

On a Problem of Oppenheim concerning "Factorisatio Numerorum"

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Communicated by P. T. Bateman

Received January 14, 1981

Let $f(n)$ denote the number of factorizations of the natural number n into factors larger than 1 where the order of the factors does not count. We say n is "highly factorable" if $f(m) < f(n)$ for all $m < n$. We prove that $f(n) = n \cdot L(n)^{-1+o(1)}$ for n highly factorable, where $L(n) = \exp\{\log n \log \log \log n / \log \log n\}$. This result corrects the 1926 paper of Oppenheim where it is asserted that $f(n) = n \cdot L(n)^{-2+o(1)}$. Some results on the multiplicative structure of highly factorable numbers are proved and a table of them up to 10^9 is provided. Of independent interest, a new lower bound is established for the function $\Psi(x, y)$, the number of $n \leq x$ free of prime factors exceeding y .

1. INTRODUCTION

Let $f(n)$ denote the number of factorizations of the natural number n into factors larger than 1, where the order of the factors does not count. Also let $f(1) = 1$. Thus, for example, $f(12) = 4$ since 12 has the factorizations

$$12, \quad 2 \cdot 6, \quad 3 \cdot 4, \quad 2 \cdot 2 \cdot 3.$$

*Research supported in part by grants from the National Science Foundation.

In this paper we establish a rather accurate estimate for the maximal order of $f(n)$. Roughly, we show that this maximal order is $n \cdot L(n)^{-1+o(1)}$, where

$$L(n) = \exp(\log n \cdot \log_3 n / \log_2 n)$$

and $\log_k n$ denotes the k -fold iteration of the natural logarithm. For a more explicit determination of the " $o(1)$," see our theorems in Sections 2, 4, and 5.

In [13], Oppenheim also considered the problem of the maximal order of $f(n)$, but he erroneously claimed that it was $n \cdot L(n)^{-2+o(1)}$. His error arose when he assumed uniformity in k for his estimation of the maximal order of the Piltz divisor function $d_k(n)$, the number of factorizations of n into exactly k positive factors with order counting.

We present two different proofs that there is an infinite set of n with $f(n) \geq n \cdot L(n)^{-1+o(1)}$. In the first proof (Theorem 2.1), we show that the average value of $f(n)$ for $n \leq x$ with n divisible by only very small prime factors is $x \cdot L(x)^{-1+o(1)}$. Our proof requires an accurate lower bound for the function $\Psi(z, y)$ when y is about $e^{\sqrt{\log z}}$. Here

$$\Psi(z, y) = \#\{n: 1 \leq n \leq z, P(n) \leq y\},$$

where $P(n)$ denotes the largest prime factor of n when $n > 1$, $P(1) = 1$, and where $\#A$ denotes the cardinality of the set A . Although there is a large literature on $\Psi(z, y)$, little is known about lower bounds when

$$e^{(\log z)^\epsilon} < y < e^{(\log z)^{5/8}}.$$

In Section 3 we establish a lower bound for $\Psi(z, y)$ that agrees closely with the known upper bound if $y > (\log z)^{1+\epsilon}$.

In Section 4 we present a second proof that the maximal order of $f(n)$ is at least $n \cdot L(n)^{-1+o(1)}$. We accomplish this by explicitly exhibiting integers with many factorizations. These integers have a somewhat prohibitive structure. More "natural" candidates, like the product of the primes up to k , or $k!$, or the least common multiple of the integers up to k , do not work. (We can show $f(n) = n \cdot L(n)^{-2+o(1)}$ for the first and last sequences. For $n = k!$, we have $f(n) = n \cdot L(n)^{(-1+o(1)) \log_3 n}$.) To get lower estimates for $f(n)$, we use the relationship, also exploited by Oppenheim, between $f(n)$ and $d_k(n)$. While Theorem 4.1 has the advantage of being constructive, Theorem 2.1 has its own advantage in that the result holds for the smaller function $f_0(n)$ which counts only factorizations of n into distinct factors.

In Section 5 we show that $f(n) \leq n \cdot L(n)^{-1+o(1)}$ for all n . Our proof employs a common trick that Rankin [15] and de Bruijn [2, Part II] also used to study $\Psi(x, y)$. The proof also uses the formula

$$\sum_{P(n) \leq y} f(n) n^{-s} = \prod_{\substack{P(n) \leq y \\ n > 1}} (1 - n^{-s})^{-1}, \quad (1.1)$$

which is a generalization of a formula of McMahon [11] who had no restriction on $P(n)$ on either side of the equation. Our formula is certainly valid for all s in the half plane $\operatorname{Re} s > 0$, but we shall only use it for s real and $\frac{1}{2} < s < 1$.

We say that a natural number n is *highly factorable* if $f(m) < f(n)$ for all m , $1 \leq m < n$. There is an obvious analogy with the highly composite numbers n of Ramanujan [14] which satisfies $d(m) < d(n)$ for all m , $1 \leq m < n$. It is obvious that if $n > 1$ is highly factorable, then there is some $t \geq 1$ with

$$n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}, \quad a_1 \geq a_2 \geq \cdots \geq a_t \geq 1,$$

where p_i denotes the i th prime. In Section 6 we show that $p_t > (\log n)^{1-\delta}$ for any $\delta > 0$ and all sufficiently large highly factorable n . It follows, of course, from the prime number theorem that $p_t \leq (1 + o(1)) \log n$. We also show that $p_t^2 \nmid n$, if n is sufficiently large.

It is not particularly easy to compute $f(n)$. For example, to find that $f(1800) = 137$ takes some work. In Section 7 we present an algorithm for the computation of $f(n)$. We have used this algorithm (on a computer) to find all of the highly factorable numbers below 10^9 . These numbers are listed in Table I.

We are able to show that the number of values of $f(n)$ that do not exceed x is $x^{o(1)}$, but we do not include the details here.

We now mention some related results. Oppenheim [13] also considered the average value of $f(n)$, showing

$$\frac{1}{x} \sum_{n \leq x} f(n) \sim \frac{e^{2\sqrt{\log x}}}{2\sqrt{\pi} (\log x)^{3/4}}.$$

This result was independently obtained by Szekeres and Turán [17].

There is a second function connected with the name "Factorisatio Numerorum," namely $F(n)$, the number of factorizations of n into factors larger than 1, where now different permutations of the same factorization are counted as different factorizations. Thus $F(12) = 8$ since 12 has the factorizations

$$12, \quad 2 \cdot 6, \quad 3 \cdot 4, \quad 4 \cdot 3, \quad 6 \cdot 2, \quad 2 \cdot 2 \cdot 3, \quad 2 \cdot 3 \cdot 2, \quad 3 \cdot 2 \cdot 2.$$

Kalmár [9] showed that

$$\sum_{n \leq x} F(n) \sim \frac{-x^\rho}{\rho \zeta'(\rho)},$$

where $\zeta(s)$ is the Riemann zeta functions and $\rho > 1$ is such that $\zeta(\rho) = 2$. Other papers on $F(n)$ are by Erdős [3], Evans [4], Hille [7], Ikehara [8], and Kalmár [9].

TABLE I: HIGHLY FACTORABLE INTEGERS BELOW 10^9

n	number of factorizations of n	exponents in the prime decomposition of n
1	1	none
4	2	2
8	3	3
12	4	2 1
16	5	4
24	7	3 1
36	9	2 2
48	12	4 1
72	16	3 2
96	19	5 1
120	21	3 1 1
144	29	4 2
192	30	6 1
216	31	3 3
240	38	4 1 1
288	47	5 2
360	52	3 2 1
432	57	4 3
480	64	5 1 1
576	77	6 2
720	98	4 2 1
960	105	6 1 1
1080	109	3 3 1
1152	118	7 2
1440	171	5 2 1
2160	212	4 3 1
2880	289	6 2 1
4320	382	5 3 1
5040	392	4 2 1 1
5760	467	7 2 1
7200	484	5 2 2
8640	662	6 3 1
10080	719	5 2 1 1
11520	737	8 2 1
12960	783	5 4 1
14400	843	6 2 2
15120	907	4 3 1 1
17280	1097	7 3 1
20160	1261	6 2 1 1
25920	1386	6 4 1
28800	1397	7 2 2
30240	1713	5 3 1 1
34560	1768	8 3 1

