Neoplatonic solids

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Abstract

A *6-net* is a simplicial triangulation of the 2-sphere with maximum degree ≤ 6. Experiments suggest that every 6-net admits a unique realization as an undented Euclidean polyhedron built from unit equilateral triangles, and a unique realization as an ideal equilateral hyperbolic polyhedron. We call these *neoplatonic solids* and *ideal neoplatonics*.

A net is *prime* if every 3-cycle bounds a face. A computer-assisted proof shows that every prime 6-net with $v \leq 50$ has a unique realization as a convex ideal neoplatonic. Numerical homotopy from this realization yields an approximate Euclidean neoplatonic, and a computer-assisted proof shows that a true Euclidean neoplatonic lies nearby, though we do not prove uniqueness. Using the separating-triangle decomposition, we extend Euclidean existence to all 10,412,340 6-nets with $v \leq 50$, counted up to combinatorial isomorphism.

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1 Atlas

This paper should be read alongside an online atlas of neoplatonic solids [3].

2 Introduction

A *6-net* is a simplicial triangulation of the 2-sphere with maximum degree at most 6. (Cf. Thurston [24].) A net is prime if every 3-cycle bounds a face. A 6-net is prime just if it is the tetrahedron, or if it has no vertex of degree 3 and is not a stack of two or more octahedra.

Experiments suggest:

Conjecture (Neoplatonic)**.** *Any* 6*-net* σ *has a realization, unique up to isometry, as an undented Euclidean polyhedron* $\Delta(\sigma)$ *built from equilateral triangles of side length* 1*.*

A polyhedron is *undented* if every vertex has a local support plane: From outside the object you can touch any vertex with your palm if your hand is small enough. Equivalently, at every vertex the sum of the exterior dihedral angles is ≥ 0. By Gauss–Bonnet, this guarantees that when you look from a vertex into the interior of the solid, its immediate neighborhood occupies at most half of the visual sphere. The atlas gives examples of dented polyhedra, and of polyhedra with vertices that have a local but no global support plane.

We call these realizations of 6-nets *neoplatonic solids*, as they follow in the footsteps of the tetrahedron, octahedron, and icosahedron. We call them 'solids' out of respect for tradition, though we regard a polyhedron as a 2-dimensional object rather than the region of space that it bounds. Note that the cube and dodecahedron don't count as neoplatonics: We may view them as subsidiary to their duals.

The convex gallery in the atlas starts with the eight neoplatonics that are convex without coplanar faces. They were identified by Rausenberger [17], and later by Freudenthal and van der Waerden [13]. (One of these eight, the triangular bipyramid, is non-prime.) Allowing coplanar faces gives infinitely many more convex neoplatonics. Note that we allow as a limiting case 'pancakes': polyhedra with empty interior obtained by doubling a portion of the triangular lattice. Pancakes are limits of embedded realizations of the same triangulated sphere. Here the simplicial hypothesis matters: a vertex of degree 2 would permit bending along the opposite 2-cycle without creating dents, contradicting uniqueness.

Beyond these convex examples lies a world of neoplatonic solids that are not convex but still undented. We like to think that Theaetetus, the discoverer of the icosahedron, and Euclid, who constructs the icosahedron in Book XIII, Proposition 16, of the *Elements* [16], would be pleased to see the tetrahedron, octahedron, and icosahedron as the eldest siblings in a distinguished infinite family.

Proposition 1. *There are 8,239,684 prime 6-nets with* $v \leq 50$*. Every one has a neoplatonic realization.*

Proof. We generate prime 6-nets using a variant of the method of Goedgebeur [14]. To prove they have neoplatonic realizations we use the method of Ellison [12]. We compute an approximate neoplatonic, then use an effective version of the inverse function theorem combined with arithmetic with guaranteed error bounds to prove that an exact neoplatonic lies nearby. In cases where the limit lies flat at one or more degree-6 vertices, some of the equations specifying edge lengths get replaced by coplanarity conditions.

The software used for this computation is archived in [10]. □

3 Non-prime neoplatonics

The atlas includes a gallery of non-prime neoplatonics. Cutting along all separating triangles decomposes a non-prime net into prime factors. For $v \leq 50$, the non-prime nets fall into three disjoint classes.

