

CONSIDERATIO
PROGRESSIONIS CUIUSDAM AD
CIRCULI QUADRATURAM INVE-
NIENDAM IDONEAE.

AUCTORE

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§. 1.

Posita arcus cuiusdam in circulo, cuius radius sit $= r$, tangente $= t$, erit ipse arcus $= \int \frac{dt}{r + \sqrt{r^2 - t^2}}$; si iam loco differentialium dt substituantur particulae tangentes finitae quidem, sed valde exiguae, atque integrationis loco actualis eiusmodi particularum additio perficiatur, expressio prodibit eo propius ad arcum propositum accedens, quo minores capiuntur particulae tangentes t . Sic diuisa tangente in n partes aequales, quarum quaelibet erit $\frac{t}{n}$, vicem differentialis dt subeunda, loco t successive poni debent valores $\frac{t}{n}, \frac{2t}{n}, \frac{3t}{n} \dots$ vsque ad $\frac{nt}{n}$; quo facto arcus cuius tangens est t aequabitur huic progressioni $\frac{nt}{nn + tt} + \frac{nt}{nn + 9tt} + \frac{nt}{nn + 25tt} + \dots + \frac{nt}{nn + n^2tt}$ quae expressio eo minus a vero arcus valore differet, quo maior capiatur numerus n . Semper autem haec expressio nimis erit parua, nisi pro n sumatur numerus reuera infinitus.

§. 2. Cum igitur sumto pro n numero finito ista progressio $\frac{nt}{n^2 + t^2} + \frac{nt}{n^2 + 4t^2} + \frac{nt}{n^2 + 9t^2} + \dots + \frac{nt}{n^2 + n^2t^2}$ eo propius exprimat arcum cuius tangens est t , quo maior fuerit numerus n ; perpetuo autem hoc modo valor
pro-

prodeat nimis parvus, inuestigabo, quantum ista expressio quouis casu a vera arcus longitudine deficiat. Quodsi enim defectus commode atque ad calculum accommodate exhiberi queat, per seriem vehementer conuergentem, ista methodus cuiusque arcus longitudinem determinandi perquam facilis et idonea videtur.

§. 3. Ad hoc inuestigandum singulos expressionis terminos methodo consueta in progressionem geometricam resoluo infinitam, vt sequitur

$$\begin{aligned} \frac{nt}{n^2+t^2} &= \frac{t}{n} - \frac{t^3}{n^3} + \frac{t^5}{n^5} - \frac{t^7}{n^7} + \text{etc.} \\ \frac{nt}{n^2+4t^2} &= \frac{t}{n} - \frac{2^2t^3}{n^3} + \frac{2^4t^5}{n^5} - \frac{2^6t^7}{n^7} + \text{etc.} \\ \frac{nt}{n^2+9t^2} &= \frac{t}{n} - \frac{3^2t^3}{n^3} + \frac{3^4t^5}{n^5} - \frac{3^6t^7}{n^7} + \text{etc.} \\ \frac{nt}{n^2+16t^2} &= \frac{t}{n} - \frac{4^2t^3}{n^3} + \frac{4^4t^5}{n^5} - \frac{4^6t^7}{n^7} + \text{etc.} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \frac{nt}{n^2+n^2t^2} &= \frac{t}{n} - \frac{n^2t^3}{n^3} + \frac{n^4t^5}{n^5} - \frac{n^6t^7}{n^7} + \text{etc.} \end{aligned}$$

§. 4. Ponamus progressionis nostrae oblatae

$$\frac{nt}{n^2+t^2} + \frac{nt}{n^2+4t^2} + \frac{nt}{n^2+9t^2} + \dots + \frac{nt}{n^2+n^2t^2}$$

valorem iam esse actu determinatum, eumque esse = s; ac transformatio facta sequentem suppeditabit aequationem:

