Taut fillings of the 2-sphere

Peter Doyle Matthew Ellison Zili Wang

Version 1.0 dated 10 May 2025

Abstract

Let σ be a simplicial triangulation of the 2-sphere, X the associated integral 2-cycle. A filling of X is an integral 3-chain Y with $\partial Y = X$; a taut filling is one with minimal L_1-norm. We show that any taut filling arises from an extension of σ to a shellable simplicial triangulation of the 3-ball. The key to the proof is the general fact that any taut filling of an n-cycle splits under disjoint union, connected sum, and more generally what we call almost disjoint union, where summands are supported on sets that overlap in at most $n+1$ vertices. Despite the generality of this result, we have nothing to say about optimal fillings of spheres of dimension 3 or higher.

1 Overview

Let Δ be the abstract $|V|-1$-simplex with vertices V, viewed as a simplicial complex. Let C_n and Z_n be its integral n-chains and n-cycles. Here and throughout we'll take $n \geq 1$. A *filling* of an n-cycle $X \in Z_n$ is any $n+1$-chain $Y \in C_{n+1}$ with $\partial Y = X$. Let $\mathrm{Zvol}(X)$ be the minimum L_1-norm of a filling of X, and call Y *taut* if $|Y| = \mathrm{Zvol}(\partial Y)$.

For $X \in C_n$ let $\mathrm{vertices}(X)$ be the set of all the vertices of all the n-simplices to which X assigns non-0 weight. For a taut filling Y of X we have $\mathrm{vertices}(Y) = \mathrm{vertices}(X)$, because projecting V onto $\mathrm{vertices}(X)$ by mapping $V \setminus \mathrm{vertices}(X)$ to an arbitrary $x \in \mathrm{vertices}(X)$ will push any filling with an internal vertex to a smaller filling. (Cf. Proposition 4 below.) This is why we write C_n and Z_n without reference to V.

Call an n-cycle $X = X_1 + X_2 \in Z_n$ an *almost disjoint union* if

$$|\mathrm{vertices}(X_1) \cap \mathrm{vertices}(X_2)| \leq n+1.$$

This notion generalizes both disjoint union and connected sum along an n-simplex. Ellison [2] showed the Zvol adds under almost disjoint union: $\mathrm{Zvol}(X) = \mathrm{Zvol}(X_1) + \mathrm{Zvol}(X_2)$. This means that *some* taut filling Y of X splits into a sum $Y = Y_1 + Y_2$ of taut fillings of X_1, X_2. Here we amplify Ellison's Corollary 2 to show (Theorem 1) that when $n \geq 2$, *any* taut filling of X splits.

In place of integral chains we can use chains with coefficients in **Q** or **R**. Evidently Rvol = Qvol. Computing Rvol is a linear programming problem with rational coefficients, so Qvol = Rvol. Ellison used LP duality to prove that Qvol adds under almost disjoint union. Now for $n \geq 2$ we get the stronger result (Corollary 1) that taut **Q**-fillings split: Multiplying a taut **Q**-filling Y of X by a common denominator q yields a taut **Z**-filling qY of qX: Split the **Z**-filling qY and divide by q to get a splitting of the **Q**-filling Y.

Now let σ be a simplicial triangulation of S^2; let $X(\sigma) \in Z_2$ be either one of the 2-cycles that arises from σ by orienting its 2-simplices; and write $\mathrm{Zvol}(\sigma) = \mathrm{Zvol}(X(\sigma))$. Let $\mathrm{tetvol}(\sigma)$ be the minimum number of 3-simplices required to extend σ to a simplicial triangulation τ of B^3. We will show that

$$\mathrm{Zvol}(\sigma) = \mathrm{tetvol}(\sigma).$$

Certainly $\mathrm{Zvol} \leq \mathrm{tetvol}$, because any extension τ induces a filling of $X(\sigma)$. Theorem 2 states that any taut filling of $X(\sigma)$ arises in this way.