1. All prime factors are tetrahedra. With n tetrahedral factors, the numbers for $n = 2, 3, 4, 5$ are respectively $1, 1, 3, 3$. For each $6 \leq n \leq 47$ there is one net, the helical stack called '3NA' in the atlas.
2. There are two or more non-tetrahedral factors. They are octahedra arranged in a stack, with a tetrahedral cap of depth 0, 1, or 2 at each end.
3. There is exactly one non-tetrahedral prime factor, with one to four tetrahedra attached.

The first class contains

$$1 + 1 + 3 + 3 + 42 = 50$$

nets with $v \leq 50$. For the second class, let $m \geq 2$ be the number of octahedral factors, and let t be the total depth of the two caps. Then

$$v = 3 + 3m + t.$$

For $t = 0, 1, 2, 3, 4$, the numbers of cap types are respectively $1, 1, 2, 1, 2$. The final 2 counts the two inequivalent relative attachments when both caps have depth 2. Thus the second class contains

$$14 + 14 + 2 \cdot 14 + 13 + 2 \cdot 13 = 95$$

nets with $v \leq 50$.

The first two classes have direct realizations by gluing regular cells. To check them without roundoff, scale so that the squared edge length is 8. If g is the centroid of an outward-oriented exposed face (a, b, c) and $n = (b - a) \times (c - a)$, the new tetrahedral vertex is $g + n/3$, while the center of an attached octahedron is $g + n/6$; its three new vertices are the reflections of a, b, c through that center. Thus all coordinates are rational. An exact check of all 145 nets proves that boundary triangles meet only in their common simplices and that every vertex has a local support plane.

For the third class, extend the certified intervals for the exterior dihedral angles across the attached tetrahedra. After identifying combinatorial duplicates, this yields 2,172,511 distinct nets with $v \leq 50$. An interval embeddedness check proves that each has an exact embedded realization.

These attachments preserve undentedness. Indeed, let $A \leq 2\pi$ be the area inside the spherical link of a core vertex of degree d. The link has perimeter $d\pi/3$, so spherical isoperimetry gives

$$2\pi - A \;\geq\; \frac{\pi}{3}\sqrt{36 - d^2}.$$

Each incident tetrahedral attachment adds $E = 3\arccos(1/3) - \pi < \pi/2$ to A, and the degree bound allows at most $6 - d$ such attachments. For $d = 4, 5$ the displayed lower bound exceeds respectively $2E, E$; for $d = 6$ no attachment is possible. New vertices have link area E or $2E$. Thus every attached realization is undented. The software and check outputs for these computations are archived in [10].

Corollary 2. *Every* 6*-net with* $v \leq 50$ *has a neoplatonic realization. Up to combinatorial isomorphism, there are* 10,412,340 *such nets:* 8,239,684 *prime and* 2,172,656 *non-prime.*

Proof. The proposition handles the prime case. The three classes above contain

$$50 + 95 + 2{,}172{,}511 = 2{,}172{,}656$$

non-prime nets, and the preceding constructions realize each of them. For every $v \leq 50$, these counts together with the prime count sum to the census of $\{3, 4, 5, 6\}$-triangulations in [7, Tables 1 and 2]. $\square$

4 Ideal neoplatonics

Experiments suggest that Euclidean neoplatonics are the shriveled remains of equilateral ideal hyperbolic polyhedra, which we call *ideal neoplatonics*. By *equilateral* we mean that the ideal triangle faces meet midpoint-to-midpoint, where the midpoint of an infinite edge is the foot of the perpendicular dropped from the opposite vertex. (Equivalently, after adjacent faces are unfolded into a common plane, their incircles are tangent.)

Conjecture (Ideal neoplatonic)**.** *Every* 6*-net* σ *has a realization* $\Delta(\sigma, \infty)$*, unique up to isometry, as an ideal neoplatonic.*

Conjecture (Ideal prime)**.** *If the* 6*-net* σ *is prime, its ideal neoplatonic realization* $\Delta(\sigma, \infty)$ *is convex.*

Proposition 3. *Every prime* 6*-net with* $v \leq 50$ *has a convex ideal neoplatonic realization, unique up to isometry.*

Proof. We use the approach of Bobenko, Pinkall, and Springborn [5]. Existence is proved using a Perron-type method of super- and subsolutions for the equations of Springborn [23]; for details see § 6. Uniqueness comes from Rivin [18], who proves that any net can have at most one convex equilateral ideal realization. □

Rivin [18] shows that a convex ideal equilateral polyhedron maximizes volume among convex ideal realizations of its net. It is natural to ask whether convexity of the competitors can be dropped. For an oriented ideal cycle, we use algebraic volume.