P 3

s =

$$s = \left\{ \begin{array}{l} + \frac{1}{n} (1^0 + 2^0 + 3^0 + \dots + n^0) \\ - \frac{1^5}{n^5} (1^2 + 2^2 + 3^2 + \dots + n^2) \\ + \frac{1^5}{n^5} (1^4 + 2^4 + 3^4 + \dots + n^4) \\ - \frac{1^7}{n^7} (1^6 + 2^6 + 3^6 + \dots + n^6) \\ + \frac{1^9}{n^9} (1^8 + 2^8 + 3^8 + \dots + n^8) \\ - \frac{1^{11}}{n^{11}} (1^{10} + 2^{10} + 3^{10} + \dots + n^{10}) \\ \text{etc. in infinitum} \end{array} \right.$$

§. 5. Quoniam in hac expressione coefficientes terminorum $\frac{1}{n}$, $\frac{1^5}{n^5}$, $\frac{1^5}{n^5}$, etc. sunt summae progressionum potestatum parium seriei numerorum naturalium; summae hae autem se habent sequenti modo

$$\begin{aligned} 1^0 + 2^0 + \dots + n^0 &= n \\ 1^2 + 2^2 + \dots + n^2 &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \\ 1^4 + 2^4 + \dots + n^4 &= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} \\ 1^6 + 2^6 + \dots + n^6 &= \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42} \\ 1^8 + 2^8 + \dots + n^8 &= \frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} - \frac{7n^5}{45} + \frac{2n^3}{9} - \frac{n}{30} \\ &\text{etc.} \end{aligned}$$

substituuntur hi valores definiti loco indefinitorum, ac pro-
dibit sequens aequatio

$$s = \left\{ \begin{array}{l} + t \\ - \frac{t^5}{5} - \frac{t^5}{2n} - \frac{t^5}{6n^2} \\ + \frac{t^5}{5} + \frac{t^5}{2n} + \frac{t^5}{3n^2} - \frac{t^5}{30n^4} \\ - \frac{t^7}{7} - \frac{t^7}{2n} - \frac{t^7}{2n^2} + \frac{t^7}{6n^4} - \frac{t^7}{42n^6} \\ + \frac{t^9}{9} + \frac{t^9}{2n} + \frac{2t^9}{3n^2} - \frac{7t^9}{135n^4} + \frac{2t^9}{9n^6} - \frac{t^9}{302n^8} \\ \text{etc. etc.} \end{array} \right.$$

cuius

cuius lex processus vltioris pendet a coefficientibus formulae generalis series summandi. Praecipue autem ad continuandam hanc seriem notari conuenit coefficientes vltimorum terminorum in quaque expressione, quae hanc tenent progressionem: $\frac{1}{6}$; $\frac{1}{30}$; $\frac{1}{42}$; $\frac{1}{30}$; $\frac{5}{84}$; $\frac{601}{19.210}$; $\frac{7}{6}$; $\frac{1617}{17.30}$; $\frac{43867}{39.42}$; $\frac{174611}{330}$; $\frac{854917}{6.22}$; $\frac{236164091}{5.546}$; quam hucusque produxisse sufficit.

§. 6. Disponantur termini inuentae expressionis secundum columnas a summo ad imum. extensas, atque ad legem, qua singulae columnae progrediuntur, ordinentur; quo facto erit $s =$

$$\begin{aligned} &+ t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \frac{t^{11}}{11} + \text{etc.} \\ &- \frac{t^2}{2n} (t - t^3 + t^5 - t^7 + t^9 - t^{11} + \text{etc.}) \\ &- \frac{t^2}{6n^2} (t - 2t^3 + 3t^5 - 4t^7 + 5t^9 - 6t^{11} + \text{etc.}) \\ &- \frac{t^4}{30n^4} (t - 5t^3 + 14t^5 - 30t^7 + 55t^9 - 91t^{11} + \text{etc.}) \\ &- \frac{t^6}{4.2n^6} (t - \frac{29}{5}t^3 + 42t^5 - 132t^7 + \frac{1001}{5}t^9 - 728t^{11} + \text{etc.}) \\ &\text{etc.} \end{aligned}$$

quae series omnes hanc tenent legem, vt potestas

$$\frac{t^m}{Nn^m} \text{ multiplicari debeat per istam seriem } t - \frac{(m+1)(m+2)}{2.} t^3 + \frac{(m+1)(m+2)(m+3)(m+4)}{2. 3. 4. 5.} t^5 + \text{etc.}$$

