

Animadversiones

in

Rectificationem Ellipsis.

§. I.

Tab. I.

Ellipsis rectificatio tot jam variis methodis est frustra tentata, ut non solum comparationem arcuum ellipticorum cum lineis rectis, sed etiam ne cum circularibus quidem aut parabolicis expectare nequeamus. Cum enim formula illa differentialis, cujus integrale arcum ellipticum indefinitum exprimit, nullo modo ab irrationalitate liberari queat; certum hoc est signum, ejus integrationem non solum non algebraice, sed etiam ne concessis quidem circuli & hyperbolæ quadraturis perfici posse. Quod cum tenendum sit de rectificatione ellipsis indefinita, hinc adhuc minime sequitur, arcum quempiam definitum veluti totam perimetrum ellipsis omnem comparationem cum lineis vel rectis vel circularibus penitus respuere: propterea quod jam innumerabiles curvæ assignari possunt indefinite æque parum rectificabiles atque ellipsis, in quibus tamen arcus definiti per lineas rectas mensurari queant.

§. II. Missa igitur rectificatione ellipsis indefinita, definitam potius sum aggressus, experturus, utrum tota cujusque ellipsis perimenter non commode possit ad mensuras cognitatas, quorum etiam logarithmos & arcus circulares refero, per expressiones finitas revocari. Quanquam autem in hac investigatione nihil admodum sum consecutus, quod scopo meo satisfacisset; tamen præter expectationem nonnulla se mihi obtulerunt phænomena

Euleri Opuscula Tom. II.

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satis singularia, quibus theoria linearum curvarum non mediocriter promoveri videtur. Tum vero etiam difficultates, quæ in toto hoc calculo occurrerunt, ansam mihi præbuerunt quædam insignia artificia inveniendi, quæ tam in calculo integrali, quam in theoria serierum infinitarum ingentem utilitatem sæpius afferre posse videntur. Quamobrem operæ pretium fore existimavi, si has speculationes totumquæ quasi filum calculorum meorum dilucide exposuero.

Propositio.

Fig. 1. §. III. Super data recta AC tanquam altero semiaxe d. scriptos principio infinitos quadrantes ellipticos AP, AB, Ap , quorum ergo omnium est centrum C , alteri vero semiaxes conjugati sunt CP, CB, Cp . Tum ex singulis punctis P, B, p arcus elliptici PA, BA, pA in directam extendantur, ita ut qualibet PQ sit rectæ CA parallela & quadranti elliptico PTA æqualis: quod si ubique fieri concepiatur, puncta hæc Q sita erunt in linea quadam curva $AQDp$, cujus naturam investigare constitui.

Ad genesis hujus curvæ vel leviter attendenti mox patebit, eam sequentes habere proprietates, quas evolvam, antequam in ipsam hujus curvæ indolem diligentius inquiram; ut ejus figura & ductus saltem obiter perspiciatur.

§. IV. Primum igitur si in recta indefinita CBp quæ ad datam CA est normalis, capiatur quævis abscissa CP , applicata PC , quæ ei respondet, erit æqualis quadranti perimetri ellipsis, cujus semiaxes conjugati sunt, recta data CA & ipsa abscissa CP . Hinc si capiatur abscissa $CB = CA$, quo casu quadrans ellipticus abibit in quadrantem circulem AB , applicata respondens BD æqualis erit quartæ parti peripheriæ circuli radio AC descripti. Unde si ratio diametri ad peripheriam ponatur $= 1: \pi$, erit ista applicata

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$BD = \frac{1}{2} \cdot AC$: sive ob $\ast = 3, 1415926535897932$ erit $BD = 1, 5707963267948961 \cdot AC$.

§. V. Secundo: Si abscissa CP evanescat, ellipsis evadet infinite angusta, atque cum linea recta confunderetur. Hoc ergo casu quadrans ellipticus abibit in ipsam lineam AC, cui propterea applicata abscissæ evanescenti respondens erit æqualis. Quare ipsa recta CA erit applicata puncto C respondens, & curva quæsitæ per punctum A transibit. Hujus ergo curvæ jam duo habemus puncta cognita A & D, quorum alterum A geometricè datur, alterum vero D per rationem diametri ad peripheriam definitur.

§. VI. Tertio: Ex cognito quovis curvæ puncto Q intra A & D sito, semper aliud quoddam curvæ punctum q ultra D situm definiiri potest. Capiatur enim Cp tertia proportionalis ad CP & CA, ut sit $Cp = \frac{CA \cdot CA}{CP}$, quia est $CP : CA = CA : Cp$, erit quadrans ellipticus Ap similis quadranti elliptico AP, cum utrinque eadem sit ratio inter semiaxes conjugatos. Hinc erit arcus Ap ad arcum AP ut AC ad CP, ideoque $pq : PQ = AC : CP$ seu $pq = \frac{AC \cdot PQ}{CP}$. Consequenter si curvæ quæsitæ arcus AD tantum jam fuerit descriptus, ex eo reliqua curvæ pars Dq in infinitum extensa definietur.

§. VII. Quarto: Hinc jam insignis proprietas æquationis, qua natura curvæ AQDq exprimitur, agnoscitur: Si enim recta data AC unitate designetur, ut sit $AC = 1$; abscissa autem quævis unitate minor $CP = p$, eique respondens applicata $PQ = q$; tum vero ponatur abscissa illa altera $Cp = P$ & applicata $pq = Q$; erit

$$P = \frac{1}{p} \quad \& \quad Q = \frac{q}{p}.$$

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esse debeat æquatio quæ est inter p & q , patet æquationem inter p & q nullam mutationem esse subituram, si in ea loco p ubique scribatur $\frac{1}{p}$ & $\frac{q}{p}$ loco q . Unde qualis ipsius p functio sit q conijcere licet.