Conjecture (Volume max)**.** *For a prime* 6*-net* σ*, its ideal neoplatonic realization maximizes volume among all ideal geodesic* 2*-cycles with combinatorics* σ*.*

The corresponding assertion for non-prime 6-nets follows by splitting along separating triangles and maximizing the resulting cycles independently. Without the degree bound, maximizing volume is more problematic. A regular ideal 12-gonal bipyramid maximizes volume among convex realizations, but twelve regular ideal tetrahedra can wind twice around a common axis, producing a branched realization of maximal possible volume.

5 Neoplatonic homotopy

Conjecture (Neoplatonic homotopy)**.** *Every* 6*-net* σ *has a family of realizations* $\Delta(\sigma, l)$*, each unique up to isometry, as undented hyperbolic polyhedra of side length* l*,* $0 < l \leq \infty$*, where* $\Delta(\sigma, \infty)$ *is an equilateral ideal hyperbolic polyhedron. After rescaling all lengths by* $1/l$*,* $\Delta(\sigma, l)$ *converges as* $l \to 0$ *to the Euclidean neoplatonic realization* $\Delta(\sigma)$*.*

The atlas includes animations showing the deformation from ideal to Euclidean realizations.

The intuition behind this conjecture is that if we begin with the unique ideal realization $\Delta(\sigma, \infty)$ and continue it through a family $\Delta(\sigma, l)$, then it is unable to develop a dent, because one cannot slip a curve whose length remains $< 2\pi$ over the unit sphere. The issue is therefore whether this homotopy can be continued all the way down to $l = 0$. If it can, the Euclidean neoplatonic exists. If in addition there can be no bifurcation along the way, then it is unique.

For every prime 6-net with $v \leq 50$, we have rigorous existence proofs for both Euclidean and ideal neoplatonics, but our evidence for the homotopy conjecture is experimental. The approximate neoplatonics used in the proof of Proposition 1 were obtained by numerical homotopy from ideal neoplatonics.

6 Realizing ideal neoplatonics

To realize ideal neoplatonics we use the approach of Bobenko, Pinkall, and Springborn [5] and Springborn [23]. Figure 1 (top) shows a convex ideal hyperbolic polyhedron M viewed through one of its ideal vertices v_0. What one sees is a triangulated Euclidean polygon P. It lives in a visual horosphere about v_0, which can be identified with the sphere at infinity with v_0 removed. The symmetry axes of the ideal triangle faces not incident with v_0 appear as the "symmedians" of the triangles of P, shown in red.

We can tell that M is convex for two reasons:

1. The polygon P is convex;

2. The triangulation satisfies the Delaunay empty-circle condition: the circumcircle of each triangle of P contains no vertex of P in its interior.

We can tell that M is equilateral for three reasons:

1. The boundary edges of P all have the same length. This means that there is no shearing along edges incident with v_0;

2. The triangles adjacent to the boundary are isosceles. This means that there is no shearing along the remaining edge of any triangle incident with v_0;

3. The red symmedian lines match up where they meet the interior edges of P. This means there is no shearing along edges between triangles not incident with v_0.

