This is the third and final issue of our series meant to assist in listening to Peter Sarnak's lecture on the Riemann Hypothesis available at the MSRI website. Parts I and II related to the first part of Sarnak's talk in which Sarnak discusses the history of the Riemann Hypothesis. In the second part of his talk, Sarnak discussed the remarkable connections between the zeros of the zeta function, energy levels in quantum theory, and the eigenvalues of random matrices. Here we will provide some background information for understanding these connections. This part was written jointly by Dan Rockmore and Laurie Snell with a lot of help with programming from Peter Kostelec and Charles Grinstead.

It all begins with a chance meeting at Princeton's Institute for Advanced Study between number theorist Hugh Montgomery, a visitor from the University of Michigan to the Institute's School of Mathematics, and Freeman Dyson, a member of the Institute's Faculty of the School of Natural Sciences, the same department which claimed Albert Einstein as one of its first members. Dyson is perhaps best known as the mathematical physicist who turned Feynman diagrams, the intuitive and highly personal computational machinery of Richard Feynman, into rigorous mathematics, thereby laying the mathematical foundations for a working theory of quantum electrodynamics.

Montgomery had been working for some time on various aspects of the Riemann Hypothesis. In particular he had been investigating the *pair correlation* of the zeros which we now describe.

Recall that the Prime Number Theorem states that, if $\pi(n)$ is the number of primes not exceeding n, then

$$\pi(n) \sim \frac{n}{\log(n)}. \tag{1}$$

It follows from this that if $p(n)$ is the nth prime then

$$p(n) \sim n\log(n). \tag{2}$$

A proof of (2) can be found in [1] page 10. Thus the primes get further apart and the nth prime becomes significantly bigger than n as $n \to \infty$. Exactly the opposite happens with the zeros of the zeta function.

Recall that the non-trivial zeros of the zeta function all lie in the critical strip of the complex plane:

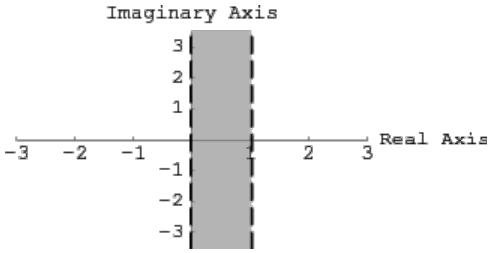

Figure 1: The critical strip.

Let $N(t)$ be the number of zeros in the upper half of the critical strip with imaginary part at most T. Of course, if the Riemann Hypothesis (RH) is true then the zeros in the critical strip all lie on the line $z = 1/2 + i\sigma$ Then the analog of the prime number theorem for N(T) is

$$N(T) \sim \frac{T\ \log(T)}{2\pi} \tag{3}$$

known apparently even to Riemann.

We now assume the RH and denote by $\gamma_1 \leq \gamma_2 \leq \gamma_3 \cdots \leq \gamma_n$ the imaginary parts of the first n zeros in the critical region. Then it follows from (3) that

$$\gamma_n \sim \frac{2\pi n}{\log n}. \tag{4}$$

For a proof of (4) see [2] section 9.3. Thus, while the primes get less dense as $n \to \infty$, the zeros of the zeta function get more dense as $T \to \infty$.

Montgomery studied the consecutive spacings $\delta_j = \lambda_{j+1} - \lambda_j$ between the zeros. He effectively spread them out to obtain normalized spacings $\hat{\delta}_j$ with mean spacing asymptotically 1. There are two ways that we can do this. We can normalize the zeros in such a way that the gaps will have their mean asymptotically 1 or we can normalize the gaps directly. Following Montgomery [3], Katz and Sarnak [5] use the first method normalizing the zeros by:

$$\hat{\gamma}_j = \frac{\gamma_j \ \log(\gamma_j)}{2\pi} \text{ for } j \ \geq 1$$

Then the normalized consecutive spacings $\hat{\delta}_j$ are defined by

$$\hat{\delta}_j = \hat{\gamma}_{j+1} - \hat{\gamma}_j \text{ for } j \ \geq 1.$$

Odlyzko uses the second method. He starts with the unnormalized consecutive spacings δ_j and normalized these by

$$\hat{\delta}_j = \delta_j \frac{\log(\frac{\gamma_j}{2\pi})}{2\pi}.$$

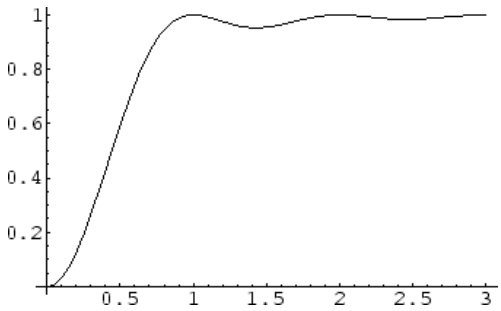

Figure 2: The pair correlation function in Montgomery's conjecture.

