Lecture 17 Activity: Mean Value Theorem and the Shape of Graphs

Originally Written by Nadia Lafreniére and Bosu Choi
Updated by Ben Logsdon
Math 3, Fall 2023

October 18, 2023

math.dartmouth.edu/~blogsdon/activity17.pdf

1. Sketch the graphs of the following functions:
   1.1 $f(x) = (x-2)(x-1)^2$
   1.2 $g(x) = \frac{x+1}{(x-2)^2}$
2. For each of the following two sets of conditions, sketch the graph of a function that satisfies all of the given conditions:
   2.1 $f'(0) = f'(2) = f'(4) = 0$,
   $f'(x) > 0$ if $x < 0$ or $2 < x < 4$,
   $f'(x) < 0$ if $0 < x < 2$ or $x > 4$,
   $f''(x) > 0$ if $1 < x < 3$,
   $f''(x) < 0$ if $x < 1$ or $x > 3$
   2.2 $f'(x) < 0$ and $f''(x) > 0$ for all x.
3. Suppose you drive for 0.5 hours, covering a distance of 35 miles. Is there any time during this drive when your speed has exceeded 65 mph? Why?

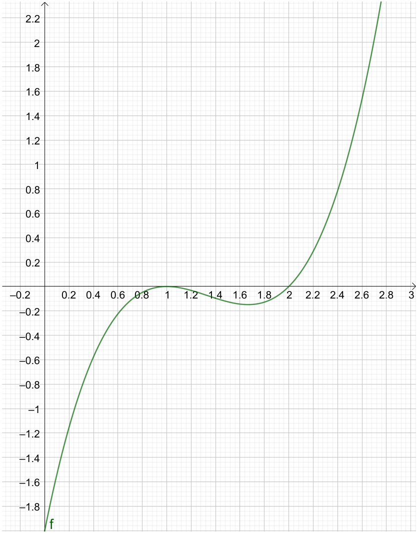
Solutions

Sketch the graph of $f(x) = (x-2)(x-1)^2$.
Domain is $(-\infty, \infty)$, and there is no asymptote because it is a polynomial.
$f(0) = -2$, $f(x) = 0$ exactly when x is 1 or 2.
$f(x) = x^3 - 4x^2 + 5x - 2$. Therefore, $f'(x) = 3x^2 - 8x + 5$ and $f''(x) = 6x - 8$.
$f'(x) = 0$ when $x = \frac{8 \pm \sqrt{64-60}}{6}$, so when $x = 1$ or $x = \frac{5}{3}$.
$f''(x) = 0$ when $x = \frac{4}{3}$.

	$-\infty$	$\dots$	0	$\dots$	1	$\dots$	$\frac{4}{3}$	$\dots$	$\frac{5}{3}$	$\dots$	2	$\dots$	∞
f	$-\infty$	-2			0			<0			0	>0	∞
f'		>0			0	<0			0		>0		
f''		<0					0			>0			
		$\nearrow$				$\searrow$		$\searrow$			$\nearrow$		
		CD				*CD*		*CU*			*CU*		

Solutions

Sketch the graph of $f(x) = (x-2)(x-1)^2$.



Solutions

Sketch the graph of $g(x) = \frac{x+1}{(x-2)^2}$.

Domain is $(-\infty, 2) \cup (2, \infty)$. There is a vertical asymptote at $x = 2$, since 2 is not in the domain, and as defined by the limit at 2 below.

$g(0) = \frac{1}{4}$, $f(x) = 0$ exactly when x is -1.

$g'(x) = \frac{(x-2)^2 - 2(x-2)(x+1)}{(x-2)^4} = \frac{-x-4}{(x-2)^3}$ using quotient rule.

$g'(x) = 0$ when $x = -4$, and is undefined where at $x = 2$. The derivative is negative to the left of $x = -4$ and to the right of $x = 2$, and positive elsewhere.

$g''(x) = \frac{-(x-2)^3 - 3(x-2)^2(-x-4)}{(x-2)^6} = \frac{-(x-2) - 3(-x-4)}{(x-2)^4} = \frac{2x+14}{(x-2)^4}$ which is 0 when $x = -7$, and undefined at $x = 2$. (The function is concave down to the left of $x = -7$, and concave up elsewhere).

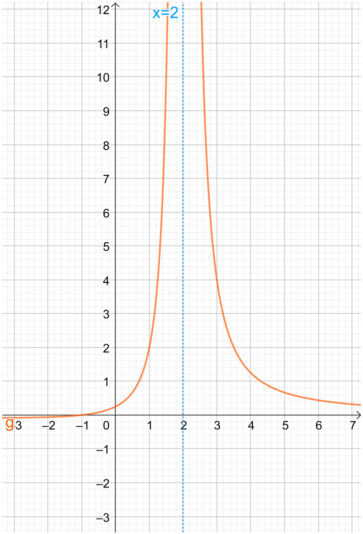
The limits at infinity are $\lim_{x \to -\infty} g(x) = 0 = \lim_{x \to -\infty} g(x)$ (horizontal asymptote).

Around the vertical asymptote:

$$\lim_{x \to 2^-} g(x) = +\infty = \lim_{x \to 2^-} g(x)$$

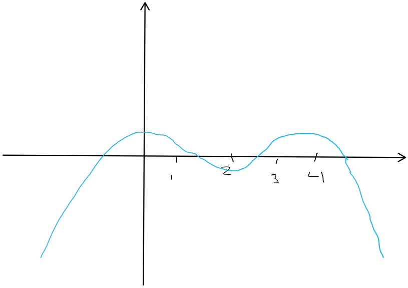
Solutions

Sketch the graph of $f(x) = \frac{x+1}{(x-2)^2}$.



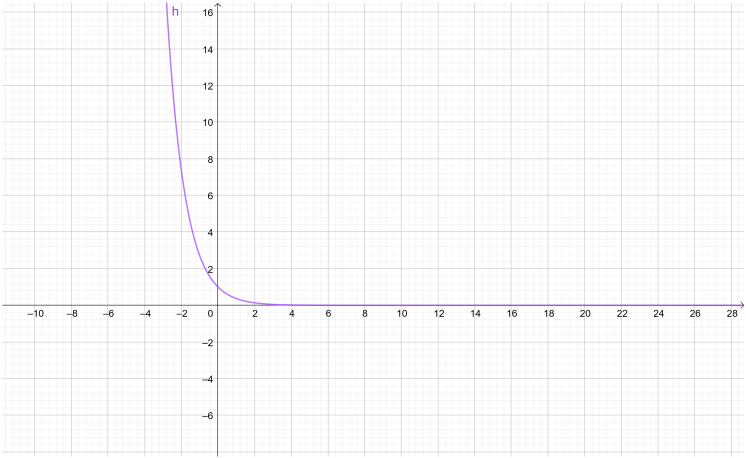
Solutions

Sketch the graph of a function that satisfies all of the given conditions:
$f'(0) = f'(2) = f'(4) = 0,$
$f'(x) > 0$ *if* $x < 0$ *or* $2 < x < 4,$
$f'(x) < 0$ *if* $0 < x < 2$ *or* $x > 4,$
$f''(x) > 0$ *if* $1 < x < 3,$
$f''(x) < 0$ *if* $x < 1$ *or* $x > 3.$



Solutions

Sketch the graph of a function that satisfies all of the given conditions: $f'(x) < 0$ *and* $f''(x) > 0$ *for all* x.



$f(x) = e^{-x}$ is an example of such a function.