The proof of Theorem 2 relies on Theorem 1. We use induction, with base case σ a tetrahedron. In any taut filling Y of $X(\sigma)$, thought of as a multiset of oriented 3-simplices, some $t \in Y$ meets σ in at least two faces. Ignoring the base case, $Y - t$ is a taut filling of $X(\sigma) - \partial t = X(\sigma')$, where σ' is either a simplicial triangulation of S^2, or two such triangulations σ_1, σ_2 fused along an edge. In the first case induction yields a triangulation of B^3, and we glue on t. In the second case by Theorem 1 $Y - t$ splits into taut fillings of $X(\sigma_1), X(\sigma_2)$; by induction both give triangulations of B^3, which we glue onto t. So we wind up with some kind of triangulation τ of B^3, built from the simplices of Y. The wrinkle is that we need to ensure that in Y there are no identifications beyond those in τ.

We don't know what happens for spheres of dimension 3 or greater.

2 Background

Our interest in Zvol stems from the work of Sleator, Tarjan, and Thurston [5]. They observed that coning from a vertex of maximal degree shows that a v-vertex triangulation σ of S^2 has tetvol $\leq 2v - 10$. They then produced examples for which tetvol $= 2v - 10$, provided v is larger than some unspecified bound, and conjectured that such examples should exist for any $v \geq 13$. Their approach was to realize σ as an ideal hyperbolic polyhedron with volume $2v - O(\log(v))$ when measured in bushels, a *bushel* being volume of an equilateral ideal tetrahedron, which is maximal. This implies tetvol $\geq 2v - O(\log(v))$. From there they worked their way up to showing that tetvol $= 2v - 10$. They suggested [5, p. 697] that in fact Qvol $= 2v - 10$, which would immediately imply tetvol $= 2v - 10$. Mathieu and Thurston [3] produced a different class of examples with Qvol $= 2v - 10$, still under the assumption that v is sufficiently large. In [1] we produced examples for any $v \geq 13$ with Qvol $= 2v - 10$, confirming the conjecture of Sleator, Tarjan, and Thurston. Theorem 2 here now tells us that whenever we have Qvol $= 2v - 10$, any taut filling must arise from a triangulation of the ball.

About Qvol versus Zvol. In [1] we describe triangulations of S^2 with Qvol $<$ Zvol. Since Qvol and Zvol add under connected sum, the gap between them can be arbitrarily large (Ellison [2]). Some examples with a gap have maxdeg $= 6$, but in all such examples that we have seen, the gap is < 1, so that $\lceil \text{Qvol} \rceil =$ Zvol. Taking connected sums of these examples produces vertices of degree ≥ 7.

3 Taut fillings

As long as $n > 0$, as we will continue to assume throughout, we can think of an n-chain $X \in C_n$ as a multiset of non-cancelling oriented simplices:

$$X = \sum_{t \in X} t.$$

In this sum t denotes an oriented simplex

$$t = [x_0, \ldots, x_n] = [x_{\pi(0)}, \ldots, x_{\pi(n)}], \quad \pi \text{ an even permutation,}$$

and each unoriented simplex contributes a number of terms corresponding to its multiplicity.

The size of X as a multiset is its L_1-norm $|X|$. Write $U \subset X$ if U is a sub-multiset of X. This happens just when $|X| = |U| + |X - U|$.

Proposition 1. *If Y is taut and $U \subset Y$ then U is taut.*

Proof. This is clear, but let's drag it out. If U is not taut, there is a smaller filling V of ∂U. Take $Y' = Y - U + V$. In terms of multisets, this means that we substitute V for U, and then do any required cancellation of oppositely oriented simplices of $Y - U$ and V. We have