TABLE I: HIGHLY FACTORABLE INTEGERS BELOW 10^9

n	number of factorizations of n	exponents in the prime decomposition of n
40320	2116	7 2 1 1
50400	2179	5 2 2 1
51840	2343	7 4 1
60480	3079	6 3 1 1
80640	3444	8 2 1 1
90720	3681	5 4 1 1
100800	3930	6 2 2 1
120960	5288	7 3 1 1
151200	5413	5 3 2 1
161280	5447	9 2 1 1
172800	5653	8 3 2
181440	6756	6 4 1 1
201600	6767	7 2 2 1
241920	8785	8 3 1 1
302400	10001	6 3 2 1
362880	11830	7 4 1 1
453600	12042	5 4 2 1
483840	14166	9 3 1 1
604800	17617	7 3 2 1
725760	20003	8 4 1 1
907200	22711	6 4 2 1
1088640	24270	7 5 1 1
1209600	29945	8 3 2 1
1451520	32789	9 4 1 1
1814400	40774	7 4 2 1
2177280	41702	8 5 1 1
2419200	49320	9 3 2 1
2903040	52412	10 4 1 1
3326400	54613	6 3 2 1 1
3628800	70520	8 4 2 1
4838400	79177	10 3 2 1
5322240	79459	9 3 1 1 1
5443200	86222	7 5 2 1
6652800	99235	7 3 2 1 1
7257600	118041	9 4 2 1
9676800	124207	11 3 2 1
9979200	129296	6 4 2 1 1
10886400	151500	8 5 2 1
13305600	173377	8 3 2 1 1
14515200	192371	10 4 2 1
18144000	199668	8 4 3 1
19958400	239312	7 4 2 1 1
21772800	257381	9 5 2 1
25401600	259906	8 4 2 2

TABLE I: HIGHLY FACTORABLE INTEGERS BELOW 10^9

n	number of factorizations of n	exponents in the prime decomposition of n
26611200	292951	9 3 2 1 1
29030400	306091	11 4 2 1
31933440	313907	10 4 1 1 1
36288000	340413	9 4 3 1
39916800	425240	8 4 2 1 1
43545600	425254	10 5 2 1
50803200	443995	9 4 2 2
53222400	481392	10 3 2 1 1
59875200	525030	7 5 2 1 1
72576000	564234	10 4 3 1
76204800	574761	8 5 2 2
79833600	729916	9 4 2 1 1
101606400	737393	10 4 2 2
106444800	771932	11 3 2 1 1
119750400	947375	8 5 2 1 1
152409600	996347	9 5 2 2
159667200	1217160	10 4 2 1 1
199584000	1262260	8 4 3 1 1
217728000	1279554	10 5 3 1
239500800	1649624	9 5 2 1 1
279417600	1653287	8 4 2 2 1
304819200	1677259	10 5 2 2
319334400	1978932	11 4 2 1 1
399168000	2205059	9 4 3 1 1
479001600	2787810	10 5 2 1 1
558835200	2894057	9 4 2 2 1
638668800	3148035	12 4 2 1 1
718502400	3470553	9 6 2 1 1
798336000	3737489	10 4 3 1 1
838252800	3786089	8 5 2 2 1
958003200	4590111	11 5 2 1 1

The function $f(n)$ is related to the concept of partitions of a multiset (or multipartite partitions). For example, $f(2^n) = p(n)$, the number of numerical partitions of n , and $f(p_1 p_2 \dots p_n) = B_n$, the n th Bell number, that is, the number of partitions of an n -element set. In general $f(p_1^{a_1} p_2^{a_2} \dots p_n^{a_n})$ is the number of partitions of the multiset which has a_i copies of p_i for each i (or equivalently, the number of partitions of the vector (a_1, \dots, a_n) into lattice point summands (b_1, \dots, b_n) with each $b_i \geq 0$). There is a large literature on the subject of partitions of a multiset. The interested reader is referred to Section P64 of W. J. Leveque's "Reviews in Number Theory." Our algorithm in Section 7 for the computation of $f(n)$ appears to be the first practical algorithm for computing the number of partitions of a multiset.

Throughout the paper the letters p and q always denote primes. Also we shall let $\log_k^j x$ denote $(\log_k x)^j$, where \log_k represents the k -fold iteration of the natural logarithm. We shall continue to let $P(n)$ denote the largest prime factor of n if $n > 1$ and $P(1) = 1$.

2. A LOWER BOUND FOR THE MAXIMAL ORDER OF $f_0(n)$

Recall that $f_0(n)$ denotes the number of factorizations of n into distinct factors greater than 1, order of factors not counting.

THEOREM 2.1. *There is a constant C such that for infinitely many n ,*

$$f_0(n) \geq n \cdot \exp \left\{ -\frac{\log n}{\log_2 n} \left(\log_3 n + \log_4 n + \frac{\log_4 n - 1}{\log_3 n} + C \frac{\log_4^2 n}{\log_3^2 n} \right) \right\}.$$

Proof. Let x be large and let A denote the set of integers a , $1 < a \leq \exp(\log_2^2 x)$ with $P(a) \leq \log x / \log_2 x$. Then from the Corollary to Theorem 3.1 we have

$$\begin{aligned} \#A &= \Psi(\exp(\log_2^2 x), \log x / \log_2 x) - 1 \\ &= \exp \left\{ \log_2^2 x - \log_2 x \left(\log_3 x + \log_4 x - 1 + \frac{\log_4 x - 1}{\log_3 x} + O \left(\frac{\log_4^2 x}{\log_3^2 x} \right) \right) \right\}. \end{aligned}$$

Let $k = [\log x / \log_2^2 x]$ and let B denote the set of k -element subsets of A . Then

$$\begin{aligned} \#B &= \binom{\#A}{k} \geq \left(\frac{\#A}{k} \right)^k \\ &> \frac{1}{\#A} \left(\frac{\#A}{\log x / \log_2^2 x} \right)^{\log x / \log_2^2 x} \\ &= x \cdot \exp \left\{ -\frac{\log x}{\log_2^2 x} \left(\log_3 x + \log_4 x + \frac{\log_4 x - 1}{\log_3 x} + O \left(\frac{\log_4^2 x}{\log_3^2 x} \right) \right) \right\}. \end{aligned}$$

Consider the mapping $\Pi: B \rightarrow \mathbb{Z}$, where if $S \in B$, then $\Pi(S)$ is the product of the members of S . Note that

$$0 < \Pi(S) \leq x \quad \text{and} \quad P(\Pi(S)) \leq \log x / \log_2 x.$$

Moreover S corresponds to a factorization of $\Pi(S)$ into exactly k distinct factors. Thus

$$\sum_{\substack{n \leq x \\ P(n) < \log x / \log_2 x}} f_0(n) \geq \sum_{n \in \Pi(B)} f_0(n) \geq \#B.$$

We conclude that there is an $n \leq x$ with

$$f_0(n) \geq \#B / \Psi(x, \log x / \log_2 x).$$

But Theorem 1 in de Bruijn [2, Part II] contains the assertion that

$$\Psi(x, \log x / \log_2 x) = \exp\{(1 + o(1)) \log x \cdot \log_3 x / \log_2^2 x\}.$$

Thus there is an $n \leq x$ with

$$f_0(n) \geq x \cdot \exp \left\{ -\frac{\log x}{\log_2 x} \left(\log_3 x + \log_4 x + \frac{\log_4 x - 1}{\log_3 x} + O\left(\frac{\log_4^2 x}{\log_3^2 x}\right) \right) \right\}, \quad (2.1)$$

which proves the theorem.

3. INTEGERS FREE OF LARGE PRIME FACTORS

If $u \geq 1$ is fixed, it is well known that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \Psi(x, x^{1/u}) = \rho(u) > 0, \quad (3.1)$$

where $\rho(u)$ is the Dickman-de Bruijn function. The best result in this direction is that if $x^2 + u^2 \rightarrow \infty$ subject to the constraint $1 \leq u \leq (\log x)^{3/8 - \epsilon}$, then $\Psi(x, x^{1/u}) \sim x\rho(u)$ (de Bruijn [2, Part I] plus the best known results on the error term in the prime number theorem). From de Bruijn [1] we have for $u \geq 3$

$$\rho(u) = \exp \left\{ -u \left(\log u + \log_2 u - 1 + \frac{\log_2 u - 1}{\log u} + O\left(\frac{\log_2^2 u}{\log^2 u}\right) \right) \right\}. \quad (3.2)$$

For each $u \geq 1$, let

$$D(u) = \inf_{x > 1} \frac{1}{x} \Psi(x, x^{1/u}).$$

Thus from (3.1) it follows that $0 < D(u) \leq \rho(u)$. We shall show in this section that the right side of (3.2) is also a valid estimation for $D(u)$.