§. VIII. Quinto: Patet crescentibus abscissis CP applicatas continuo crescere, cum semper sint majores quam abscissæ. Verum si abscissæ statuuntur infinitæ, applicatæ ipsis fient æquales: discrimin enim prodibit infinite parvum; unde colligimus quæsitam curvam habere asymptotam, & quidem rectam CV angulum rectum ACB bisecantem. Forma igitur hujus curvæ similis erit hyperbolæ æquilateræ centrum in C , axem CA & asymptotam CV habentis. Ex descriptione porro intelligitur, curvam infra rectam CA productam sui similem fore, ideoque rectam CA ejus fore diametrum perinde atque hyperbolæ. Verumtamen hoc facile perspicitur, nostram curvam multo lentius ad asymptotam suam CV appropinquare quam hyperbolam. Nam in hyperbola æquilatera, cui nostram curvam comparamus, quævis applicata PQ æqualis est rectæ lineæ AP ; unde cum applicata nostræ curvæ arcus AP sit æqualis, patet hyperbolam nostræ curvæ fore circum scriptam, ita tamen ut in initio A , & in spatio infinito se mutuo tangant.

§. IX. His affectionibus latius patentibus in genere notatis, in ipsam hujus curvæ naturam accuratius inquiramus, ac proposita quacunque abscissa $CP = p$, valorem respondentis applicatæ $PQ = q$ investigemus; qui cum expressione finita contineri nequeat, per seriem infinitam exhiberi debet. Sequens igitur resolvi oportet

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Problema.

Ex datis semiaxibus CA & CP quadrantis elliptici CAP per sortem infinitam definire longitudinem arcus quadrantis ATP.

Solutio.

§. X. Cum vocatus sit alter semiaxis AC = 1, alter vero CP = p, & arcus AYP = q, quaeratur primo arcus quivis indefinitus PY, qui vocetur = s. Jam ducta ad CP applicata normali YX; sit CX = x & XY = y, erit ex natura ellipsis $x = p \sqrt{1 - yy}$,

hincque $dx = \frac{-pydy}{\sqrt{1-yy}}$. Fiet ergo ob $ds = \sqrt{dx^2 + dy^2}$

$$ds = \frac{dy \sqrt{1 - yy + ppyy}}{\sqrt{1 - yy}}$$

unde integrando erit arcus $s = \int \frac{dy \sqrt{1 - yy + ppyy}}{\sqrt{1 - yy}}$

quæ integratio ita institui debet, ut posito $y = 0$ fiat quoque $s = 0$, quia evanescente applicata XY = y simul PY = s evanescit. Hoc igitur integrali invento si ponatur $y = CA = 1$, arcus indefinitus PY abibit in longitudinem quadrantis elliptici PYA = q, quem quaerimus, ita ut sit

$$q = \int \frac{dy \sqrt{1 - yy + ppyy}}{\sqrt{1 - yy}}$$

siquidem perfecta integratione ponatur $y = 1$.

§. XI. Ad institutum ergo nostrum non est necesse, ut quaeramus valorem integralis hujus indefiniti, sed eum tantum, quem induit, si post integrationem variabili y tribuatur valor determinatus = 1: quo pacto series multo simplicior valorem q exprimens obtineri poterit. Ponatur enim brevitatis gratia

$1 - pp = nn$, ut sit $\sqrt{1 - yy + ppyy} = \sqrt{1 - nnyy}$
eritque hanc formulam in seriem evolvendo.

$$\sqrt{1 - nnyy} = 1 - \frac{1}{2} nnyy - \frac{1 \cdot 1}{2 \cdot 4} n^4 y^4 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} n^6 y^6 \text{ \&c.}$$

Quo valore substituto pro $\sqrt{1 - yy + ppyy}$, arcus q ita
exprimetur ut sit:

$$q = \int \frac{dy}{\sqrt{1 - yy}} - \frac{1}{2} nn \int \frac{yydy}{\sqrt{1 - yy}} - \frac{1 \cdot 1}{2 \cdot 4} n^4 \int \frac{y^4 dy}{\sqrt{1 - yy}} \\ - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \int \frac{y^6 dy}{\sqrt{1 - yy}} \text{ \&c.}$$

si quidem in singulis his integralibus post integrationem ponatur
 $y = 1$.

§. XII. Evolvamus ergo singula hæc integralia; ac pri-
mo quidem ex circulo manifestum est, formulam $\int \frac{dy}{\sqrt{1 - yy}}$ ex-

primere arcum circuli cujus sinus $= y$ pro radio $= 1$: unde po-
sito $y = 1$, hæc formula dabit quartam peripheriæ partem, cu-
jus radius $= 1$. Idemque posita ratione diametri ad peripheriam
 $= 1 : \pi$, erit $\int \frac{dy}{\sqrt{1 - yy}} = \frac{\pi}{2}$; sicque jam adepti sumus valo-
rem primi termini in serie nostra ante inventa.