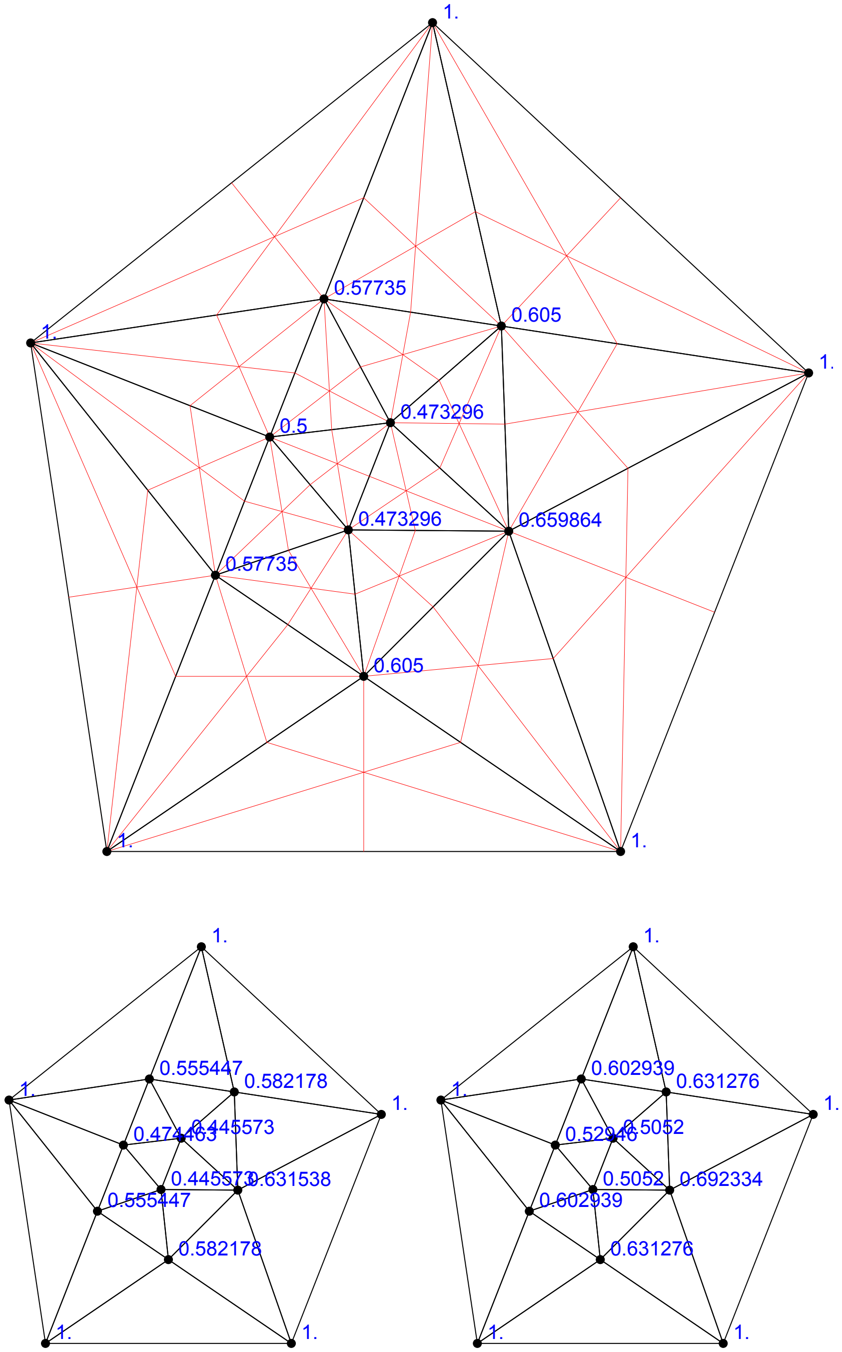

Figure 1: Realizing an ideal neoplatonic.

There is an algebraic way to express the absence of shearing. In Figure 1 the vertices of P are labeled by a function $u : V' \to (0, \infty)$. The edge ab has length

$$\mathbf{len}(ab) = u(a)u(b).$$

At the boundary vertices of P (those adjacent to v_0 in M) the function u takes the value 1. These properties imply that the quadrilaterals $abcd$ formed by triangles abc and acd adjacent along an interior edge ac are conformal rhombi:

$$\mathbf{len}(ab)\mathbf{len}(cd) = \mathbf{len}(ad)\mathbf{len}(bc) = u(a)u(b)u(c)u(d).$$

If we think of P as living in the complex plane, this means that the appropriate cross-ratio of the points a, b, c, d has modulus 1. This is an algebraic way of saying that there is no shearing along ac.

As explained by Springborn [23], to construct such a realization for a given triangulation of S^2 one seeks an appropriate function u from which to build P by gluing triangles with edge lengths $\mathbf{len}(ab) = u(a)u(b)$. We need the following properties:

1. For every triangle abc, we require the triangle inequalities:

   $$u(a)(u(b) + u(c)) > u(b)u(c)$$

   and its cyclic variants.