$$\partial Y' = \partial Y - \partial U + \partial V = \partial Y$$

and

$$|Y'| \leq |Y - U| + |V| = |Y| - |U| + |V| < |Y|,$$

contradiction. $\square$

4 Coning

For $x \in V$, $U \in C_n$ let

$$\text{nbhd}(x, U) = \sum_{t \in U : x \in \text{vertices}(t)} t,$$

$$\deg(x, U) = |\text{nbhd}(x, U)|,$$

$$\text{maxdeg}(U) = \max_x \deg(x, U).$$

The *cone from x to U* is the $(n+1)$-chain

$$\text{cone}(x, U) = \sum_{t \in U} \text{adj}(x, t),$$

consisting of all the non-trivial oriented $(n+1)$-simplices $\text{adj}(x, t) = [x x_0 \ldots x_n]$ obtained by adjoining x to $t = [x_0 \ldots x_n] \in U$. If $x \in \text{vertices}(t)$ then t doesn't contribute to the sum, so $|\text{cone}(x, U)| = |U| - \deg(x, U)$.

If X is closed then $\partial\, \text{cone}(x, X) = X$, so

Proposition 2.

$$\text{Zvol}(X) \leq |X| - \text{maxdeg}(X) \qquad \square$$

If $U \in Z_n$ and $x \notin \text{vertices}(U)$ then $\deg(x, U) = 0$ and $|\text{cone}(x, U)| = |U|$. In this case we call $\text{cone}(x, U)$ a *complete cone*. By Proposition 2 a non-trivial complete cone is not taut. (This is a long-winded way of saying that instead of coning from $x \notin U$, we could have coned from some $x \in U$.) Thus:

Proposition 3. *If Y is taut it contains no non-trivial complete cone.* $\square$

Call $x \in \text{vertices}(Y) \setminus \text{vertices}(\partial Y)$ an *internal vertex* of $Y \in C_{n+1}$. If x is internal to Y then $\text{nbhd}(x, Y)$ is a complete cone, so as observed above:

Proposition 4. *If Y is taut then it has no internal vertices.* $\square$

5 Almost disjoint unions

Recall that if $X, Y \in Z_n$ we call $X + Y$ an almost disjoint union if

$$|\text{vertices}(X) \cap \text{vertices}(Y)| \leq n + 1.$$

The most interesting special case is a connected sum, where $t = [c_0, c_1, \ldots, c_n]$ occurs once in X and $-t = [c_1, c_0, \ldots, c_n]$ occurs once in Y. For example, if $n = 1$ with X a cycle of length p and Y a cycle of length q the connected sum $X + Y$ is a cycle of length $p + q - 2$. Zvol adds, because $\text{Zvol}(X) = p - 2$, $\text{Zvol}(Y) = q - 2$, and

$$\text{Zvol}(X + Y) = (p + q - 2) - 2 = \text{Zvol}(X) + \text{Zvol}(Y).$$

In this case not every filling of $X + Y$ splits. We want to show that fillings do split when $n \geq 2$.

For $A \subset V$, $p \in A$ define

$$\pi_{A,p} : V \to A$$

$$\pi_{A,p}(x) = x \textbf{ if } x \in A \textbf{ else } p$$

and let

$$K_*(A, p) : C_*(V) \to C_*(A)$$

be the induced chain map. This is a projection of $C_*(V)$ onto $C_*(A)$.

Let A, B be finite vertex sets sharing the vertices $C = A \cap B$. For practice with this overloading of the letter 'C', observe that $C_{|C|}(C)$ is trivial, as is $Z_{|C|+1}(C)$.

Let

$$C_*(A,B) = C_*(A) \oplus C_*(B).$$

For $p,q \in C$ define

$$g_*(p,q) = K_*(A,p) \oplus K_*(B,q) : C_*(A \cup B) \to C_*(A,B).$$

Proposition 5. *Let $(X,Y) \in C_n(A,B)$. If $|C| \leq n$ we can recover (X,Y) from $X+Y$. If $(X,Y) \in Z_n(A,B)$ this holds also when $|C| = n+1$.*

