A 27-vertex graph that is vertex-transitive and edge-transitive but not l-transitive

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Version dated 1985 *
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Abstract

I describe a 27-vertex graph that is vertex-transitive and edge-transitive but not 1-transitive. Thus while all vertices and edges of this graph are similar, there are no edge-reversing automorphisms.

A graph (undirected, without loops or multiple edges) is said to be *vertex-transitive* if its automorphism group acts transitively on the set of vertices, *edge-transitive* if its automorphism group acts transitively on the set of undirected edges, and *1-transitive* if its automorphism group acts transitively on the set of paths of length 1. If a graph is edge-transitive but not 1-transitive then any edge can be mapped to any other, but in only one of the two possible ways. In my Harvard senior thesis [2], I described a graph that is vertex-transitive and edge-transitive but not 1-transitive. It has 27 vertices, and is regular of degree 4. This beautiful graph was also discovered by Derek Holt [4]. It seems likely that this is the smallest graph that is vertex-transitive and edge-transitive but not 1-transitive.

The question of the existence of graphs that are vertex-transitive and edge-transitive but not 1-transitive was raised by Tutte [5], who showed that any such graph must be regular of even degree. The first examples were given

*Derived from the Harvard senior thesis of Peter G. Doyle, dated June 1976.

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by Bouwer [1]. Bouwer's smallest example has 54 vertices, and is regular of degree 4. While Bouwer's method of construction differs from the method used here, Ronald Foster has pointed out to me that the 27-vertex graph described here can be obtained from Bouwer's 54-vertex graph by identifying pairs of diametrically opposed vertices.

Recall that given a group G and a set $H \subseteq G - \{1\}$ such that $H = H^{-1}$, we construct the group-graph $\Gamma_{G,H}$ by taking G as the set of vertices, and connecting every $g \in G$ to every element of the set gH. The idea, which is inspired by work of Watkins [6], will be to find a group G and a set of generators $K \subseteq G - \{1\}$ such that:

1. $K \cap K^{-1} = \emptyset$.
2. For any $k_l, k_2 \in K$, there is an automorphism ϕ of G such that $\phi(k_1) = k_2$.
3. If ϕ is an automorphism of G such that $\phi(K \cup K^{-1}) = K \cup K^{-1}$, then $\phi(K) = K$.

The group-graph $\Gamma_{G,K\cup K^{-1}}$ will then be vertex-transitive and edge-transitive, and we may hope that conditions 1–3 will preclude its being 1-transitive.

For the group G we take the non-abelian group of order 27 with generators a, b and relations

$$a^9 = 1, \quad b^3 = 1, \quad b^{-1}ab = a^4.$$

(Cf. Hall [3], p. 52.) Setting $c = ba^{-1}$, we find that G can be described as the group with generators a, c and relations

$$a^9 = 1, \quad c^9 = 1,$$

$$c^3 = a^{-3}, \quad a^3 = c^{-3},$$

$$c^{-1}ac = a^4, \quad a^{-1}ca = c^4.$$

These relations are not independent. Their redundancy allows us to see at a glance that there is an automorphism ϕ of G such that $\phi(a) = c, \phi(c) = a$. Setting $K = \{a, c\}$, we see that K satisfies conditions 1 and 2, and it is easy to show that condition 3 also holds.

The graph $\Gamma = \Gamma_{G,K\cup K^{-1}}$ is shown in Figure 1. Although it is not obvious from the drawing, we know that this graph is vertex-transitive and edge-transitive. To see that it is not 1-transitive, consider the subgraph Γ'