2. The total angle at every interior vertex is 2π, so that P lies flat.

3. The total angle at every boundary vertex is $\leq \pi$, so that the polygon is convex.

4. The triangulation is Delaunay: for quadrilaterals $abcd$ as above, writing $U(x) = u(x)^2$, we require

   $$2U(b)U(d)(U(a) + U(c)) - U(a)U(c)(U(b) + U(d)) \geq 0.$$

One possible approach is to compute u numerically to high precision, recover from it approximate coordinates for the vertices of P, recognize those coordinates as algebraic numbers, prove the rhombus equalities algebraically, and verify the Delaunay inequalities using arithmetic with guaranteed error bounds. We ran into trouble recognizing the algebraic numbers once the number of vertices got bigger than 11 or so. Where we succeeded, fields of definition were not special in any obvious way.

There is a looser approach: the method of super- and subsolutions, familiar from the Dirichlet problem and circle packing. Constructing P resembles circle packing, except that circle packing assigns the edge ab the length $u(a) + u(b)$ rather than $u(a)u(b)$ and has no corresponding issue with triangle inequalities.

Let us illustrate this by example. Figure 1 (bottom) shows two functions $u_0 \leq u_1$ for an example with $v = 14$. On the boundary $u_0 = u_1 = 1$. At interior vertices, u_1 has defect ≥ 0 and u_0 has defect ≤ 0. (In other words, u_1 puts too little angle at interior vertices, u_0 puts too much.) Starting with $u = u_1$, we can pick an interior vertex a and adjust the value of $u(a)$ downward to add angle at a and make the defect disappear there. The defect diffuses to neighboring vertices, whose defects all increase. We can then go from vertex to vertex, pushing u down, down, down. The function u_0 acts as a floor. If $u \geq u_0$, lowering $u(a)$ all the way to $u_0(a)$ would make the defect at a negative. Thus the defect vanishes before $u(a)$ reaches $u_0(a)$, and the inequality $u \geq u_0$ is preserved.

What about the triangle inequality? Could this fail and prevent us from adjusting $u(a)$ to remove the defect? No, because we can check that as long as $u_0 \leq u \leq u_1$ the triangle inequalities hold for all triangles of P. So we never run off the rails during the descent.

This produces a solution u, bracketed by u_0 and u_1, for which P lies flat. To prove convexity we check that any function bracketed in this way must have total angle $< \pi$ at boundary vertices, and satisfy the Delaunay inequality at every interior edge.

This approach works to prove existence of convex ideal neoplatonics for all prime 6-nets with $4 \leq v \leq 50$. For $v = 4, \ldots, 24$, we chose super- and subsolutions with all defects equal to $\pm 1/500$. For $v = 25$ we used $\pm 1/1000$; by $v = 50$ we were using $\pm 1/4000$.

We think the Perron approach could lead to a general proof. It reduces the problem to constructing super- and subsolutions whose entire bracket satisfies the triangle, Delaunay, and boundary inequalities. The computations show that these bounds must become tight: at $v = 50$ the defects were $\pm 1/4000$.

The ideal icosahedron

We close this section, as Euclid closed Book XIII, with the icosahedron, only now in its incarnation as an ideal neoplatonic. We get this from the functions u_0, u_1 shown at the bottom of Figure 2. The exact values for u, shown at the top of the figure, are

$$(\sqrt{5} - 1)/2 = .61803\ldots; \ \sqrt{(5 - \sqrt{5})/10} = .52573\ldots.$$

A direct verification is also available. It suffices to check that when we inscribe a pentagram in a regular pentagon and subdivide the interior pentagon, every pair of