Proof. We can assume $|C| \geq 1$. (Add a brand new point to C if necessary.) We claim that for any $p,q \in C$ (not necessarily distinct) we have

$$g_n(p,q)(X+Y) = (X,Y).$$

Indeed,

$$K_n(A,p)(X+Y) = X + K_n(A,p)(Y).$$

The second term belongs to $C_n(C)$, which is trivial when $|C| \leq n$. If $Y \in Z_n(B)$ the second term belongs to $Z_n(C)$, which is trivial when $|C| \leq n+1$. $\square$

Theorem 1. *If $|C| \leq n+1$, for all $(X,Y) \in Z_n(A,B)$ we have*

$$\mathrm{Zvol}(X+Y) = \mathrm{Zvol}(X) + \mathrm{Zvol}(Y).$$

And as long as $n \geq 2$, for any $Z \in \mathrm{taut}(X+Y)$ we have $Z = Z_X + Z_Y$ with $Z_X \in \mathrm{taut}(X)$, $Z_Y \in \mathrm{taut}(Y)$.

Note. This result resembles Theorem 6.2 of Pournin and Wang [4] about flip paths. Like theirs, our proof uses a variation on the normalization technique of STT [5, Lemma 7]. It would be nice to fit these results under one roof.

Proof. We can assume $|C| = n+1$, as this is the hardest case. And we might as well go ahead and take $n = 2$, $|C| = 3$, as this case illustrates all the issues.

Take any $Z \in \mathrm{taut}(X+Y) \subset C_3(A \cup B)$. Pick distinct points $p,q \in C$, and let

$$(Z_X, Z_Y) = g_{n+1}(p,q)(Z).$$

(For now we'll suppress the dependence of Z_X, Z_Y on p,q.)

Because $g_*(p,q)$ is a chain map we have $\partial_{n+1} Z_X = X$, $\partial_{n+1} Z_Y = Y$:

$$(\partial_{n+1} Z_X, \partial_{n+1} Z_Y) = \partial_{n+1}(g_{n+1}(p,q)(Z)) = g_n(p,q)(\partial_{n+1} Z) = g_n(p,q)(X+Y) = (X,Y).$$

We want to show that

$$|Z| \geq |Z_X| + |Z_Y|$$

because then we'll be done:

$$\mathrm{Zvol}(X+Y) = |Z| \geq |Z_X| + |Z_Y| \geq \mathrm{Zvol}(X) + \mathrm{Zvol}(Y)$$

To prove that $|Z| \geq |Z_X| + |Z_Y|$, we'll show that under the map $g_{n+1}(p, q)$, any $t \in Z$ dies either in Z_X or in Z_Y.

We'll call any 2-simplex $t \in Z$ a 'tet', short for 'tetrahedron'.

Say that a tet $t \in Z$ has type $CCXY$ if $t = \pm[c_1 c_2 x_1 y_1]$ for $c_1, c_2 \in C$, $x_1 \in A \setminus C$, $y_1 \in B \setminus C$. Similarly for types $XXXX$, $CXXX$, $CXXY$, $XXXY$, etc. The first two are pure X cases, meaning that they live in $C_3(A)$; the last two are hybrid cases.

Any XX.. tet dies in Z_Y; any YY.. tet dies in Z_X.

The remaining cases to check are the hybrid case $CCXY$ and the pure cases $CCCX$, $CCCY$. The more interesting case is $CCXY$: The key is that since $|C| = 3$, $\{c_1, c_2\}$ cannot be disjoint from $\{p, q\}$. As for $CCCX$, these must die in Z_Y because $q \in \{c_1, c_2, c_3\} = C$. Ditto for $CCCY$.

So

$$|Z| = |Z_X(p, q)| + |Z_Y(p, q)|,$$

and Zvol adds.

Note how we're now emphasizing the possible dependence of Z_X, Z_Y on p, q. When $n = 1$ the choice of p, q can indeed make a difference: We can get a different pair (Z_X, Z_Y) if we switch p and q. (Think about the connected sum of two cycle graphs.)

But when $n \geq 2$ we will now show that all tets are pure X or pure Y, so Z_X consists of all the pure X tets, and ditto for Z_Y. This will make $Z = Z_X + Z_Y$ with $Z_X \in \mathrm{taut}(X)$, $Z_Y \in \mathrm{taut}(Y)$.