§. XIII. Reliqui termini pari modo per valorem π com-
mode poterunt exprimi; cujusvis enim termini integratio ad in-
tegrationem præcedentis reducitur: quod quo facilius intelligatur

consideremus formulam quamcunque $\int \frac{y^m dy}{\sqrt{1 - yy}}$; erit sequens

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$$\int \frac{y^{\mu}}{\sqrt{1 - y^2}}$$

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$\int \frac{y^{\mu+2} dy}{V(1-yy)}$. Jam assumamus hanc formulam algebraicam

$y^{\mu+1} V(1-yy)$, cujus differentiale cum sit =

$$\frac{(\mu+1)y^{\mu} dy - (\mu+2)y^{\mu+2} dy}{V(1-yy)} \text{ erit vicissim}$$

$$(\mu+1) \int \frac{y^{\mu} dy}{V(1-yy)} - (\mu+2) \int \frac{y^{\mu+2} dy}{V(1-yy)} = y^{\mu+1} V(1-yy)$$

unde colligimus fore

$$\int \frac{y^{\mu+2} dy}{V(1-yy)} = \frac{\mu+1}{\mu+2} \int \frac{y^{\mu} dy}{V(1-yy)} - \frac{1}{\mu+2} y^{\mu+1} V(1-yy)$$

Quare invento integrali $\int \frac{y^{\mu} dy}{V(1-yy)}$ ex eo facile elicetur integra-

le sequens $\int \frac{y^{\mu+2} dy}{V(1-yy)}$

§. XIV. Quoniam vero eos tantum horum integralium valores desideramus, qui prodeunt posito $y = 1$; hoc casu quan-

titas algebraica $\frac{1}{\mu+1} y^{\mu+1} V(1-yy)$ evanescit, eritque generatim pro casu $y = 1$

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$$\int \frac{y^{\mu+2} dy}{\sqrt{(1-yy)}} = \frac{\mu+1}{\mu+2} \int \frac{y^{\mu} dy}{\sqrt{(1-yy)}}$$

substituamus jam pro μ successive valores 0, 2, 4, 6, 8 &c. & quoniam vidimus esse $\int \frac{dy}{\sqrt{(1-yy)}} = \frac{\pi}{2}$, erit, ut sequitur:

$$\mu = 0; \int \frac{y^2 dy}{\sqrt{(1-yy)}} = \frac{1}{2} \int \frac{dy}{\sqrt{(1-yy)}} = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\mu = 2; \int \frac{y^4 dy}{\sqrt{(1-yy)}} = \frac{3}{4} \int \frac{y^2 dy}{\sqrt{(1-yy)}} = \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\pi}{2}$$

$$\mu = 4; \int \frac{y^6 dy}{\sqrt{(1-yy)}} = \frac{5}{6} \int \frac{y^4 dy}{\sqrt{(1-yy)}} = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{\pi}{2}$$

$$\mu = 6; \int \frac{y^8 dy}{\sqrt{(1-yy)}} = \frac{7}{8} \int \frac{y^6 dy}{\sqrt{(1-yy)}} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{\pi}{2}$$

unde lex, qua sequentes progrediuntur, sponte elucet.

§. XV. Quodsi jam isti valores pro formulis integralibus, ex quibus longitudo quadrantis elliptici q constari inventus est, substituuntur, reperietur

$$q = \frac{\pi}{2} - \frac{1}{2} \pi n^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1 \cdot 1}{2 \cdot 4} n^4 \cdot \frac{1 \cdot 3}{2 \cdot 4} \frac{\pi}{2} - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} n^6 \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{\pi}{2} \dots$$

quæ ad sequentem seriem satis concinnam revocatur

$$q = \frac{\pi}{2} \left(1 - \frac{1 \cdot 1}{2 \cdot 2} n^2 - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} n^4 - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} n^6 - \dots \right)$$

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cujus lex progressionis est manifesta. Restituatur ergo pro nn suus valor $1 - pp$, eritque

$$q = \frac{\pi}{2} \left(1 - \frac{1 \cdot 1}{2 \cdot 2} (1 - pp) - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} (1 - pp)^2 - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} (1 - pp)^3 - \&c. \right)$$

§. XVI. Cum pro curva nostra $AQDq$ littera p exhibeat abscissam CP & littera q applicatam PQ ; jam adepti sumus pro ista curva æquationem inter ejus-coordinatas p & q , quæ etsi serie constat infinita, tamen non solum ejus naturam in se complectitur, sed etiam valores applicatæ q mox satis accurate exhibet, si abscissa p parum ab unitate differat: hoc est, cum sit $CB = CA = 1$, si punctum P ipsi B fuerit proximum; cum enim ob $1 - pp = nn$ quantitatem valde parvam series inventa valde convergit.

§. XVII. Hinc igitur indolem nostræ curvæ prope punctum D , hoc est ejus directionem & curvaturam definire poterimus.

Primo enim patet uti jam vidimus, si $p = 1$ fore $q = \frac{\pi}{2}$, ita ut

summa abscissa $CB = 1$ sit applicata $BD = \frac{\pi}{2} = 1,570796326$

7948961. Deinde ad positionem tangentis inveniendam, queratur ratio differentialium $dq: dp$, quæ per differentiationem reperitur:

$$\frac{dq}{dp} = \frac{\pi}{2} p \left(\frac{1}{2} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4} (1 - pp) + \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6} (1 - pp)^2 + \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8} (1 - pp)^3 + \&c. \right)$$

Posito jam $p = 1$ fiet $\frac{dq}{dp} = \frac{\pi}{4}$. Unde si DG sit tangens
 curvæ in puncto D, cum sit BD: BG = $dq: dp$, erit $BG = \frac{dp}{dq}$
 $BD = \frac{4}{\pi}$. BD & ob $BD = \frac{\pi}{2}$, fiet BH = 2 = 2EC, & CG
 = BC. Sicque hoc casu subtangens BG erit dupla abscissæ BC:
 & cum anguli BGD tangens sit = $\frac{dq}{dp} = \frac{\pi}{4} = 0,78539816$
 erit angulus BGD = 38, 8, 45, 41, 51.

