METHODVS FACILIS COMPVTANDI ANGVLORVM SINVS AC TANGENTES TAM NATVRALES QVAM ARTIFICIALES

AVCTORE
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§. 1.

Exposui anno praeterito methodum inueniendi valores eiusmodi expressionum, quae sint producta ex infinitis factoribus certa quadam lege progredientibus, eaque methodus deducta erat ex formulis integralibus, quarum integratio a se inuicem pendet. Nunc autem, cum nuper exposuissem modum summandi huiusmodi series

$$\frac{1}{1 \pm p} \pm \frac{1}{4 \pm p} + \frac{1}{9 \pm p} \pm \frac{1}{16 \pm p} + \frac{1}{25 \pm p} \pm \text{etc.}$$

ex eo nactus sum commodam atque aptam methodum quam plurimorum productorum, ex infinitis factoribus constantium, valores determinandi, eiusque beneficio mihi licuit innumerabiles istiusmodi expressiones definire, quae per alteram mothodum vel omnino tractari non poterant, vel saltem tam expedite et concinne non absoluuntur. Quod negotium, quo clarius ob oculos ponatur, in sequentibus problematis sum complexurus.

Problema. 1.

§. 2. Inuenire valorem huius expressionis per continuos factores in infinitum progredientis.

1 +

$\frac{1+p}{1} \cdot \frac{4+p}{4} \cdot \frac{9+p}{9} \cdot \frac{16+p}{16} \cdot \frac{25+p}{25} \cdot \frac{36+p}{36} \cdot$ etc.

Solutio.

Ponatur huius expreſſionis propoſitae valor quaeſitus $= s$, et ſumtis logarithmis, erit $ls = l(1+p) + l(1+\frac{p}{4}) + l(1+\frac{p}{9}) + l(1+\frac{p}{16}) + l(1+\frac{p}{25}) + l(1+\frac{p}{36}) +$ etc. His igitur logarithmis per ſeries notas expreſſis habebitur

$$ls = +\frac{p}{1} - \frac{p^2}{2} + \frac{p^3}{3} - \frac{p^4}{4} + \frac{p^5}{5} - \frac{p^6}{6} + \text{etc.}$$
$$+\frac{p}{4} - \frac{p^2}{2\cdot 4^2} + \frac{p^3}{3\cdot 4^3} - \frac{p^4}{4\cdot 4^4} + \frac{p^5}{5\cdot 4^5} - \frac{p^6}{6\cdot 4^6} + \text{etc.}$$
$$+\frac{p}{9} - \frac{p^2}{2\cdot 9^2} + \frac{p^3}{3\cdot 9^3} - \frac{p^4}{4\cdot 9^4} + \frac{p^5}{5\cdot 9^5} - \frac{p^6}{6\cdot 9^6} + \text{etc.}$$
$$+\frac{p}{16} - \frac{p^2}{2\cdot 16^2} + \frac{p^3}{3\cdot 16^3} - \frac{p^4}{4\cdot 16^4} + \frac{p^5}{5\cdot 16^5} - \frac{p^6}{6\cdot 16^6} + \text{etc.}$$
$$\text{etc.}$$

Sumantur differentia; eritque

$$\frac{ds}{sdp} = 1 - p + p^2 - p^3 + p^4 - p^5 + \text{etc.}$$
$$+\frac{1}{4} - \frac{p}{4^2} + \frac{p^2}{4^3} - \frac{p^3}{4^4} + \frac{p^4}{4^5} - \frac{p^5}{4^6} + \text{etc.}$$
$$+\frac{1}{9} - \frac{p}{9^2} + \frac{p^2}{9^3} - \frac{p^3}{9^4} + \frac{p^4}{9^5} - \frac{p^5}{9^6} + \text{etc.}$$
$$+\frac{1}{16} - \frac{p}{16^2} + \frac{p^2}{16^3} - \frac{p^3}{16^4} + \frac{p^4}{16^5} - \frac{p^5}{16^6} + \text{etc.}$$
$$\text{etc.}$$

Cum nunc hae ſeries omnes ſint geometricae, ſummari poterunt, hocque facto prodibit

$$\frac{ds}{sdp} = \frac{1}{1+p} + \frac{1}{4+p} + \frac{1}{9+p} + \frac{1}{16+p} + \frac{1}{25+p} + \text{etc.}$$

Huius autem ſeriei ſummam nuper exhibui; vnde ſi circuli cuius diameter $= 1$, peripheria ponatur $= \pi$ erit $\frac{ds}{sdp} = \frac{\pi\sqrt{p}-1}{2p} + \frac{\pi\sqrt{p}}{p(e^{2\pi\sqrt{p}}-1)}$. Ponatur facilitatis gratia $p = qq$, erit $dp = 2qdq$, atque aequatio inuenta abibit in hanc

 $\frac{ds}{s} =$

$\frac{ds}{s} = \pi dq - \frac{dq}{q} + \frac{2\pi dq}{e^{2\pi q} - 1} = - \pi dq - \frac{dq}{q} + \frac{2 e^{2\pi q}\pi dq}{e^{2\pi q} - 1}$ Cuius integrale eſt $ls = lC - \pi q - lq + l(e^{2\pi q} - 1)$ ſeu $s = \frac{C(e^{2\pi q} - 1)}{e^{\pi q}q} = \frac{C(e^{2\pi\sqrt{p}} - 1)}{e^{\pi\sqrt{p}}\sqrt{p}}$, vbi conſtantem C ita determinari oportet, vt poſito p vel $q = 0$ fiat $ls = 0$. At facto $q = 0$, fit $e^{2\pi q} - 1 = 2\pi q$, ideoque $ls = 0 = lC - \pi q - lq + l\,2\pi q = lC + l\,2\pi$, ergo $C = \frac{1}{2\pi}$. Conſequenter expreſſionis propoſitae

$\frac{1+p}{1} \cdot \frac{4+p}{4} \cdot \frac{9+p}{9} \cdot \frac{16+p}{16} \cdot \frac{25+p}{25} \cdot \frac{36+p}{36}$ etc.

valor erit $= \frac{e^{2\pi\sqrt{p}} - 1}{2e^{\pi\sqrt{p}}\pi\sqrt{p}}$. Q. E. I.

Coroll. 1.

§. 3. Quodſi loco p ponatur $4p$, habebitur iſta expreſſio:

$\frac{1+4p}{1} \cdot \frac{1+p}{1} \cdot \frac{9+4p}{9} \cdot \frac{4+p}{4} \cdot \frac{25+4p}{25} \cdot \frac{9+p}{9}$ etc.

cuius igitur valor erit $\frac{e^{4\pi\sqrt{p}} - 1}{4e^{2\pi\sqrt{p}}\pi\sqrt{p}}$.

Coroll. 2.

§. 4. Cum iam in hac expreſſione praecedens contineatur, diuidatur haec per illam, prodibitque

$\frac{1+4p}{1} \cdot \frac{9+4p}{9} \cdot \frac{25+4p}{25} \cdot \frac{49+4p}{49}$ etc.

cuius proinde valor eſt $\frac{e^{2\pi\sqrt{p}} + 1}{2e^{\pi\sqrt{p}}}$.

Coroll.

Coroll. 3.

§. 5. Hinc igitur nanciſcimur valorem huius expreſſionis propoſitae affinis :

$$\frac{1+p}{1} \cdot \frac{9+p}{9} \cdot \frac{25+p}{25} \cdot \frac{49+p}{49} \cdot \frac{81+p}{81} \cdot \text{etc.}$$

quippe cuius valor erit $= \frac{e^{\pi\sqrt{p}}+1}{2e^{\frac{1}{2}\pi\sqrt{p}}}$.

Coroll. 4.

§. 6. Diuidatur per hanc ipſa expreſſio propoſita, fiet

$$\frac{4+p}{4} \cdot \frac{16+p}{16} \cdot \frac{36+p}{36} \cdot \frac{64+p}{64} \cdot \frac{100+p}{100} \cdot \text{etc.}$$

huius ſcilicet valor erit $= \frac{e^{\pi\sqrt{p}}-1}{e^{\frac{1}{2}\pi\sqrt{p}}\pi\sqrt{p}}$.

Coroll. 5.

§. 7. Si nunc expreſſio §. 5. per expreſſionem §. 6. diuidatur, prodibit haec forma

$$\frac{1+p}{1} \cdot \frac{4}{4+p} \cdot \frac{9+p}{9} \cdot \frac{16}{16+p} \cdot \frac{25+p}{25} \cdot \frac{36}{36+p} \cdot \text{etc.}$$

cuius valor erit $= \frac{(e^{\pi\sqrt{p}}+1)\pi\sqrt{p}}{2(e^{\pi\sqrt{p}}-1)}$.

Coroll. 6.