This normalization is arrived at by considering more accurate asymptotic expressions for $N(T)$. (See Odlyzko [4]).

Montgomery proved a series of results about zeros of the zeta function which led him to the following conjecture. Let α and β be two positive numbers with $\alpha < \beta$. Montgomery conjectured that

$$n^{-1}|(j,k) : 1 \leq j \leq n, k \geq 0, \hat{\delta}_j + \hat{\delta}_{j+1} + \cdots + \hat{\delta}_{j+k} \in [\alpha, \beta]| \sim \int_\alpha^\beta \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right) du \, .$$

where the left side of the equation uses the first n normalized zeros. Note that:

$$\hat{\delta}_j + \hat{\delta}_{j+1} + \cdots + \hat{\delta}_{j+k} = \hat{\gamma}_{j+k} - \hat{\gamma}_j.$$

The right side $\hat{\gamma}_{j+k} - \hat{\gamma}_j$ is called a kth-consecutive spacing. Thus, for the normalized spacings, we are counting the number of kth-consecutive spacings that lie in the interval $[\alpha, \beta]$. Of course this number can be bigger than 1. The function

$$1 - \left(\frac{\sin \pi u}{\pi u}\right)^2$$

is called the *pair correlation function*. In Figure 2 we give a graph of the pair correlation function.

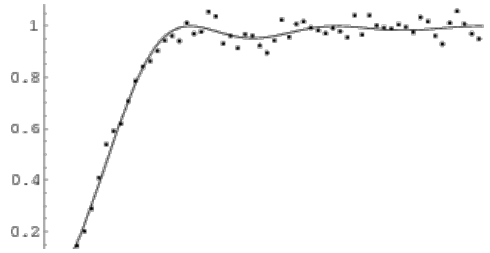

Figure 3: Testing the Montgomery's conjecture using the zeros from zero 10^{21} to zero $10^{21} + 10,000$.

If the zeros were to occur at random points on the critical line, i.e. where one zero occurs has no bearing on where another would occur, then the zeros would act like a Poisson process. In this case the pair correlation function would be the constant function $f(x) \equiv 1$. On the other hand, Montgomery's conjecture implies that, if we are sitting at a zero on the critical line, it is unlikely that we will find another zero close by. An analogy is with the prime numbers themselves - that two is prime implies that it is impossible that any two numbers separated by one (other than two and three) can both be prime.

Of course, it is natural to ask if the data supports Montgomery's conjecture. Thanks to the remarkable computations of Andrew Odlyzko we have a large amount of data. Since Montgomery's conjecture is an asymptotic results Odlyzko concentrated on computing the zeros near a very large zero. He has some of this data available on his web site. For example he has the zeros number 10^{21+1} through $10^{21} + 10,000$ of the zeta function. In Figure 3 we show the empirical results from the data compared with the theoretical limiting density.

We have divided the interval from 0 to 3.05 into 61 equal subintervals and used these for our intervals $[\alpha, \beta]$. We see that the fit is very good for intervals between 0 and 1, but it is less convincing for the other intervals.

In computations in the 1980's Odlyzko computed 175,587,726 zeros near zero 10^{20}. More specifically he started with zero $10^{20} - 30,769,710$ and ended with zero number $10^{20} + 144,818,015$. Using this data, Odlyzko used 70 million zeros near zero 10^{20} to check Montgomery's conjecture obtaining a remarkable fit as shown in Figure 4.

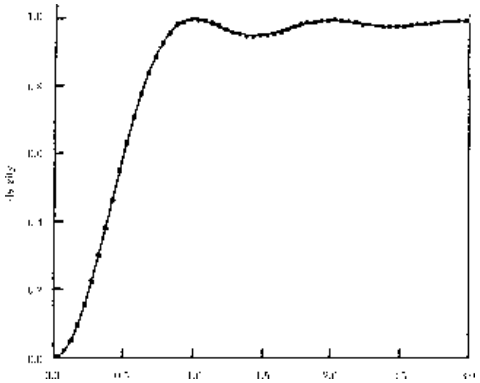

Figure 4: Testing Montgomery's conjecture using 70 million zeros near zero 10^{20}.

