Lecture 19 Activity: Maximum and Minimum Values

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Compute the following limits.

1. $\lim_{x\to 0^+} x^2 \ln x$
2. $\lim_{x\to \frac{\pi}{2}^+} \frac{\tan x}{\sec x}$
3. $\lim_{x\to 0} \frac{1-\cos x}{x}$
4. $\lim_{x\to \pi} \frac{\sin x}{\cos x - 1}$
5. $\lim_{x\to \infty} \frac{e^x}{x^3}$
6. $\lim_{x\to \infty}(e^x - x^4)$
7. $\lim_{x\to \infty} \frac{\ln x}{\sqrt[n]{x}}$, where $n \geq 1$
8. $\lim_{x\to \infty} \frac{e^{(x^2)}}{e^x}$
9. $\lim_{x\to -\infty}(e^x)^{1/x}$
10. $\lim_{x\to 0}(x+1)^{1/x}$

Solution 1

This is an indeterminate form $0 \cdot \infty$. Thus, we will try to apply l'Hospital's Rule. We have

$$\lim_{x \to 0^+} x^2 \ln x = \lim_{x \to 0^+} \frac{\ln x}{x^{-2}}.$$

Since this now has indeterminate form ∞/∞, we can apply l'Hospital's Rule:

$$\lim_{x \to 0^+} \frac{x^{-1}}{-2x^{-3}} = \lim_{x \to 0^+} \frac{x^2}{-2}.$$

We can evaluate this by plugging in, and we get 0.

Solution 2

This is an indeterminate form ∞/∞. We could apply l'Hospital's Rule, but it is easier to evaluate by simplifying:

$$\lim_{x\to\frac{\pi}{2}^+} \frac{\tan x}{\sec x} = \lim_{x\to\frac{\pi}{2}^+} \frac{\sin x/\cos x}{1/\cos x}$$
$$= \lim_{x\to\frac{\pi}{2}^+} \sin x.$$

We can evaluate this by plugging in; we get 1.

Solution 3

This is an indeterminate form $0/0$. Thus, we can apply l'Hospital's Rule:

$$\begin{aligned} \lim_{x \to 0} \frac{1 - \cos x}{x} &= \lim_{x \to 0} \frac{\sin x}{1} \\ &= \lim_{x \to 0} \sin x. \end{aligned}$$

We can evaluate this by plugging in; we get 0.

Solution 4

If we plug in π, we get $0/-2=0$. This is **not** and indeterminate form—therefore, we cannot apply l'Hospital's Rule! Instead, this is a limit that we can just evaluate by plugging in; the answer is 0.

Solution 5

This is an indeterminate form ∞/∞. So we can apply l'Hospital's rule:

$$\lim_{x\to\infty} \frac{e^x}{x^3} = \lim_{x\to\infty} \frac{e^x}{3x^2}.$$

This is still ∞/∞, so we can apply l'Hospital's Rule again:

$$\lim_{x\to\infty} \frac{e^x}{6x}.$$

This is *still* ∞/∞, so we apply l'Hospital's Rule *again*:

$$\lim_{x\to\infty} \frac{e^x}{6}.$$

Now we can see that the limit is ∞.

Solution 6

This is an indeterminate form $\infty - \infty$. We'll first rewrite it like this:

$$\lim_{x \to \infty} e^x - x^4 = \lim_{x \to \infty} x^4 (\frac{e^x}{x^4} - 1)$$

Now focus on e^x/x^4. This is the indeterminate form ∞/∞. If we apply l'Hospital's rule a few times, we can see that this limit is ∞. (See Solution 5.) Thus, our original limit is now written as $\infty \cdot \infty$, which means that it is just ∞.

Solution 7

Regardless of the value of n, this is the indeterminate form ∞/∞. We can rewrite it as

$$\lim_{x\to\infty} \frac{\ln x}{x^{1/n}}$$

and apply l'Hospital's Rule:

$$\begin{aligned}
\lim_{x\to\infty} \frac{1/x}{\frac{1}{n}x^{1/n-1}} &= \lim_{x\to\infty} \frac{nx^{1-1/n}}{x} \\
&= \lim_{x\to\infty} nx^{-1/n} \\
&= \lim_{x\to\infty} \frac{n}{\sqrt[n]{x}} \\
&= 0.
\end{aligned}$$

Solution 8

This is an indeterminate form ∞/∞. We could apply l'Hospital's Rule, but it turns out to make the limit more complicated rather than less. We will start by simplifying and continue:

$$\begin{aligned}\lim_{x\to\infty} e^{x^2-x} &= e^{\left(\lim_{x\to\infty}(x^2-x)\right)} \\ &= e^{\infty} \\ &= \infty.\end{aligned}$$

The inner limit can be evaluated by noticing that $x^2 - x = x(x-1)$, which is $\infty \cdot \infty$.

Solution 9

This is the indeterminate form 0^0. We try to turn this into a quotient as follows:

$$
\begin{aligned}
\lim_{x\to-\infty}(e^x)^{1/x} &= \lim_{x\to-\infty} e^{\ln\left((e^x)^{1/x}\right)} \\
&= \lim_{x\to-\infty} e^{\frac{1}{x}\ln((e^x))} \\
&= \lim_{x\to-\infty} e^{x/x} \\
&= \lim_{x\to-\infty} e \\
&= e.
\end{aligned}
$$

We ended up not even using l'Hospital's Rule!

Solution 10

This is the indeterminate form 1^∞. We try to turn this into a quotient:

$$\begin{aligned}\lim_{x\to 0}(x+1)^{1/x} &= \lim_{x\to 0} e^{\ln\left((x+1)^{1/x}\right)} \\ &= \lim_{x\to 0} e^{\frac{1}{x}\ln((x+1))} \\ &= e^{\lim_{x\to 0}\frac{\ln(x+1)}{x}}.\end{aligned}$$

The limit on the inside has indeterminate form $0/0$. Thus, we can use l'Hospital's Rule, and we get that it is $\lim_{x\to 0}\frac{1}{x+1} = 1$. (Try working it out in detail.)
Thus, the original limit is $e^1 = e$.