§. 8 Si ſumantur binae huiusmodi ſeries, atque altera per alteram diuidatur, obtinebuntur ſequentes ſummationes.

$$\frac{1+p}{1+q} \cdot \frac{4+p}{4+q} \cdot \frac{9+p}{9+q} \cdot \frac{16+p}{16+q} \cdot \text{etc.} = \frac{e^{\pi\sqrt{q}}(e^{2\pi\sqrt{p}}-1)\sqrt{q}}{e^{\pi\sqrt{p}}(e^{2\pi\sqrt{q}}-1)\sqrt{p}}$$

$$\frac{1+p}{1+q} \cdot \frac{4+q}{4+p} \cdot \frac{9+p}{9+q} \cdot \frac{16+q}{16+p} \cdot \text{etc.} = \frac{(e^{\pi\sqrt{p}}+1)(e^{\pi\sqrt{q}}-1)\sqrt{p}}{(e^{\pi\sqrt{p}}-1)(e^{\pi\sqrt{q}}+1)\sqrt{q}}$$

$$\frac{1+p}{1+q} \cdot \frac{9+p}{9+q} \cdot \frac{25+p}{25+q} \cdot \text{etc.} = \frac{e^{\frac{1}{2}\pi\sqrt{q}}(e^{\pi\sqrt{p}}+1)}{e^{\frac{1}{2}\pi\sqrt{p}}(e^{\pi\sqrt{q}}+1)}$$

$$\frac{4+p}{4+q} \cdot \frac{16+p}{16+q} \cdot \frac{36+p}{36+q} \cdot \text{etc.} = \frac{e^{\frac{1}{2}\pi\sqrt{q}}(e^{\pi\sqrt{p}}-)\sqrt{q}}{e^{\frac{1}{2}\pi\sqrt{p}}(e^{\pi\sqrt{q}}-1\sqrt{p}}.$$

§. 9. Ex ſolutione igitur huius primi problematis conſequuti ſumus valores eiusmodi productorum infinitis fractionibus contentorum, quarum tam numeratores quam denominatores ſunt quadrata vel numerorum omnium in ſerie naturali prpgredientium, vel imparium tantum vel parium, eaque datis numeris aucta. Cum igitur iſtiusmodi factores in ſimplices reales, qui arithmeticam teneant progreſſionem, reſolui nequeant, iſtae ſummationes methodo iam ante expoſita abſolui non poterunt. At viciſſim hinc non intelligitur, quinam prodituri ſint valores, ſi vel p vel q negatiue accipiatur ob exponentes $\pi\sqrt{p}$ et $\pi\sqrt{q}$, qui hoc caſu fiunt imaginarii. Quamobrem hos caſus in ſequenti problemate euoluemus.

Problema 2.

§. 1. Inuenire valorem huius expreſſionis per continuos factores in infinitum progredientis

$$\frac{1-p}{1} \cdot \frac{4-p}{4} \cdot \frac{9-p}{9} \cdot \frac{16-p}{16} \cdot \frac{25-p}{25} \cdot \frac{36-p}{36} \cdot \text{etc.}$$

Solutio.

Ponatur valor quaeſitus $= s$, eritque logarithmis ſumendis,

$$l\,s = l(1-p) + l(1-\tfrac{p}{4}) + l(1-\tfrac{p}{9}) + l(1-\tfrac{p}{16}) + l(1-\tfrac{p}{25}) + \text{etc.}$$

His

His vero logarithmis in ſeries conuerſis habebitur:

$$l\,s = -\frac{p}{1} - \frac{p^2}{2} - \frac{p^3}{3} - \frac{p^4}{4} - \frac{p^5}{5} - \text{etc.}$$
$$- \frac{p}{4} - \frac{p^2}{2\cdot 4^2} - \frac{p^3}{3\cdot 4^3} - \frac{p^4}{4\cdot 4^4} - \frac{p^5}{5\cdot 4^5} - \text{etc.}$$
$$- \frac{p}{9} - \frac{p^2}{2\cdot 9^2} - \frac{p^3}{3\cdot 9^3} - \frac{p^4}{4\cdot 9^4} - \frac{p^5}{5\cdot 9^5} - \text{etc.}$$
$$\text{etc.}$$

Sumtisque differentialibus prodibit:

$$\frac{-ds}{s\,dp} = +1 + p + p^2 + p^3 + p^4 + \text{etc.}$$
$$+ \frac{1}{4} + \frac{p}{4^2} + \frac{p^2}{4^3} + \frac{p^3}{4^4} + \frac{p^4}{4^5} + \text{etc.}$$
$$+ \frac{1}{9} + \frac{p}{9^2} + \frac{p^2}{9^3} + \frac{p^3}{9^4} + \frac{p^4}{9^5} + \text{etc.}$$
$$\text{etc.}$$

Quae ſeries cum ſingulae ſint geometricae, ſummae illarum loco ſubſtituantur, hincque erit

$$\frac{-ds}{s\,dp} = \frac{1}{1-p} + \frac{1}{4-p} + \frac{1}{9-p} + \frac{1}{16-p} + \frac{1}{25-p} + \text{etc.}$$

At iſtius ſeriei ſummam nuper elicui, quae, ſi ſubſtituatur, orietur

$$-\frac{ds}{s\,dp} = \frac{1}{2p} - \frac{\pi\sqrt{p}}{2p\,\text{tang.}\,A.\pi\sqrt{p}}$$

Sit commodi ergo $p = qq$ eritque

$$-\frac{ds}{s} = \frac{dq}{q} - \frac{\pi dq}{\text{tang.}\,A.\pi q} = \frac{dq}{q} - \frac{\pi dq\,\text{coſ.}\,A.\pi q}{\text{ſin.}\,A.\pi q}$$

Quoniam nunc eſt $d.\,\text{ſin.}\,A.\pi q = \pi dq\,\text{coſ.}\,A.\pi q$, erit integrale aequationis inuentae,
$l\,C - l\,s = l\,q - l\,\text{ſin.}\,A.\pi q$; conſtante autem C ita definita vt facto p vel $q = 0$ euaneſcat $l\,s$ prodibit $l\,C = l\,q - l\,\pi q = -l\,\pi$. Quocirca erit $\frac{1}{\pi s} = \frac{q}{\text{ſin.}\,A.\pi q} = \frac{\sqrt{p}}{\text{ſin.}\,A.\pi\sqrt{p}}$ hincque $s = \frac{\text{ſin.}\,A.\pi\sqrt{p}}{\pi\sqrt{p}}$ ſiue

$$\frac{1-p}{1} \cdot \frac{4-p}{4} \cdot \frac{9-p}{9} \cdot \frac{16-p}{16} \cdot \frac{25-p}{25} \cdot \text{etc.} = \frac{\text{ſin.}\,A.\pi\sqrt{p}}{\pi\sqrt{p}}.$$

Q. E. I.

Coroll. 1.

Coroll. 1.

§. 11. Quodsi loco p ponatur $4p$, habebitur ista expressio:

$\frac{1-4p}{1} \cdot \frac{1-p}{1} \cdot \frac{9-4p}{9} \cdot \frac{4-p}{4} \cdot \frac{25-4p}{25}$. etc.

cuius valor erit $= \frac{\text{sin. A} \cdot 2\pi\sqrt{p}}{2\pi\sqrt{p}} = \frac{\text{sin. A} \cdot \pi\sqrt{p} \cdot \text{cos. A} \cdot \pi\sqrt{p}}{\pi\sqrt{p}}$.

Coroll. 2.

§. 12. Diuidatur haec series per illam, prodibitque

$\frac{1-4p}{1} \cdot \frac{9-4p}{9} \cdot \frac{25-4p}{25}$. etc. $=$ cos. A . $\pi\sqrt{p}$ siue

$\frac{1-p}{1} \cdot \frac{9-p}{9} \cdot \frac{25-p}{25}$. etc. $=$ cos. A . $\frac{\pi\sqrt{p}}{2}$.

Coroll. 3.

§. 13. Cum iam sit $\frac{1-p}{1} \cdot \frac{4-p}{4} \cdot \frac{9-p}{9} \cdot \frac{16-p}{16}$. etc. $=$

$$\frac{\text{sin. A} \cdot \pi\sqrt{p}}{\pi\sqrt{p}} = \frac{2\ \text{sin. A}\ \frac{1}{2}\pi\sqrt{p} \cdot \text{cos. A} \cdot \frac{1}{2}\pi\sqrt{p}}{\pi\sqrt{p}}$$ erit

$$\frac{4-p}{4} \cdot \frac{16-p}{16} \cdot \frac{36-p}{36} \cdot \frac{64-p}{64} = \frac{2\ \text{sin. A} \cdot \frac{1}{2}\pi\sqrt{p}}{\pi\sqrt{p}}.$$

Coroll. 4.

§. 14. Diuidatur per hanc expressionum praecendens orietur.

$\frac{1-p}{1} \cdot \frac{4}{4-p} \cdot \frac{9-p}{9} \cdot \frac{16}{16-p} \cdot \frac{25-p}{25} \cdot \frac{36}{36-p}$ etc.

cuius valor erit $= \frac{\pi\sqrt{p}}{2\ \text{tang. A}\ \frac{1}{2}\pi\sqrt{p}}$.

Coroll.

Coroll. 5.

§. 15. Si ſumantur binae huiusmodi ſeries, earumque altera per alteram diuidatur, obtinebuntur ſequentes ſummationes.

$$\frac{1-p}{1-q}\cdot\frac{4-p}{4-q}\cdot\frac{9-p}{9-q}\cdot\frac{16-p}{16-q}\text{ etc} = \frac{\sqrt{q}\,\text{ſin. A. }\pi\sqrt{p}}{\sqrt{p}\,\text{ſin. A. }\pi\sqrt{q}}$$

$$\frac{1-p}{1-q}\cdot\frac{4-q}{4-p}\cdot\frac{9-p}{9-q}\cdot\frac{16-q}{16-p}\text{ etc.} = \frac{\sqrt{p}\,\text{tang. A. }\frac{1}{2}\pi\sqrt{q}}{\sqrt{q}\,\text{tang. A. }\frac{1}{2}\pi\sqrt{p}}$$

$$\frac{1-p}{1-q}\cdot\frac{9-p}{9-q}\cdot\frac{25-p}{25-q}\cdot\frac{45-p}{49-q}\text{. etc.} = \frac{\text{coſ. A. }\frac{1}{2}\pi\sqrt{p}}{\text{coſ. A. }\frac{1}{2}\pi\sqrt{q}}$$

$$\frac{4-p}{4-q}\cdot\frac{16-p}{16-q}\cdot\frac{36-p}{36-q}\cdot\frac{64-p}{46-q}\text{. etc.} = \frac{\sqrt{q}\,\text{ſin. A. }\frac{1}{2}\pi\sqrt{p}}{\sqrt{p}\,\text{ſin. A. }\frac{1}{2}\pi\sqrt{q}}$$