Again our test case is $n = 2$.

The key observation is that, now that we know that every tet dies on one side or the other, we know that no tet can die on both sides, because that would make $|Z| > \mathrm{Zvol}(X) + \mathrm{Zvol}(Y)$.

An $XXYY$ tet would die on both sides, so there can be none of these.

Any $pqXY$, $pXXY$, or $qXYY$ would die on both sides, and p, q are arbitrary, so this rules out all $CCXY, CXXY, CXYY$.

The only remaining hybrids are $XXXY$ and $XYYY$. Let's rule out $XXXY$, leaving $XYYY$ to symmetry.

Fix any $y_0 \in B \setminus C$, and suppose there is some $XXXy_0$ tet in Z. Let $U \subset Z$ consist of all such $XXXy_0$ tets. We claim that $y_0 \notin \text{vertices}(\partial U)$. For let s be any 2-simplex of the form XXy_0, the only kind of 2-simplex containing y_0 that could belong to ∂U. Any $t \in Z$ of which s or $-s$ is a face must be a hybrid tet, and the only possibility is $XXXy_0$, so $t \in U$. Because $\partial Z = 0$ the signed multiplicity of s vanishes in ∂Z, hence also in ∂U. So $y_0 \notin \text{vertices}(\partial U)$, making U a complete cone on y_0, contradiction. So there is no such $XXXy_0$ tet, ruling out $XXXY$, and with it $XYYY$.

So there are no hybrid tets, meaning that Z splits, as claimed.

That takes care of $n = 2$. Let's quickly look at $n = 3$. As $CCXY$ was the crux for $n = 2$, $CCCXY$ is the crux for $n = 3$. $pqCXY$ dies on both sides, and p, q are arbitrary, so $CCCXY$ dies.

Finally, let's consider what goes wrong when $n = 1$. Here the only hybrid type is CXY, and neither pXY nor qXY dies on both sides. so we can't rule this type out. Failing that, the XXy_0 tets needn't form a complete cone, because $[x_0x_1y_0)$ can continue across $[x_1y_0]$ to $-[px_1y_0]$, so we can't rule out XXY either. $\square$

The foregoing proof works equally well over $\mathbf{Q}$, providing we permit ourselves to work with fractional multisets. Alternatively, we can clear denominators as in the introduction above. Either way, we have:

Corollary 1. Qvol *adds under almost disjoint union, and for $n \geq 2$ taut $\mathbf{Q}$-fillings split.* $\square$

6 Triangulations

We turn now to filling simplicial triangulations of S^2. Let's begin by establishing terminology.

An *n-simplex* s is simply a set of size $n+1$. Its *faces* are its subsets, which are k-simplices with $-1 \leq k \leq n$, -1 being the dimension of the empty simplex. A *simplicial complex* σ is a finite collection of simplices closed under taking faces: If $t \subset s \in \sigma$ then $t \in \sigma$.

For any k-simplex $s \in \sigma$ define the *link*

$$\text{link}(s, \sigma) = \{t : t \cap s = \emptyset, s \cup t \in \sigma\}.$$

This is a simplicial complex, but (except for $\text{link}(\emptyset, \sigma) = \sigma$) it is not a subcomplex of σ, because we are taking simplices in σ and knocking down their dimension by $k+1$.

We are interested in particularly nice simplicial complexes called normal pseudomanifolds, or as we will prefer to say, *clean n-complexes.*

1. To start with, a clean complex must be *pure*, meaning that every simplex of σ belongs to some n-simplex in σ. To determine σ we can specify its n-simplices, and then throw in all their subsets. So we can confound a pure complex with the set of its n-simplices.

