Solution of Montmort's Problême du Treize

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Abstract

We determine the exact value to the dealer of the game of Treize. This solves a problem posed about 300 years ago by Pierre Remond de Montmort.

The game of Treize

'Treize', which is French for 'thirteen', is the name of a card game that features prominently in Pierre Remond de Montmort's 'Essay d'Analyse sur les Jeux de Hazard'. This famous book was first published in 1708; a second edition, revised and augmented by extensive correspondence between Montmort and Jean and Nicolas Bernoulli, appeared in 1713. Here, from the second edition, is Montmort's description of the game.

> The players draw to see who will be the dealer. Let's call the dealer 'Pierre', and let's suppose that there are as many other players as you like. Pierre takes a full deck of 52 cards, shuffles them, and deals them out one after the other, calling out 'un' as he turns over the first card, 'deux' as he turns over the second, 'trois' as he turns over the third, and so on up to the thirteenth, which is a king. ['un', 'deux', 'trois', 'quatre', 'cinq', 'six', 'sept',

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'huit', 'neuf', 'dix', 'onze', 'douze', 'treize'.] Now if, in this whole series of cards, he never once turns over the card he is naming, he pays out what each other player has put up for the game, and the deal passes to the player sitting to his right.

But if in this sequence of thirteen cards he happens to turn over the card he is naming, for example, if he turns over an ace as he calls out 'un', or a two as he calls out 'deux', or a three as he calls out 'trois', etc., he collects all the money that is in play, and begins over as before, calling out 'un', and then 'deux', etc.

It may happen that Pierre, having won several times, and beginning again at 'un', does not have enough cards in his hand to reach all the way to 'treize'; then he must, when he runs out of cards, shuffle the cards, allow them to be cut, and take from the deck as many cards as he needs to continue the game, starting again where he left off in the preceeding hand. For example, if in turning over the last card he called out 'sept', in turning over the first card of the new deck, after it has been cut, he must call out 'huit', and then 'neuf', etc., up to 'treize'—unless he wins again, in which case he begins over, calling out 'un', and then 'deux', and so forth, as has already been explained. Thus it appears that Pierre may win several hands in a row, and may even continue the game ad infinitum.

Montmort's problem

Montmort was interested in the following problem, which is the first of the four 'problêmes à résoudre' that Montmort set as a challenge at the end of part 4 of his book.

Premier problême. SUR LE JEU DU TREIZE. Déterminer generalement quel est à ce jeu l'avantage de celui qui tient les cartes.

In English, 'find in general the expected value to the dealer of the game of Treize'. The assumption is that the amount of money in play each time is always the same: If the dealer wins, the other players ante up the same amount as before. Montmort denotes the amount at stake each time by A. Thus if Pierre loses on the first round, he wins $-A$. If Pierre wins the first

two rounds and then loses, he wins A. If the expected number of rounds he wins before finally losing is T, then the expected value of the game to him will be $(T-1)A$, so Montmort's problem amounts to determining T. Note that we may as well assume that Pierre has only one opponent (inevitably named 'Paul').

Montmort solved a number of simpler, related problems, beginning with the now-familiar 'hat-check problem'. However, the general version of the 'Problème du Treize' remained open. Montmort's correspondents had this to say about the problem.

> Jean Bernoulli to Montmort, 17 March 1710:
> The four problems that you propose at the end of your treatise are interesting, but the first appears to me to be unsolvable due to the length of the calculation which it would require, and a lifetime would not suffice to complete it.

> Nicolas Bernoulli to Montmort, 26 February 1711:
> I have not yet attempted the general solution of the Problem on the Game of Treize, because it appears to me to be nearly impossible.

Fortunately for us, modern electronic computers can carry out calculations that are 'nearly impossible'. Thus we are able here to present the general solution to Montmort's Problème du Treize.

Can the dealer quit playing?

As we have indicated above, we interpret the 'expected value to the dealer' to mean the expected winning of Pierre while he remains dealer. Nicolas Bernoulli objected to this, on the grounds that as soon as the dealer loses a round, the deal passes to the next player, and now the dealer is at a disadvantage. We can't resist reproducing the back-and-forth between Montmort and Bernoulli on this question.

> Nicolas Bernoulli to Montmort, 26 February 1711:
> It is pertinent to observe here, that according to the rules of the game, we cannot suppose that the game is over when the dealer loses, because then he must cede the deal to another player,

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and the game continues; correcting for the disadvantage to the dealer once he loses the deal, the expected value to the dealer is ...half of the value found [assuming there is only one other player]. If we suppose there to be several players against the dealer, and if their number is n, the dealer's expected value [is further multiplied by] n, and that of the other players [by] either $n-2$, or $n-4$, or $n-6$, etc., according to how far each one sits to the right of the dealer. This remark applies to all games in which the deal passes from one player to another.

