Lecture 26 Activity: u-Substitution

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`math.dartmouth.edu/~blogsdon/activity26.pdf`

1. Calculate the following integrals with u-substitution using the given u.
   1.1 $\int 2x\cos(x^2)\,dx,\ u = x^2$
   1.2 $\int x^2 e^{(x^3)}\,dx,\ u = x^3$
   1.3 $\int \frac{x^3}{1+(x^4-1)^2}\,dx,\ u = x^4 - 1$
   1.4 $\int_0^2 (5x-10)^{10}\,dx,\ u = 5x - 10$
   1.5 $\int_6^9 (x+1)\sqrt{x-5}\,dx,\ u = x - 5$
2. Calculate the following integrals.
   2.1 $\int \cos x\, e^{\sin x}\,dx$
   2.2 $\int_0^1 x\sqrt{1-x^2}\,dx$
   2.3 $\int \cos x(\sin(\sin x))\,dx$
   2.4 $\int_0^{\pi/4} \sec x \tan x\,dx$

Solution 1.1

We have $u = x^2$, so $du = 2x\,dx$ and $dx = du/2x$. So

$$\begin{aligned}
\int 2x\cos(x^2)\,dx &= \int 2x\cos(u)\,\frac{du}{2x} && \text{(substitute)} \\
&= \int \cos u\,du && \text{(cancel out the } 2x\text{)} \\
&= \sin u + C && \text{(integral of cos is sin))} \\
&= \sin(x^2) + C && \text{(plug } u \text{ back in).}
\end{aligned}$$

Solution 1.2

Since $u = x^3$, we have $du = 3x^2\,dx$ and $dx = du/(3x^2)$. So

$$
\begin{aligned}
\int x^2 e^{x^3}\,dx &= \int x^2 e^u \,\frac{du}{3x^2} && \text{(substitute)} \\
&= \frac{1}{3}\int e^u\,du && \text{(cancel } x^2\text{, move } 1/3 \text{ to front)} \\
&= \frac{1}{3}e^u + C && \text{(integral of } e^u \text{ is } e^u\text{)} \\
&= \frac{e^{x^3}}{3} + C && \text{(plug } u \text{ back in)}
\end{aligned}
$$

Solution 1.3

Since $u = x^4 - 1$, we have $du = 4x^3\,dx$ and $dx = du/(4x^3)$. So

$$\begin{aligned}
\int \frac{x^3}{1+(x^4-1)^2}\,dx &= \int \frac{x^3}{1+u^2}\,\frac{du}{4x^3} && \text{(substitute)} \\
&= \frac{1}{4}\int \frac{1}{1+u^2}\,du && \text{(cancel } x^3\text{, move } 1/4 \text{ to front)} \\
&= \frac{1}{4}\arctan u + C && \text{(integral of } 1/(1+u^2) \text{ is } \arctan u\text{)} \\
&= \frac{\arctan(x^4-1)}{4} + C && \text{(plug } u \text{ back in)}
\end{aligned}$$

Solution 1.4

Since $u = 5x - 10$, we have $du = 5\,dx$ and $dx = du/5$. When $t = 0$, $u = -10$, and when $t = 2$, $u = 0$, so So

$$\begin{aligned}
\int_0^1 (5x-10)^{10}\,dx &= \int_{-10}^{0} u^{10}\,\frac{du}{5} && \text{(substitute)} \\
&= \frac{1}{5}\int_{-10}^{0} u^{10}\,du && \text{(move 1/5 to front)} \\
&= \frac{1}{5}\left(\frac{u^{11}}{11}\Big|_{-10}^{0}\right) && \text{(take antiderivative)} \\
&= \frac{1}{5}\left(\frac{0^{11}}{11} - \frac{(-10)^{11}}{11}\right) \\
&= -\frac{10^{11}}{55}.
\end{aligned}$$

Solution 1.5

Since $u = x - 5$, we have $du = dx$. When $t = 6$, $u = 1$, and when $t = 9$, $u = 4$. So

$$\begin{aligned}
\int_6^9 (x+1)\sqrt{x-5}\,dx &= \int_1^4 ((u+5)+1)\sqrt{u}\,du \\
&= \int_1^4 (u+6)\sqrt{u}\,du \\
&= \int_1^4 (u^{3/2} + 6u^{1/2})\,du \\
&= \left(\frac{2}{5}\cdot u^{5/2} + \frac{2}{3}\cdot 6u^{3/2}\right)\Bigg|_1^4 \\
&= \left(\frac{2}{5}\cdot 4^{5/2} + \frac{2}{3}\cdot 6\cdot 4^{3/2}\right) - \left(\frac{2}{5}\cdot 0^{5/2} + \frac{2}{3}\cdot 6\cdot 0^{3/2}\right) \\
&= \frac{2}{5}\cdot 2^5 + \frac{12}{3}\cdot 2^3 \\
&= \frac{64}{5} + \frac{32}{3}.
\end{aligned}$$

Solution 2.1

We pick $u = \sin x$, so $du = \cos x \, dx$. So

$$\begin{aligned}
\int \cos x \, e^{\sin x} dx &= \int e^{\sin x} (\cos x \, dx) \\
&= \int e^u \, du && \text{(substitute } u = \sin x \text{ and } du = \cos x \, dx\text{)} \\
&= e^u + C \\
&= e^{\sin x} + C.
\end{aligned}$$

Solution 2.2

We pick $u = 1 - x^2$, so $du = -2x\,dx$ and $du/(-2) = x\,dx$. When $x = 0$, $u = 1$, and when $x = 1$, $u = 0$. So

$$\begin{aligned}
\int_0^1 x\sqrt{1-x^2}\,dx &= \int_0^1 \sqrt{1-x^2}\,(x\,dx) \\
&= \int_1^0 \sqrt{u}\,\frac{du}{-2} \\
&= -\frac{1}{2}\int_1^0 \sqrt{u}\,du \\
&= -\frac{1}{2}\left(\frac{2}{3}u^{3/2}\Big|_1^0\right) \\
&= -\frac{1}{2}\left(\frac{2}{3}(0)^{3/2} - \frac{2}{3}(1)^{3/2}\right) \\
&= -\frac{1}{2}(-\frac{2}{3}) \\
&= \frac{1}{3}.
\end{aligned}$$

Solution 2.3

We pick $u = \sin x$, so $du = \cos x\, dx$. So

$$\begin{aligned}
\int \cos x(\sin(\sin x))\, dx &= \int (\sin(\sin x))\,(\cos x\, dx) \\
&= \int \sin u\, du \\
&= -\cos u + C \\
&= -\cos(\sin x) + C.
\end{aligned}$$

Solution 2.4

This problem does not require u-substitution. The antiderivative of $\sec x \tan x$ is $\sec x$. (You can check this by confirming that the derivative of $\sec x$ is $\sec x \tan x$.) So

$$\begin{aligned} \int_0^{\pi/4} \sec x \tan x \, dx &= \sec x 0^{\pi/4} \\ &= \sec \frac{\pi}{4} - \sec 0 \\ &= \sqrt{2} - 1. \end{aligned}$$