§. 16. In his expreſſionibus ſinus, coſinus et tangentes referuntur ad ſinum totum $= 1$, ſeu arcus circulares in circulo ſunt capiendi, cuius ſemidiameter eſt $= 1$. In tali igitur circulo exprimet π ſemiſſem peripheriae ſeu arcum 180. graduum. In numeris autem proximis erit, vt conſtat,

$$\pi = 3,14159265357989$$

Quodſi vero $\sqrt{p}$ et $\sqrt{q}$ fuerint numeri rationales, tum ſinus et tangentes, geometrice poterunt exhiberi, erit ſcilicet

ſin. A. $\pi = 0$	ſin. A. $\frac{1}{2}\pi = 1$	ſin. A. $\frac{1}{3}\pi = \frac{\sqrt{3}}{2}$	ſin. A. $\frac{1}{4}\pi = \frac{1}{\sqrt{2}}$
coſ. A. $\pi = -1$	coſ. A. $\frac{1}{2}\pi = 0$	coſ. A. $\frac{1}{3}\pi = \frac{1}{2}$	coſ. A. $\frac{1}{4}\pi = \frac{1}{\sqrt{2}}$
tang. A. $\pi = 0$	tang. A. $\frac{1}{2}\pi = \infty$	tang. A. $\frac{1}{3}\pi = \sqrt{3}$	tang. A. $\frac{1}{4}\pi = 1$.

§. 17. Expreſſionum harum vſus primum in hoc conſiſtit, vt earum ope peripheria circuli multifariam

per

per iſtiusmodi producta continua concinne poſſit exhiberi. Quod vt appareat ponamus $p = \frac{m^2}{n^2}$ et cum π ſit arcus 180. graduum erit per §. 10.

$$\frac{n^2-m^2}{n^2} \cdot \frac{4n^2-m^2}{4n^2} \cdot \frac{9n^2-m^2}{9n^2} \cdot \frac{16n^2-m^2}{16n^2} \text{ etc.} = \frac{n \text{ ſin. A. } \frac{m}{n} 180^\circ}{m\pi}$$

ſeu

$$\pi = \frac{n}{m} \text{ ſin. A. } \frac{m}{n} 180^\circ \frac{n^2}{n^2-m^2} \cdot \frac{4n^2}{4n^2-m^2} \cdot \frac{9n^2}{9n^2-m^2} \cdot \frac{16n^2}{16n^2-m^2} \cdot \text{etc.}$$

vnde emergunt ſequentes pro valore ipſius π expreſſiones.

Si $m = 1$, $n = 2$

$$\pi = 2 \cdot \frac{4}{3} \cdot \frac{16}{15} \cdot \frac{36}{35} \cdot \frac{64}{63} \cdot \frac{100}{99} \cdot \frac{144}{143} \cdot \text{ etc.}$$

ſeu $$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12 \cdot 12}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11 \cdot 13} \text{ etc.}$$

quae eſt ipſa expreſſio Walliſii alibi a me demonſtrata.

Si $m = 1$ et $n = 3$

$$\pi = \frac{3\sqrt{3}}{2} \cdot \frac{9}{8} \cdot \frac{36}{35} \cdot \frac{81}{80} \cdot \frac{144}{143} \cdot \frac{225}{224} \text{ etc. ſeu}$$

$$\pi = \frac{3\sqrt{3}}{2} \cdot \frac{3 \cdot 3 \cdot 6 \cdot 6 \cdot 9 \cdot 9 \cdot 12 \cdot 12 \cdot 15 \cdot 15}{2 \cdot 4 \cdot 5 \cdot 7 \cdot 8 \cdot 10 \cdot 11 \cdot 13 \cdot 14 \cdot 16} \text{ etc.}$$

Si $m = 1$ etc. $n = 4$

$$\pi = 2\sqrt{2} \cdot \frac{16}{15} \cdot \frac{64}{63} \cdot \frac{144}{143} \cdot \frac{256}{255} \cdot \text{etc. ſeu}$$

$$\pi = 2\sqrt{2} \cdot \frac{4 \cdot 4 \cdot 8 \cdot 8 \cdot 12 \cdot 12 \cdot 16 \cdot 16 \cdot 20 \cdot 20}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17 \cdot 19 \cdot 21} \text{ etc.}$$

Si $m = 1$ et $n = 6$

$$\pi = 3 \cdot \frac{36}{35} \cdot \frac{144}{143} \cdot \frac{324}{323} \cdot \frac{576}{575} \text{ etc. ſeu}$$

$$\pi = 3 \cdot \frac{6 \cdot 6 \cdot 12 \cdot 12 \cdot 18 \cdot 18 \cdot 24 \cdot 24}{5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 25} \text{ etc.}$$

§. 18. Expreſſiones hae, quanquam ſatis cito convergunt, tamen ſunt aptiores ad logarithmum ipſius π inueniendum, quam ad ipſum valorem π. Ita erit ex vltima expreſſione

$$l\pi = l3 + l\frac{36}{35} + l\frac{144}{143} + l\frac{324}{323} + \text{ etc. ſeu}$$

$$l\pi = l3 + \frac{1}{6^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.}\right)$$
$$+ \frac{1}{2 \cdot 6^4}\left(1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} + \frac{1}{25^2} + \text{etc.}\right)$$

+

$$+ \frac{1}{3\cdot 6^6}\left(1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} + \frac{1}{25^3} + \text{etc.}\right)$$
$$+ \frac{1}{4\cdot 6^8}\left(1 + \frac{1}{4^4} + \frac{1}{9^4} + \frac{1}{16^4} + \frac{1}{25^4} + \text{etc.}\right)$$
etc.

vnde calculus ſequenti modo inſtituetur ad logarithmum hyperbolicum ipſius π inueniendum

$l\,3 = 1,0986122888668$
$l\,\frac{36}{35} = 0,0281708376966$

$0,0179148835217 = \frac{1}{6^2}\left(\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}\right)$
$0,0000317605507 = \frac{1}{2\cdot 6^4}\left(\frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} + \text{etc.}\right)$
$0,0000000123907 = \frac{1}{3\cdot 6^6}\left(\frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} + \text{etc.}\right)$
$0,0000000000607 = \frac{1}{4\cdot 6^8}\left(\frac{1}{4^4} + \frac{1}{9^4} + \frac{1}{16^4} + \text{etc.}\right)$

4

$l\,\pi = 1,1447298858879$

Logarithmus hic hyperbolicus ſi multiplicetur per

0,4342944819032 51

prodibit logarithmus communis valoris π ſeu numeri

3,14159265357989 etc.

qui logarithmus a Cl. Sharpio in Tabulis mathematicis computatus eſt

0,49714,98726,94133,85435,12682,88290 etc.

§. 19. Cum autem peripheria circuli per ſe ſatis ſit cognita ex approximationibus iam diligentiſſime peractis, vſui harum expreſſionum in hoc negotio ſuperſedebimus. Alter autem vſus, qui ex his expreſſionibus duci poteſt, conſiſtit in inueniendis ſinibus et tangentibus et ſecantibus quorumcunque angulorum, qua quidem in re opus eſt noſſe

nosse valorem ipsius π. Ita si ponamus $\pi = 2\,q$ ita vt sit q arcus 90 graduum erit

$$\text{sin. A}\,.\,\frac{m}{n}q = \frac{m}{n}q\,.\,\frac{4n^2-m^2}{4n^2}\,.\,\frac{16n^2-m^2}{16n^2}\,.\,\frac{36n^2-m^2}{36n^2}\,.\,\frac{64n^2-m^2}{64n^2}\ \text{etc.}$$

hincque

$$\text{cosec. A}\,.\,\frac{m}{n}q = \frac{n}{mq}\,.\,\frac{4n^2}{4n^2-m^2}\,.\,\frac{16n^2}{16n^2-m^2}\,.\,\frac{36n^2}{36n^2-m^2}\,.\,\frac{64n^2}{64n^2-m^2}\ \text{etc.}$$

Porro ex §. 12. posito $\sqrt{p} = \frac{m}{n}$ habebitur

$$\text{cos. A}\,.\,\frac{m}{n}q = \frac{n^2-m^2}{n^2}\,.\,\frac{9n^2-m^2}{9n^2}\,.\,\frac{25n^2-m^2}{25n^2}\,.\,\frac{49n^2-m^2}{49n^2}\ \text{etc.}$$

hincque

$$\text{sec. A}\,.\,\frac{m}{n}q = \frac{n^2}{n^2-m^2}\,.\,\frac{9n^2}{9n^2-m^2}\,.\,\frac{25n^2}{25n^2-m^2}\,.\,\frac{49n^2}{49n^2-m^2}\ \text{etc.}$$

Denique ex §. 14. deducitur pari modo

$$\text{tang. A}\,.\,\frac{m}{n}q = \frac{m}{n}q\,.\,\frac{n^2}{n^2-m^2}\,.\,\frac{4n^2-m^2}{4n^2}\,.\,\frac{9nn}{9n^2-m^2}\,.\,\frac{16n^2-m^2}{16n^2}\ \text{etc.}$$

hincque

$$\text{cot. A}\,.\,\frac{m}{n}q = \frac{n}{mq}\,.\,\frac{n^2-m^2}{n^2}\,.\,\frac{4n^2}{4n^2-m^2}\,.\,\frac{9n^2-m^2}{9n^2}\,.\,\frac{16n^2}{16n^2-m^2}\ \text{etc.}$$

Hae vero formulae, etsi vehementer conuergunt, tamen multo sunt aptiores ad logarithmos sinuum, tangentium et secantium inueniendos; quem vsum singularem antequam exponamus, methodum facilem aperiemus, ipsos sinus et tangentes expedite computandi: idque sine consuetis subsidiis ex multiplicatione arcuum, aliisque huc pertinentibus theorematis.