§. 7. Quanquam haec series ob m numerum integrum affirmatiuum in infinitum excurrit, tamen semper habet summam finitam, quae sequenti modo inuenietur. Ponatur tantisper seriei illius summa $= v$ erit $m v =$

$$\frac{m}{2} - \frac{m(m+1)(m+2)}{2. 2. 3.} t^3 + \frac{m(m+1)(m+2)(m+3)(m+4)}{2. 2. 3. 4. 5.} t^5 - \text{etc.}$$

$$= \frac{(1-tV-1)^{-m} - (1+tV-1)^{-m}}{2V-1} \quad \text{Haec autem expressio}$$

transmutatur in istam $mv = \frac{(1+tV-1)^m - (1-tV-1)^m}{2(1+tt)^m V-1}$. At

binomiis his actu ad potestatem exponentis m euectis pro-

$$\text{hibit per aliam seriem } mv = \frac{1}{(1+tt)^m} \left(\frac{mt}{1} - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} \right.$$

$t^3 + \frac{m(m-1)(m-2)(m-3)(m-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} t^5 - \text{etc.}$) quae ad nostrum in-

stitutum maxime est accommodata, cum sponte abrum-

patur, quando m est numerus integer affirmatiuus.

§. 8. Series ergo v , per quam terminus quisque $\frac{t^m}{n^m}$ multiplicari debet, nunc transmutata est in hanc $\frac{1}{m(1+tt)^m}$

$$\left(\frac{mt}{1} - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} t^3 + \text{etc.} \right); \text{quamobrem habebitur } s =$$

$$t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \frac{t^{11}}{11} + \text{etc.}$$

$$- \frac{t^3}{2n(1+tt)}$$

$$- \frac{t^5}{2 \cdot 6 n^2 (1+tt)^2} \cdot \frac{2t}{3}$$

$$- \frac{t^7}{4 \cdot 30 n^4 (1+tt)^4} \left(\frac{4t}{1} - \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} t^3 \right)$$

$$- \frac{t^9}{6 \cdot 42 n^6 (1+tt)^6} \left(\frac{6t}{1} - \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} t^3 + \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} t^5 \right)$$

$$- \frac{t^{11}}{8 \cdot 30 n^8 (1+tt)^8} \left(\frac{8t}{1} - \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} t^3 + \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} t^5 - \text{etc.} \right)$$

$$- \frac{t^{13}}{10 \cdot 66 n^{10} (1+tt)^{10}} \left(\frac{10t}{1} - \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} t^3 + \text{etc.} \right)$$

$$- \frac{t^{15}}{12 \cdot 13 \cdot 210 n^{12} (1+tt)^{12}} \left(\frac{12t}{1} - \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3} t^3 + \text{etc.} \right)$$

$$- \frac{t^{17}}{14 \cdot 6 n^{14} (1+tt)^{14}} \left(\frac{14t}{1} - \frac{14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3} t^3 + \text{etc.} \right)$$

$$- \frac{t^{19}}{16 \cdot 57 \cdot 30 n^{16} (1+tt)^{16}} \left(\frac{16t}{1} - \frac{16 \cdot 15 \cdot 14}{1 \cdot 2 \cdot 3} t^3 + \text{etc.} \right)$$

etc.

§. 9.