§. XVIII. Ad radius osculi seu evolutæ in puncto D de-
 finiendum, cum sit ob $\frac{dq}{dp} = \frac{\pi}{4}$, elementum curvæ $\sqrt{dp^2 + dq^2}$
 = $dp \sqrt{1 + \frac{\pi^2}{16}}$, erit radius osculi = $(1 + \frac{\pi^2}{16})^{\frac{3}{2}} dp : ddq$.

At sumendis differentialibus secundis erit

$$\frac{ddq}{dp^2} = \frac{\pi}{2} \left(\frac{1}{2} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4} (1 - pp) + \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6} (1 - pp)^2 + \&c. \right)$$

$$- \frac{\pi}{2} pp \left(\frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 6} (1 - pp) + \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 7} (1 - pp)^2 + \&c. \right)$$

Posito ergo $p = 1$, erit $\frac{ddq}{dp^2} = \frac{\pi}{2} \left(\frac{1}{2} - \frac{3}{8} \right) = \frac{\pi}{16}$. Unde in
 puncto curvæ D erit radius evolutæ = $\frac{16}{\pi} \left(1 + \frac{\pi^2}{16} \right) \sqrt{1 + \frac{\pi^2}{16}}$,
 qui valor in numeris proxime reperitur = 10, 470672.

§. XIX.

§. XIX. Potest hinc adhuc alia series inveniri, quæ valorum applicatæ PQ = q exprimat. Consideretur enim illud alterum curvæ punctum q, pro quo sit abscissa Cp = P & applicata pq = Q, erit quoque ob P > 1.

$$Q = \frac{q}{2} \left(1 + \frac{1 \cdot 1}{2 \cdot 2} (PP - 1) - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} (PP - 1)^2 + \right. \\ \left. \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} (PP - 1)^3 - \&c. \right)$$

Jam vero supra notavimus, si sit $P = \frac{1}{p}$, fore $Q = \frac{q}{p}$; quare his valoribus substitutis impetrabimus novam æquationem inter p & q, qua natura curvæ pariter exprimeretur.

$$q = \frac{q}{2} p \left(1 + \frac{1 \cdot 1 (1 - pp)}{2 \cdot 2 pp} - \frac{1 \cdot 1 \cdot 1 \cdot 3 (1 - pp)^2}{2 \cdot 2 \cdot 4 \cdot 4 p^2} + \right. \\ \left. \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 (1 - pp)^3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 p^3} - \&c. \right)$$

quæ si cum ante inventa combinetur, innumerabiles alia novæ æquationes obtineri poterunt. Veluti si prior per p multiplicata ab hac subtrahatur, prodibit.

$$q - pq = \frac{q}{2} p \left(\frac{1 \cdot 1 (1 - pp) (1 + pp)}{2 \cdot 2 pp} - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} \right. \\ \left. \frac{(1 - pp)^2 (1 - p^2)}{p^2} + \&c. \right)$$

quæ reducit ad hanc:

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$$q = \frac{1}{4} (1+p) \left(\frac{1}{2} \cdot \frac{1+pp}{p} - \frac{1 \cdot 1 \cdot 3 (1-p^4) (1-pp)}{2 \cdot 4 \cdot 4 p^3} + \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 (1+p^4) (1-pp)^2}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 p^5} - \&c. \right)$$

vel cum series adhuc sit divisibilis per $\frac{1+pp}{2p}$ erit

$$q = \frac{1}{8} \frac{(1+p)(1+pp)}{p} \left(1 - \frac{1 \cdot 3 (1-pp)}{4 \cdot 4 pp} (1-pp) + \frac{1 \cdot 3 \cdot 3 \cdot 5}{4 \cdot 4 \cdot 6 \cdot 6} \frac{(1-pp+p^4)}{p^4} (1-pp)^2 - \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 (1-pp+p^4-p^8)}{4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 p^5} (1-pp)^3 + \&c. \right)$$

§. XX. Manifestum autem est has series parum subsidii afferre, si applicatas invenire velimus, quæ longius a BD, quæ abscissæ $p = 1$ respondet, sint remotæ, si enim pro p ponatur numerus vel valde magnus vel valde parvus, series inventa vel parum admodum convergit vel etiam divergit. Si enim inde longitudinem primæ applicatæ CA, quæ abscissæ $p = 0$ respondet, definire velimus, serie primum inventa uti conveniet, quia in reliquis termini evadunt infinite magni. Habebimus igitur pro hoc casu $p = 0$;

$$q = \frac{1}{2} \left(1 - \frac{1 \cdot 1}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 1 \cdot 2}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8} - \&c. \right)$$

quæ tam lente convergit, ut etiamsi plurimi termini actu colligerentur, tamen verus ipsius q valor, quem novimus esse $\frac{1}{2}$, inde difficillime agnosci posset.