Problema 3.

§. 20. Inuenire canonem generalem, ad sinus et cosinus angulorum quorumcunque inueniendos idoneum.

Solutio.

Formulae, quas hic pro sinibus et cosinibus exhibuimus, si euoluantur, recidunt ad formulas iam pridem notas; scilicet posito arcu circuli $= s$, fit

sin. A

$$\text{ſin. A} \,.\, s = s - \frac{s^3}{1 \cdot 2 \cdot 3} + \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^7}{1 \cdot 2 \cdots 7} + \text{etc,}$$

$$\text{coſ. A} \,.\, s = 1 - \frac{s^2}{1 \cdot 2} + \frac{s^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{s^6}{1 \cdot 2 \cdots 6} + \text{etc.}$$

poſito ſinu toto $= 1$. Quodſi ergo ponatur q pro arcu 90. graduum, ſumaturque arcus propoſitus $s = \frac{m}{n} q$, fiet

$$\text{ſin. A} \,.\, \frac{m}{n} q = \frac{m}{n} \cdot q - \frac{m^3}{n^3} \cdot \frac{q^3}{1 \cdot 2 \cdot 3} + \frac{m^5}{n^5} \cdot \frac{q^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \text{etc.}$$

$$\text{coſ. A} \,.\, \frac{m}{n} q = 1 - \frac{m^2}{n^2} \cdot \frac{q^2}{1 \cdot 2} + \frac{m^4}{n^4} \cdot \frac{q^4}{1 \cdot 2 \cdot 3 \cdot 4} - \text{etc.}$$

Cum igitur ſit $q = \frac{\pi}{2}$ erit

$q = 1,570796326794896619231313216916$

Hoc vero valore loco poteſtatum ipſius q computato ac ſubſtituto, obtinebuntur formulae numericae, quibus tam ſinus quam coſinus arcus $\frac{m}{n} q$ exprimentur. Quoniam vero tantum pro arcubus 45° minoribus ſinus et coſinus deſiderantur erit $\frac{m}{n} < \frac{1}{2}$, et hanc ob rem ſeries datae maxime conuergent. Supputaui ego autem ſingulos horum ſerierum terminos a ſolo q pendentes in fractionibus decimalibus ad 28. figuras, quas, vt alios calculo tam taedioſo liberem, hic appono.

Erit igitur ſinus arcus $\frac{m}{n}$ 90 graduum $=$

$+ \frac{m}{n} \,.\, 1,5707963267948966192313216916$
$- \frac{m^3}{n^3} \,.\, 0,6459640975062462536557565636$
$+ \frac{m^5}{n^5} \,.\, 0,0796926262461670451205055487$
$- \frac{m^7}{n^7} \,.\, 0,0046817541353186881006854633$
$+ \frac{m^9}{n^9} \,.\, 0,0001604411847873598218726605$
$- \frac{m^{11}}{n^{11}} \,.\, 0,0000035988432352120853404580$
$+ \frac{m^{13}}{n^{13}} \,.\, 0,0000000569217292196792681170$
$- \frac{m^{15}}{n^{15}} \,.\, 0,0000000006688035109811467225$

$+\frac{m^{17}}{n^{17}}$. 0,0000000000060669357311061950

$-\frac{m^{19}}{n^{19}}$. 0,0000000000000437706546731370

$+\frac{m^{21}}{n^{21}}$. 0,0000000000000002571422892855

$-\frac{m^{23}}{n^{23}}$. 0,0000000000000000012538995403

$+\frac{m^{25}}{n^{25}}$. 0,0000000000000000000051564550

$-\frac{m^{27}}{n^{27}}$. 0,0000000000000000000000181239

$+\frac{m^{29}}{n^{29}}$. 0,0000000000000000000000000549

$-\frac{m^{31}}{n^{31}}$. 0,0000000000000000000000000001

Atque ſimili modo erit coſinus arcus $\frac{m}{n}$ 90 grad. =

$+$ 1,0000000000000000000000000000

$-\frac{m^2}{n^2}$. 1,2337005501361698273543113745

$+\frac{m^4}{n^4}$. 0,2536695079010480136365633659

$-\frac{m^6}{n^6}$. 0,0208634807633529608730516364

$+\frac{m^8}{n^8}$. 0,0009192602748394265802417158

$-\frac{m^{10}}{n^{10}}$. 0,0000252020423730606054810526

$+\frac{m^{12}}{n^{12}}$. 0,0000004710874778818171503665

$-\frac{m^{14}}{n^{14}}$. 0,0000000063866030837918522408

$+\frac{m^{16}}{n^{16}}$. 0,0000000000656596311497947230

$-\frac{m^{18}}{n^{18}}$. 0,0000000000005294400200734620

$+\frac{m^{20}}{n^{20}}$. 0,0000000000000034377391790981

$-\frac{m^{22}}{n^{22}}$. 0,0000000000000000183599165212

$+\frac{m^{24}}{n^{24}}$. 0,0000000000000000000820675327

$-\frac{m^{26}}{n^{26}}$. 0,0000000000000000000003115285

$+\frac{m^{28}}{n^{28}}$. 0,0000000000000000000000010165

$-\frac{m^{30}}{n^{30}}$, 0,0000000000000000000000000026

Quo-

Quocunque igitur angulo propoſito, eius ratio ad 90° eſt primum quaerenda, quae ſit vt m ad n, qua inuenta, ſi in his formulis fiat ſubſtitutio debito modo, reperietur tam ſinus quam coſinus anguli propoſiti.

Q. E. I.

§. 21. Quodſi igitur ponatur $\frac{m}{n} = 1$, prior formula dare debebit ſinum totum $= 1$, quod vt appareat calculum ſubiiciamus.

+	1,5707963267948966192313216916
−	0,6459640975062462536557565636
+	0,9248322292886503655755651280
+	0,0796926262461670451205055487
+	1,0045248555348174106960706767
−	4681754135318688100685463 3
+	0,9998431013994987225953852134
+	1604411847873598218726605
+	1,0000035425842860824172578739
−	3598843235212085340458 0
+	0,9999999437410508703319174159
+	5692172921967926811 70
+	1,0000000006627800900111855329
−	6688035109811467225
+	0,9999999999993976579030038 8104
+	6066935731 1061950
+	1,0000000000000435147611450054
−	43770654673 1370

$$
\begin{array}{lr}
+ & 0,9999999999999997441064718684 \\
+ & 2571422892855 \\
\hline
+ & 1,0000000000000000012487611539 \\
- & 12538995403 \\
\hline
+ & 0,9999999999999999999948616136 \\
+ & 51564550 \\
\hline
+ & 1,0000000000000000000000180686 \\
- & 181239 \\
\hline
+ & 0,9999999999999999999999999447 \\
 & 549 \\
\hline
+ & 0,9999999999999999999999999996
\end{array}
$$

vnde intelligitur errorem tantum 5 vnitatum in vltimis figuris esse commissum, qui ob totuplices additiones et subtractiones euitari omnino non potuit.

§. 22. Exemplum hoc adieci, vt appareat in computo harum formularum errorem a me non esse commissum, easque ideo tuto adhiberi posse. Quod idem vt clarius perspiciatur, calculum etiam cosinus anguli 90° hic apponam, qui debet esse $= 0$.

$$
\begin{array}{lr}
+ & 1,0000000000000000000000000000 \\
- & 1,2337005501361698273543113745 \\
\hline
- & 0,2337005501361698273543113745 \\
+ & 2536695079010480136365633659 \\
\hline
+ & 199689577648781862822519914 \\
- & 208634807633529608730516364 \\
\hline
\end{array}
$$

−	8945229984747745907996450
+	9192602748394265802417158
+	2473727636465198944207O8
−	2520204237306060548105260
−	4647660084080100389818
+	4710874778818171503665
+	632146947320111113847
−	638660308379185224O8
−	6513301059074O8561
+	656596311497947230
+	5260205590538669
−	529440020073462O
−	3419401019595I
+	3437739179098I
+	182781595030
−	183599165212
−	817570182
+	820675327
+	3105145
−	3115285
−	10140
+	10165
+	25
−	26
−	1

Qui consensus cum veritate tantus est, vt de veritate datarum formularum dubitare amplius non liceat.

§. 23. Quaeramus speciminis loco sinum et cosinum anguli 9 graduum, qui casus est facilis ob valorem $\frac{m}{n} = \frac{1}{10}$. Ac primo quidem pro sinu erunt termini affirmatiui

0, 15707963267948966192313216 91
796926262461670450 2050
160441184787359 8
56921729

\+ 0, 1570804290059125647840629069

Termini vero negatiui sunt.

0, 000645964097506246253655 7565
468175413531868 8100
35988432352 1
6688

− 0, 0006459645656816957739575874
\+ 0, 1570804296059125647840629069

0, 1564344650402308690101053195 = sin. 9°

Pro cosinu autem sunt termini affirmatiui

1, 000000000000000000000000000
2536695079010480136365 63
919260274839426 58
4710874778
65

\+ 1, 0000253669599827080208454065

Termini vero negatiui

0,

$$
\begin{array}{r}
0,01233700550136169827354311 37 \\
20863480763352960873 0 \\
2520204237306 0 \\
63866 0 \\
\hline
-\ 0,01233702630484449818308051587 \\
+\ 1,00002536695998270802084 54065 \\
\hline
0,98768834059513772619004024 78 = \text{cof. } 9^\circ
\end{array}
$$

Hoc autem exemplum, etſi in ſuo genere eſt facillimum, tamen abunde declarat vtilitatem formularum datarum, atque compendium, quod illae calculo alias operoſiſſimo afferunt.