§. 9. Cum nunc huius expressionis prima series $t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \dots$ etc. illum ipsum circuli arcum denotet, cuius tangens est t , quem quaerere instituimus, fit z iste arcus atque manente $s = \frac{nt}{n^2+1^2} + \frac{nt}{n^2+4^2} + \frac{nt}{n^2+9^2} + \dots + \frac{nt}{n^2+n^2i^2}$, reperietur arcus $z = s + \frac{t^3}{2n(1+tt)} + \frac{1}{5} \cdot \frac{t^5}{2nn(1+tt)^2} \cdot 2t + \frac{1}{35} \cdot \frac{t^7}{4n^4(1+tt)^2} (4t - 4t^3) + \frac{t^9}{576(1+tt)^6} (6t - 20t^3 + 6t^5) + \frac{1}{35} \cdot \frac{t^9}{8n^8(1+tt)^6} (8t - 56t^3 + 56t^5 - 8t^7) + \frac{5}{65} \cdot \frac{t^{10}}{10n^{10}(1+tt)^{10}} (10t - 120t^3 + 252t^5 - 120t^7 + 10t^9) + \frac{691}{13 \cdot 210} \cdot \frac{t^{12}}{12n^{12}(1+tt)^{12}} (12t - 220t^3 + 792t^5 - 792t^7 + 220t^9 - 12t^{11}) + \frac{7}{7} \cdot \frac{t^{14}}{34n^{14}(1+tt)^{14}} (14t - 364t^3 + 2002t^5 - 3432t^7 + 2002t^9 - 364t^{11} + 14t^{13}) + \dots$

§. 10. Expressio haec commodissime accommodabitur ad casum, quo est $t = 1$, cum alterni seriei termini evanescent, atque insuper arcus z abeat in quartam semiperipheriae circuli partem, posita ergo semiperipheria circuli $= \pi$, ita vt fit $z = \frac{\pi}{4}$, sumtoque quocunque numero integro affirmatio pro n erit $\frac{\pi}{4} = \frac{n}{n^2+1} + \frac{n}{n^2+4} + \frac{n}{n^2+9} + \frac{n}{n^2+16} + \dots + \frac{n}{n^2+n^2} + \frac{1}{4n} + \frac{1}{5} \cdot \frac{1}{2 \cdot 2n^2} - \frac{1}{42} \cdot \frac{1}{2^3 \cdot 6n^6} + \frac{5}{80} \cdot \frac{1}{2^5 \cdot 10 \cdot n^{10}} - \frac{7}{8} \cdot \frac{1}{2^7 \cdot 14 \cdot n^{14}} + \frac{43967}{19 \cdot 42} \cdot \frac{1}{2^9 \cdot 18 \cdot n^{18}} - \frac{854513}{6 \cdot 23} \cdot \frac{1}{2^{11} \cdot 22 \cdot n^{22}} + \dots$ etc. Hinc igitur erit $\pi = \frac{4n}{n^2+1} + \frac{4n}{n^2+4} + \frac{4n}{n^2+9} + \dots + \frac{4n}{n^2+n^2} + \frac{1}{n} + \frac{1}{5} \cdot \frac{1}{1n^2} - \frac{1}{42} \cdot \frac{1}{2^3 \cdot 6n^6} + \frac{5}{80} \cdot \frac{1}{2^5 \cdot 10 \cdot n^{10}} - \frac{7}{8} \cdot \frac{1}{2^7 \cdot 14 \cdot n^{14}} + \frac{43967}{19 \cdot 42} \cdot \frac{1}{2^9 \cdot 18 \cdot n^{18}} - \frac{854513}{6 \cdot 23} \cdot \frac{1}{2^{11} \cdot 22 \cdot n^{22}} + \dots$ etc. quae series eo magis conuergit, quo maior numerus pro n accipiatur.