There are at least two other papers where a lower bound for $\Psi(x, x^{1/u})$ is established. In [5], Fainleib shows that

$$\frac{1}{x} \Psi(x, x^{1/u}) \geq \exp \left\{ -u \left(\log u + \log_2 u - 1 + c \frac{\log_2 u}{\log u} \right) \right\}$$

for some absolute constant c and for $3 \leq u < \log x / \log_2 x$. His method is to use an asymptotic result (stated without proof) for certain differential delay equations that are similar to equations studied by Levin. In [6], Halberstam uses the Buchstab identity and an induction argument to show that for $3 \leq u < u_0(x)$

$$\frac{1}{x} \Psi(x, x^{1/u}) \geq 2e^{-10} \cdot \exp \{ -u(\log u + \log_2 u + \eta(u)) \},$$

where $\eta(u)$ is an explicit function that is asymptotic to $\log_2 u / \log u$. The function $u_0(x)$ is not explicitly given, but tracing it through the proof, we find that the Halberstam inequality is claimed only for a region where the asymptotic relation (3.1) is already known. However, it is possible to tighten the estimates in Halberstam's proof and establish his inequality for the larger region $3 \leq u \leq c \log x / (\log_2 x)^{5/3+\epsilon}$.

Our method of proof is to produce a succession of increasingly sharp estimates for $D(u)$ using the inequality

$$\Psi(x, x^{1/u}) \geq \sum_i \Psi(x/m_i, w),$$

where the m_i run over certain integers composed solely of primes in the interval $(w, x^{1/u}]$ and where $w \approx x^{(1-\epsilon)/u}$. We begin with a crude estimate that is essentially implicit in de Bruijn [2, Part II].

LEMMA. *There is a constant c_1 such that if $u \geq c_1$ and $x \geq 1$, then*

$$\Psi(x, x^{1/u}) > x/u^{3u}.$$

Proof. Since $\Psi(x, x^{1/u}) \geq 1$, the result is trivial if $u^{3u} > x$. So assume $x \geq u^{3u}$. From what we have said above, we also may assume $u > (\log x)^{3/8-\epsilon}$ (if u is sufficiently large).

Thus, we suppose $c_1 \leq u$, $(\log x)^{3/8-\varepsilon} < u$, $u^{3u} \leq x$. Then $x^{1/u} \geq c_1^3$, so that

$$\pi(x^{1/u}) > ux^{1/u}/(2 \log x),$$

if c_1 is large enough. Let $\pi'(y)$ denote $\pi(y)$ if $y \geq 2$ and $\pi'(y) = 1$ otherwise. Let $u = m + \theta$, where $m = [u]$. We evidently have

$$\begin{aligned} \Psi(x, x^{1/u}) &\geq \pi(x^{1/u})^m \pi'(x^{\theta/u})/(m+1)! \\ &> \left(\frac{ux^{1/u}}{2 \log x}\right)^m \left(\frac{x^{\theta/u}}{2 \log x}\right)/u^m \\ &= x/(2 \log x)^{m+1} \\ &\geq x \cdot \exp\{-(u+1)(\log_2 x + \log 2)\} \\ &> x \cdot \exp\{-3u \log u\} = x/u^{3u}, \end{aligned}$$

where the last inequality is valid for $u > (\log x)^{3/8-\varepsilon}$ and u sufficiently large.

THEOREM 3.1. *If $x \geq 1$ and $u \geq 3$, we have*

$$\Psi(x, x^{1/u}) \geq x \cdot \exp \left\{ -u \left(\log u + \log_2 u - 1 + \frac{\log_2 u - 1}{\log u} + C \frac{\log^2 u}{\log^2 u} \right) \right\},$$

where C is an absolute constant.

Proof. It suffices to show the theorem for all $u \geq c_2$, where c_2 is an arbitrary absolute constant. Since $\Psi(x, x^{1/u}) \geq 1$, we may assume

$$x > u^u. \quad (3.3)$$

Consider the intervals

$$I_j = (x^{(1/u)(1-(k+1-j)/\log^3 u)}, x^{(1/u)(1-(k-j)/\log^3 u)})$$

for $j = 1, \dots, k$, where $k = [\log^2 u \log_2 u]$. Let

$$\alpha_j = \frac{\exp(1/\log^2 u) - 1}{\exp(k/\log^2 u) - 1} \exp((j-1)/\log^2 u)$$

for $j = 1, \dots, k$. Note that

$$\begin{aligned} \exp(k/\log^2 u) &= \exp(\log_2 u + O(1/\log^2 u)) \\ &= \log u + O(1/\log u). \end{aligned} \quad (3.4)$$

Let $m_{j,1}, m_{j,2}, \dots$, denote the integers composed of exactly $[a_j u]$ primes (not

necessarily distinct) from I_j . Let m_1, m_2, \dots , denote the integers of the form $m_{1,i_1} m_{2,i_2} \dots m_{k,i_k}$. Then we evidently have

$$\Psi(x, x^{1/u}) \geq \sum_i \Psi(x/m_i, w), \quad w = x^{(1/u)(1-(k/\log^3 u))}. \quad (3.5)$$

Note that for each m_i ,

$$\frac{\log m_i}{\log x} \geq \sum_{j=1}^k \frac{[\alpha_j u]}{u} \left(1 - \frac{k+1-j}{\log^3 u}\right) = \sum_j \alpha_j \left(1 - \frac{k+1-j}{\log^3 u}\right) + O\left(\frac{k}{u}\right). \quad (3.6)$$

Now

$$\sum_j \alpha_j = 1 \quad (3.7)$$

and from (3.4),

$$\begin{aligned} \sum_j \alpha_j (k+1-j) &= \alpha_1 \sum_j (k+1-j) \exp((j-1)/\log^2 u) \\ &= \alpha_1 \left\{ \exp\left(\frac{1}{\log^2 u}\right) \cdot \left(\exp\left(\frac{k}{\log^2 u}\right) - 1\right) \right. \\ &\quad \left. - k \left(\exp\left(\frac{1}{\log^2 u}\right) - 1\right) \right\} / \left\{ \exp\left(\frac{1}{\log^2 u}\right) - 1 \right\}^2 \\ &= \frac{\exp(1/\log^2 u)}{\exp(1/\log^2 u) - 1} - \frac{k}{\exp(k/\log^2 u) - 1} \\ &= \log^2 u \cdot \left(1 + O\left(\frac{1}{\log^2 u}\right)\right) - \frac{\log^2 u \log_2 u}{\log u - 1 + O(1/\log u)} \\ &= \log^2 u - \log u \log_2 u + O(\log_2 u). \end{aligned} \quad (3.8)$$

Thus from (3.6)–(3.8) we have

$$\frac{\log m_i}{\log x} \geq 1 - \frac{1}{\log u} + \frac{\log_2 u}{\log^2 u} + O\left(\frac{\log_2 u}{\log^3 u}\right).$$

Since $x/m_i > 1$, we may define v_i so that $w = (x/m_i)^{1/v_i}$, that is,

$$\begin{aligned} v_i &= \frac{\log(x/m_i)}{\log w} \\ &\leq \left\{ \frac{1}{\log u} - \frac{\log_2 u}{\log^2 u} + O\left(\frac{\log_2 u}{\log^3 u}\right) \right\} / \left\{ \frac{1}{u} \left(1 - \frac{\log_2 u}{\log u} + O\left(\frac{1}{\log^3 u}\right)\right) \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{u}{\log u} \left\{ 1 - \frac{\log_2 u}{\log u} + O\left(\frac{\log_2 u}{\log^2 u}\right) \right\} \cdot \left\{ 1 + \frac{\log_2 u}{\log u} + O\left(\frac{\log_2^2 u}{\log^2 u}\right) \right\} \\
&= \frac{u}{\log u} \left(1 + O\left(\frac{\log_2^2 u}{\log^2 u}\right) \right).
\end{aligned}$$

Thus if we let $v = \max\{v_i\}$, we have

$$v \leq \frac{u}{\log u} \left(1 + O\left(\frac{\log_2^2 u}{\log^2 u}\right) \right). \quad (3.9)$$

Since $w \geq (x/m_i)^{1/v}$, we have from (3.5) that

$$\Psi(x, x^{1/u}) \geq \sum_i \Psi(x/m_i, (x/m_i)^{1/v}) \geq xD(v) \sum_i 1/m_i. \quad (3.10)$$