§. 24. Labor autem iſtius computi multo fiet minor, ſi ſinus et coſinus non ad tot figuras in fractionibus decimalibus deſiderentur. Ponamus igitur ſinum totum ſeu radium eſſe

10000000000

atque pro hoc radio erit

$$
\begin{array}{rl}
\text{ſin. A.} \frac{m}{n} 90^\circ = & + \frac{m}{n} \,.\, 15707963267,94 \\
& - \frac{m^3}{n^3} \quad 6459640975,06 \\
& + \frac{m^5}{n^5} \,..\, 796926162,46 \\
& - \frac{m^7}{n^7} \,...\, 46817541,35 \\
& + \frac{m^9}{n^9} \,....\, 1604411,84 \\
& - \frac{m^{11}}{n^{11}} \,.....\, 35988,43 \\
& + \frac{m^{13}}{n^{13}} \,.......\, 569,21 \\
& - \frac{m^{15}}{n^{15}} \,.........\, 6,68 \\
& + \frac{m^{17}}{n^{17}} \,..........\, ,06
\end{array}
$$

Atque pari modo erit

 cof.

$$\begin{aligned}\text{cos. A.}\,\tfrac{m}{n}\,90^\circ = &+ 10000000000,00\\ &-\tfrac{m^2}{n^2}\,12337005501,36\\ &+\tfrac{m^4}{n^4}\,.2536695079,01\\ &-\tfrac{m^6}{n^6}\,..208634807,63\\ &+\tfrac{m^8}{n^8}\,...9192602,74\\ &-\tfrac{m^{10}}{n^{10}}\,...252020,42\\ &+\tfrac{m^{12}}{n^{12}}\,.....4710,87\\ &-\tfrac{m^{14}}{n^{14}}\,.......63,86\\ &+\tfrac{m^{16}}{n^{16}}\,..........,65\end{aligned}$$

vbi partes centesimas adiecimus, vt de vltimis figuris penitus certi esse queamus.

Problema 4.

§. 25. Inuenire canonem generalem pro inueniendis tangentibus et cotangentibus omnium angulorum.

Solutio.

Quod primum ad tangentes attinet, ponatur angulus rectus seu $90^\circ = q$, propositusque sit angulus $\frac{m}{n}q$ graduum seu $\frac{m}{n}\,90^\circ$, erit posito sinu toto $= 1$, tang. A . $\frac{m}{n}\,90^\circ$ $= \frac{2m}{nq}\left(\frac{n^2}{n^2-m^2} + \frac{n^2}{9n^2-m^2} + \frac{n^2}{25n^2-m^2} + \text{etc.}\right)$

$$\begin{aligned}\text{tang. A}\,.\,\tfrac{m}{n}\,90^\circ = &\tfrac{2m}{nq}\left(1+\tfrac{1}{3^2}+\tfrac{1}{5^2}+\tfrac{1}{7^2}+\text{etc.}\right)\\ &+\tfrac{2m^3}{n^3q}\left(1+\tfrac{1}{3^4}+\tfrac{1}{5^4}+\tfrac{1}{7^4}+\text{etc.}\right)\\ &+\tfrac{2m^5}{n^5q}\left(1+\tfrac{1}{3^6}+\tfrac{1}{5^6}+\tfrac{1}{7^6}+\text{etc.}\right)\\ &+\tfrac{2m^7}{n^7q}\left(1+\tfrac{1}{3^8}+\tfrac{1}{5^8}+\tfrac{1}{7^8}+\text{etc.}\right)\\ &\qquad\text{etc.}\end{aligned}$$

Quodsi

Quodſi autem termini primi harum ſerierum ſeorſim capiantur, vt reliqui eo magis conuergant erit

$$\text{tang. A} . \tfrac{m}{n} 90^\circ = + \frac{2mn}{(nn-mm)q}$$
$$+ \frac{2m}{nq}\left(\tfrac{1}{3^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \text{etc.}\right.$$
$$+ \frac{2m^3}{3n^3}\left(\tfrac{1}{3^4} + \tfrac{1}{5^4} + \tfrac{1}{7^4} + \text{etc.}\right.$$
$$+ \frac{2m^5}{n^5 q}\left(\tfrac{1}{3^6} + \tfrac{1}{5^6} + \tfrac{1}{7^6} + \text{etc.}\right.$$
$$+ \frac{2m^7}{n^7 q}\left(\tfrac{1}{3^8} + \tfrac{1}{5^8} + \tfrac{1}{7^8} + \text{etc.}\right.$$
$$\text{etc.}$$

Eſt vero $\frac{1}{\pi} = \frac{1}{2}q = 0,318309886183 79$ hincque $\frac{2}{q} = 1,273239544735 16$. Quare ſi ſummae iſtarum ſerierum quae proxime habentur, per hunc valorem multiplicentur prodibit

$$\text{tang. A} . \tfrac{m}{n} 90^\circ = + \frac{m}{n-m} . 0,6366197723675$$
$$+ \frac{m}{n+m} . 0,6366197723675$$
$$+ \frac{m}{n} . 0,2975567820597$$
$$+ \frac{m^3}{n^3} . 0,0186886502773$$
$$+ \frac{m^5}{n^5} . 0,0018424752034$$
$$+ \frac{m^7}{n^7} . 0,0001975800714$$
$$+ \frac{m^9}{n^9} . 0,0000216977245$$
$$+ \frac{m^{11}}{n^{11}} . 0,0000024011370$$
$$+ \frac{m^{13}}{n^{13}} . 0,0000002664132$$
$$+ \frac{m^{15}}{n^{15}} . 0,0000000295864$$
$$+ \frac{m^{17}}{n^{17}} . 0,0000000032867$$
$$+ \frac{m^{19}}{n^{19}} . 0,0000000003651$$
$$+ \frac{m^{21}}{n^{21}} . 0,0000000000405$$
$$+ \frac{m^{23}}{n^{23}} . 0,0000000000045$$
$$+ \frac{m^{25}}{n^{26}} . 0,0000000000000$$

 cuius

cuius formulae ope tangentes in fractionibus decimalibus ad 12. figuras facile computari poterunt posito sinu toto $= 1$.

Quod secundo ad cotangentes attinet, erit iisdem positis

$$\text{cotang. A} \,.\, \tfrac{m}{n}\, 90^\circ = \frac{n}{mq} - \frac{m}{2nq}\left(\frac{4n^2}{4n^2-m^2} + \frac{4n^2}{16n^2-m^2} + \frac{4n^2}{36n^2-m^2} + \frac{4n^2}{64n^2-m^2} + \text{etc.}\right)$$

seu

$$\begin{aligned}\text{cot. A} \,.\, \tfrac{m}{n}\, 90^\circ = \frac{n}{mq} &- \frac{m}{2nq}\left(1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \text{etc.}\right)\\ &- \frac{m^3}{8n^3q}\left(1 + \tfrac{1}{4^2} + \tfrac{1}{9^2} + \tfrac{1}{16^2} + \text{etc.}\right)\\ &- \frac{m^5}{32n^5q}\left(1 + \tfrac{1}{4^3} + \tfrac{1}{9^3} + \tfrac{1}{16^3} + \text{etc.}\right)\\ &- \frac{m^7}{128n^7q}\left(1 + \tfrac{1}{4^4} + \tfrac{1}{9^4} + \tfrac{1}{16^4} + \text{etc.}\right)\\ &\quad \text{etc.}\end{aligned}$$

Additis autem terminis primis erit

$$\begin{aligned}\text{cot. A} \,.\, \tfrac{m}{n}\, 90^\circ = \frac{n}{mq} - \frac{m}{2n-m}\cdot\frac{1}{2q} &- \frac{m}{2n+m}\cdot\frac{1}{2q}\\ &- \frac{2m}{nq}\cdot\tfrac{1}{4}\left(\tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} + \text{etc.}\right)\\ &- \frac{2m^3}{n^3q}\cdot\tfrac{1}{4^2}\left(\tfrac{1}{4^2} + \tfrac{1}{9^2} + \tfrac{1}{16^2} + \tfrac{1}{25^2} + \text{etc.}\right)\\ &- \frac{2m^5}{n^5q}\cdot\tfrac{1}{4^3}\left(\tfrac{1}{4^3} + \tfrac{1}{9^3} + \tfrac{1}{16^3} + \tfrac{1}{25^3} + \text{etc.}\right)\\ &\quad \text{etc.}\end{aligned}$$

At tam serierum loco summis substituendis, quam loco q valore debito, obtinebitur

$$\begin{aligned}\text{cot. A} \,.\, \tfrac{m}{n}\, q = &+ \frac{n}{m}\,.\,0,636619772367 5\\ &- \frac{m}{2n-m}\,.\,0,3183098861837\\ &- \frac{m}{2n+m}\,.\,0,3183098861837\\ &- \frac{m}{n}\,.\,0,2052888894145\\ &- \frac{m^3}{n^3}\,.\,0,0065510747882\\ &- \frac{m^5}{n^5}\,.\,0,0003450292554\\ &- \frac{m^7}{n^7}\,.\,0,0000202791060\\ &- \frac{m^9}{n^9}\,.\,0,0000012366527\end{aligned}$$

$$-\frac{m^{11}}{n^{11}} \cdot 0,0000000764959$$
$$-\frac{m^{13}}{n^{13}} \cdot 0,0000000047597$$
$$-\frac{m^{15}}{n^{15}} \cdot 0,0000000002969$$
$$-\frac{m^{17}}{n^{17}} \cdot 0,0000000000186$$
$$-\frac{m^{19}}{n^{19}} \cdot 0,0000000000011$$

Huiusque formulae ope cotangentes angulorum omnium 90. gradibus minorum expedite reperiri poterunt.