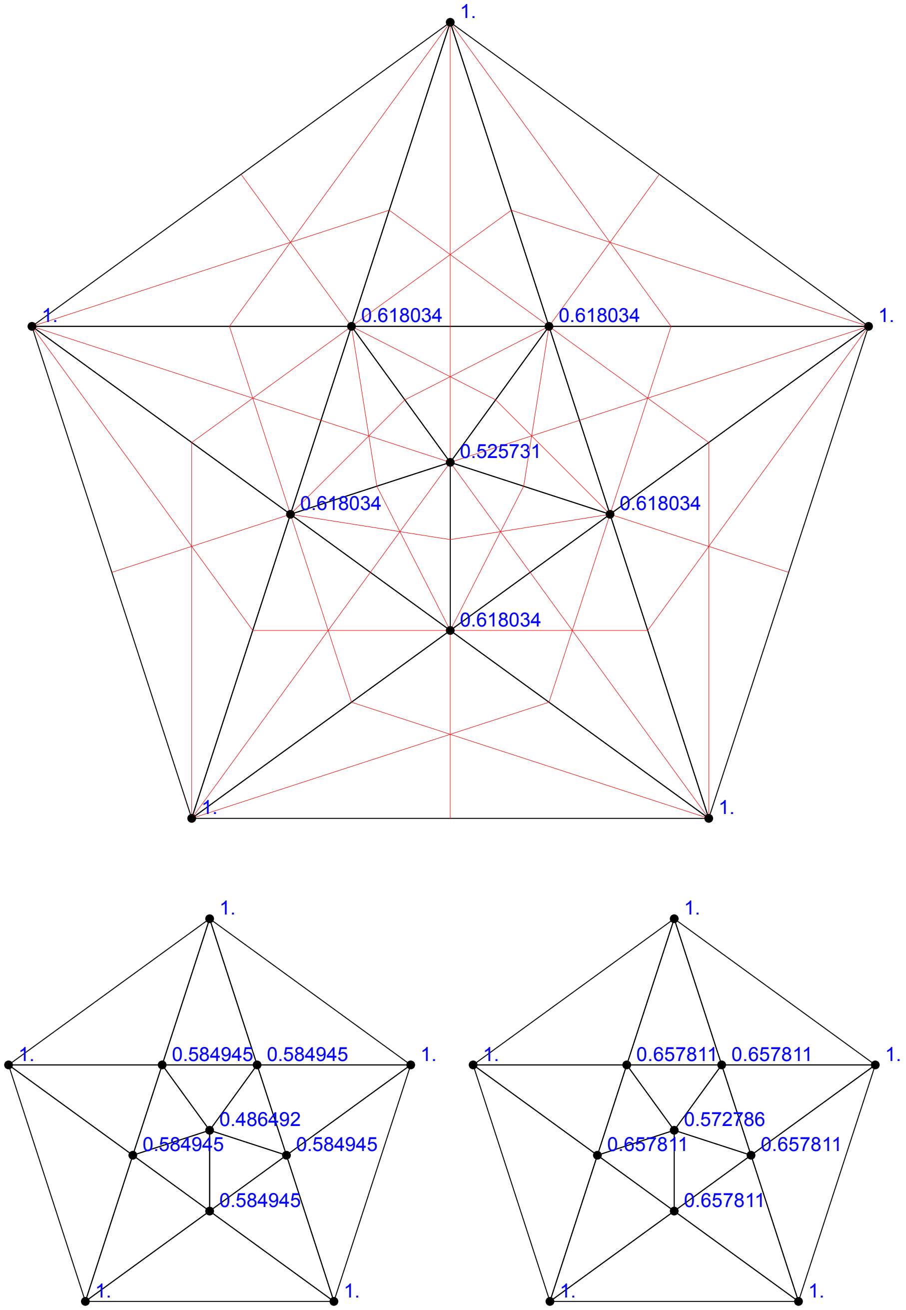

Figure 2: Realizing the ideal icosahedron.

adjacent triangles yields a conformal rhombus. This is clear by inspection.

Standard methods (Cf. Vinberg [25, Sec. 5.2]) give the volume of the ideal icosahedron as

$$13.5296281984835045317869052968492406346019656356 62\ldots.$$

We do not know a standard reference in which this value is tabulated.

7 Further conjectures

Neoconvex bodies

If neoplatonics are unique, it is natural to ask whether uniqueness holds more generally.

Call a hyperbolic or Euclidean polyhedron *neoconvex* if it is *intrinsically convex* (all angle defects are ≥ 0) and undented. This is a discrete analogue of having nonnegative Gaussian curvature and nonnegative mean curvature, which for a smooth surface would be equivalent to convexity.

Hypothesis (Neoconvex rigidity). *Two neoconvex polyhedra with the same combinatorics and isometric faces are ambiently isometric.*

This would imply uniqueness of neoplatonics. If true, neoconvex rigidity would strengthen Cauchy rigidity [8] in a direction orthogonal to Alexandrov-type uniqueness [2, Lemma 1, p. 62]. Alexandrov allows faces to bend while insisting on global convexity. Here the faces remain rigid, but nonconvexity is allowed, provided that the vertices remain undented. Even an ordinary convex polyhedron would then be unique among all undented polyhedra with the same metric net, not merely among convex ones.