It remains to estimate $D(v)$ and $\sum 1/m_i$. For the latter, note that

$$\sum_i \frac{1}{m_i} = \prod_{j=1}^k \sum_i \frac{1}{m_{j,i}} \geq \prod_j \left(\left(\sum_{p \in I_j} \frac{1}{p} \right)^{[\alpha_j u]} / [\alpha_j u]! \right). \quad (3.11)$$

Now from (3.3)

$$\sum_{p \in I_j} \frac{1}{p} = \log \log x^{(1/u)(1-(k-j)/\log^3 u)} - \log \log x^{(1/u)(1-(k+1-j)/\log^3 u)},$$

$$+ O\left(\left(\frac{u}{\log x}\right)^{10}\right)$$

$$= \log \left(1 - \frac{k-j}{\log^3 u} \right) - \log \left(1 - \frac{k+1-j}{\log^3 u} \right) + O\left(\frac{1}{\log^{10} u}\right)$$

$$= \log \left\{ \left(1 - \frac{k+1-j}{\log^3 u} + \frac{1}{\log^3 u} \right) / \left(1 - \frac{k+1-j}{\log^3 u} \right) \right\}$$

$$+ O\left(\frac{1}{\log^{10} u}\right)$$

$$= \frac{1}{\log^3 u} \left(1 + \frac{k+1-j}{\log^3 u} \right) + O\left(\frac{\log_2^2 u}{\log^5 u}\right).$$

Then using (3.7) and (3.8) we have

$$\begin{aligned}
 & \log \left(\prod_j \left(\sum_{p \in I_j} \frac{1}{p} \right)^{[\alpha_j u]} \right) \\
 &= \sum_j [\alpha_j u] \log \left(\sum_{p \in I_j} \frac{1}{p} \right) \\
 &= \sum_j [\alpha_j u] \left(-3 \log_2 u + \frac{k+1-j}{\log^3 u} + O \left(\frac{\log_2^2 u}{\log^2 u} \right) \right) \\
 &= u \sum_j \alpha_j \left(-3 \log_2 u + \frac{k+1-j}{\log^3 u} + O \left(\frac{\log_2^2 u}{\log^2 u} \right) \right) + O(k \log_2 u) \\
 &= u \left(-3 \log_2 u + \frac{1}{\log u} + O \left(\frac{\log_2^2 u}{\log^2 u} \right) \right). \tag{3.12}
 \end{aligned}$$

From (3.7) and Stirling's formula, we have

$$\begin{aligned}
 \log \left(\prod_j [\alpha_j u]! \right) &= \sum_j [\alpha_j u] (\log [\alpha_j u] - 1) + O(k \log u) \\
 &= \sum_j \alpha_j u (\log(\alpha_j u) - 1) + O(k \log u) \\
 &= u \left(\log u - 1 + \sum_j \alpha_j \log \alpha_j \right) + O(k \log u). \tag{3.13}
 \end{aligned}$$

To estimate this last sum, we use (3.4), (3.7), (3.8) to get

$$\begin{aligned}
 \sum_j \alpha_j \log \alpha_j &= \sum_j \alpha_j \left(\log \alpha_1 + \frac{j-1}{\log^2 u} \right) \\
 &= \log \alpha_1 - \frac{1}{\log^2 u} \sum_j \alpha_j (k+1-j) + \frac{k}{\log^2 u} \sum_j \alpha_j \\
 &= \log \alpha_1 - 1 + \frac{\log_2 u}{\log u} + O \left(\frac{\log_2 u}{\log^2 u} \right) + \frac{k}{\log^2 u} \\
 &= \log \left(\exp \left(\frac{1}{\log^2 u} \right) - 1 \right) - \log \left(\exp \left(\frac{k}{\log^2 u} \right) - 1 \right) \\
 &\quad - 1 + \frac{\log_2 u}{\log u} + \log_2 u + O \left(\frac{\log_2 u}{\log^2 u} \right) \\
 &= -2 \log_2 u - \log_2 u + \frac{1}{\log u} - 1 + \frac{\log_2 u}{\log u} + \log_2 u + O \left(\frac{\log_2 u}{\log^2 u} \right) \\
 &= -2 \log_2 u - 1 + \frac{\log_2 u + 1}{\log u} + O \left(\frac{\log_2 u}{\log^2 u} \right).
 \end{aligned}$$

With this result and (3.13), we have

$$\log \left(\prod_j [\alpha_j u]! \right) = u \left(\log u - 2 \log_2 u - 2 + \frac{\log_2 u + 1}{\log u} + O \left(\frac{\log_2^2 u}{\log^2 u} \right) \right). \quad (3.14)$$

Thus following from (3.10)–(3.12) and (3.14) we have

$$\Psi(x, x^{1/u}) \geq x D(v) \cdot \exp \left\{ -u \left(\log u + \log_2 u - 2 + \frac{\log_2 u}{\log u} + O \left(\frac{\log_2^2 u}{\log^2 u} \right) \right) \right\}. \quad (3.15)$$

From the lemma and (3.9) we have for large u

$$\log D(v) \geq -3v \log v \geq -3u,$$

so that (3.15) becomes

$$\Psi(x, x^{1/u}) \geq x \cdot \exp \{ -u (\log u + \log_2 u + O(1)) \}.$$

Using this result with (3.9) we have

$$\begin{aligned} \log D(v) &\geq -v(\log v + \log_2 v + O(1)) \\ &\geq -\frac{u}{\log u} \left(1 + O \left(\frac{\log_2^2 u}{\log^2 u} \right) \right) (\log u + O(1)) \\ &= -u \left(1 + O \left(\frac{1}{\log u} \right) \right), \end{aligned}$$

so that from (3.15) we now obtain

$$\Psi(x, x^{1/u}) \geq x \cdot \exp \left\{ -u \left(\log u + \log_2 u - 1 + \frac{\log_2 u}{\log u} + O \left(\frac{1}{\log u} \right) \right) \right\}.$$

We iterate our procedure one more time using this last result with (3.9) to get

$$\begin{aligned} \log D(v) &\geq -v \left(\log v + \log_2 v - 1 + O \left(\frac{\log_2 v}{\log v} \right) \right) \\ &\geq -\frac{u}{\log u} \left(1 + O \left(\frac{\log_2^2 u}{\log^2 u} \right) \right) \left(\log u - 1 + O \left(\frac{\log_2 u}{\log u} \right) \right) \\ &= -u \left(1 - \frac{1}{\log u} + O \left(\frac{\log_2^2 u}{\log^2 u} \right) \right), \end{aligned}$$

so that (3.15) at last gives

$$\Psi(x, x^{1/u}) \geq x \cdot \exp \left\{ -u \left(\log u + \log_2 u - 1 + \frac{\log_2 u - 1}{\log u} + O \left(\frac{\log_2^2 u}{\log^2 u} \right) \right) \right\},$$

which was to be shown.

COROLLARY. *If $\varepsilon > 0$ is arbitrary and $3 \leq u \leq (1 - \varepsilon) \log x / \log_2 x$, then*

$$\Psi(x, x^{1/u}) = x \cdot \exp \left\{ -u \left(\log u + \log_2 u - 1 + \frac{\log_2 u - 1}{\log u} + E(x, u) \right) \right\},$$

where

$$|E(x, u)| \leq c_\varepsilon \frac{\log_2^2 u}{\log^2 u},$$

where c_ε is a constant that depends only on the choice of ε .

Proof. Theorem 3.1 is half of the corollary. The other half follows from Theorem 2 in de Bruijn [2, Part II].

4. AN EXPLICIT EXAMPLE

In this section we explicitly describe an infinite set of integers, each of which has many factorizations.

