Lecture 15 Activity: Linear Approximation and Hyperbolic Functions

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`math.dartmouth.edu/~blogsdon/activity15.pdf`

1. Use linear approximation / differentials to estimate the number 9.9^4 without using a calculator.
2. Suppose that a cube has a side length of 20 cm $\pm$ 0.3 cm. What is the approximate margin of error in the volume of the cube? What is the relative error and the percentage error?
3. Use the definitions of the hyperbolic functions and the derivative rules to find the derivatives of $\tanh x$, $\operatorname{csch} x$, $\operatorname{sech} x$, and $\coth x$.
4. Find a formula for $\cosh^{-1} x$.

Solution 1

Let $f(x) = x^4$. Then $\frac{df}{dx} = 4x^3$ and $df = 4x^3 dx$. Since $9.9 = 10 - 0.1$, we'll use $x = 10$ and $dx = -0.1$.

$$\begin{aligned} f(9.9) &\approx f(10) + df \\ &= f(10) + 4(10)^3 \cdot (-0.1) \\ &= 10,000 - 4 \cdot 10^2 \\ &= 10,000 - 400 \\ &= 9,600. \end{aligned}$$

Indeed, the actual value, rounded to the nearest integer, is 9,606, so this approximation is reasonable.

Solution 2

Let $V(x) = x^3$, where x is the side length. We have $\frac{dV}{dx} = 3x^2$, so $dV = 3x^2 dx$. Since the side length is 20 ± 0.3, we let $x = 20$ and $dx = 0.3$, so $dV = 3(20)^2 \cdot 0.3 = 360\text{cm}^3$. This is the approximate margin of error. It seems like a lot, but the relative error is about

$$\frac{dV}{V} = \frac{360}{20^3} = 0.045,$$

which is a percentage error of 4.5%. Even though $360\,\text{cm}^3$ seems large, it is small compared to the actual volume of the cube.

Solution 3

We have

$$\begin{aligned}\frac{d}{dx}(\tanh x) &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)\\ &= \frac{(\cosh x)\frac{d}{dx}(\sinh x) - (\sinh x)\frac{d}{dx}(\cosh x)}{\cosh^2 x}\\ &= \frac{(\cosh x)(\cosh x) - (\sinh x)(\sinh x)}{\cosh^2 x}\\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}.\end{aligned}$$

This answer is correct. We can simplify further by using the identity $\cosh^2 x - \sin^2 x = 1$, arriving at

$$\frac{1}{\cosh^2 x} = \operatorname{sech}^2 x.$$

The others are similar.

Solution 4

Write $y = \cosh^{-1} x$, so $\cosh y = x$. Thus,

$$\begin{aligned} x &= \frac{e^y + e^{-y}}{2} \\ 2x &= e^y + e^{-y} \\ 0 &= e^y - 2x - e^{-y} \\ 0 \cdot e^y &= (e^y - 2x - e^{-y})e^y \\ 0 &= e^y \cdot e^y - 2xe^y - e^{-y} \cdot e^y \\ 0 &= e^{2y} - 2xe^y - 1. \end{aligned}$$

Continued on the next page.

Solution 4 cont.

Viewing this as a quadratic equation in terms of e^y, we can use the quadratic formula:

$$\begin{aligned} e^y &= \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} \\ &= \frac{2x \pm \sqrt{4x^2 - 4}}{2} \\ &= \frac{2x \pm 2\sqrt{x^2 - 1}}{2} \\ &= x \pm \sqrt{x^2 - 1}. \end{aligned}$$

Therefore, we have

$$\cosh^{-1} x = y = \ln(x \pm \sqrt{x^2 - 1}).$$

Almost there! Continued on the next page.

Solution 4 cont.

We have

$$\cosh^{-1} x = y = \ln(x \pm \sqrt{x^2 - 1}).$$

Now we need to figure out whether to make the $\pm$ a $+$ or a $-$. It turns out that $+$ is right, so $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}$.