The Dyson connection

Montgomery's work was very exciting, since proving any sort of structure about the zeros - if they are where we think they are - is exciting. But things like this can provide subtle clues towards a proof. Perhaps it is possible to find a way of producing numbers that have the same properties, and if so then maybe, just maybe, you can find some direct link between Riemann's zeta function and the process, and then maybe, really - just maybe you can prove the Riemann Hypothesis. So, did this pair correlation arise in other places?

Freeman Dyson had been working on a problem in statistical mechanics - this is the physics and mathematics which attempt to model the behavior of huge systems, like all the atoms in a liter of gas - all 6×10^{23} of them. For systems of such huge populations it is impossible to model each particle individually and what statistical mechanics tries to do is to find some predictable average behavior, or to work out the rough distribution of what the particles are doing, say 20% are moving at such and such a speed, in such and such a direction, so much percent of the time.

Dyson, along with his collaborator Mehta, had been working out the theory of random matrices. Building on work of another Princetonian (and Nobel laureate) Eugene Wigner, they were looking to gain some insight (and prove some theorems) about the statistical properties of the eigenvalues of a random matrix, which would then cast some light on the problem of the prediction and distribution of *energy levels* in the nuclei of heavy atoms.

The connection between eigenvalues and nuclei is a bit beyond this discussion; but suffice it to say that the classical many-body formulation of the subatomic dynamics (electrons spinning around a positively charged nucleus) fails. This led Heisenberg to develop a quantization of this classical setting in which a deterministic Hamiltonian, a scalar function describing the energy of a system in terms of the positions and momenta of all the acting agents, is replaced by a matrix acting on a ``wave function'' which encapsulates our uncertain or probabilistic knowledge of the state of our energetic system. It is interesting to note that Heisenberg did not even know what matrices were when he developed his theory. He was completely led to their constructions by physical theories.

In this matrix mechanics, setting the *spectral lines* which are observed when a nucleus is bombarded with a particular kind of radiation (which corresponds to an effect of resonant radiation) may in fact be predicted as eigenvalues of certain matrices which model the physics. For heavy nuclei the models are too

difficult to construct (too many interactions to write down), so the hope is that the behavior of this model might be like that of an average model of the same basic type (symmetry). These average or *ensemble* behaviors could be calculated and give some insight into the particular situation, or so it was hoped. These were the things that Dyson and Mehta were studying.

In a brief conversation in the Institute tea room, where the ghosts of Einstein, Godel, Von Neumann and Oppenheimer still held forth (although did not eat many of the cookies) Dyson and Montgomery discovered that it looked as though they had been studying the same thing!

Random Matrices

To understand why Dyson recognized the pair correlation function, we need to understand some results about *random matrices*. A *random matrix* is a matrix whose entries are random variables. In the areas of quantum mechanics and number theory, there are several classes of random matrices that are of interest; they are referred to by certain acronyms, such as GUE, CUE, GOE, and GSE. In Sarnak's lectures, he discusses the CUE class which is asymptotically the same as the GUE class. Here, we will discuss the GUE class because that is the one chosen by Odlyzko but it turns out that the mathematics is the same.

The acronym GUE stands for Gaussian Unitary Ensemble. By *ensemble* we mean a collection of matrices and a method for choosing one of them at random. The entries in an $n \times n$ GUE matrix are complex numbers chosen so that the matrix is Hermitian, i.e. so that if the jk th entry is $a+ib$ then the kj th entry is $a-ib$. For diagonal entries this means that $a+jb = a-jb$ so the diagonal entries must be real. Thus to pick a random $n \times n$ GUE matrix, we only have to specify how we pick the entries on or above the main diagonal. For a GUE ensemble, we choose entries α_{jj} on the main diagonal from a normal distribution with mean 0 and standard deviation $\sqrt{2}$. For entries $\alpha_{jk} + i\beta_{jk}$ with $j < k$ (above the main diagonal) α_{jk} and β_{jk} are chosen from a normal distribution with mean 0 and standard deviation 1. Then the entries below the diagonal are determined by the condition that the matrix be Hermitian.