Q. E. I.

§. 26. Quanquam ex datis anguli ſinu et coſinu eiusdem tangens et cotangens inueniri poſſunt, tamen diuiſio, quae adhiberi debet, plerumque nimis moleſta eſſe ſolet. Quamobrem formulas hic datas multo aptiores eſſe merito arbitramur ad tangentes et cotangentes quorumuis angulorum inueniendas. Vt autem veritas harum regularum perſpiciatur, eiusmodi exempla tangentium et cotangentium euoluamus, quae per ſe ſint cognita. Quaeratur itaque tangens anguli ſemirecti ſeu 45°, quam conſtat eſſe aequalem ſinui toti ſeu 1. Erit igitur $m = 1$ et $n = 2$: vnde termini prodibunt ſequentes addendi:

0,6366197723675
0,2122065907891
1487783910298
23360812847
575773501
15435943
423784
11724
325
9
1,0000000000000

vti

vbi in additione vltimae columnae tres vnitates ſunt adiectae, quippe quae prodituræ fuiſſe cenſendae ſunt ex ſequentibus columnis, ſi affuiſſent. Ceterum ex formula manifeſtum eſt tangentem anguli recti fore infinitam ob $n-m=0$. Pro cotangente ſumamus exemplum anguli recti, cuius cotangens eſt $=0$. Cum igitur expreſſio noſtra cotangentis omnes terminos praeter primum habeat negatiuos addamus terminos negatiuos ſeorſim, qui ob $m = n = 1$ ita ſe habebunt.

0, 3183098861837
0, 1061032953946
0, 2052888894145
65510747882
3450292554
202791060
12366527
764959
47597
2969
186
11
0, 6366197723675

Terminus autem affirmatiuus, a quo haec ſumma auferri debet eſt

0, 6366197723675

ita vt cotangens anguli recti actu reperiatur $=0$.

§. 27. Etſi autem haec, quae de inuentione ſinuum et tangentium attulimus ex ſeriebus meis nuper expoſitis conſequuntur, tamen eaedem hae formulae ex aliis iam dudum cognitis ſeriebus deduci potuiſſent. His igitur re-

lictis

lictis grogredior ad ea, quae huic methodo ſummandi ſeries ſunt propria, atque modum docebo facilem inueniendi logarithmos ſinuum, et tangentium quorumcunque angulorum; qui eo magis eſt notatu dignus, quod logarithmos ſiue ſinuum ſiue tangentium praebeat, ſine praeuia ipſorum ſinuum ac tangentium cognitione. Cum autem logarithmi ſint duplices, vel naturales ſeu hyperbolici, vel decadici, in quibus logarithmus 10 ponitur $= 1$, vtriusque generis logarithmos hic inuenire docebo.

Problema. 5.

§ 28. Definire logarithmum tam naturalem quam conſuetum ſiue ſinus ſiue coſinus anguli cuiuscunque propoſiti.

Solutio.

Ex paragr. 19. capiatur pro logarithmo ſinus inueniendo expreſſio haec

$$\text{ſin. A.}\,\frac{m}{n}q = \frac{m}{n}q \cdot \frac{4n^2-m^2}{4n^2} \cdot \frac{16n^2-m^2}{16n^2} \cdot \frac{36n^2-m^2}{36n^2} \cdot \text{etc.}$$

quae in logarithmos conuerſa ſtatim dat

$$l\,\text{ſin. A.}\,\frac{m}{n}q = l\,\frac{m}{n} + \left(1 \frac{-m^2}{4n^2}\right) + l\left(1 \frac{-m^2}{16n^2}\right) + \text{etc.}$$

Quaeratur primo logarithmus naturalis ſinus anguli $\frac{m}{n}q$ ſeu $\frac{m}{n}90^{\circ}$, eritque logarithmis per ſeries expreſſis

$$\begin{aligned} l\,\text{ſin. A.}\,\frac{m}{n}q = l\,q + l\,\frac{m}{n} & \\ & - \frac{m^2}{4n^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}\right) \\ & - \frac{m^4}{2\cdot 4^2 n^4}\left(1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} + \text{etc.}\right) \\ & - \frac{m^6}{3\cdot 4^3 n^6}\left(1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} + \text{etc.}\right) \\ & - \frac{m^8}{4\cdot 4^4 n^8}\left(1 + \frac{1}{4^4} + \frac{1}{9^4} + \frac{1}{16^4} + \text{etc.}\right) \\ & \quad \text{etc.} \end{aligned}$$

fiue l fin. A $\frac{m}{n}\,q = l\,q - l\,\frac{n}{m} - l\,\frac{4n^2}{4n^2-m^2}$

$$- \frac{m^2}{1\cdot 4n^2}\left(\tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} + \text{etc.}\right)$$
$$- \frac{m^4}{2\cdot 4^2n^4}\left(\tfrac{1}{4^2} + \tfrac{1}{9^2} + \tfrac{1}{16^2} + \tfrac{1}{25^2} + \text{etc.}\right)$$
$$- \frac{m^6}{3\cdot 4^3n^6}\left(\tfrac{1}{4^3} + \tfrac{1}{9^3} + \tfrac{1}{16^3} + \tfrac{1}{25^3} + \text{etc.}\right)$$
$$- \frac{m^8}{4\cdot 4^4n^8}\left(\tfrac{1}{4^4} + \tfrac{1}{9^4} + \tfrac{1}{16^4} + \tfrac{1}{25^4} + \text{etc.}\right)$$

etc.

Quodfi nunc loco harum ferierum fummae proximae fubftituantur, eaeque per coefficientes numericas multiplicentur prodibit l fin. A $\frac{m}{n}\,90^\circ =$

$$l\,q - 2\,l\,2 - 3\,l\,n + l\,m + l(2n-m) + l(2n+m)$$
$$- \frac{m^2}{n^2}\cdot 0{,}16123351671205660911810379$$
$$- \frac{m^4}{n^4}\cdot 0{,}00257260105347306848487511$$
$$- \frac{m^6}{n^6}\cdot 0{,}00009032844783567260267978$$
$$- \frac{m^8}{n^8}\cdot 0{,}00000398179316205501892449$$
$$- \frac{m^{10}}{n^{10}}\cdot 0{,}00000019425295465196979631$$
$$- \frac{m^{12}}{n^{12}}\cdot 0{,}00000001001328748812045486$$
$$- \frac{m^{14}}{n^{14}}\cdot 0{,}00000000053404135618987888$$
$$- \frac{m^{16}}{n^{16}}\cdot 0{,}00000000002914859658937808$$
$$- \frac{m^{18}}{n^{18}}\cdot 0{,}000000000000161797979778706$$
$$- \frac{m^{20}}{n^{20}}\cdot 0{,}000000000000009097690905311$$
$$- \frac{m^{22}}{n^{22}}\cdot 0{,}00000000000000051682754587$$
$$- \frac{m^{24}}{n^{24}}\cdot 0{,}00000000000000002960770778$$
$$- \frac{m^{26}}{n^{26}}\cdot 0{,}00000000000000000170813$$
$$- \frac{m^{28}}{n^{28}}\cdot 0{,}00000000000000000009913$$
$$- \frac{m^{30}}{n^{30}}\cdot 0{,}00000000000000000000578$$
$$- \frac{m^{32}}{n^{32}}\cdot 0{,}0000000000000000000034$$

qui

qui termini poſtemi etſi non eousque ſint continuati ac priores, tamen poteſtate vlterius porriguntur niſi ſit $\frac{m}{n} = 1$. Dat autem haec forma logarithmum hyperbolicum ſinus anguli $\frac{m}{n} 90^\circ$, poſito ſinu toto $= 1$ eiusque logarithmo $= 0$. Debent autem pro hoc negotio etiam numerorum $2, n, m, 2n - m$ et $2n + m$ logarithmi hyperbolici accipi, itemque ipſius q, quem ſupra indicauimus. Hac vero ipſa methodo poterit lq accuratius exhiberi. Quodſi enim ponatur $m = 1$ et $n = 2$ erit $l\,\text{ſin. A}. 45' = l\frac{1}{\sqrt{2}} = -\frac{1}{2}l2$: ſerierum vero ſummae additae conficient vt ſequitur.

$$\begin{array}{r}
0,04030837917801415227952595 \\
16078756584206678030469 \\
141113819974323844 1687 \\
15553879539277741767 \\
1897001510273 1425 \\
24446502656544 1 \\
325952976 1901 \\
44477228683 \\
617210311 \\
8676234 \\
123221 \\
1765 \\
25 \\
\hline
0,04047059387191103834465527
\end{array}$$

qui valor ponatur tantiſper $= a$ eritque $lq = a + 5\,l2 - \frac{1}{2}l2 - l3 - l5$ eſt vero