Montmort to Nicolas Bernoulli, 10 April 1711:
You say that according to the rules of the game, we must not suppose that the game is over when the dealer has just lost, because then, you say, he must cede the deal to another player. ...You then extend this remark to the game of Lansquenet, and it appears that you would wish to apply this to all sorts of games. For my part, I believe that I have reasons to think otherwise: I will explain them to you. First, in regard to Treize, it is certain that the dealer upon losing the hand is in no way obliged to continue to play, and in addition he is not obliged to bet the same amount; on the contrary, it happens in the game that those who have observed, as it is easy to observe in practice, that the dealer has an advantage, ...bet little money when they are not the dealer. It remains to remark that in this game the size of the bets goes up and down, as well as the number of players, and that in Lansquenet the number of players can decrease from one hand to another of the same dealer. Thus in my opinion one cannot say anything useful and certain about these games, except by deciding to determine for each play the advantage or disadvantage to the dealer with respect to bets of fixed size by the players. When I considered in Lansquenet the expected value to the dealer, it was only for the sake of elegance, because in the end this result is only justified if the number of players is always the same while Pierre retains the deal, which is not certain. It seems to me that in order to know as well as it is possible to know the chances of a game, for example Lansquenet, it suffices to know for a given number of players and size of bets that there is such-and-such an advantage or disadvantage to each of the players,

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according to the different seats they occupy.

Nicolas Bernoulli to Montmort, 10 November 1711:

I am surprised, Monsieur, at your objections to my remarks on games in which the deal passes from one player to another. It seems to me that you are wrong to raise against me objections which argue just as much against you as against me; For if you are prepared to suppose, for example in Lansquenet, that the number of players and bets is always the same, and that the game continues as long as Pierre has the deal, why should I not suppose the same thing, even after Pierre loses the deal? You say that one can say nothing useful or certain about these games, because the bets and the number of players may constantly vary: This is true, and this is precisely the reason why one must make a certain hypothesis which one can hold to in making the calculation. I have therefore made this hypothesis, namely that the game continues after the dealer loses the hand, because it is more natural and in keeping with what ordinarily happens than is your hypothesis, which supposes that the game continues only as long as Pierre has the deal, which is something that almost never happens in real games, above all when the players know that it is advantageous to have the deal. But you argue against me further that, for example in Treize, the dealer upon losing the deal is not at all obliged to keep playing, to which I reply that an honest man must consider himself obligated to keep playing, even if it has not been expressly agreed upon; for it is certain that ordinarily one begins a game with the idea of playing a considerable number of rounds, and not to quit after the first loss, and thus the players tacitly agree to continue the game for a certain period. It would therefore not be permissible to leave the game immediately, after having had the advantage of the deal—that is, assuming that one does not wish to be known as a man who thinks more of making off with other people's money than of having a good time with them. Thus you see, Monsieur, that you would do well to take into consideration, not only the advantage to the dealer while retaining the deal, but also the disadvantage after losing it.

Montmort to Nicolas Bernoulli, 1 March 1712:

> I will not reply to what you have said about games where the deal rotates, as I don't believe I have anything to add to what I have already said . . . .

Outline of the solution

It is the periodic reshufflings that make it possible to analyze this game completely. Pierre's expected winnings after a shuffle depend only on the point at which he will be starting the count. In particular, if Pierre happens to win a point as the last card is turned over, after the shuffle he will begin counting again from 'un', and his expected winnings from here on are equal to his expected winnings when he first got the deal; if Pierre calls out 'sept' as he turns over the last card, and the card is not a seven, after the shuffle he will begin counting again from 'huit', and his expected winnings from here on are just what they would be if for some strange reason the custom were that when you first got the deal you started counting from 'huit', but then continued exactly as specified by the real rules of the game.

Since Pierre's winnings after a shuffle depend only on where he will be beginning the count, we can analyze Treize in terms of a Markov chain. Each step of the chain corresponds to one pass through the deck. On each pass through the deck, Pierre either loses the deal, or survives to make another pass through the deck. And when a new pass through the deck begins, all that matters is where Pierre is in the count—that's what makes this a Markov chain. Of course what we are really interested in is what happens when the count begins at 'un', but this is bound together with what happens when it begins at 'deux', or 'trois', or 'quatre', etc.

Specifically, we consider a Markov chain with 14 states, The first 13 states, labelled $0, 1, \ldots, 12$, are 'live' states, where Pierre continues as dealer, beginning the next pass through the deck counting from $i + 1$. The remaining state, labelled 13, is the 'dead' state, where Pierre has lost the deal, and the game is over; this is what is called an 'absorbing' state, and once the chain reaches this state it stays there forever.

Let $P_{ij}, 0 \leq i, j \leq 13$ denote the matrix of transition probabilities for this Markov chain, and let Q denote the 13-by-13 submatrix of P giving the transition probabilities between the 13 live states: $Q_{ij} = P_{ij}, 0 \leq i, j \leq 12$. Thus Q_{ij} is the probability that Pierre begins the count after the shuffle at $i + 1$, survives until the next shuffle, and begins counting again at $j + 1$,

and $\sum_j Q_{ij}$ is the total probability that when Pierre starts in state i he will survive until the next shuffle, i.e. fail to get 'absorbed'.

Next let $e_i, 0 \le i \le 12$ denote the expected number of points Pierre scores on one step of the chain—that is, in one pass through the deck—starting from state i.

To determine the expected number of points scored throughout the whole game, rather than on a single round, let $N = (I - Q)^{-1} = I + Q + Q^2 + \ldots$, and let $v = Ne$. The entry $N_{ij}, 0 \le i, j \le 12$ equals the expected number of times in state j when the Markov chain is started in state i; the component $v_i = \sum_j N_{ij} e_j$ equals the expected number of points scored throughout the rest of the game when starting from state i. Starting from state 0, the expected number of points scored is v_0.

The value of the game of Treize to the dealer is one less than the number of points scored, so with the notation we've established, the expected value to the dealer is $v_0 - 1$. To determine this, we need to compute the entries of the 13-by-13 matrix P, and the vector e.