It should be noted that some authors choose the entries $\alpha_{jj}.$ on the main diagonal to be a standard normal distribution and the entries above the main diagonal α_{ij} and β_{ij} to have a normal distribution with mean 0 and standard deviation $1/\sqrt{2}$. This choice does not make an essential difference in the results that we will discuss but makes annoying differences in normalization factors.

It is well-known from linear algebra that Hermitian matrices have real eigenvalues The eigenvalues of a matrix M are numbers λ such that for each such λ, there exists a non-zero vector v with the property that $Mv = \lambda v$.The first question one might consider is how these eigenvalues are distributed along the real line. More precisely, if we choose a random $n \times n$ GUE matrix, can we say anything about the probability of finding an eigenvalue of this matrix in the interval $[a,b]$?

The physicist Eugene Wigner worked with random matrices in connection with problems in quantum mechanics. He proved that as $n \to \infty$, the density function of the positions of the eigenvalues approaches a semicircle with center at the origin. In Figure 5, we show the fit for eigenvalues of 50 random 40×40 GUE matrices, along with the semicircle of radius $2\sqrt{2 \cdot 40}$. In the $n \times n$ case, the eigenvalues lie in the interval $[-2\sqrt{2n}, 2\sqrt{2n}]$. We denote these eigenvalues by $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$.

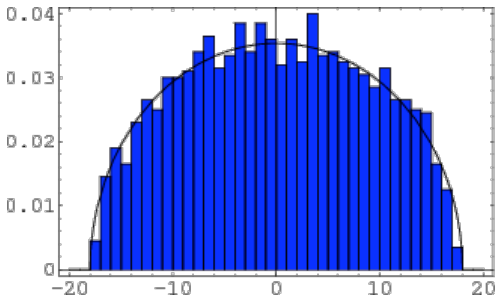

Figure 5: Eigenvalue distribution in 40×40 GUE matrices.

The eigenvalues of the matrix are called the ``spectrum" of the matrix. From this histogram we see that there are many more eigenvalues near the middle of the spectrum than at the extreme values.

One can also ask about the distribution of the gaps $\delta_n = \lambda_{n+1} - \lambda_n$ between successive eigenvalues of a random GUE matrix. It turns out that there are different limit theorems that apply to the middle of the spectrum (called the ``bulk spectrum") and to the extreme values (called the``edge" spectrum). We shall consider only the limit theorem for the bulk spectrum. For this theorem we again normalize the gaps so that asymptotically they have mean 1. The appropriate normalization for our definition of GUE matrices is:

$$\hat{\delta}_i = \delta_i \frac{\sqrt{8n - \lambda_i^2}}{4\pi}.$$

(Here is an example where the normalization depends on the particular choice we made for the definition of GUE random matricies.)

It has been proven that, in the bulk spectrum, the normalized spacings have a limiting distribution as $n \to \infty$, called the *Gaudin distribution*. The density of the Gaudin distribution cannot be written in closed form, although it can be computed numerically. In Figure 6, we show a simulation of the normalized spacing distribution for 500 random GUE matrices of size 100×100. We have also superimposed the Gaudin density. We see that the fit is quite good.

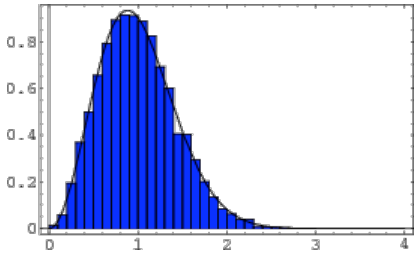

Figure 6: Normalized spacing distribution.

In Figure 7 we compare the Gaudin density with the Poisson density. We note that in the Poisson case, small values are quite probable, while in the normalized spacing case, such values are much less probable. This is sometimes described by saying that the eigenvalues `repulse' one another.

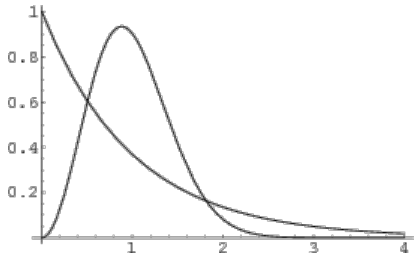

Figure 7: the Gaudin density compared to Poisson density with mean 1.

The analog of Montgomery's conjecture for the gaps for the normalized eigenvalues is:

$$E(|\{\hat{\delta}' : \hat{\delta}'. > \hat{\delta}, \hat{\delta}' - \hat{\delta} \in [\alpha, \beta]\}|) \sim \int_{\alpha}^{\beta} \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right) du$$

for any nonnegative numbers $\alpha < \beta$ where E means expected value.