2. A clean complex is a *pseudomanifold*: Every n-simplex abuts at most one other n-simplex across any given $n-1$-simplex. Another way to say this is that the link $\mathrm{link}(s, \sigma)$ of any $n-1$-simplex s consists of either a single point (in which case $s \in \partial\sigma$ is a *boundary simplex*), or two points ($s \in \sigma \setminus \partial\sigma$ is an *interior simplex.*)

3. A clean complex is *normal*: For any simplex s of dimension $0, \ldots, n-2$, $\mathrm{link}(s, \sigma)$ is connected. (Some definitions extend this requirement to the empty simplex of dimension -1; this forces σ to be connected.) Being normal means that σ is just what you get by gluing its n-simplices along shared faces, without extra identifications. This rules out, for example, an icosahedron with a pair of opposite vertices identified.

If σ is clean, $\mathrm{link}(s, \sigma)$ is clean. The boundary $\partial\sigma$ of σ is clean, and closed: $\partial\partial\sigma = \{\emptyset\}$.

If M is an n-manifold, say that a simplicial complex σ is a *simplicial triangulation* of M if the geometric carrier of σ is homeomorphic to S^2. In this case σ is necessarily a clean complex. Secretly we imagine that we've prescribed a specific homeomorphism, at least up to the point of picking out one particular orientation $X(\sigma) \in C_n$ if M is oriented, but we don't insist upon this because nothing will depend on which orientation we pick.

This notion of triangulation is not as general as you might want for some purposes. The double cover of a triangle is not a simplicial triangulation of S^2, because that would requires two distinct 2-simplices to have the same three edges. Nor can any 2-simplex have two of its edges glued to one another. Thurston [6] allows such triangulations, but it is unclear how important these are to his theory of shapes of surfaces. We don't allow them.

We continue to think of a chain $Y \in C_n$ as a multiset of oriented n-simplices. If this is only nominally a multiset (all multiplicites are 1) we'll call Y *simplicial*, and view it as a pure simplicial complex. (To stickle, in doing this we are viewing oriented simplices as a subclass of simplices, and confounding a set of n-simplices with a pure n-complex.) We'll call Y *clean* if it is simplicial and the associated simplicial complex is clean.

7 Filling a triangulation of the 2-sphere

Theorem 2. *Let σ be a simplicial triangulation of S^2, and Y a taut filling of σ. Then Y is clean, and arises from a simplicial triangulation of B^3.*

Proof.

Let v, e, f count the vertices, edges, and faces of σ. We have $e = 3v - 6$, $f = 2v - 4$. (Check: $3f = 2e$, $v - e + f = 2$.)

Consider a counter-example pair (σ, Y) for which $\mathrm{Zvol}(\sigma) = |Y|$ is minimal. Obviously $v > 4$. We can assume that σ is prime (not a connected sum along a triangle), and in particular (this is all we will need) that there is no vertex of degree 3.

Call a tet $t \in Y$ *eligible* if it shares two faces with σ. (It can't share more, since σ has no vertex of degree 3.) Since

$$|Y| = \mathrm{Zvol}(\sigma) \leq f - \mathrm{maxdeg}(\sigma)$$

there must be at least $\mathrm{maxdeg}(\sigma)$ disjointly eligible tets, but we'll only need two: One with faces $s_1, s_2 \in \sigma$, the other with disjoint faces $s_3, s_4 \in \sigma$. The s_3, s_4 tet can't also have s_1 or s_2 as a face, as then there would be a degree-3 vertex in σ. So for any face s of σ there is an eligible tet without s as a face.

Now as indicated in the introduction above, when we remove an eligible tet t, we get a taut filling $Y - t$ of σ', where σ' is either (1) a triangulation of S^2, or (2) the almost disjoint union of two triangulations σ_1, σ_2 of S^2 that are joined along an edge of t. We claim $Y - t$ is simplicial: In the case (1) $Y - t$ is actually clean, by minimality of $|Y|$. In case (2) $Y - t$ is a taut filling of an almost disjoint union, and as such splits $Y - t = Y_1 + Y_2$ where Y_i is a clean taut filling of σ_i. In this case $Y - t$ isn't clean, because the link of the common edge shared by σ_1 and σ_2 is disconnected, but $Y - t$ is still simplicial.