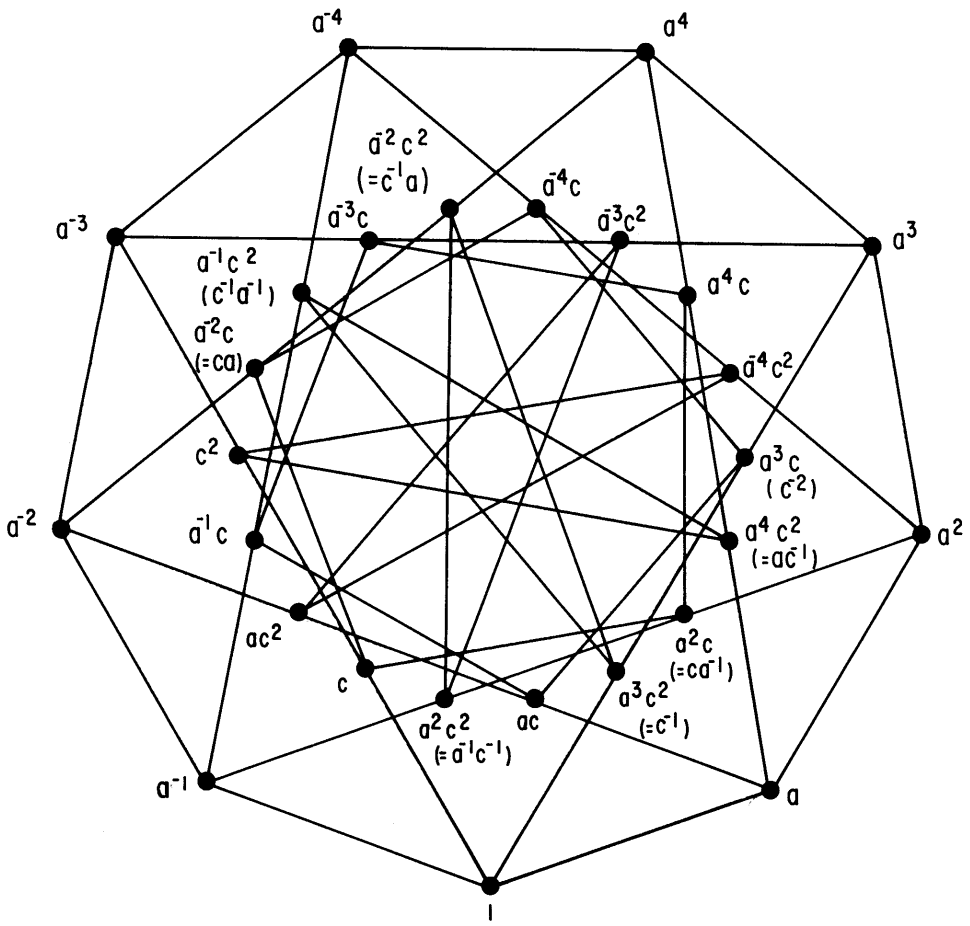

Figure 1: The graph Γ.

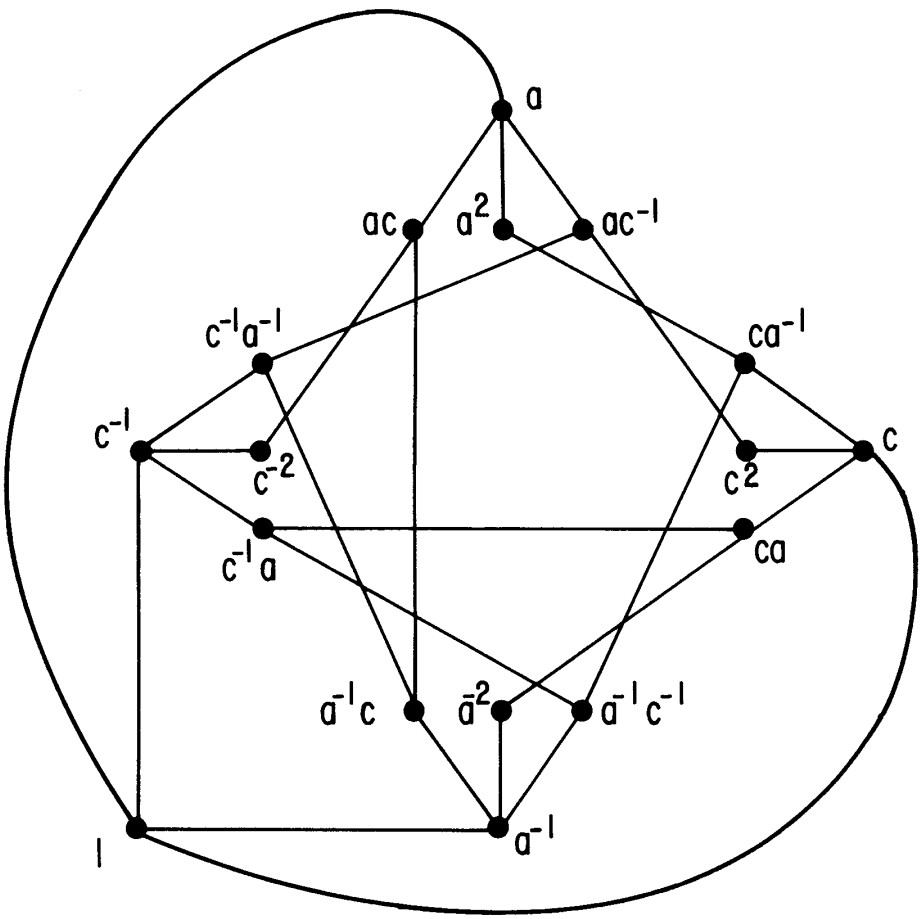

Figure 2: The subgraph Γ'.

obtained by removing all vertices whose distance from the identity is > 2. (See Figure 2.) If there were a graph-automorphism ϕ of Γ such that $\phi(1) = 1, \phi(a) = a^{-1}$, the restriction ϕ' of ϕ to Γ' would be an automorphism of Γ' such that $\phi'(1) = 1, \phi'(a) = a^{-1}$, but it is easy to verify that no such automorphism exists. Hence Γ is not 1-transitive.

Acknowledgements. I would like to thank W. T. Tutte and Ronald Foster for helpful correspondence.

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