Lecture 21 Activity: Optimization

Math 3, Fall 2022

October 27, 2023

1. You want to build a rectangular garden with a fence that costs \$1 per foot. The garden needs to have an area of 100 square feet. What shape should you make the garden to minimize the cost, and how much will the fence cost?
2. Uh-oh! Your neighbor on the east side purchased an attack dog trained to attack due west at all times. You'll need a stronger fence on the east side costing \$7 per square foot. What shape should you make the garden to minimize the cost, and what will be the total fence cost?
3. Your neighbor kindly apologizes for the attack dog and offers to subsidize your garden fence up to \$100 to offset the cost of the stronger fence. What is the maximum area you can enclose with this money, and what shape will the garden be?
4. How would your answer to question 3 change if the fence must be placed in whole 1-foot segments?

Solution 1

Let x and y be the side lengths of the garden in feet. Since the garden must have an area of 100, we need $xy = 100$. The cost is $c = 2x + 2y$. Substituting $y = \frac{100}{x}$, we have $c = 2x + \frac{200}{x}$. The domain of c is $(0, \infty)$.
We want to minimize c. Since $c' = 2 - \frac{200}{x^2}$, we want to solve:

$$\begin{aligned} 0 &= 2 - \frac{200}{x^2} \\ 0 &= 2x^2 - 200 \\ 200 &= 2x^2 \\ 100 &= x^2 \\ 10 &= x. \end{aligned}$$

So one side length is 10 feet. The other side length must also be 10 feet. Thus, the perimeter is 40 feet, so the cost is \$40.

Solution 2

We still want $xy = 100$. This time, however, the cost is different. One of the y sides of the fence costs a total of $7y$ rather than just y. So the total cost is $c = 2x + 8y$. Substituting $y = \frac{100}{x}$, we get $c = 2x + \frac{800}{x}$. Once again, the domain of c is $(0, \infty)$.
We want to minimize c. Since $c' = 2 - \frac{800}{x^2}$, we want to solve:

$$\begin{aligned} 0 &= 2 - \frac{800}{x^2} \\ 0 &= 2x^2 - 800 \\ 800 &= 2x^2 \\ 400 &= x^2 \\ 20 &= x. \end{aligned}$$

So one side length is 20 feet. The other side length must be $100/20 = 5$ feet. The total cost is $2 \cdot 20 + 8 \cdot 5$, or \$80.

Solution 3

Now the cost is fixed and the area is allowed to change. So $A = xy$ and $100 = 2x + 8y$. Solving the second equation for x, we get $x = 50 - 4y$. Plugging this into the area formula, we get $A = y(50 - 4y) = 50y - 4y^2$. The domain of A is $[0, \frac{50}{4}]$.

We want to maximize A. Since $\frac{dA}{dy} = 50 - 8y$, we want to solve $0 = 50 - 8y$, which gives $y = \frac{25}{4}$. Plugging this into $100 = 2x + 8y$, we have $100 = 2x + 50$, so $x = 25$.

We need to check the endpoints of the domain; however, at the endpoints, $A = 0$. Thus, the fence will be 25 ft by $\frac{25}{4} = 6.25$ ft, with an area of $\frac{625}{4} = 156.25$ sq. ft.

Solution 4

Now, our x and y must be whole numbers. However, the optimal solution using whole numbers will be close to the original optimal solution.

Our solution to the previous problem had $x = 25$ and $y = 6.25$. One straightforward option is to decrease y to 6, which would give an area of $A = 25 \cdot 6 = 150$. However, we could also increase y to 7. Then we would have

$$\begin{aligned} 100 &= 2x + 56 \\ 44 &= 2x \\ x &= 22, \end{aligned}$$

giving an area of $A = 7 \cdot 22 = 154$, which is slightly greater. This is the optimal solution if the fence must be placed in one-foot segments.