§. 11. Quamuis autem haec series eo magis conuergere videatur, quo maior fit numerus n tamen perpetuo ad certum vsque terminum tantum conuergit, post quem

termini crescent iterum; hancque ob causam non iuvat seriem eo vsque adhibere, quoad termini diuergere incipiunt, sed expediet operationem ibi finire, vbi maxima obseruatur conuergentia. Namque si fractionum $\frac{1}{6}$; $\frac{1}{36}$; $\frac{1}{12}$; $\frac{1}{54}$; $\frac{5}{81}$; etc. ea quae indicem habet v ponatur $= X$, atque sequens $= Y$ erit semper $\frac{Y}{X} > \frac{(v-1)(2v-3)}{2v^2}$, atque v in infinitum crescente fiet $\frac{Y}{X} = \frac{v^2}{\pi^2}$. Ex quo apparet terminos istius seriei continuo magis crescere, atque nullam progressionem geometricam quantumuis conuergentem cum ea coniunctam eam reddere posse conuergentem. Hinc autem concluditur in serie paragr. praec. plures terminos accipi non licere quam ad summum $\frac{\pi\pi}{\sqrt{2}}$ hoc est proxime 2π : etiam si enim sumerentur plures termini, summa non ad veram propior accedens reperiretur.

§. 12. Ex hoc vero ipso subsidium ad valorem ipsius π propius inueniendum ope seriei paragraphi 10, consequitur. Ponamus enim seriei: $\frac{1}{2} \cdot \frac{1}{1\pi^2} - \frac{1}{4} \cdot \frac{1}{2^2 \cdot 3\pi^4} + \frac{5}{8} \cdot \frac{1}{2^4 \cdot 5 \cdot 7\pi^6} -$ etc. iam actu esse additos μ terminos, ac sequentem terminum esse $= P$, eius loco sumatur ista expressio $\frac{\pi^{4\mu+P}}{\pi^{4\mu+1} + \mu^4}$, isque loco omnium reliquorum addatur vel subtrahatur, prout terminus P habuerit signum $+$ vel $-$. Est vero proxime $\pi^4 = 90, 740909$, unde loco termini P substitui poterit $\frac{P}{1 + \frac{16\mu^4}{367\pi^4}}$. Hocque modo eo propius ad verum valorem ipsius π accedetur, quo maior fuerit numerus μ : hoc est quo plures termini iam fuerint additi.

§. 13. His tamen non obstantibus series paragrapho decimo data semper dat valorem ipsius π nimis magnum, quic-

quicquid pro n substituatur; eo propius autem acceditur, quo maior numerus pro n substituatur. Sumto enim 1 pro n prodit $\pi = 3, 1646 +$ quae expressio iam in figura secunda a vero valore $3, 1415926535897932$ aberrat. Si ponatur $n = 3$, prodibit $\pi = 3, 1415927216 +$ a vero valore in octava figura discrepans. At si ponatur $n = 5$ reperietur per eandem methodum

$$\begin{array}{r} \pi = 3, 1415926535900726 + \\ \quad 3, 1415926535897932 \\ \hline \quad 0, 0000000000002794 \end{array}$$

cuius numeri excessus in decima tertia demum figura conspicitur. Haecque aberratio a veritate eo magis est notatu digna, quo minus vitium in ratiocinio instituto deprehendi potest. Ad quod accedit ut ista formula aberratione hac non obstante commode ad valorem ipsius π inveniendum inferuire queat, substituendo scilicet maiores numeros loco n .

§. 14. Ex his exemplis quibus 1, 3, et 5 loco n substituimus per inductionem concludi posse videtur, valorem ipsius π in fractionibus decimalibus fere ad triplo plures figuras iustum repertum iri, quam n contineat unitates, siquidem prima figura 3 computetur; prima autem hac figura non computata videtur numerus figurarum iustarum fore $= 2 \frac{1}{2} n$. Sic si ponatur $n = 2$ reperitur $\pi = 3, 141635$ cuius quinta figura quaternario nimis est magna. Ac posito $n = 4$ prodit $\pi = 3, 14159265374 +$ cuius decima figura binario maior est vera. Posito autem $n = 6$ reperitur, $\pi = 3, 141592653589793558 +$ cuius figura demum decima sexta a veritate recedit.

