Lecture 13 Activity: Exponential Growth and Decay

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math.dartmouth.edu/~blogsdon/activity13.pdf

1. 100 rabbits are released into a large forest. After 10 years, there are 500 rabbits. Assume the population grows exponentially. How many rabbits will there be after 20 years? What will be the rate of growth of the rabbits after 20 years? How long will it take to reach a population of 10,000 rabbits?
2. Suppose that the half-life of caffeine in a person's body is 6 hours. If the person drinks a cup of coffee with 120 mg of caffeine at 8 AM, how much caffeine will remain in their body at 8 PM?
3. There are three savings accounts available:
   - Account A offers 5% interest compounded annually.
   - Account B offers 5% interest compounded quarterly (i.e. every three months).
   - Account C offers 4.9% interest compounded continuously.

   Suppose that you invest $1000 in each account. How much money in total will you have after 10 years? Which account is the best?

Solution 1

Information

- Independent variable: time t, in years
- Dependent variable: population p, in rabbits
- Facts: $p(0) = 100$, $p(10) = 500$
- Goals: $p(20) = ?$, $p'(20) = ?$, $p(?) = 10,000$

Equations We'll use the exponential growth formula $p(t) = Ae^{kt}$. Using the fact $p(0) = 100$ and solving, we get $A = 100$. Using the fact $p(10) = 500$ and solving, we get $k = \frac{1}{10}\ln(5)$. So the equation is $p(t) = 100e^{kt}$, where $k = \frac{1}{10}\ln(5)$.

Solve

- After 20 years, there will be $p(20) = 100e^{k\cdot 20}$ rabbits.
- To determine $p'(20)$, we first need to figure out $p'(t)$. Differentiation gives us $p'(t) = \frac{k}{10}e^{kt}$. So $p'(20) = \frac{k}{10}e^{k\cdot 20}$ rabbits per year.
- For the third part, we want to solve $p(t) = 10,000$. Setting $10,000 = 100e^{kt}$ and solving, we get $t = \frac{1}{k}\ln(100)$.

Solution 2

Information

- Independent variable: time t, in hours
- Dependent variable: mass m, in mg
- Facts: We'll use $t = 0$ for 8 AM. So $m(0) = 120$, and since the half-life is 6 hours, $m(6) = 120/2 = 60$.
- Goals: $m(12) = ?$

Equations We'll use the exponential growth formula $p(t) = Ae^{kt}$. Using the fact that $p(0) = 120$ and solving, we get $A = 120$. Using the fact that $m(6) = 60$ and solving, we get $k = \frac{1}{6} \ln \frac{1}{2}$. So the formula is $p(t) = 120e^{kt}$, where $k = \frac{1}{6} \ln \frac{1}{2}$.

Solve To find $m(12)$, we can just plug in 12: $m(12) = 120e^{k \cdot 12}$, for $k = \frac{1}{6} \ln \frac{1}{2}$.