Activity 14: Related Rates

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1. A stone dropped in a pond sends out a circular ripple whose radius increases at a constant rate of 4 ft/sec. After 12 seconds, how rapidly is the area enclosed by the ripple increasing?
2. The volume of a cylinder is decreasing at a rate of 20 m^3 per hour, and the height of the cylinder is decreasing at a rate of 4 meters per hour. At a certain instant, the base radius is 5 meters and the height is 8 meters. What is the rate of change of the radius of the cylinder at the instant? (The volume of a cylinder is $V = \pi r^2 h$, where r is the radius and h is the height.)
3. A person who is 6 feet tall is walking away from a lamp post at the rate of 40 feet per minute. When the person is 10 feet from the lamp post, their shadow is 20 feet long. Find the rate at which the length of the shadow is increasing when they are 30 feet from the lamp post.

Solution 1

Note that

$$\frac{dr}{dt} = 4.$$

We want to find

$$\frac{dA}{dt}$$

when $t = 12$. We use the area formula for a circle

$$A = \pi r^2.$$

Differentiate both sides with respect to t:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

Plug in $\frac{dr}{dt} = 4$. When t=12 seconds, $r = 4 * 12 = 48$. Eventually,

$$\frac{dA}{dt} = 2\pi(48) * 4 = 384\pi \ ft^2/sec.$$

Solution 2

Let $r(t)$ and $h(t)$ denote the radius and height at time t respectively. Note that $\frac{dV}{dt} = -20$ and $\frac{dh}{dt} = -4$. We want to find $\frac{dr}{dt}$ when $r = 5$ and $h = 8$. We use the volume formula for a cylinder

$$V = \pi r^2 h.$$

Differentiate both sides with respect to t:

$$\frac{dV}{dt} = (\pi r^2) * \left(\frac{dh}{dt}\right) + h\left(2\pi r \frac{dr}{dt}\right).$$

Plug in $\frac{dV}{dt} = -20$, $\frac{dh}{dt} = -4$, $r = 5$ and $h = 8$. Eventually,

$$\begin{aligned} -20 &= \pi * 5^2 * (-4) + 8 * 2\pi * 5 * \frac{dr}{dt} \\ \frac{dr}{dt} &= \frac{-20 + 100\pi}{80\pi} = \frac{-1 + 5\pi}{4\pi} \ m/hour. \end{aligned}$$

Solution 3

Let $x(t)$ denote the distance from a person and the lamp post, and $s(t)$ denote the length of shadow at time t. Note that $\frac{dx}{dt} = 40$ when $x = 10$ and $s = 20$. We want to find $\frac{ds}{dt}$ when $x = 30$. We set up a ratio of similar triangles:

$$\frac{x+s}{h} = \frac{s}{6}.$$

The height, h, of the pole is a constant. We solve for h by using that when $x = 10$ and $s = 20$:

$$\frac{10+20}{h} = \frac{20}{6}$$
$$6(30) = 20h$$
$$h = 180/20 = 9.$$

Rewrite the original ratio equation with the constant height solved for:

$$\frac{x+s}{h} = \frac{s}{6}$$
$$6(x+s) = 9s$$

Solution 3 (cont.)

Differentiate both sides with respect to t and solve for $\frac{ds}{dt}$:

$$6\frac{dx}{dt} = 3\frac{ds}{dt}.$$

Plug in $\frac{dx}{dt} = 40$, and solve for $\frac{ds}{dt}$:

$$\frac{ds}{dt} = 80 \; ft/min.$$