Lecture 23 Activity: Areas Under Curves

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In this problem, we will approximate and find the area under the curve $f(x) = x^2$ from $x = 0$ to $x = 2$.

1. Approximate the area using four rectangles with a right endpoint approximation using the following steps.
   1.1 What is Δx, the width of each triangle?
   1.2 Find the height of each triangle.
   1.3 Find the area of each triangle.
   1.4 Add all the areas together (You should get 3.75).
2. Approximate the area using n rectangles with a right endpoint approximation using the following steps.
   2.1 What is Δx, the width of each triangle? (Your answer should have "n" in it.)
   2.2 Consider the triangle number k. What is its left endpoint? What is its right endpoint? What is its height?
   2.3 What is the area of triangle number k?
   2.4 Write an expression that adds all the areas together. (You'll need Σ-notation for this.)
   2.5 We can make this approximation better by making n bigger, and we can make it perfect by taking the limit as $n \to \infty$. Write down this limit and evaluate it. (Your answer should be 8/3.)

Solution 1.1, 1.2, 1.3

1. We are using four rectangles from $x = 0$ to $x = 2$. Thus, each rectangle has a width of $\frac{1}{2}$.
2. We will use the right endpoint for each rectangle. The first rectangle is from $x = 0$ to $x = 1/2$. Thus, its height is $f(1/2) = (1/2)^2 = 1/4$. Similarly, the heights of the others are the following:

$$f(1) = 1^2 = 1$$
$$f(3/2) = (3/2)^2 = 9/4$$
$$f(2) = 2^2 = 4.$$

3. Each rectangle has the same width $(1/2)$, but they have different heights. To get the area of each, we multiply the width by the height. The four areas are the following:

$$(1/2) \cdot (1/4) = 1/8$$
$$(1/2) \cdot 1 = 1/2$$
$$(1/2) \cdot (9/4) = 9/8$$
$$(1/2) \cdot 4 = 2.$$

Solution 2.1

The total width is 2, and there are n rectangles. So the width of each rectangle is $\Delta x = 2/n$.

Solution 2.2

First, think about the first triangle, where $k = 1$. The left endpoint is $x = 0$, and the right endpoint is $x = 2/n$. For triangle 2, the left endpoint is $x = 2/n$, and the right endpoint is $x = 4/n$. For triangle 3, the left endpoint is $x = 4/n$, and the right endpoint is $x = 6/n$. These can all be captured by the following formulas. The right endpoint of triangle k is $2k/n$, and the left endpoint of triangle k is $(2k - 2)/n$. To get the height of triangle k, we plug the right endpoint into the function f. So we get $f(2k/n) = \left(\frac{2k}{n}\right)^2$ as the height.

Solution 2.3

To get the area of rectangle k, we multiply its width by its height. The width of each rectangle is $2/n$. The height of rectangle k is $\left(\frac{2k}{n}\right)^2$. So the area of rectangle k is $\frac{2}{n} \cdot \left(\frac{2k}{n}\right)^2$. We can rewrite this as $\frac{8}{n^3} \cdot k^2$.

Solution 2.4

We need to add up the area of each rectangles. The rectangles are numbered from 1 to n, so we use sigma notation:

$$\sum_{k=1}^{n} \left(\frac{8}{n^3} \cdot k^2 \right).$$

Solution 2.5

We want to calculate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{8}{n^3} \cdot k^2 \right).$$

Since the $\frac{8}{n^3}$ doesn't have a k in it, we can pull it out of the sum.

$$\lim_{n\to\infty} \frac{8}{n^3} \sum_{k=1}^{n} k^2.$$

Using the formula $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, we get

$$\lim_{n\to\infty} \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}.$$

Now we can just treat this as a normal limit. The answer winds up being 8/3.