The hypothesis that the zeros of the zeta function and the eigenvalues of GUE matrices have the same pair correlation function is called the *GUE hypothesis* (or the *Montgomery-Odlyzko law*). Odlyzko contributed to this hypothesis by calculating a very large number of zeros of the Riemann zeta function and using these to support the GUE Hypothesis. As we have seen earlier in Figure 4, Odlyzko compared the empirical pair correlation for the normalized zeros of the zeta function to the conjectured limiting distribution and found a remarkably good fit.

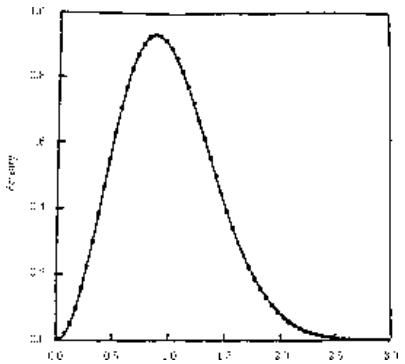

Figure 8: Pair correlation of zeros of the zeta function based on 8×10^6 zeros near 10^{20}-th zero, compared with the GUE conjectured density $1 - (\frac{\sin \pi x}{\pi x})^2$.

Odlyzko also compared the limiting distributions for the normalized spacings of the eigenvalues of GUE matrices with the distribution of the normalized spacings of the zeros of the zeta function as shown in Figure 8. We see that the fits again remarkably good and this has led to the hope that the use of the theory of random matrices will lead to a proof of the Riemann hypothesis.

Related Work

In a series of papers, Craig Tracy and Harold Widom made major contributions to the study of random matrices and their applications. This started with their finding limiting distribution for the largest eigenvalue for three ensembles of random matrices studied by Wigner and Dyson: the Gaussian Orthogonal Ensemble (GOE), Gaussian Unitary Ensemble (GUE), and Gaussian Symplectic Ensemble (GSE). For each of the ensembles there is a unique probability measure, called Haar measure, that is invariant under transformations associated with the ensemble. A random matrix for a particular ensemble means a matrix chosen according to the Haar measure. For the GUE ensemble, Haar measure amounts to choosing random matrices by normal distributions, as described earlier. For the three ensembles, the probability densities for the eigenvalues of an $n \times n$ matrices to lie in infinitesimal intervals about the points $x_1, ..., x_n$ are:

$$P_{n\beta}(x_1, ..., x_n) = C_{n\beta} e^{-\frac{1}{2}\beta \sum {x_i}^2} \prod_{j<k} |x_j - x_k|^2,$$

where $C_{n\beta}$ is a normalizing constant and $\beta = 1, 2, \text{or} 4$ according to whether we are considering the GOE, GUE, or GSE ensemble.

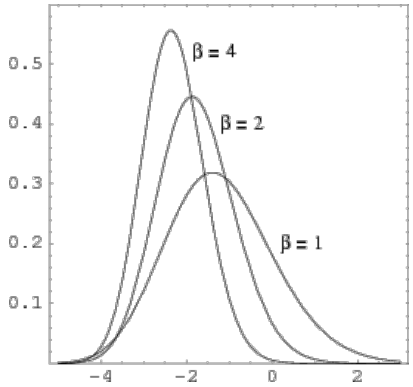

Figure 9: Probability density of scaled largest eigenvalue.

Tracy and Widom [9] showed that the largest eigenvalue, properly normalized, tends to a limiting distribution. These limiting distributions were different for the three possible values of $\beta = 1, 2, 4$ corresponding to the three different ensembles. Their densities are shown in Figure 9. They are denoted by F_1, F_2, F_4 and they are now called *Tracy-Widom distributions*. As with the Gaudin distribution there is no formula for these densities though they can be computed numerically. These distributions have been found to be the limiting distributions in many different fields and appear to indicate a new *central limit theorem*.

Let m_n be the largest eigenvalue of a random $n \times n$ GUE matrix. Then Tracy and Widom proved that

$$\frac{m_n - 2\sqrt{n}}{n^{1/6}}$$

has limiting distribution F_2. In Figure 10, we show the results of simulating 1000 random 100×100 GUE matrices and scaling their largest eigenvalues.