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§. XXI. Quamquam autem hunc quidem novimus esse

$$1 - \frac{1.1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} - \&c. = \frac{2}{\pi}$$

tamen inventio summæ hujus seriei non parum ardua videtur, si a priori tentetur. Veritatem quidem ex formula, quam quondam Wallisius pro circuli quadratura dedit, intelligere licet, si termini ab initio in unum colligantur, sic enim prodit

$$1 - \frac{1.1}{2.2} = \frac{1.3}{2.2}$$

$$\frac{1.3}{2.2} - \frac{1.1.1.3}{2.2.4.4} = \frac{1.3.(4.4 - 1.1)}{2.2.4.4} = \frac{1.3.3.5}{2.2.4.4}$$

$$\frac{1.3.3.5}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} = \frac{1.3.3.5.5}{2.2.4.4.6.6}$$

unde valor seriei in infinitum continuatæ erit

$$\frac{1.3.3.5.5.7.7.9.9.11.11.13.13}{2.2.4.4.6.6.8.8.10.10.12.12.14} - \&c.$$

quæ expressio cum sit ipsa Wallisiana, patet summam nostræ seriei esse $= \frac{2}{\pi}$. Interim tamen juvabit tradere methodum hanc seriei aliasque similes a priori summandi.

Problema.

Invenire summam hujus seriei infinitæ:

$$1 - \frac{1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.5}{2.2.4.4.6.6} - \&c.$$

cujus lex progressionis primo intuitu est manifesta.

Solutio.

§. XXII. Ponatur summa hujus seriei, quæ quæritur $\equiv P$, ut sit

$$P = 1 - \frac{1.1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} - \&c.$$

Jam eligatur series, cujus summa constat, & cujus coefficientes jam in his terminis contineantur. Cujusmodi est hæc

$$\frac{1}{\sqrt{(1-xx)}} = 1 + \frac{1}{2} xx + \frac{1.3}{2.4} x^4 + \frac{1.3.5}{2.4.6} x^6 + \frac{1.3.5.7}{2.4.6.8} x^8 + \&c.$$

Erit ergo per differentiale quodpiam dP multiplicando & integrando

$$\int \frac{dP}{\sqrt{(1-xx)}} = P + \frac{1}{2} \int xx dP + \frac{1.3}{2.4} \int x^4 dP + \frac{1.3.5}{2.4.6} \int x^6 dP + \&c.$$

Nunc differentiale hoc dP ita definiatur, ut si post integrationem ponatur $x=1$. fiat

$$\int x dP = -\frac{1}{2} P$$

$$\int x^4 dP = +\frac{1}{2} \int x x dP = -\frac{1.1}{2.4} P$$

$$\int x^6 dP = +\frac{1}{3} \int x^4 dP = -\frac{1.1.3}{2.4.6} P$$

$$\int x^8 dP = +\frac{1}{4} \int x^6 dP = -\frac{1.1.3.5}{2.4.6.8} P$$

quo facto si hi valores substituantur habebitur;

$$\int \frac{dP}{\sqrt{(1-xx)}} = P \left(1 - \frac{1.1}{2.2} - \frac{1.1.3}{2.2.4.4} - \frac{1.1.3.3.5}{2.2.4.4.6.6} - \&c. \right)$$

ideoque

ideoque $\int \frac{dP}{\sqrt{(1-xx)}} = P$, si quidem post integrationem statuitur $x = 1$.

§. XXIII. Huc ergo res redit, ut quaeratur formula differentialis dP , ut superioribus conditionibus satisfiat.

seu ut in genere fit $\int x^{\mu+2} dP = \frac{\mu-1}{\mu+2} \int x^{\mu} dP$, si quidem post integrationem utramque ponatur $x = 1$. Omissa igitur hac conditione fit

$$\int x^{\mu+2} dP = \frac{\mu-1}{\mu+2} \int x^{\mu} dP + \frac{Qx^{\mu+2}}{\mu+2}$$

ubi Q ejus modi sit functio ipsius x , quae evanescat posito $x = 1$.

Capiantur ergo differentia, eritque per x^{μ} dividendo

$$xxdP = \frac{\mu-1}{\mu+2} dP + \frac{xdQ + (\mu+1)Qdx}{\mu+2}$$

seu $0 = (\mu-1)dP - (\mu+2)xxdP + xdQ + (\mu+1)Qdx$ quae aequatio, cum locum habere debeat pro omni valore ipsius μ , resolvetur in has duas:

$$0 = dP - xxdP + Qdx$$

$$0 = -dP - 2xxdP + xdQ + Qdx$$

$$\text{unde fit } dP = \frac{-Qdx}{1-xx} = \frac{xdQ + Qdx}{1+2xx}$$

$$\& xxdQ(1-xx) = -Qdx(2+xx)$$

Quae

Quare cum sit $\frac{dQ}{Q} = \frac{dx(1+xx)}{x(1-xx)} = \frac{2dx}{x} - \frac{3xdx}{1-xx}$

erit $Q = \frac{-(1-xx)^{\frac{3}{2}}}{xx}$ & $dP = \frac{dx}{xx} \sqrt{1-xx}$

§. XXIV. Verum hic notandum est, et si valor ipsius Q evanescat posito $x = 1$; tamen casu $\mu = 0$, quantitatem algebraicam

$\frac{Qx}{\mu + 1}$ non evanescere, si ponatur $x = 0$, quæ tamen conditio æque est necessaria atque altera, ita ut hoc casu non sit $\int xxxdP = -\frac{1}{2}P$. Cum autem reliquæ formulæ quibus $\mu > 0$ locum habeant, a formula $\int xxxdP$ erit incipiendum, eritque

$\int x^4 dP = \frac{1}{4} \int xxxdP$

$\int x^6 dP = \frac{3}{6} \int x^4 dP = \frac{1 \cdot 3}{4 \cdot 6} \int xxxdP$

$\int x^8 dP = \frac{5}{8} \int x^6 dP = \frac{1 \cdot 3 \cdot 5}{4 \cdot 6 \cdot 8} \int xxxdP$

&c.

unde habebitur

$\int \frac{dP}{\sqrt{1-xx}} = P + \int xxxdP \left(\frac{1}{2} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 4} + \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} + \&c. \right)$

At est $\frac{1}{2} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 4} + \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} + \&c = 2(1-x)$; ideoque

$\int dP$

$\int \frac{dP}{\sqrt{1-xx}}$

erit P =

f.