§. 15. Si nunc in causam huius a veritate aberrationis calculi inquiramus, aliam detegere non valemus, nisi diuergentiam seriei §. 10. allatae; reliqua enim omnia prorsus se recte habere deprehenduntur. Namque si t unitatem excedat, eo maior reperietur aberratio a veritate, quo minor accipiatur numerus n ; id quod clarissime se manifestabit si t ponatur infinitum atque simul $n =$ numero infinito. Ponamus enim $t = \infty$, quo casu in §. 9. abibit z in quartam peripheriae partem, eritque ideo $z = \frac{\pi}{2}$. Sit insuper $n = p^2$, denotante p numerum quemcunque affirmatiuum siue integrum siue fractum, eritque ob $z = \frac{\pi}{2} = s + \frac{1}{2p}$ ac reliqui termini omnes negligi posse videntur, quod tamen in terminis infinitesimis perperam fit, quippe qui tandem ad finitam magnitudinem excrefcere possunt.

§. 16. Interim tamen notari meretur errorem satis esse exiguum, nisi p fit numerus unitate minor, atque quo maior valor ipsi p tribuatur eo minorem fore aberrationem a veritate. Cum enim hoc casu fit $s = \frac{p}{p^2+1} + \frac{p}{p^2+9} + \frac{p}{p^2+16} + \frac{p}{p^2+25} + \dots$ etc. in infinitum; videatur huius seriei summa posse per quadraturam circuli definiri, quod tamen secus se habet. Per ultimam enim aequationem foret $s = \frac{\pi}{2} - \frac{1}{2p}$ seu $\frac{\pi}{2p} - \frac{1}{2pp} = \frac{1}{p^2+1} + \frac{1}{p^2+9} + \frac{1}{p^2+16} + \dots$ etc. cuius quidem aequationis falsitas si $p = 0$ sponte elucet. At sumto $p = 1$ foret $\frac{1}{2} + \frac{1}{8} + \frac{1}{18} + \frac{1}{32} + \frac{1}{50} + \dots = \frac{\pi}{2} - \frac{1}{2}$. Vera autem summa per alias regulas reperitur $= \frac{\pi}{2} - 0,4941222793$ ita vt illa summa sit iusto minor, idque parte 0,0058777206 si autem ponatur $p = 2$, habebitur ista
series

series $\frac{1}{5} + \frac{1}{8} + \frac{1}{13} + \frac{1}{20} + \text{etc.}$ cuius summa per viam hanc erroneam prodit $= \frac{\pi}{4} - \frac{1}{8} = \frac{\pi}{4} - 0, 125$; cum tamen constet veram summam esse $= \frac{\pi}{4} - 0, 124994522075$, ita vt illius defectus tantum sit $= 0, 000005477924$. Multo autem adhuc minor erit aberratio si maiores numeri pro p accipiantur: sic si $p = 3$, in nona demum figura accidet aberratio, atque quocunque numero pro p sumto prodibit summa iusta ad 3 p figuras.

§. 17. Ex his satis perspicitur, quam caute circa summationem serierum diuergentium versari oporteat, praesertim si eiusmodi series diuergentes occurrant infinitae. Huiusque rei adhuc vnum exemplum afferre visum est, ex quo necessitas summae circumspectionis clarius elucebit. Proposita sit series quaecunque $\frac{1}{a} + \frac{2}{b} + \frac{3}{c} + \frac{4}{d} + \frac{5}{e} + \frac{6}{f} + \frac{7}{g} + \frac{8}{h} + \text{etc.}$ cuius constat terminum quemcunque indicis x fore $= a + \frac{(x-1)}{1} (b-a) + \frac{(x-1)(x-2)}{1 \cdot 2} (c-2b+a) + \frac{(x-1)(x-2)(x-3)}{1 \cdot 2 \cdot 3} (d-3c+3b-a) + \text{etc.}$ Ex hac forma definiantur omnes termini praecedentes versus sinistram in infinitum progredientes, eritque vt sequitur