THEOREM 4.1. *Let x be large and let*

$$\varepsilon = \frac{1}{\log_2 x} \left(\log_3 x + \log_4 x + \frac{\log_4 x}{\log_3 x} \right),$$

$$t = (1 + \varepsilon \log_2^2 x)^{1/\varepsilon}, \quad k = \log x / \log_2^2 x,$$

$$n = \prod_{p < t} p^{\lfloor kp^{\varepsilon-1} \rfloor}.$$

Then there is an absolute constant C such that

$$f(n) \geq n \cdot \exp \left\{ -\frac{\log n}{\log_2 n} \left(\log_3 n + \log_4 n + \frac{\log_4 n - 1}{\log_3 n} + C \frac{\log_4 n}{\log_3^2 n} \right) \right\}.$$

Proof. We first show that $\log n$ cannot be too much bigger than $\log x$. In fact, we show

$$\log n \leq \log x + O(\log x / \log_2^2 x). \tag{4.1}$$

To see this, note that

$$\log n \leq \sum_{p < t} kp^{\varepsilon-1} \log p. \quad (4.2)$$

Now if we let $\pi(s) = li(s) + \Delta(s)$, then

$$\begin{aligned} \sum_{p < t} p^{\varepsilon-1} \log p &= \int_{2^-}^t s^{\varepsilon-1} \log s \, d\pi(s) \\ &= \int_2^t s^{\varepsilon-1} \, ds + \int_{2^-}^t s^{\varepsilon-1} \log s \, d\Delta(s). \end{aligned} \quad (4.3)$$

We shall show that the last integral in (4.3) is $O(1)$. First note that

$$\begin{aligned} \log t &= \frac{1}{\varepsilon} \left(\log \varepsilon + 2 \log_3 x + O \left(\frac{1}{\log_2 x \log_3 x} \right) \right) \\ &= \frac{\log_2 x}{\log_3 x + \log_4 x + \log_4 x / \log_3 x} \\ &\quad \times \left(\log_3 x + \log_4 x + \frac{\log_4 x}{\log_3 x} - \frac{\log_4^2 x}{2 \log_3^2 x} + O \left(\frac{\log_4 x}{\log_3^2 x} \right) \right) \\ &= \log_2 x \left(1 - \frac{\log_4^2 x}{2 \log_3^2 x} + O \left(\frac{\log_4 x}{\log_3^2 x} \right) \right). \end{aligned} \quad (4.4)$$

With this estimate and the fact that $t^\varepsilon \sim \log_2 x \log_3 x$, we have for $2 \leq s \leq t$ and t large,

$$s^\varepsilon = (\log s)^{\varepsilon \log s / \log \log s} \leq (\log s)^{\varepsilon \log t / \log \log t} = (\log s)^{1+o(1)}.$$

Also using $|\Delta(s)| \ll s / \log^4 s$, we have

$$\begin{aligned} \int_{2^-}^t s^{\varepsilon-1} \log s \, d\Delta(s) &= t^{\varepsilon-1} \log t \Delta(t) - 2^{\varepsilon-1} \log 2 \Delta(2) \\ &\quad - \int_2^t s^{\varepsilon-2} ((\varepsilon-1) \log s + 1) \Delta(s) \, ds \\ &= O(1) + O \left(\int_2^t \frac{s^\varepsilon}{s \log^3 s} \, ds \right) \\ &= O \left(\int_2^t \frac{1}{s \log^{3/2} s} \, ds \right) \\ &= O(1). \end{aligned} \quad (4.5)$$

Using (4.5) in (4.3) we have

$$\begin{aligned}
 \sum_{p \leq t} p^{\varepsilon-1} \log p &= \int_2^t s^{\varepsilon-1} ds + O(1) \\
 &= \frac{1}{\varepsilon} t^{\varepsilon} - \frac{1}{\varepsilon} 2^{\varepsilon} + O(1) \\
 &= \frac{1}{\varepsilon} (t^{\varepsilon} - 1) + O(1) \\
 &= \log_2^2 x + O(1).
 \end{aligned} \tag{4.6}$$

Thus (4.1) follows from (4.2) and (4.6).

Recall that the Piltz divisor function $d_l(n)$ counts the number of factorizations of n into exactly l positive factors, where 1 is allowed as a factor and different permutations of a single factorization count separately. It is easily shown that $d_l(n)$ is multiplicative and that

$$d_l(p^a) = \binom{l+a-1}{a-1}.$$

Moreover, we evidently have for any choice of l that

$$f(n) \geq d_l(n)/l!.$$

Thus

$$\begin{aligned}
 \log f(n) &\geq \log d_{[k]}(n) - \log [k]! \\
 &= \sum_{p \leq t} \log \binom{[k] + [kp^{\varepsilon-1}] - 1}{[kp^{\varepsilon-1}] - 1} - \log [k]!.
 \end{aligned} \tag{4.7}$$

Now if $a, b \geq 2$, then

$$\log \binom{[a] + [b] - 1}{[b] - 1} = (a+b) \log(a+b) - a \log a - b \log b + O(\log(a+b))$$

so that

$$\begin{aligned}
 &\log \binom{[k] + [kp^{\varepsilon-1}] - 1}{[kp^{\varepsilon-1}] - 1} \\
 &= k(1+p^{\varepsilon-1})(\log k + \log(1+p^{\varepsilon-1})) - k \log k \\
 &\quad - kp^{\varepsilon-1}(\log k + (\varepsilon-1) \log p) + O(\log k) \\
 &= k(1+p^{\varepsilon-1}) \log(1+p^{\varepsilon-1}) + k(1-\varepsilon) p^{\varepsilon-1} \log p + O(\log k).
 \end{aligned} \tag{4.8}$$

Now

$$\begin{aligned}
 \sum_{p < t} (1 + p^{\varepsilon-1}) \log(1 + p^{\varepsilon-1}) &= \sum_{p < t} p^{\varepsilon-1} + O(1) \\
 &= \int_{2^-}^t s^{\varepsilon-1} d\pi(s) + O(1) \\
 &= \int_2^t \frac{s^{\varepsilon-1}}{\log s} ds + \int_{2^-}^t s^{\varepsilon-1} d\Delta(s) + O(1). \quad (4.9)
 \end{aligned}$$

The last integral is

$$\begin{aligned}
 t^{\varepsilon-1} \Delta(t) - 2^{\varepsilon-1} \Delta(2) - \int_2^t (\varepsilon - 1) s^{\varepsilon-2} \Delta(s) ds \\
 = O(1) + O\left(\int_2^t \frac{s^{\varepsilon}}{s \log^4 s} ds\right) = O(1)
 \end{aligned}$$

by (4.5). Also

$$\begin{aligned}
 \int_2^t \frac{s^{\varepsilon-1}}{\log s} ds &= \int_{2^\varepsilon}^{t^\varepsilon} \frac{du}{\log u} - li(t^\varepsilon) + O\left(\int_{2^\varepsilon}^{t^\varepsilon} \frac{du}{\log u}\right) \\
 &= \frac{t^\varepsilon}{\varepsilon \log t - 1} \left(1 + O\left(\frac{1}{\varepsilon^2 \log^2 t}\right)\right) + O(|\log \varepsilon|).
 \end{aligned}$$

Thus using (4.4), we have

$$\begin{aligned}
 \sum_{p < t} (1 + p^{\varepsilon-1}) \log(1 + p^{\varepsilon-1}) \\
 &= \frac{\log_2 x (\log_3 x + \log_4 x + \log_4 x / \log_3 x) (1 + O(1/\log_3^2 x))}{\log_3 x + \log_4 x + \log_4 x / \log_3 x - 1 + O(\log_4^2 x / \log_3^2 x)} \\
 &= \log_2 x \left(1 + \frac{1}{\log_3 x} + O\left(\frac{\log_4 x}{\log_3^2 x}\right)\right) \left(1 + O\left(\frac{1}{\log_3^2 x}\right)\right) \\
 &= \log_2 x \left(1 + \frac{1}{\log_3 x} + O\left(\frac{\log_4 x}{\log_3^2 x}\right)\right). \quad (4.10)
 \end{aligned}$$

Thus from (4.1), (4.6)–(4.8), and (4.10), we have

$$\begin{aligned}
 \log f(n) &\geq \frac{\log x}{\log_2 x} \left(1 + \frac{1}{\log_3 x} + O\left(\frac{\log_4 x}{\log_3^2 x}\right)\right) \\
 &\quad + \log x - \frac{\log x}{\log_2 x} \left(\log_3 x + \log_4 x + \frac{\log_4 x}{\log_3 x}\right) + O\left(\frac{\log x}{\log_2^2 x}\right)
 \end{aligned}$$

$$\begin{aligned}
& -\frac{\log x}{\log_2^2 x} (\log_2 x + O(\log_3 x)) \\
&= \log x - \frac{\log x}{\log_2 x} \left(\log_3 x + \log_4 x + \frac{\log_4 x - 1}{\log_3 x} + O\left(\frac{\log_4 x}{\log_3^2 x}\right) \right) \\
&\geq \log n - \frac{\log n}{\log_2 n} \left(\log_3 n + \log_4 n + \frac{\log_4 n - 1}{\log_3 n} + O\left(\frac{\log_4 n}{\log_3^2 n}\right) \right),
\end{aligned}$$

which proves the theorem.

5. AN UPPER BOUND FOR $f(n)$

In this section, to get an upper bound for $f(n)$, we employ a formula of MacMahon and a method that Rankin and de Bruijn used to get upper bounds for $\Psi(x, y)$.