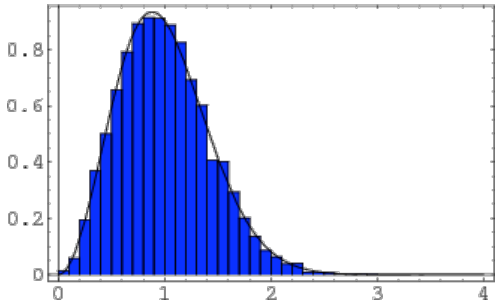

Figure 10: Simulated distribution of largest eigenvalue for 100×100 GUE matrices compared to Tracy Widom limiting distribution.

The discovery of these distributions has led Tracy and Widom and others to find these distributions as limiting distributions for phenomena in a number of different areas outside of the study of random matrices. We now discuss two such applications.

Around 1960, Stanislaw Ulam considered the following problem: Given a positive integer n, let π be a random permutation of the integers from 1 to n. Let

$X_n(\pi)$ be the length of the longest increasing subsequence in π. For example, if $\pi = 10,3,5,2,1,7,9,6,4,5$ the longest increasing sequence is $3,5,7,9$, so $X_{10}(\pi) = 4$. Ulam asked: what is the average value of X_n? Based upon Monte Carlo simulations, Ulam conjectured that the average length is asymptotic to $2\sqrt{n}$. This conjecture was proven by Vershik and Kerov [6]. Recently the distribution of X_n has been the subject of much research. In particular, Baik, Deift, and Johansson [7] have shown that if X_n is scaled in the appropriate way, then the distribution of X_n approaches a limiting distribution as $n \to \infty$. More precisely, they show that the random variable

$$\frac{X_n - 2\sqrt{n}}{n^{1/6}}$$

converges in distribution to F_2 as $n \to \infty$. In Figure 11, we show the results of simulating 1000 random permutations of length 100, and scaling the sizes of their longest increasing subsequences.

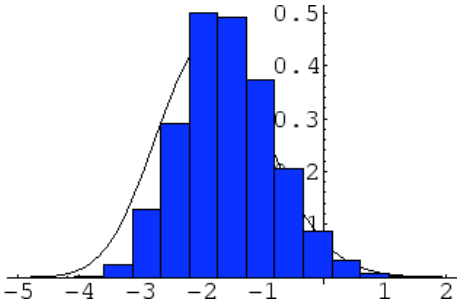

Figure 11: Simulation results for longest increasing subsequences of permutations.

Another interesting example is based on a model for the spread of a fire studied by Gravner, Tracy and Widom (GTW) [8]. We imagine a large sheet of paper and a fire starting in the bottom left corner. The fire moves deterministically to the right but also moves up by a random process described next.

We start with a grid of unit squares on the paper. We label the rows $0,1,2,3,\cdots$ and the columns $0,1,2,3,\cdots$. We represent the spread of the fire by a placing letters H or D in the squares of the grid. We start at $t = 0$ with a letter D at (0,0). To get the configuration of the fire at time $t+1$ we look at the fire in columns 0 to t+1 at time t. If the height of the fire at a column is less than the height of the fire in the column to its left, we add a D to the column. If the height of the fire is greater than or equal to the height of the fire in the column to its left, we toss a coin. If heads comes up we add an H to the height of the column. For column 0 which has no column to the left we always toss a coin to see if we should add an H to this column.

Let's see in more detail how this works. We start at time 0 and put a D in column 0 and so the height of the fire in this column is 1.

```
t = 0     D
          0   1   2   3   4   5   6.
```

At time t = 1 we look at columns 0 and 1 at time 0. The height of the fire in column 1 is less than in column 0 so we add a D to this column. In column 0 we toss a coin. It came up heads so we added an H to the first blank square in this column. Thus for time t = 1 we have:

```
t = 1     H
          D   D
          0   1   2   3   4   5   6.
```

Now at time $t = 2$ we look at columns 0,1,2 at time t = 1. The height of the fire in column 2 is less than that in column 1 so we add a D to column 2. Similarly, the height of the fire in column 1 is less than the height of column 0 so we add a D to column 1. For column 0 we tossed a coin to decide if we should add an H to column 0. It came up tails so we did not add an H. Thus for time t = 2 we have:

```
          H   D
t = 2     D   D   D
          0   1   2   3   4   5   6.
```

Continuing in this way we have:

```
              H
t = 3     H   D   D
          D   D   D   D
          0   1   2   3   4   5   6

              H
              H   D
 t = 4    H   D   D   D
          D   D   D   D   D
          0   1   2   3   4   5   6

              H   D
```

```
          H   H   D   D
t = 5     H   D   D   D   D
          D   D   D   D   D   D
          0   1   2   3   4   5   6

                  H
              H   D   D
t = 6         H   H   D   D   D
              H   D   D   D   D   D.
```

Let $h_t(x)$ be the height of the fire in column x after time t. We wrote a Mathematica program to simulate the spread of the fire. We ran the program for t = 500, t = 1000 and t = 2000. In Figure 12 we show plots of the height of the fire, $h_t(x)$, for these three times.