term. indicis 0 $= a + (a-b) + (a-2b+c) + (a-3b+3c-d) + \text{etc.}$
 term. indic. -1 $= a + 2(a-b) + 3(a-2b+c) + 4(a-3b+3c-d) + \text{etc.}$
 term. indic. -2 $= a + 3(a-b) + 6(a-2b+c) + 10(a-3b+3c-d) + \text{etc.}$
 term. ind. -3 $= a + 4(a-b) + 10(a-2b+c) + 20(a-3b+3c-d) + \text{etc.}$
 etc.

§. 18. Colligantur omnes hi termini antecedentes in infinitam, reperieturque omnium summa $= \frac{a}{1-1} + \frac{a-b}{(1-1)^2} + \frac{a-2b+c}{(1-1)^3} + \frac{a-3b+3c-d}{(1-1)^4} + \text{etc.}$ quae in series innumera-biles secundum litteras a, b, c, d, e , etc. resoluta abibit in hanc formam:

$$\begin{aligned}
 &+ a \left(\frac{1}{(1-x)} + \frac{1}{(1-x)^2} + \frac{1}{(1-x)^3} + \frac{1}{(1-x)^4} + \text{etc.} \right) \\
 &- b \left(\frac{1}{(1-x)^2} + \frac{2}{(1-x)^3} + \frac{3}{(1-x)^4} + \frac{4}{(1-x)^5} + \text{etc.} \right) \\
 &+ c \left(\frac{1}{(1-x)^3} + \frac{3}{(1-x)^4} + \frac{6}{(1-x)^5} + \frac{10}{(1-x)^6} + \text{etc.} \right) \\
 &- d \left(\frac{1}{(1-x)^4} + \frac{4}{(1-x)^5} + \frac{10}{(1-x)^6} + \frac{20}{(1-x)^7} + \text{etc.} \right)
 \end{aligned}$$

§. 19. Series hac singulae autem summationem admittunt; atque summis earum loco substitutis prodibit aggregatum omnium terminorum antecedentium versus sinistram in infinitum, vt sequitur

$$\begin{aligned}
 &+ a \cdot \frac{1}{(1-x)-1} = a \\
 &- b \cdot \frac{1}{((1-x)-1)^2} = -b \\
 &+ c \cdot \frac{1}{((1-x)-1)^3} = c \\
 &- d \cdot \frac{1}{((1-x)-1)^4} = -d \\
 &+ \text{etc.}
 \end{aligned}$$

Ex quo videtur terminorum horum antecedentium summa fore $= -a - b - c - d - \text{etc.}$ Quare si series quaecunque infinita $a + b + c + d + e + \text{etc.}$ etiam versus sinistram in infinitum continuaretur, foret totius seriei vtrique in infinitum abeuntis summa semper $= 0$; si quidem ratiocinium hoc esset iustum.

§. 20. Neque vero hoc ratiocinium semper fallit, sed in innumerabilibus seriebus veritati consentaneum deprehenditur. Primo enim omnes progressionis geometricae hac gaudent proprietate vt in infinitum vtrique progredientes summam habeant $= 0$. Scilicet seriei $n + n^2 + n^3 + \text{etc.}$ summa est $= \frac{n}{1-n}$ partis autem praecedentis $1 + \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \text{etc.}$ summa est $= \frac{n}{n-1}$, quae cum illa iuncta producit nihil. In infinitis autem seriebus

seriebus aliis ratiocinium hoc maxime a veritate recedit, cuiusmodi est series $1 + \frac{1}{9} + \frac{1}{25} + \text{etc.}$ quae antrosum continuata sui fit similis et aequalis, scilicet $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \text{etc.}$ cuius adeo totius summa non fit 0 sed potius duplo maior. Haec igitur proposuisse non minoris utilitatis esse arbitror, quam summo rigore demonstratas veritates.