 $\frac{9}{8}l2$

$$\tfrac{9}{2}\,l\,2 = 3,\,1191623125197538923775 4454$$
$$l\,3 = 1,\,0986122886681096913952452 6$$
$$l\,5 = 1,\,6094379124341003746007593 5$$
$$\tfrac{9}{2}\,l2 - l15 = 0,\,4111121114175438263815399 3$$
$$\alpha = 0,\,\underline{0404705938719110383446552 7}$$
$$lq = 0,\,4515827052894548647261952 0$$
$$l2 = 0,\,\underline{6931471805599453094172321 2}$$
$$l2\,q = 1,\,1447298858494001741434273 2$$

qui eſt valor pro logarithmo hyperbolico ipſius π, quem ſupra minus accurate §. 18. definiuimus. Quare ſi iſte valor loco lq ſubſtituatur, facili negotio logarithmi hyperbolici ſinuum quorumuis angulorum reperiri poterunt, vbi hoc tantum eſt monendum, numerorum $2, m, n, 2n - m$ et $2n + m$ logarithmos quoque hyperbolicos ſumi debere; qui vel facile computantur vel paſſim computati reperiuntur. Ex logarithmis autem hyperbolicis inueniuntur logarithmi communes, ſi illi multipliceutur per

$$0,\,4342944819325182$$

Fiat igitur haec multlplicatio, et tum addatur 10, eo quod in tabulis ordinariis logarithmus ſinus totius poni ſolet $= 10$, quo facto erit

$$\text{log. ſin. A.}\,\tfrac{m}{n}\,90^\circ = l(2n+m) + l(1n-m) + lm - 3\,ln$$
$$+\; 9,\,5940598857021801 7$$
$$-\,\tfrac{m^2}{n^2}\cdot\; 0,\,0700228266059019 1$$
$$-\,\tfrac{m^4}{n^4}\cdot\; 0,\,0011172664416618 4$$
$$-\,\tfrac{m^6}{n^6}\cdot\; 0,\,0000392291464539 1$$
$$-\,\tfrac{m^8}{n^8}\cdot\; 0,\,0000017292707983 6$$
$$-\,\tfrac{m^{10}}{n^{10}}\cdot\; 0,\,0000000843629862 9$$
$$-\,\tfrac{m^{12}}{n^{12}}\cdot\; 0,\,0000000043487155 0$$

$$-\frac{m^{14}}{n^{14}} \cdot 0,00000000023193121$$
$$-\frac{m^{16}}{n^{16}} \cdot 0,00000000001265907$$
$$-\frac{m^{18}}{n^{18}} \cdot 0,00000000000070268$$
$$-\frac{m^{20}}{n^{20}} \cdot 0,00000000000003951$$
$$-\frac{m^{22}}{n^{22}} \cdot 0,00000000000000224$$
$$-\frac{m^{24}}{n^{24}} \cdot 0,0000000000000013$$

Huius igitur expreſſionis ope logarithmi ſinuum ad duodecim atque etiam plures figuras computari poterunt: ſi quidem $\frac{m}{n}$ ſit $< \frac{1}{2}$ quibus caſibus termini duplo pauciores ſufficiunt.

Pergamus ergo ad logarithmos coſinuum definiendos, id quod commodiſſime fiet ex aequatione coſ. A. $\frac{m}{n}\, q = \frac{n^2 - m^2}{n^2} \cdot \frac{9n^2 - m^2}{9n^2} \cdot \frac{25n^2 - m^2}{25n^2}$. etc.
ex qua fit

$$l\,\text{coſ. A.}\, \frac{m}{n}\, 90^\circ = l\, \frac{n^2 - m^2}{n^2}$$
$$-\frac{m^2}{n^2} \cdot \left(\tfrac{1}{9} + \tfrac{1}{25} + \tfrac{1}{49} + \text{etc.}\right)$$
$$-\frac{m^4}{2n^4} \cdot \left(\tfrac{1}{9^2} + \tfrac{1}{25^2} + \tfrac{1}{49^2} + \text{etc.}\right)$$
$$-\frac{m^6}{3n^6} \cdot \left(\tfrac{1}{9^3} + \tfrac{1}{25^3} + \tfrac{1}{49^3} + \text{etc.}\right)$$
$$-\frac{m^8}{4n^8} \cdot \left(\tfrac{1}{9^4} + \tfrac{1}{25^4} + \tfrac{1}{49^4} + \text{etc.}\right).$$

etc.

ſeu ſummis proxime ſumendis erit

$$l\,\text{coſ. A.}\, \frac{m}{n}\, 90^\circ = l(n-m) + l(n+m) - 2\,l n$$
$$-\frac{m^2}{n^2} \cdot 0,23370055013616982735$$
$$-\frac{m^4}{n^4} \cdot 0,00733901580209602727$$
$$-\frac{m^6}{n^6} \cdot 0,00048235888031404063$$
$$-\frac{m^8}{n^8} \cdot 0,00003879475632402982$$
$$-\frac{m^{10}}{n^{10}} \cdot 0,00000340827260896510$$

$$
\begin{aligned}
&- \tfrac{m^{12}}{n^{12}} \cdot 0,00000031430809718659 \\
&- \tfrac{m^{14}}{n^{14}} \cdot 0,00000002989150274450 \\
&- \tfrac{m^{16}}{n^{16}} \cdot 0,00000000290464467239 \\
&- \tfrac{m^{18}}{n^{18}} \cdot 0,00000000028682639518 \\
&- \tfrac{m^{20}}{n^{20}} \cdot 0,00000000002868076974 \\
&- \tfrac{m^{22}}{n^{22}} \cdot 0,00000000000289697956 \\
&- \tfrac{m^{24}}{n^{24}} \cdot 0,00000000000029506024 \\
&- \tfrac{m^{36}}{n^{26}} \cdot 0,00000000000003026249 \\
&- \tfrac{m^{28}}{n^{28}} \cdot 0,00000000000000312232 \\
&- \tfrac{m^{30}}{n^{30}} \cdot 0,00000000000000032379 \\
&- \tfrac{m^{32}}{n^{32}} \cdot 0,00000000000000003373 \\
&- \tfrac{m^{34}}{n^{34}} \cdot 0,00000000000000000352 \\
&- \tfrac{m^{36}}{n^{36}} \cdot 0,00000000000000000037 \\
&- \tfrac{m^{38}}{n^{38}} \cdot 0,00000000000000000004
\end{aligned}
$$

Hoc modo igitur reperitur logarithmus hyperbolicus cofinus cuiusque anguli, exiftente logarithmo finus totius $= 0$. At logarithmus ordinarius obtinebitur, fi ifte logarithmus multiplicetur per

$$0,434294481903251 82$$

atque ad eum addatur 10. logarithmus fcilicet finus totius in tabulis receptus: erit igitur

$$
\begin{aligned}
\log.\ \mathrm{cof.}\ A \cdot \tfrac{m}{n} 90^\circ = {} & 10,000000000000000 \\
& - 2 ln + l(n-m) + l(n+m) \\
& - \tfrac{m^2}{n^2} \cdot 0,10149485934189 2 \\
& - \tfrac{m^4}{n^4} \cdot 0,00318729406545 1 \\
& - \tfrac{m^6}{n^6} \cdot 0,00020948580001 7 \\
& - \tfrac{m^8}{n^8} \cdot 0,00001684834859 7
\end{aligned}
$$

—

$$- \frac{m^{10}}{n^{10}} \cdot 0,000001480193986$$
$$- \frac{m^{12}}{n^{12}} \cdot 0,000000136502272$$
$$- \frac{m^{14}}{n^{14}} \cdot 0,000000012981715$$
$$- \frac{m^{16}}{n^{16}} \cdot 0,000000001261471$$
$$- \frac{m^{18}}{n^{18}} \cdot 0,000000000124567$$
$$- \frac{m^{20}}{n^{20}} \cdot 0,000000000012456$$
$$- \frac{m^{22}}{n^{22}} \cdot 0,000000000001258$$
$$- \frac{m^{24}}{n^{24}} \cdot 0,000000000000128$$
$$- \frac{m^{26}}{n^{26}} \cdot 0,000000000000013$$
$$- \frac{m^{28}}{n^{28}} \cdot 0,000000000000001$$

Hinc igitur inuenientur logarithmi vulgares cofinuum quorumcunque angulorum, idque ad 14 figuras in fractionibus decimalibus.

Q E I.

§. 29. Ex datis logarithmis finuum et cofinuum inveniuntur primo ftatim logarithmi fecantium et cofecantium. Deinde cum tangentis logarithmus prodeat, fi ab aggregato logarithmorum finus totius et finus anguli dati fubtrahatur logarithmus cofinus, erit pro logarithmis hyperbolicis pofito logarithmo finus totius $= 0$;

$$l \text{ tang. A} . \frac{m}{n} 90^\circ = l \frac{2n+m}{n+m} + l \frac{2n-m}{n-m} - l \frac{n}{m}$$
$$- 0,93471165583043 5$$
$$+ \frac{m^2}{n^2} \cdot 0,07246703342410 3$$
$$+ \frac{m^4}{n^4} \cdot 0,00476641474862 3$$
$$+ \frac{m^6}{n^6} \cdot 0,00039203043247 8$$
$$+ \frac{m^8}{n^8} \cdot 0,00003481296316 2$$

+

$$+ \frac{m^{10}}{n^{10}} \cdot 0,000003214019654$$
$$+ \frac{m^{12}}{n^{12}} \cdot 0,000000304294809$$
$$+ \frac{m^{14}}{n^{14}} \cdot 0,000000029357461$$
$$+ \frac{m^{16}}{n^{16}} \cdot 0,000000002875496$$
$$+ \frac{m^{18}}{n^{18}} \cdot 0,000000000285208$$
$$+ \frac{m^{20}}{n^{20}} \cdot 0,000000000028589$$
$$+ \frac{m^{22}}{n^{22}} \cdot 0,000000000002891$$
$$+ \frac{m^{24}}{n^{24}} \cdot 0,000000000000294$$
$$+ \frac{m^{26}}{n^{26}} \cdot 0,000000000000030$$
$$+ \frac{m^{28}}{n^{28}} \cdot 0,000000000000003$$

Huius expreſſionis autem negatiuum dabit cotangentis anguli $\frac{m}{n}$ 90° logarithmum hyperbolicum. Haecque expreſſio magnam afferet vtilitatem in Hydrographia, in quam ab Halleio logarithmi tangentium ſunt introducti.