A stronger version is:

Hypothesis (Dent location). *An embedded polyhedron with no vertices of negative angle defect is unique up to isometry among embedded realizations of its metric net with the same set of dented vertices.*

Embeddedness is essential: denting a vertex of the regular octahedron gives the flexible "cootie catcher," which can fold onto a single triangle.

Under this hypothesis, a polyhedron with no vertex of negative angle defect could at most have 2^v embedded realizations up to isometry. Thus a continuous flex through embedded polyhedra with no vertices of negative angle defect would disprove the hypothesis.

8 Acknowledgments and computations

These conjectures emerged from playing with models built from Polydrons™ and Geometiles™, in the context of our work with Zili Wang on the conjecture of Sleator, Tarjan, and Thurston [22]. See, for example, [11, Fig. 15]. We are deeply indebted to Zili for her support and encouragement throughout.

To construct and prove the existence of ideal realizations, we used the approach of Springborn [23], which builds on work of Luo [15] and Bobenko, Pinkall, and Springborn [5].

Further inspiration came from Agol [1, 'frog eggs', p. 5]; Sabitov [21]; Connelly [9]; Audet [4]; and extensive discussions with Yana Mohanty and Richard Schwartz.

Computations were carried out primarily by Claude Code, with supervision and advice from ChatGPT. Both contributed to the development of ideas, and the exposition benefited from ChatGPT. The proof programs and certificate formats are distributed on GitHub in the repositories `peterdoyle1717/bendprover` and `peterdoyle1717/ideal`.

We encoded nets using the edgebreaker algorithm of Rossignac, Safonova, and Szymczak [19, 20]. We assigned each net a canonical CLERS name by changing the edgebreaker code letters from CLERS to CBEAD and taking the lexicographically first representative.

We generated fullerene duals using the buckygen software of Brinkmann, Goedgebeur, and McKay [6], and promoted them to a catalog of prime 6-nets by a variant of the method of Goedgebeur [14], using the canonical CLERS name to identify isomorphic nets.

We are grateful to all these researchers for sharing their ideas, algorithms, and code.

A Counts of prime 6-nets

The table gives the numbers p_v of prime 6-nets with v vertices, $1 \leq v \leq 100$. They were computed using a variant of the method of Goedgebeur [14], and can be deduced from the tables presented there.

v	p_v	v	p_v	v	p_v	v	p_v
1	0	26	6 539	51	1 651 488	76	47 782 118
2	0	27	8 750	52	1 945 387	77	53 374 432
3	0	28	11 854	53	2 279 167	78	59 595 319
4	1	29	15 629	54	2 668 297	79	66 400 484
5	0	30	20 755	55	3 109 939	80	73 920 746
6	1	31	26 924	56	3 619 603	81	82 155 286
7	1	32	35 070	57	4 196 607	82	91 253 047
8	2	33	44 866	58	4 862 665	83	101 140 713
9	3	34	57 649	59	5 608 787	84	112 064 305
10	7	35	72 749	60	6 464 974	85	123 946 248
11	10	36	91 995	61	7 428 133	86	136 995 975
12	22	37	114 966	62	8 523 556	87	151 178 908
13	32	38	143 725	63	9 748 741	88	166 745 838
14	59	39	177 439	64	11 142 291	89	183 587 072
15	92	40	219 402	65	12 692 741	90	202 085 816
16	153	41	268 496	66	14 449 523	91	222 059 220
17	230	42	328 571	67	16 402 234	92	243 882 591
18	373	43	398 578	68	18 597 312	93	267 490 334
19	536	44	483 170	69	21 036 300	94	293 261 943
20	820	45	581 293	70	23 778 320	95	320 966 986
21	1 174	46	699 570	71	26 798 812	96	351 250 688
22	1 735	47	835 175	72	30 186 590	97	383 795 613
23	2 418	48	996 468	73	33 926 502	98	419 202 743
24	3 442	49	1 182 607	74	38 091 396	99	457 264 253
25	4 711	50	1 401 622	75	42 676 406	100	498 626 241

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