = y\left(\frac{20x}{x^2 + 5} + \frac{\cos x}{2\sin x}\right)$$
$$y' = (x^2 + 5)^{10}\sqrt{\sin x}\left(\frac{20x}{x^2 + 5} + \cdot\frac{\cos x}{2\sin x}\right)$$

Solution 2.4

First, take the natural log of both sides and simplify:

$$y = \frac{\sin^2 x \cdot \sqrt[3]{e^x}}{x^3 - 5x^2 + 4}$$

$$\ln y = 2\ln(\sin x) + \frac{1}{3}\ln(e^x) - \ln(x^3 - 5x^2 + 4)$$

$$\ln y = 2\ln(\sin x) + \frac{1}{3}x - \ln(x^3 - 5x^2 + 4)$$

Differentiate both sides:

$$\frac{y'}{y} = 2\frac{\cos x}{\sin x} + \frac{1}{3} - \frac{3x^2 - 10x}{x^3 - 5x^2 + 4}$$

Solve for y' and plug in y:

$$y' = y\left(2\frac{\cos x}{\sin x} + \frac{1}{3} - \frac{3x^2 - 10x}{x^3 - 5x^2 + 4}\right)$$

$$y' = \frac{\sin^2 x \cdot \sqrt[3]{e^x}}{x^3 - 5x^2 + 4}\left(2\frac{\cos x}{\sin x} + \frac{1}{3} - \frac{3x^2 - 10x}{x^3 - 5x^2 + 4}\right)$$