THEOREM 5.1. *There is a constant C such that for all large n*

$$f(n) \leq n \cdot \exp \left\{ -\frac{\log n}{\log_2 n} \left(\log_3 n + \log_4 n + \frac{\log_4 n - 1}{\log_3 n} + C \frac{\log_4^2 n}{\log_3^2 n} \right) \right\}.$$

Proof. Since $f(n)$ depends only on the array of exponents in the prime factorization of n and not on the choice of the primes themselves, to prove the theorem it is sufficient to consider only integers n that are divisible by all the primes up to some point. Let $l(n) = \log n + \log n / \log_2^{10} n$. Since

$$\sum_{p \leq l(n)} \log p > \log n$$

for all large n , we may assume $P(n) \leq l(n)$. From (1.1) we have for any choice of $c > 0$,

$$f(n) \leq n^c \sum_{P(m) \leq l(n)} f(m)/m^c = n^c \prod_{\substack{P(m) \leq l(n) \\ m > 1}} (1 - m^{-c})^{-1}. \quad (5.1)$$

We shall choose

$$c = 1 - \frac{1}{\log_2 n} \left(\log_3 n + \log_4 n + \frac{\log_4 n - 1}{\log_3 n} - \frac{\log_4^2 n}{\log_3^2 n} \right).$$

Thus to prove the theorem it is sufficient to show that

$$A \stackrel{\text{def}}{=} \log \prod_{\substack{P(m) \leq l(n) \\ m > 1}} (1 - m^{-c})^{-1} = O\left(\frac{\log n}{\log_2 n} \cdot \frac{\log_4^2 n}{\log_3^2 n}\right). \quad (5.2)$$

Now

$$A = \sum_{P(m) \leq l(n)} m^{-c} + O(1) = \prod_{p \leq l(n)} (1 - p^{-c})^{-1} + O(1),$$

and

$$B \stackrel{\text{def}}{=} \log \prod_{p \leq l(n)} (1 - p^{-c})^{-1} = \sum_{p \leq l(n)} p^{-c} + O(1).$$

By an argument similar to (4.9) and the subsequent calculations we have

$$\begin{aligned} B &= \frac{l(n)^{1-c}}{(1-c)\log l(n) - 1} \{1 + O((1-c)^{-2} \log^{-2} l(n))\} + O(|\log(1-c)|) \\ &= \frac{\exp\{\log_3 n + \log_4 n + ((\log_4 n - 1)/\log_3 n) - (\log_4^2 n / \log_3^2 n)\}}{\log_3 n + \log_4 n - 1 + ((\log_4 n - 1)/\log_3 n) - (\log_4^2 n / \log_3^2 n)} \\ &\quad \times \left\{1 + O\left(\frac{1}{\log_3^2 n}\right)\right\} \\ &= \log_2 n \left[\exp\left\{\frac{\log_4 n - 1}{\log_3 n} - \frac{\log_4^2 n}{\log_3^2 n}\right\} / \left(1 + \frac{\log_4 n - 1}{\log_3 n} + O\left(\frac{\log_4 n}{\log_3^2 n}\right)\right) \right] \\ &= \log_2 n \left\{1 - \frac{\log_4^2 n}{2 \log_3^2 n} + O\left(\frac{\log_4 n}{\log_3^2 n}\right)\right\} \\ &< \log_2 n - \log_2^{1/2} n, \end{aligned}$$

for all large n . Thus

$$A = e^B + O(1) \leq (\log n) e^{-\log_2^{1/2} n} + O(1) = o\left(\frac{\log n \log_4^2 n}{\log_2 n \log_3^2 n}\right)$$

which establishes (5.2) and thus the theorem.

6. THE LARGEST PRIME FACTOR OF A HIGHLY FACTORABLE NUMBER AND OTHER PROBLEMS

If n is highly factorable (that is, $f(m) < f(n)$ for all m , $1 \leq m < n$) and n is large, then we saw in the proof of Theorem 5.1 that $P(n) < \log n +$

$\log n / \log_2^{10} n$. In this section we use Theorem 2.1 and the method of Theorem 5.1 to show that for each $\delta > 0$ we have $P(n) > (\log n)^{1-\delta}$ for all sufficiently large highly factorable numbers n .

THEOREM 6.1. *For all large highly factorable numbers n we have*

$$P(n) > (\log n)^{1-(\log_3 n)^{-2}} \tag{6.1}$$

Proof. Our strategy is to get an upper bound result for $f(n)$ for those n which do not satisfy (6.1). This upper bound will be smaller than our lower bound result for highly factorable numbers (Theorem 2.1). We then conclude that these n are not highly factorable.

Let $l(n) = (\log n)^{1-(\log_3 n)^{-2}}$. If $P(n) < l(n)$, then the argument of (5.1) shows that

$$f(n) \leq n^c \prod_{\substack{P(m) \leq l(n) \\ m > 1}} (1 - m^{-c})^{-1}$$

for any $c > 0$. We shall choose

$$c = 1 - \frac{1}{\log_2 n} \left(\log_3 n + \log_4 n + \frac{\log_4 n}{\log_3 n} - \frac{1}{2 \log_3 n} \right).$$

It thus follows from the proof of Theorem 2.1 (by applying (2.1) with x replaced by n) that n will not be highly factorable if

$$A \stackrel{\text{def}}{=} \log \prod_{\substack{P(m) \leq l(n) \\ m > 1}} (1 - m^{-c})^{-1} = o \left(\frac{\log n}{\log_2 n \log_3 n} \right). \tag{6.2}$$

As in Section 5 we may argue that

$$\log A = \frac{l(n)^{1-c}}{(1-c) \log l(n) - 1} (1 + O((1-c)^{-2} \log^{-2} l(n))) + O(|\log(1-c)|). \tag{6.3}$$

Now

$$(1-c) \log l(n) = \left(1 - \frac{1}{\log_2 n} \right) \left(\log_3 n + \log_4 n + \frac{\log_4 n}{\log_3 n} - \frac{1}{2 \log_3 n} \right),$$

so that

$$l(n)^{1-c} = \frac{\log_2 n \log_3 n (1 + \log_4 n / \log_3 n + 1/2 \log_3 n + O(\log_4^2 n / \log_3^2 n))}{(1 + 1/\log_3 n + O(\log_4 n / \log_3^2 n))}.$$

Thus from (6.3) we have

$$\begin{aligned} \log A &= \frac{\log_2 n (\log_3 n + \log_4 n - \frac{1}{2} + O(\log_4^2 n / \log_3 n))}{\log_3 n + \log_4 n + O(\log_4 n / \log_3 n)} \left(1 + O\left(\frac{1}{\log_3^2 n}\right) \right) \\ &= \log_2 n \left(1 - \frac{1}{2 \log_3 n} + O\left(\frac{\log_4 n}{\log_3^2 n}\right) \right), \end{aligned}$$

which gives (6.2).

The following lemma will help us prove that $P(n) \parallel n$ if n is a large highly factorable number.

LEMMA. *Suppose p, q are primes and n is an integer with $p^2 | n$, $p^2 \neq n$, $q \nmid n$. Then $f(qn/p) \geq \frac{6}{3} f(n)$.*

Proof. Let $\mathcal{E}(n)$ denote the set of factorizations of n . Thus an element $\varphi \in \mathcal{E}(n)$ is a multiset of integers exceeding 1 whose product is n . If $\varphi \in \mathcal{E}(n)$, let $|\varphi|_p$ denote the number of unequal factors in φ which are multiples of p . For example, if $p = 2$ and $\varphi = \{4, 4, 6, 11\}$ is a factorization of 1056, then $|\varphi|_2 = 2$.

Given $\varphi \in \mathcal{E}(n)$ we can transform φ into a factorization of qn/p by changing one p to a q . Thus φ corresponds to $|\varphi|_p$ different factorizations of qn/p . Moreover, every factorization of qn/p arises in exactly one way in this fashion. Thus

$$f(qn/p) = \sum_{\varphi \in \mathcal{E}(n)} |\varphi|_p.$$

Let $f_p(n) = \#\{\varphi \in \mathcal{E}(n) : |\varphi|_p = 1\}$. Thus

$$\begin{aligned} f(qn/p) &= f_p(n) + \sum_{\substack{\varphi \in \mathcal{E}(n) \\ |\varphi|_p \geq 2}} |\varphi|_p \geq f_p(n) + \sum_{\substack{\varphi \in \mathcal{E}(n) \\ |\varphi|_p \geq 2}} 2 \\ &= 2f(n) - f_p(n). \end{aligned} \tag{6.4}$$

Hence to prove the lemma it suffices to show $f_p(n) \leq \frac{4}{3} f(n)$.

