Lecture 24 Activity: Definite Integrals

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1. Calculate the integral $\int_0^1 x^3 dx$ using the definition. You may use the formula $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$. (The answer is 1/4).
2. Calculate $\int_{-1}^1 \sqrt{1-x^2}$. **Hint:** Use geometry for this one, not the definition. What shape is this function?
3. What is $\int_0^{2\pi} \sin x \, dx$? **Hint:** Draw the graph of $\sin x$ and make an educated guess.
4. Suppose $\int_0^5 f(x)dx = 3$, $\int_0^5 g(x)dx = -2$, $\int_0^3 h(x)dx = 10$.
   - 4.1 What is $\int_0^5 (f(x) - 3g(x))dx$?
   - 4.2 If $\int_0^5 (f(x) + h(x))dx = 5$, what is $\int_3^5 h(x)dx$?
5. **Challenge problem:** The formula we used to define $\int_a^b f(x)dx$ used Riemann sums with the right endpoint approximation. What would the formula look like if we used a left endpoint approximation instead? What about a midpoint approximation? Do you think the value of the definite integral changes based on which version we use?

Solution 1

For this definite integral, $a = 0$, $b = 1$, and $f(x) = x^3$. So

$$\begin{aligned}
\int_0^1 x^3 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f\left(\frac{(b-a)i}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^3 \\
&= \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3 \\
&= \lim_{n\to\infty} \frac{1}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\
&= \lim_{n\to\infty} \frac{(n+1)^2}{4n^2} \\
&= \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2}.
\end{aligned}$$

There are a few ways to figure out this limit. The shortcut is to notice that the leading terms of the top and bottom are $1n^3$ and $4n^2$, so the limit is $1/4$.

Solution 2

This curve is the upper half of the unit circle. Therefore, the integral itself is the area of the upper half of the unit circle. Since the area of the unit circle is π, this integral is $\pi/2$.

Solution 3

The graph of $\sin x$ from 0 to 2π is positive from 0 to π and negative from π to 2π. These sections look like they have the same size. Since one is positive and one is negative, we might guess that they cancel out and the whole integral is 0.

Indeed, this does turn out to be true.

Solution 4

For the first part, we can split up using the properties of integrals:

$$\begin{aligned}\int_0^5 (f(x) - g(x))dx &= \int_0^5 f(x)dx - \int_0^5 g(x)dx \\ &= 3 - (-2) \\ &= 5.\end{aligned}$$

For the second part, since $\int_0^5 f(x)dx = 3$ and $\int_0^5 (f(x) + h(x))dx = 5$, it must be that $\int_0^5 h(x)dx = 2$. Therefore,

$$\begin{aligned}\int_0^5 h(x)dx &= \int_0^3 h(x)dx + \int_3^5 h(x)dx \\ 2 &= 10 + \int_3^5 h(x)dx \\ -8 &= \int_3^5 h(x)dx.\end{aligned}$$