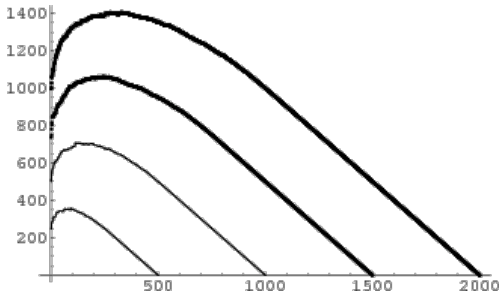

Figure 12: Simulation for height of the fire for t = 500,1000,1500,2000.

Note that up to time $t/2$ the graphs have a curved shape and after $t/2$ they seem to follow a straight line. This behavior is completely described by the limiting results obtained by GTW.

In our simulations we have assumed that the coin that is tossed is a fair coin. The authors consider more generally that the probability that the coin turns up head is p. The authors prove limit theorems for the height of the fire after time t under the assumption that x and $t \to \infty$ in such a way that their ratio x/t is a positive constant $1 - p_c$ for $0 < p_c < 1$. Their limit theorems use two constants, c_1 and c_2, which depend on p and p_c. These constants are:

$$c_1(p, p_c) = 2p_c p - p + 2\sqrt{p p_c (1-p)(1-p_c)},$$

$$c_2(p, p_c) = (p_c(1-p_c))^{1/6}(p(1-p))^1/2\left[\left(1 + \sqrt{\frac{(1-p)(1-p_c)}{p p_c}}\right)\left(\sqrt{\frac{p_c}{1-p_c}} - \sqrt{\frac{p}{1-p}}\right)\right]^{2/3}.$$

The authors prove the following limit theorems showing what happens when the ratio x/t = c is kept fixed with c a positive constant. They have limit theorems for four different ranges for c.

1. GUE Universal Regime: x/t = c with $0 < c < 1 - p$

$$\lim_{x \to \infty \; y \to \infty \; x/t = c} \text{Prob}\left(\frac{h_t(x) - c_1 t}{c_2 \, t^{1/3}} < s\right) = F_2(s)$$

   Thus in this regime, the limiting distribution is the Tracy-Widom distribution for the the limiting distribution for of largest eigenvalue for the GUE ensemble and also the limiting distribution for the length of the longest increasing subsequence in a random permutation.

2. Critical Regime: x/t = c with $c = 1 - p$.

$$\lim_{x \to \infty \; x/t = c} \text{Prob}(h_t(x) - (t - x) \leq -\Delta) = H(\Delta)$$

   where $H(\Delta)$ is a $\Delta \times \Delta$ determinate. Actually it is enough to assume that $x/t = 1 - p + o(\sqrt{t})$.

Note that, since it takes x time units for the fire to reach column $x, h_t(x)$ can be at most $t - x$. The authors provide the following numerical values of the limiting distribution for negative integers.

Δ	$H(\Delta)$
0	1
1	.5
2	.0908
3	.0056
4	.0001
5	.0000008

Thus, when $x/t = c$ with $c = 1 - p$, the fire at time t will be close to it's maximum possible height t-x.

3. Deterministic Regime: x/t = c with $1 - p < c < 1$.

$$\text{Prob}(h_t(x) = t - x) \to 1 \text{ exponentially fast.}$$

Note again that $h_t(x)$ can be at most $t - x$ so this says that, in this regime, in the limit, the fire in column x reaches its maximum possible height.

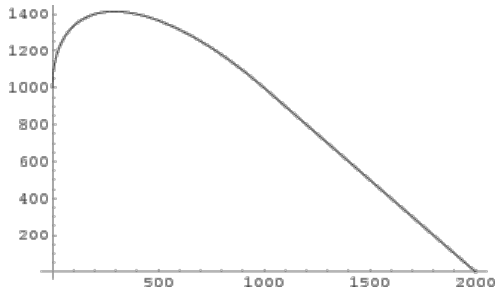

Figure 13: $h_{2000}(x)$ as estimated from limit theorems.