Say $p^k \parallel n$. If $\varphi \in \mathcal{E}(n)$ and $|\varphi|_p = 1$, then for some $j | k$ and some d , φ contains k/j copies of the factor $p^j d$. Let A, B, C, D respectively denote the number of $\varphi \in \mathcal{E}(n)$ with $|\varphi|_p = 1$ and

- for A : φ contains k/j copies of $p^j d$, where $j > 1$ and $p^k d \neq p^2$,
- for B : φ contains k copies of pd , where $p^k d \neq p^2$,
- for C : φ contains p^2 ,
- for D : φ contains two copies of p .

Then $A + B + C + D = f_p(n)$. Note that $C = D = 0$ unless $k = 2$. We now show that each of B, C, D is at most A .

If $\varphi \in \mathcal{E}(n)$ is counted by B , we can let $\varphi' \in \mathcal{E}(n)$ be the same factorization except that the pd 's are consolidated into one factor $p^k d^k$. Then φ' is counted by A and the mapping $\varphi \rightarrow \varphi'$ is one to one, so $B \leq A$.

Suppose now $k = 2$ so that $C, D > 0$. Each type C factorization can have the p^2 consolidated with one of the other factors in φ (using $n \neq p^2$) to form a type A factorization. Thus $C \leq A$. Obviously $C = D$, so $D \leq A$ as well.

We now show that $A \leq f(n) - f_p(n)$. Indeed, if $\varphi \in \mathcal{E}(n)$ is counted by A , we let $\varphi' \in \mathcal{E}(n)$ be the same factorization except that one of the factors $p^j d$ is split into $p, p^{j-1}d$. It is evident that the mapping $\varphi \rightarrow \varphi'$ is one to one. Moreover, $|\varphi'|_p \geq 2$. For if $p = p^{j-1}d$, then $p^j d = p^2$ occurs at least twice in φ (if not, then φ would be a type C factorization), so that p^2 occurs at least once in φ' .

Thus

$$f_p(n) = A + B + C + D \leq 4A \leq 4f(n) - 4f_p(n),$$

so that $f_p(n) \leq \frac{4}{3}f(n)$ and $f(qn/p) \geq \frac{6}{5}f(n)$ from (6.4).

THEOREM 6.2. *There is an $\varepsilon > 0$ such that if n is a large highly factorable number and $(1 - \varepsilon)P(n) < p \leq P(n)$, then $p \parallel n$.*

Proof. Say n is a large highly factorable number with the prime factorization

$$n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}.$$

Say for some $p_s, (1 - \varepsilon)p_t < p_s \leq p_t$, we have $a_s \geq 2$. Let $k = [6 \log_2 n]$ and let

$$\gamma_k = \frac{p_{t+1} p_{t+2} \cdots p_{t+k}}{p_s p_{s-1} \cdots p_{s-k+1}}.$$

We now estimate γ_k . From Theorem 6.1, we have $p_t > (\log n)^{1-\delta}$, where $\delta > 0$ is small. Thus from the prime number theorem with error term, we have

$$p_{t+k} < p_t(1 + 1/\log_2 n), \quad p_{s-k} > (1 - \varepsilon - 1/\log_2 n)p_t.$$

Thus

$$\begin{aligned} \log \gamma_k &< k(\log(1 + 1/\log_2 n) - \log(1 - \varepsilon - 1/\log_2 n)) \\ &= k(-\log(1 - \varepsilon) + O(1/\log_2 n)) \\ &\leq -6 \log(1 - \varepsilon) \log_2 n + O(1). \end{aligned}$$

We now choose $\varepsilon = \frac{1}{7}$. Thus for large n we have

$$\gamma_k < (\log n)^{13/14} < (1 - \varepsilon) p_t < p_s.$$

Thus the integer $n' = n\gamma_k/p_s$ is smaller than n .

We now show $f(n') > f(n)$, thus contradicting the choice of n as a highly factorable number. Indeed, using the lemma k times we have

$$f(n\gamma_k) \geq \left(\frac{6}{5}\right)^k f(n).$$

Also, if $|\varphi|$ denotes the number of unequal factors in the factorization φ , we have, using the notation of the lemma,

$$\begin{aligned} f(n\gamma_k) &\leq \sum_{\varphi \in \mathcal{E}(n')} (|\varphi| + 1) \\ &\leq f(n') \cdot \max\{|\varphi| + 1 : \varphi \in \mathcal{E}(n')\} \\ &< f(n') \cdot \left(\frac{\log n}{\log 2} + 1\right). \end{aligned}$$

Thus

$$f(n') > \left(\frac{\log n}{\log 2} + 1\right)^{-1} f(n\gamma_k) \geq \left(\frac{\log n}{\log 2} + 1\right)^{-1} \left(\frac{6}{5}\right)^{\lfloor 6 \log_2 n \rfloor} f(n) > f(n).$$

This contradiction proves the theorem.

Remark. Our proof has us taking $\varepsilon = \frac{1}{7}$. Being a little more careful, we could actually choose any $\varepsilon < \frac{1}{6}$. Proving a better lemma will allow even larger choices for ε . Indeed, with more effort it is possible to replace the $\frac{6}{5}$ of the lemma with $2 - \delta$, where $\delta > 0$ is arbitrarily small, provided we assume m has many prime factors. With such an improved lemma, we could then prove Theorem 6.2 for any $\varepsilon < \frac{1}{2}$. We conjecture that this result is best possible, that is, that asymptotically 50% of the primes in a highly factorable number appear with exponent one.

We next might ask how many primes, if any, appear with exponent 2, 3, etc. We can prove that if $p^2 | m$, $q || m$, then $f(qm) > \left(\frac{3}{2} - \delta\right) f(pm)$ provided m has many prime factors. If our conjecture that asymptotically $\frac{1}{2}$ of the exponents are 1 is correct, then we can argue similarly as in Theorem 6.2 to show that there are asymptotically at least (and we conjecture at most) $\frac{1}{6}$ of the exponents equal to 2. Continuing with such a chain of conjectures, we conjecture that for each fixed k there are asymptotically exactly $1/k(k+1)$ of the exponents equal to k . Note that numbers of the form $n!$ also have this property. Also note that in Table I there are many numbers of the form $n!$ which are highly factorable, namely for $n = 1, 4, 5, 6, 7, 8, 9, 10, 11, 12$.

However, this is only a temporary phenomenon; that is, if n is sufficiently large, then $n!$ is not highly factorable. We know this, since we can show, using inequality (1.52) in Oppenheim [9], that

$$\log f(n!) = n \log n - (1 + o(1)) n \log_2^2 n,$$

while if $n!$ were highly factorable, then we would have

$$\log f(n!) = n \log n - (1 + o(1)) n \log_2 n.$$

It is somewhat a mystery to us why $n!$ has so few factorizations. Indeed if m is the product of the primes up to $n \log n - 2n$, then $m < n!$, m is of course square-free, and yet m has far more factorizations than $n!$ (if n is large). Probably the "fault" with $n!$ is that the exponents on the small primes are wastefully large. Another possibility is that our conjecture above that a large highly factorable number has asymptotically $1/k(k+1)$ of the exponents equal to k is wrong.

We now mention a few additional problems.

(1) From Table I we see that if n is highly factorable and $4 < n < 10^9$, then there is a prime p with n/p highly factorable. Does this remain true for all highly factorable numbers $n > 4$? For infinitely many? See Robin [16] for examples of highly composite numbers n such that n/p is not highly composite for all primes p .

(2) Let $N(x)$ denote the number of highly factorable numbers $n \leq x$. It is easy to see that $N(x) \gg \log x$ since if $n > 1$ is highly factorable and if n' is the next highly factorable number, then $n' \leq 2n$. Does $\log N(x)/\log \log x$ tend to a limit larger than 1? Can it at least be shown that there are quantities α, β with $1 < \alpha \leq \beta < \infty$ such that

$$\alpha < \log N(x)/\log \log x < \beta$$

for all large x ?

(3) If n, n' are consecutive highly factorable numbers, does $n'/n \rightarrow 1$? Does $f(n')/f(n) \rightarrow 1$?

(4) Find asymptotic formulas for the exponents on the small primes of a highly factorable number.

(5) A highly factorable number is a "champion" for the function $f(n)$. What do the champions for $f_0(n)$ or $F(n)$ look like? What is the maximal order of $F(n)$? Some work has been done on this: see Erdős [3], Evans [4], Hille [7], and Kalmár [9].

