Lecture 16 Activity: Maximum and Minimum Values

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Math 3, Fall 2023

October 16, 2023

`math.dartmouth.edu/~blogsdon/activity16.pdf`

1. Find the local extrema of the function $f(x) = x^3 - 12x^2 + 45x - 1$ and the x-coordinates at which they occur.
2. Find the critical point(s) of the function $f(x) = ax^2 + bx + c$ when $a \neq 0$. What does this tell you about the graph of f?
3. Find the global extrema of the function $f(x) = xe^{2x}$ on the interval $[-1, 0]$.
4. Find the critical point(s) of the function $f(x) = ax^3 + bx^2 + cx + d$ when $a \neq 0$. What does this tell you about the graph of f?
5. **Challenge problem:** The acceleration due to gravity on Earth is about $-9.8\text{m}/\text{sec}^2$. Suppose that I throw a ball straight upward. I release the ball from my hand at precisely 1.5 m above ground level at a speed of 5m/sec. How high will the ball travel?

Solution 1

Find critical points We have $f'(x) = 3x^2 - 24x + 45$. Solving this equation, we get $x = 3, 5$. These are critical points of f. Since there is nowhere where f' doesn't exist, 3 and 5 are all the critical points of f.

First Derivative Test There are three intervals to consider: $(-\infty, 3)$, $(3, 5)$, and $(5, \infty)$. All we need to do is plug in one number for each of these intervals.
If we do this, we find that $f'(0) > 0$, $f'(4) < 0$, and $f'(6) > 0$. Therefore, f has a local maximum at 3 and a local minimum at 5.

Solution 2

We have $f'(x) = 2ax + b$. When $0 = 2ax + b$, we have $x = -\frac{b}{2a}$. Since f' exists everywhere, $-\frac{b}{2a}$ is the only critical point of f.
Recall that the graph of f is a parabola, which has a single vertex, which is either its global maximum or global minimum. This tells us that $-\frac{b}{2a}$ is the x-coordinate of the vertex.

Solution 3

Check continuity The function xe^{2x} is continuous everywhere, so it is continuous on the interval $[-1, 0]$.

Find critical points We have $f'(x) = e^{2x} + 2xe^{2x}$.. This function exists everywhere, so the only critical points of f are its roots. So

$$0 = e^{2x} + 2xe^{2x} = (1 + 2x)e^{2x}.$$

Since e^{2x} is always positive, we just have $0 = 1 + 2x$, which means $x = -\frac{1}{2}$. This is the only critical point of f.

Plug in critical points and endpoints The critical point is $-\frac{1}{2}$, and the endpoints are -1 and 0. We have

$$f(-1) = -e^{-2} = -\frac{1}{e^2}$$
$$f(-1/2) = (-1/2)e^{2(-1/2)} = -\frac{1}{e}$$
$$f(0) = 0.$$

The global max is the greatest of these, which is $f(0) = 0$. The global min is the least of these, which is $f(-1/2) = -\frac{1}{e}$.

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Solution 4

We have $f'(x) = 3ax^2 + 2bx + c$. This exists everywhere, so the only critical points of f are where $f'(x) = 0$. We can solve this using the quadratic formula, and we end up with

$$x = \frac{-(2b) \pm \sqrt{(2b)^2 - 4 \cdot (3a) \cdot c}}{2(3a)}.$$

Depending on a, b, and c, this will either be two solutions, one solution, or no solutions. If there are two solutions, these will be the x-coordinates of the peak and valley (local max and min) in the graph of f. (As an example, use a graphing calculator to graph $f(x) = x^3 - x$. It has two critical points: one local max and one local min.)

If there is one solution, it will be the x-coordinate of a critical point in f, but it will not be a local max or min. (As an example, use a graphing calculator to graph $f(x) = x^3$. It has a critical point at $x = 0$.)

If there are no solutions, then f has no critical points. (As an example, use a graphing calculator to graph $f(x) = x^3 + x$.)

Solution 5

Let $f(t)$ denote the height of the ball at time t, where $t = 0$ is the moment the ball is released from my hand. The acceleration of -9.8 means that $f''(t) = -9.8$. Since the ball is released at a height of 1.5, we have $f(0) = 1.5$. Since it is released with a speed of 5, we have $f'(0) = 5$.

We can use a quadratic equation for this function: $f(x) = ax^2 + bx + c$. Since $f''(t) = -9.8$, we have $a = -4.9$. Since $f'(0) = 5$, we have $b = 5$. Since $f(0) = 1.5$, we have $c = 1.5$. Thus, we can model the motion of the ball using the equation $f(t) = -4.9t^2 + 5t + 1.5$.

To find the height of the ball at its highest point, we must find the global maximum of f. Since f is a parabola with negative a, this will occur at the only critical point of f. From Solution 2 above, we know that this critical point occurs at $t = -\frac{b}{2a}$. Thus, the time when the ball is highest is $t = -\frac{5}{-9.8} = \frac{5}{9.8}$. To find the height, we plug this into the original function: $f(5/9.8) \approx 2.8$ meters above ground level.