Let's see what these limit theorems would predict for the height of the fire at t = 2000 when p =1/2. The GUE universal regime will apply for $0 \leq x < 1000$. Here the limiting result in this regime implies that $h_{2000}(x)$ should be approximately $2000c_1(p, p_c)$. The critical regime applies when x = 1000. Here the limit theorem says that $h_{2000}(1000)$ will be approximately 1000. The deterministic regime will apply for $1000 < x < 2000$. The limiting theorem for this regime implies that, for t, $h_t(x)$ is well approximated by the line $y = 2000 - x$.

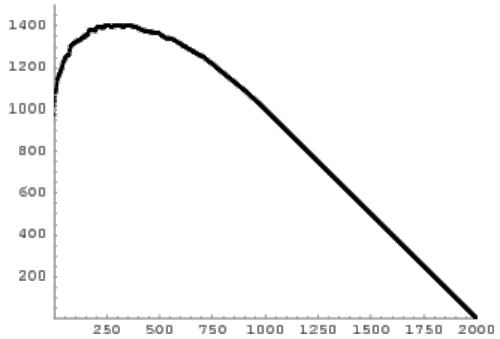

Figure 14: $h_{2000}(x)$ for a simulated fire for 2000 time units.

Putting all this together we would predict that for, t = 2000, the limiting curve should be as shown in Figure 13.

In Figure 14 we show the result of simulating the fire for 2000 time units.

These two graphs are remarkably similar. In fact, if you superimpose them you will not be able to see any difference! Thus the limit results predict the outcome for large t very well.

The authors also consider a fourth regime which they call the *finite x GUE regime* For this regime they fix x and let $t \to \infty$ Thus they look for the limiting distribution for the height of the fire in a specific column. For this they prove that

$$\frac{h_t(x) - pt}{\sqrt{p(1-p)t}}$$

has a limiting distribution F_{x+1}^{GUE} which is the distribution of the largest eigenvalue in the GUE of $(x+1)(x+1)$ Hermitian matrices. In Figure 15 we show the density for the x = 0 to x = 6.

Since for column one we are just tossing coins to determine the height of the fire, we will get the normal distribution as limiting distribution for column 0, and this is F_1^{GUE}

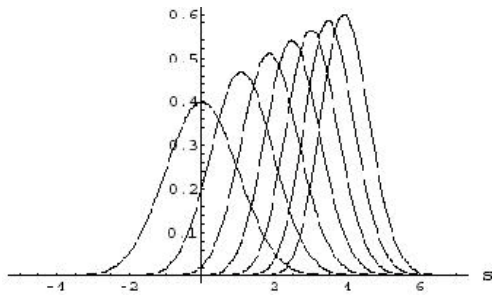

Figure 15: the densities $f_n^{GUE}, 1 \leq n \leq 7$.

Bibliography

1 G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, 1938.

2 E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Oxford University Press, 1951.

3 H. L. Montgomery, *The pair correlation of zeros of the zeta function*, Proc. Symp. Pure Math. 24, A.M.S., Providence 1973, 181-193.

4 A. M. Odlyzko,*On the distribution of spacings between zeros of the zeta function*, Math. Comp., 48 (1987), pp. 273-308.

5 Nicholas M. Katz and Peter Sarnak,*Zeroes of Zeta Functions and Symmetry*, Bulletin of the American Mathematical Society, vol. 36 Number 1 (1999), pp. 1-20.

6 Vershik, A.M. and Kerov, S.V. (1977). *Asymptotics of the Plancherel measure of the symmetric group and the limiting form of Young tableaux. Soviet Math.* Dokl. 18, 527-531. Translation of Dokl. Acad. Nauk. SSSR 32, 1024-1027.

7 J. Baik, P. Deift, K. Johansson, *On the distribution of the length of the longest increasing subsequence of random permutations*, J. Amer. Math. Soc. 12 (1999), 1119-1178.

8 J. Gravner, C .A. Tracy, H. Widom)*A Growth Model in a Random Environment*, to appear in Annals of Probability. Available at http://arxiv.org/.

9 C .A. Tracy, H. Widom)*The Distribution of the Largest Eigenvalue in the Gaussian Ensembles*, Calogero-Moser-Sutherland Models, eds. J.F. van Diejen and L. Vinet, CRM Series in Mathematical Physics 4, Springer-Verlag, New York, 2000, pp. 461-472. Available at http://arxiv.org/.

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