7. CALCULATION OF TABLE I.

In this section, we describe the algorithm used to determine the values displayed in Table I. If $n = \prod p_i^{a_i}$ (with $p_1 = 2$, $p_2 = 3$, etc.) is highly factorable, then we must have $a_1 \geq a_2 \geq \dots$. We calculated (by computer) $f(n)$ for each of the 1274 values of $n \leq 10^9$ whose prime exponents are monotone nonincreasing. Table I shows the 118 numbers found to be highly factorable. We have suppressed the values of $f(n)$ for n not highly factorable. We shall gladly send these values to any interested reader. (Knowing $f(n)$ for n satisfying $a_1 \geq a_2 \geq \dots$ and $n \leq 10^9$ allows one to readily determine $f(n)$ for any $n \leq 10^9$ and for infinitely many other n .)

The computational problem then is how to determine the number of partitions of a multiset \mathcal{M} having a_i copies of i , for $1 \leq i \leq k$. Our solution is to systematically generate each such partition, and count them in the process. To make the generation process systematic, we impose the structure of a rooted tree on the collection of all partitions of \mathcal{M} . The partitions are then enumerated by a standard tree-traversal algorithm of computer science called "preorder traversal"; for a description of this algorithm, see, for example, [10, p. 334]. Thus, our algorithm is specified by describing how the tree structure is imposed.

First, if B_1 and B_2 are submultisets of \mathcal{M} let us write " $B_1 \geq B_2$ " to mean that B_1 is lexicographically larger than B_2 , where B_1 and B_2 themselves are written with their elements in decreasing order. We agree to always write a partition π of \mathcal{M} with the blocks in order

$$\pi = (B_1, B_2, \dots, B_l), \quad B_1 \geq B_2 \geq \dots \geq B_l.$$

In the case where \mathcal{M} contains simply a_1 copies of 1, a partition is the usual notion of "numerical partition of the integer a_1 ," and the above convention agrees with the traditional way of writing numerical partitions.

Now let $\pi = (B_1, B_2, \dots, B_l)$ and $\pi' = (B'_1, B'_2, \dots, B'_{l-1})$ be two partitions of \mathcal{M} with l and $l-1$ blocks, respectively. Let us say that π' is an immediate offspring of π (or that π is the parent of π') provided these conditions are met: for some $j < l$,

- (i) $B'_i = B_i$ for all $i < j$,
- (ii) each of B'_i and B_i contains exactly one element for all $i > j$,
- (iii) $B'_j = B_j \cup B_k$ for some $k > j$, and the unique element of B_k is the smallest element of B'_j .

We check that with this definition every partition has a unique parent with one exception, namely the partition whose every block contains one element. This latter partition is the root of our tree. Finding a partition's parent is simple: with the blocks written in lexicographically decreasing order, remove

the smallest element from the rightmost nonsingleton block and let it become a singleton. Thus, for example, with $\mathcal{M} = \{3, 3, 3, 3, 2, 2, 2, 1, 1, 1\}$ the unique path from π to the root is given as follows:

$$\begin{aligned} \pi = & \{3, 3, 1\} \{3, 2, 1\} \{3, 2\} \{2, 1\} \\ & \{3, 3, 1\} \{3, 2, 1\} \{3, 2\} \{2\} \{1\} \\ & \{3, 3, 1\} \{3, 2, 1\} \{3\} \{2\} \{2\} \{1\} \\ & \{3, 3, 1\} \{3, 2\} \{3\} \{2\} \{2\} \{1\} \{1\} \\ & \{3, 3, 1\} \{3\} \{3\} \{2\} \{2\} \{2\} \{1\} \{1\} \\ & \{3, 3\} \{3\} \{3\} \{2\} \{2\} \{2\} \{1\} \{1\} \{1\} \\ & \{3\} \{3\} \{3\} \{3\} \{2\} \{2\} \{2\} \{1\} \{1\} \{1\}. \end{aligned}$$

In [12], MacMahon presents a table of values of $f(n)$ for those n which divide one of $2^{10} \cdot 3^8$, $2^{10} \cdot 3 \cdot 5$, $2^9 \cdot 3^2 \cdot 5$, $2^8 \cdot 3^3 \cdot 5$, $2^6 \cdot 3^2 \cdot 5^2$, $2^5 \cdot 3^3 \cdot 5^2$. There are four values of $f(n)$ which disagree with our computations. We double-checked our computations for these numbers by a different algorithm and have come to the conclusion that MacMahon's figures are in error. Specifically

$$\begin{aligned} f(2^{10} \cdot 3^5) &= 3804, & \text{not } 3737, \\ f(2^9 \cdot 3^8) &= 13715, & \text{not } 13748, \\ f(2^{10} \cdot 3^8) &= 21893, & \text{not } 21938, \\ f(2^4 \cdot 3 \cdot 5) &= 38, & \text{not } 28. \end{aligned}$$

The latter two discrepancies could have been typographical errors. MacMahon does not state how he prepared his table. He states as his "Cardinal Theorem" a formula for the generating function

$$\sum_{n_1=0}^{\infty} f(2^{n_1} 3^{n_2} \dots p_s^{n_s}) x^{n_1}$$

(the exponent on 2 is variable, all others fixed). The formula involves a summation over all factorizations of $3^{n_2} \dots p_s^{n_s}$, so it is probably not a better means of enumeration than what we have done.

ACKNOWLEDGMENTS

We take this opportunity to thank P. T. Bateman for his numerous suggested improvements and K. K. Norton who informed us of [5].

REFERENCES

1. N. G. DE BRUIJN, The asymptotic behaviour of a function occurring in the theory of primes, *J. Indian Math. Soc. (N.S.)* **15** (1951), 25–32.
2. N. G. DE BRUIJN, On the number of positive integers $\leq x$ and free of prime factors $> y$, *Nederl. Akad. Wetensch. Proc. Ser. A* **54** (1951), 50–60; II, **69** (1966), 239–247.
3. P. ERDÖS, On some asymptotic formulas in the theory of “factorisatio numerorum,” *Ann. of Math.* **42** (1941), 989–993; corrections to two of my papers, *Ann. of Math.* **44** (1943), 647–651.
4. R. EVANS, An asymptotic formula for extended Eulerian numbers, *Duke Math. J.* **41** (1974), 161–175.
5. A. S. FAINLEIB, On the estimation from below of the number of numbers with small prime factors, *Dokl. Akad. Nauk UzSSr* **7** (1967), 3–5. [Russian] (*MR* **46** (1973), #5265).
6. H. HALBERSTAM, On integers all of whose prime factors are small, *Proc. London Math. Soc. (3)* **21** (1970), 102–107.
7. E. HILLE, A problem in “Factorisatio Numerorum,” *Acta Arith.* **2** (1937), 134–144.
8. S. IKEHARA, On Kalmár’s problem in “Factorisatio Numerorum,” *Proc. Phys.-Math. Soc. Japan (3)* **21** (1939), 208–219; II, **23** (1941), 767–774.
9. L. KALMÁR, A “factorisatio numerorum” problémájáról, *Mat. Fiz. Lapok* **38** (1931), 1–15; Über die mittlere Anzahl der Produktdarstellungen der Zahlen. (Erste Mitteilung) *Acta Litt. Sci. Szeged* **5** (1931), 95–107.
10. D. E. KNUTH, “The Art of Computer Programming,” Vol. 1, Addison-Wesley, Reading, Mass., 1973.
11. P. A. MACMAHON, Dirichlet’s series and the theory of partitions, *Proc. London Math. Soc. (2)* **22** (1923), 404–411.
12. P. A. MACMAHON, The enumeration of the partitions of multipartite numbers, *Proc. Cambridge Philos. Soc.* **22** (1925), 951–963.
13. A. OPPENHEIM, On an arithmetic function, *J. London Math. Soc.* **1** (1926), 205–211; II, **2** (1927), 123–130.
14. S. RAMANUJAN, Highly composite numbers, *Proc. London Math. Soc. (2)* **14** (1915), 347–409.
15. R. A. RANKIN, The difference between consecutive prime numbers, *J. London Math. Soc.* **13** (1938), 242–247.
16. G. ROBIN, “Constructions of Highly Composite Numbers,” to appear.
17. G. SZEKERES AND P. TURÁN, Über das zweite Hauptproblem der “Factorisatio Numerorum,” *Acta Litt. Sci. Szeged* **6** (1933), 143–154.