The crux here is to show that Y is clean. For starters, Y is simplicial. The only way Y could fail to be simplicial is if t has multiplicity 2. But then $Y - t'$ wouldn't be simplicial for eligible t' distinct from t, contradiction.

Now, then:

1. Y is a pseudomanifold. For suppose some 2-simplex s occurs more than twice as the boundary of a 3-simplex of Y. If $s \notin \sigma$, it must occur at least twice plus and twice minus; after removing any eligible tet it still occurs either twice plus or twice minus in $Y - t$ in case (1), or in one or the other of Y_1, Y_2 in case (2), contradicting minimality. If $s \in \sigma$, it occurs at least twice plus in Y: Remove some eligible tet that does not have s as a face to get a contradiction.

2. Y has no edge $\{a, b\}$ whose link is disconnected. For the link is a 1-dimensional complex whose edges correspond to tets of Y that contain $\{a, b\}$. The only way a component of the link can have fewer than three vertices is if consists of a single edge $\{c, d\}$, corresponding to a tet $t = \{a, b, c, d\} \in Y$. In this case $\{a, b\}$ must be an edge of σ, and $\{a, b, c\}, \{a, b, d\}$ the faces of σ that are adjacent along $\{a, b\}$. These are the only faces of σ that contain $\{a, b\}$, so no other component of the link can reduce to a single edge. If the link is disconnected, removing any eligible tet other than t will leave it disconnected, contradicting minimality.

3. Y has no vertex whose link is disconnected: For if $\mathrm{link}(\{a\}, Y)$ is not connected, the only way removing a single tet t can render the link connected is if one component of the link has a single 2-simplex $\{b, c, d\}$ and $t = \{a, b, c, d\}$. In this case $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$ must all be faces of σ, making a a vertex of σ of degree 3, contradiction.

4. Hence Y is clean. $\square$

8 Shelling

A *shelling* of a simplicial triangulation τ of B^3 is an ordering $(t_1, \ldots, t_{|\tau|})$ of its tets such that gluing on one at a time maintains a triangulation of B^3, i.e., so that for $k = 1, \ldots, |\tau|$ the subcomplex τ_k generated by $\{t_1, \ldots, t_k\}$ is a simplicial triangulation of B_3. Each new tet will be glued on along one face (v increases by 1); two faces (v stays the same); or three faces (v decreases by 1). Call the shelling *monotone* if you never glue along three faces, so that v never decreases. Call τ *freely monotone shellable* if any $t \in \tau$ can serve as the initial tet of a monotone shellinb.

Theorem 3. *If σ is a simplicial triangulation of S^2, any taut filling Y of $X(\sigma)$ is a freely monotone shellable simplicial triangulation of B^3.*

Proof. This is a corollary of the proof of Theorem 2 that insists on being called a separate theorem. $\square$

References

[1] Peter Doyle, Matthew Ellison, and Zili Wang, *Filling a triangulation of the 2-sphere*, 2024, `https://doi.org/10.48550/arXiv.2303.10773`.

[2] Matthew Ellison, *Lower bounds and integrality gaps in simplicial decomposition*, 2024, `https://doi.org/10.48550/arXiv.2404.01279`.

[3] Claire Mathieu and William Thurston, *Rotation distance using flows*, 1992, `https://doi.org/10.48550/arXiv.2409.17905`.

[4] Lionel Pournin and Zili Wang, *Strong convexity in flip-graphs*, 2021, `https://doi.org/10.48550/arXiv.2106.08012`.

[5] Daniel D. Sleator, Robert E. Tarjan, and William P. Thurston, *Rotation distance, triangulations, and hyperbolic geometry*, J. Amer. Math. Soc. **1** (1988), 647–681, `https://doi.org/10.2307/1990951`.

[6] William P. Thurston, *Shapes of polyhedra and triangulations of the sphere*, 1998, `https://doi.org/10.48550/arXiv.math/9801088`.