$\frac{1}{2} A \sin x$

ubi constan-
cant posito
tamen conj

$\int \frac{dP}{\sqrt{1-x}}$

quæ ut eva

jam $x = 1$

dem hoc ca

$-1 + \frac{x}{2}$

hincque col

1 -

uti ex rei r

Exteri Op

$$\int \frac{dP}{\sqrt{(1-xx)}} = P + 2(1-s) \int xxdP. \text{ At ob } dP = \frac{dx}{xx} \sqrt{(1-xx)}$$

$$\text{erit } P = C - \frac{\sqrt{(1-xx)}}{x} - A \sin x;$$

$$\int xxdP = \int dx \sqrt{(1-xx)} =$$

$$\frac{1}{2} A \sin x + \frac{1}{2} x \sqrt{(1-xx)}, \text{ \& } \int \frac{dP}{\sqrt{(1-xx)}} = D - \frac{1}{x}$$

ubi constantes C & D ita accipi debent, ut integralia hæc evanescant posito $x = 0$: quanquam scilicet utraque seorsim sit insignis, tamen conjunctæ se mutuo destruent. Erit enim

$$\int \frac{dP}{\sqrt{(1-xx)}} - P = D - \frac{1}{x} - C + \frac{\sqrt{(1-xx)}}{x} + A \sin x$$

quæ ut evanescat posito $x = 0$, debet esse $D = C$, ideoque posito

$$\text{jam } x = 1 \text{ fiet } \int \frac{dP}{\sqrt{(1-xx)}} - P = -1 + \frac{\pi}{2}: \text{ \& quia eo-}$$

dem hoc casu est $\int xxdP = \frac{\pi}{4}$, prædabit

$$-1 + \frac{\pi}{2} = 2(1-s) \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{2} s,$$

hincque colligitur fore $\frac{\pi}{2} s = 1$ & $s = \frac{2}{\pi}$ seu

$$1 = \frac{1 \cdot 1}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} + \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - \dots \text{ \& } = \frac{2}{\pi}$$

uti ex rei natura jam conclusimus.

§. XXV. Quoniam igitur cruiamus in ipso initio esse applicatam curvæ CA = 1, indolem hujus curvæ prope punctum A indagemus, seu in valorem applicatæ q inquiramus, si abscissa p fuerit valde parva. In hunc finem ponamus iterum 1 - pp = m, & cum sit

$$q = \frac{p}{2} \left(1 - \frac{1.1}{2.2} m - \frac{1.1.1.3}{2.2.4.4} m^2 - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} m^3 - \&c. \right)$$

$$q = 1 +$$

& quia novimus fore proxime q = 1, addamus æqualitatem modo inveniam:

$$0 = 1 - \frac{p}{2} \left(1 - \frac{1.1}{2.2} m - \frac{1.1.1.3}{2.2.4.4} m^2 - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} m^3 - \&c. \right)$$

atque habebimus:

$$q = 1 + \frac{p}{2} \left(\frac{1.1}{2.2} (1 - m) + \frac{1.1.1.3}{2.2.4.4} (1 - m^2) + \frac{1.1.1.3.3.5}{2.2.4.4.6.6} (1 - m^3) + \&c. \right)$$

cujus seriei cum singuli termini sint per 1 - m = pp divisibiles, reducetur hæc expressio ad hanc:

$$q = 1 + \frac{p}{8} pp \left(1 + \frac{1.3}{4.4} (1 + m) + \frac{1.3.3.5}{4.4.6.6} (1 + m + m^2) + \frac{1.1.3.3.5.7}{4.4.6.6.8.8} (1 + m^2 + m^4 + m^6) + \&c. \right)$$

§. XXVI. Quod si in hac expressione singuli termini ad potestates ipsius n evolvantur, reperietur

+

At
 $\frac{1.1}{2.2} +$
 quæ si pri
 ciens ipse
 $\frac{1.1.1.3}{2.2.4.4}$
 hæc den
 n², nempe
 $\frac{1.1.1.3.5}{2.2.4.4.6.6}$

$$\begin{aligned}
 & \left[\begin{aligned}
 & + \frac{1.1}{2.2} + \frac{1.1.1.3}{2.2.4.4} + \frac{1.1.1.3.3.5}{1.2.4.4.6.6} + \\
 & \frac{1.1.1.3.3.5.7}{2.2.4.4.6.6.8.8} + \&c. \\
 & + n \left(\frac{1.1.1.3}{2.2.4.4} + \frac{1.1.1.3.3.4}{2.2.4.4.6.6} + \right. \\
 & \left. \frac{1.1.1.3.3.5.7}{2.2.4.4.6.6.8.8} + \&c. \right) \\
 & + n^2 \left(\frac{1.3.1.3.3.5}{2.2.4.4.6.6} + \frac{1.1.1.3.3.5.7}{2.2.4.4.6.6.8.8} + \right. \\
 & \left. \&c. \right) \\
 & + n^3 \left(\frac{1.1.1.3.3.5.7}{2.2.4.4.6.6.8.8} + \&c. \right) \\
 & \&c.
 \end{aligned} \right. \\
 & q = 1 + \frac{n}{2} pp
 \end{aligned}$$