§. 30 Simili modo logarithmi vulgares tangentium hinc inuenientur, erit ſcilicet

$$\log. \text{ tang. A.} \frac{m}{n} 90° = l\frac{2n+m}{n+m} + l\frac{2n-m}{n-m} - l\frac{n}{m}$$
$$+ 9,594059885702190$$
$$+ \frac{m^2}{n^2} \cdot 0,031472032735990$$
$$+ \frac{m^4}{n^4} \cdot 0,002070027623789$$
$$+ \frac{m^6}{n^6} \cdot 0,000170256653563$$
$$+ \frac{m^8}{n^8} \cdot 0,000015119077799$$
$$+ \frac{m^{10}}{n^{10}} \cdot 0,000001395831000$$
$$+ \frac{m^{12}}{n^{22}} \cdot 0,000000132153556$$
$$+ \frac{m^{14}}{n^{14}} \cdot 0,000000012749783$$
$$+ \frac{m^{16}}{n^{16}} \cdot 0,000000001248812$$

+

$+ \frac{m^{18}}{n^{18}} . 0,000000000123864$
$+ \frac{m^{20}}{n^{20}} . 0,0000000000012416$
$+ \frac{m^{22}}{n^{22}} . 0,0000000000001256$
$+ \frac{m^{24}}{n^{24}} . 0,0000000000000128$
$+ \frac{m^{26}}{n^{26}} . 0,0000000000000013$
$+ \frac{m^{28}}{n^{28}} . 0,0000000000000001$

Quodsi hinc quaeratur logarithmus tangentis anguli 45. graduum erit $n = 2$ et $m = 1$, fietque summa seriei

```
0,0078680081839977
      12937672648 68
       266026021 19
        5905889 76
        1363111 62
         32264 0
          778 2
           19 1
             4
```

0, 0080001056257721

logarithmi vero numerorum naturalium sunt

$l\,5 = 0,6989700043360188$
$-l\,2 = 0,3010299956639811$

0, 3979400086720377
addatur 9. 5940598857021902

9, 9919998943742279
itemque, 0, 0080001056257721

10, 0000000000000000

qui est logarithmus tangentis anguli 45. grad.

§. 32. Quodsi quis igitur voluerit tabulas sinuum et tangentium eorumque logarithmorum computare ad duodecim figuras in fractionibus decimalibus, dum tabulae vsu receptae eas tantum ad septem figuras exhibent; is sequentibus regulis vti poterit. Propositus scilicet sit angulus $\frac{m}{n}$ 90. graduum erit.

$$\begin{aligned}
\text{sin. A}. \tfrac{m}{n}\,90^\circ = &+ \tfrac{m}{n} \,.\, 1,5707963267949 \\
&- \tfrac{m^3}{n^3} \,.\, 0,6459640975062 \\
&+ \tfrac{m^5}{n^5} \,.\, 0,0796926262461 \\
&- \tfrac{m^7}{n^7} \,.\, 0,0046817541353 \\
&+ \tfrac{m^9}{n^9} \,.\, 0,0001604411848 \\
&- \tfrac{m^{11}}{n^{11}} \,.\, 0,0000035988432 \\
&+ \tfrac{m^{13}}{n^{13}} \,.\, 0,0000000569217 \\
&- \tfrac{m^{15}}{n^{15}} \,.\, 0,0000000006688 \\
&+ \tfrac{m^{17}}{n^{17}} \,.\, 0,0000000000061
\end{aligned}$$

$$\begin{aligned}
\text{cos. A}. \tfrac{m}{n}\,90^\circ = &+ \ 1,0000000000000 \\
&- \tfrac{m^2}{n^2} \,.\, 1,2337005501361 \\
&+ \tfrac{m^4}{n^4} \,.\, 0,2536695079010 \\
&- \tfrac{m^6}{n^6} \,.\, 0,0208634807633 \\
&+ \tfrac{m^8}{n^8} \,.\, 0,0009192602748 \\
&- \tfrac{m^{10}}{n^{10}} \,.\, 0,0000252020424
\end{aligned}$$

+

$$+\frac{m^{12}}{n^{12}} \cdot 0,0000004710875$$
$$-\frac{m^{14}}{n^{14}} \cdot 0,0000000063866$$
$$+\frac{m^{16}}{n^{16}} \cdot 0,0000000000656$$
$$-\frac{m^{18}}{n^{18}} \cdot 0,0000000000005$$

$$\text{tang. A.}\,\frac{m}{n}\,90^\circ = \frac{m}{n-m} \cdot 0,6366197723675$$
$$+\frac{m}{n+m} \cdot 0,6366197723675$$
$$+\frac{m}{n} \cdot 0,2975567820597$$
$$+\frac{m^3}{n^3} \cdot 0,0186886502773$$
$$+\frac{m^5}{n^5} \cdot 0,0018424752034$$
$$+\frac{m^7}{n^7} \cdot 0,0001975800714$$
$$+\frac{m^9}{n^9} \cdot 0,0000216977245$$
$$+\frac{m^{11}}{n^{11}} \cdot 0,0000024011370$$
$$+\frac{m^{13}}{n^{13}} \cdot 0,0000002664132$$
$$+\frac{m^{15}}{n^{15}} \cdot 0,0000000295864$$
$$+\frac{m^{17}}{n^{17}} \cdot 0,0000000032867$$
$$+\frac{m^{19}}{n^{19}} \cdot 0,0000000003651$$
$$+\frac{m^{21}}{n^{21}} \cdot 0,0000000000405$$
$$+\frac{m^{23}}{n^{23}} \cdot 0,0000000000045$$
$$+\frac{m^{25}}{n^{25}} \cdot 0,0000000000005$$

$$\text{cot. A.}\,\frac{m}{n}\,90^\circ = \frac{n}{m} \cdot 0,6366197723675$$
$$-\frac{m}{2m-n} \cdot 0,3183098861837$$
$$-\frac{m}{2n+m} \cdot 0,3183098861837$$
$$-\frac{m}{n} \cdot 0,2052888894145$$
$$-\frac{m^3}{n^3} \cdot 0,0065510747882$$

$$- \frac{m^5}{n^5} . 0,0003450292554$$
$$- \frac{m^7}{n^7} . 0,0000202791060$$
$$- \frac{m^9}{n^9} . 0,0000012366527$$
$$- \frac{m^{11}}{n^{11}} . 0,0000000764959$$
$$- \frac{m^{13}}{n^{13}} . 0,0000000047597$$
$$- \frac{m^{15}}{n^{15}} . 0,0000000002969$$
$$- \frac{m^{17}}{n^{17}} . 0,0000000000185$$
$$- \frac{m^{19}}{n^{19}} . 0,0000000000011$$

$$\log. \sin. A. \frac{m}{n} 90^{0} = l(2n+m) + l(2n-m) + lm - 3\,ln$$
$$+ 9,5940598857021$$
$$- \frac{m^2}{n^2} . 0,0700228266059$$
$$- \frac{m^4}{n^4} . 0,0011172664416$$
$$- \frac{m^6}{n^6} . 0,0000392291464$$
$$- \frac{m^8}{n^8} . 0,0000017292708$$
$$- \frac{m^{10}}{n^{10}} . 0,0000000843629$$
$$- \frac{m^{12}}{n^{12}} . 0,0000000043487$$
$$- \frac{m^{14}}{n^{14}} . 0,0000000002319$$
$$- \frac{m^{16}}{n^{16}} . 0,0000000000126$$
$$- \frac{m^{18}}{n^{18}} . 0,0000000000007$$

$$\log. \cos. A. \frac{m}{n} 90^{0} = 10,0000000080000$$
$$+ l(n+m) + l(n-m) - 2\,ln$$

$$-\frac{m^2}{n^2}\,.\,0{,}1014948593419$$
$$-\frac{m^4}{n^4}\,.\,0{,}0031872940654$$
$$-\frac{m^6}{n^6}\,.\,0{,}0002094858000$$
$$-\frac{m^8}{n^8}\,.\,0{,}0000168483486$$
$$-\frac{m^{10}}{n^{10}}\,.\,0{,}0000014801940$$
$$-\frac{m^{12}}{n^{12}}\,.\,0{,}0000001365023$$
$$-\frac{m^{14}}{n^{14}}\,.\,0{,}0000000129817$$
$$-\frac{m^{16}}{n^{16}}\,.\,0{,}0000000012614$$
$$-\frac{m^{18}}{n^{18}}\,.\,0{,}0000000001245$$
$$-\frac{m^{20}}{n^{20}}\,.\,0{,}0000000000126$$
$$-\frac{m^{22}}{n^{22}}\,.\,0{,}0000000000013$$

$$\text{log. tang. A}\,\frac{m}{n}\,90^{\circ} = l\,\frac{2n+m}{n+m} + l\,\frac{2n-m}{n-m} - l\,\frac{n}{m}$$
$$+\;9{,}5940598857022$$
$$+\frac{m^2}{n^2}\,.\,0{,}0314720327359$$
$$+\frac{m^4}{n^4}\,.\,0{,}0020700276238$$
$$+\frac{m^6}{n^6}\,.\,0{,}0001702566535$$
$$+\frac{m^8}{n^8}\,.\,0{,}0000151190778$$
$$+\frac{m^{10}}{n^{10}}\,.\,0{,}0000013958310$$
$$+\frac{m^{12}}{n^{12}}\,.\,0{,}0000001321535$$
$$+\frac{m^{14}}{n^{14}}\,.\,0{,}0000000127498$$
$$+\frac{m^{16}}{n^{16}}\,.\,0{,}0000000012488$$
$$+\frac{m^{18}}{n^{18}}\,.\,0{,}0000000001238$$
$$+\frac{m^{20}}{n^{20}}\,.\,0{,}0000000000124$$

$$+ \frac{m^{22}}{n^{22}} . \, 0{,}000000000012$$

$$+ \frac{m^{24}}{n^{24}} . \, 0{,}0000000000001$$

Quodſi hic logerithmus a 20. ſubtrahatur, prodibit logarithmus cotangentis eiusdem anguli $\frac{m}{n}$ 90°. Simili autem modo logarithmus coſinus a 20. ſubtractus relinquet logarithmum ſecantis, atque logarithmus ſinus a 20. ſubtractus logarithmum coſecantis.

CLAS