At ex supra inventis habemus summam primae seriei

$$\frac{1.1}{2.2} + \frac{1.1.1.3}{2.2.4.4} + \frac{1.1.1.3.3.5}{2.2.4.4.6.6} + \&c. = 1 - \frac{n}{2}$$

quæ si primo termino multetur, prodibit secunda, quæ est coefficientis ipsius n , ita ut fit

$$\frac{1.1.1.3}{2.2.4.4} + \frac{1.1.1.3.3.5}{2.2.4.4.6.6} + \&c. = \frac{1.3}{2.2} - \frac{n}{2}$$

hæc denuo primo termino multata, dabit coefficientem ipsius n^2 , nempe

$$\frac{1.1.1.3.3.5}{2.2.4.4.6.6} + \&c. = \frac{1.3.3.5}{2.2.4.4} - \frac{n}{2}$$

similique modo coefficientis ipsius n^2 erit $= \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} = \frac{2}{\dots}$

& ita porro, sicque tandem obtinebitur.

$$q = 1 + \frac{\pi}{2} pp \left(\left(1 - \frac{2}{\pi}\right) + \left(\frac{1 \cdot 3}{2 \cdot 2} - \frac{2}{\pi}\right) pp + \left(\frac{1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{2}{\pi}\right) pp^2 + \left(\frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - \frac{2}{\pi}\right) pp^3 + \dots \right)$$

vel erit;

$$q = 1 + pp \left(\left(\frac{1 \cdot 3}{2 \cdot 2} - 1\right) + \left(\frac{1 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 4} - 1\right) pp + \left(\frac{1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4} - 1\right) pp^2 + \left(\frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - 1\right) pp^3 + \dots \right)$$

§. XXVII. Ponamus jam hic $n = 1$, ut obtineamus æquationem hujus formæ $q = 1 + App$, qua natura curvæ prope punctam A exprimitur: cum enim conjecture licet veram æquationem futuram esse hujus formæ:

$$q = 1 + App + Bp^2 + Cpp^3 + Dp^4 + \dots$$

si abscissâ p æquidè parva statuitur, reliqui termini præter binos primos omitti poterunt, atque ex æquatione $q = 1 + App$, tam positio tangentis, quam curvatura in puncto A colligi poterit. Posito enim $AR = x$, $RQ = y$, erit $q = 1 + x \& p = y$, ideoque si arcus AQ fuerit minimus, is cum parabola confundetur, cujus æquatio $x = Ayy$ seu $yy = \frac{1}{A} x$, ac propterea $\frac{1}{A}$ parameter. Unde sequitur tangentem curvæ in A fore ad reſtam AC

perpendiculararem, & radium osculæ ibidem esse $= \frac{1}{2A}$.

§. XXVIII.

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§. XXVIII. Hic igitur coefficientens A renerletur, si in superiore serie, per quam quantitas pp multiplicatur, ponatur $n = 1$; ita ut sit

$$A = \left(\frac{\pi}{2} - 1\right) + \left(\frac{1 \cdot 2}{2 \cdot 2} \cdot \frac{\pi}{2} - 1\right) + \left(\frac{1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4} \cdot \frac{\pi}{2} - 1\right) + \&c.$$

que autem si ejus summatio tentetur, tam parum convergens deprehenditur, ut ejus summam adeo infinitam suspicari debeamus. In hac autem suspitione eo magis confirmamur, si seriem primo (§. 15.) inventam, secundum dimensiones ipsius p evolvamus, unde fit

$$= \frac{\pi}{2} \left\{ \begin{array}{l} 1 - \frac{1 \cdot 1}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - \&c. \\ + p \left(\frac{1 \cdot 1}{2 \cdot 2} + \frac{1 \cdot 1 \cdot 1 \cdot 2}{2 \cdot 2 \cdot 4 \cdot 4} + \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 2 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} + \&c. \right) \\ - p^2 \left(\frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} + \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} + \&c. \right) \end{array} \right.$$

§. XXIX. Hinc ergo coefficientens ipsius pp in æquatione generali pro curva $q = 1 + App + Bp^2 + Cp^3 + Dp^4 + \&c.$ erit

$$A = \frac{\pi}{2} \left(\frac{1 \cdot 1}{2 \cdot 2} + \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} + \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} + \&c. \right)$$

$$\text{seu } A = \frac{\pi}{4} \left(\frac{1}{2} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4} + \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6} + \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8} + \&c. \right)$$

similique modo & reliquos coefficientens B, C, D. &c. ex hac serie eruere licebit. Verum hoc labore supersedere poterimus, cum liquet non solum coefficientem A, sed etiam omnes reliquos produ-

dituros esse infinitos. Perspicuum hoc fiet ex solutione hujus problematis.

Problema.

Invenire summam hujus seriei infinite:

$$S = \frac{1}{2} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4} + \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6} + \&c.$$

Solutio.

§. XXX. Assumatur ad hanc summam s inveniendam hæc formula:

$$\frac{1}{\sqrt{1-xx}} = 1 + \frac{1}{2}xx + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \&c.$$

ut fit

$$\int \frac{dP}{\sqrt{1-xx}} = P + \frac{1}{2} \int xx dP + \frac{1 \cdot 3}{2 \cdot 4} \int x^4 dP + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \int x^6 dP + \&c.$$

sicque si post integrationes singulas ponatur $x = 1$

$$\int xx dP = \frac{3}{4} P$$

$$\int x^4 dP = \frac{5}{6} \int xx dP = \frac{3 \cdot 5}{4 \cdot 6} P$$

$$\int x^6 dP = \frac{7}{8} \int x^4 dP = \frac{3 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8} P$$

&c.

hincque fiet

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Fiet

$$\frac{dP}{\sqrt{(1-xx)}} = P \left(1 + \frac{1 \cdot 3}{2 \cdot 3} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 4 \cdot 6} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8} + \&c. \right)$$

sive $\int \frac{dP}{\sqrt{(1-xx)}} = 2P$; unde invento P , reperietur, si post integrationem ponatur $x = 1$.

§. XXXI. Cum igitur generaliter esse debeat.

$$\int x^{\mu+1} dP = \frac{\mu+3}{\mu+4} \int x^{\mu} dP + \frac{x^{\mu+1} Q}{\mu+4}$$

dummodo Q ejusmodi sit functio, quae evanescat posito $x = 1$, erit

$$(\mu+4) x x dP = (\mu+3) dP + x dQ + (\mu+1) Q dx$$

unde duae sequentes aequationes conficiuntur.

$$x x dP = dP + Q dx$$

$$4 x x dP = 3 dP + dQ + Q dx$$

$$\& dP = \frac{-Q dx}{1-xx} = \frac{-x dQ - Q dx}{3-4xx}$$

$$\text{hincque elicitur } \frac{dQ}{Q} = \frac{2 dx - 3 x x dx}{x(1-xx)} = \frac{2 dx}{x} - \frac{x dx}{1-xx}$$

& $Q = -xx \sqrt{(1-xx)}$. Quare habebitur

$$dP = \frac{x x dx}{\sqrt{(1-xx)}} \& \frac{dP}{\sqrt{(1-xx)}} = \frac{x x dx}{1-xx} = -dx + \frac{dx}{1-xx}$$

Fiet ergo $P = \frac{1}{2}$, si post integrationem ponatur $x = 1$

at $\int \frac{dP}{\sqrt{(1-xx)}} = x + \frac{1}{2} \frac{1+x}{1-x}$, cujus valor posito $x=1$

fit utique infinitus. Erit igitur $s = \infty$, seu summa seriei propositæ infinite magna.

§. XXXII. Quia igitur coefficientens A ipsius pp , in æquatione

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + \&c.$$

est infinitus, radius osculi curvæ in puncto A utique erit infinite parvus. Verum præterea hæc æquatio: in qua omnes omnino coefficientens A, B, C, D &c. sunt infiniti, nihil plane ad curvæ cognitionem confert. Quia enim radius osculi curvæ in A est infinite parvus, natura curvæ circa punctum A hujus modi æquatione

one $q = 1 + ap^m$ exprimetur, in qua exponens m binario sit minor, verumtamen unitate major: sed ex omnibus, quæ hætenus sunt tradita nulla via patet, qua hunc exponentem m scrutari queamus. Cum enim is numerus integer esse nequeat, nulla scriporum, quas pro q eruimus, ita est comparata, ut ex ea potestatem ipsius p irrationalem elicere liceat.

§. XXXIII. Hinc intelligimus problema esse summopere difficile, quo æquatio tantum elementaris requiritur, quæ naturam curvæ propositæ A Q D q saltem proximè circa punctum A exhibeat. Notum est enim si ponatur AR = x & RQ = y , quæcunque fuerit curva A Q, naturam minimæ ejus portiunculæ cir-

ca A semper hujusmodi æquatione $y^m = Ax$, comprehendi posse; siquidem curva sit algebraica; pro curvis autem transcendentibus certum videtur, quasvis earum minimas portiunculas cum arcibus curvarum algebraicarum comparari posse. Quare in nostra curva,

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etfi est transcendens, hoc eo magis mirum videri debet, quod nulla hujusmodi formula $y = Ax^m$ exhiberi possit, quæ saltem minimæ ejus portiunculæ circa A sitæ naturam declaret.

§. XXXIV. Hunc nodum ut resolvamus, æquationem nobis finitam inter coordinatas p & q investigare oportebit, quæ etfi, ut facile prævidere licet, ad differentialia secundi ordinis exsurget, tamen ad accuratorem curvæ cognitionem magis erit accommodata. Eliciemus autem hujusmodi æquationem, quæ numero terminorum finito constet, si seriem primo inventam ad summam revocabimus. Cum enim posito $1 - pp = n^n$ fit

$$\frac{2q}{n} = 1 - \frac{1.1}{2.2} n^n - \frac{1.1.1.3}{2.2.4.4} n^n - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} n^n - \&c.$$

Erit differentiando

$$\frac{2dq}{n dn} = -\frac{1.1}{2} n^{-1} - \frac{1.1.1.3}{2.2.4} n^{-2} - \frac{1.1.1.3.3.5}{2.2.4.4.6} n^{-3} - \&c.$$

quæ per n multiplicata deinceps differentiata dat

$$\frac{2}{n} \frac{d}{dn} \frac{ndq}{dn} = -1.1 n^{-1} - \frac{1.1}{2.2} 1.3 n^{-2} - \frac{1.1.1.3}{2.2.4.4} 3.5 n^{-3} - \&c.$$

Multiplicetur hæc per $\frac{dn}{n}$ ac rursus integratur, erit

$$\frac{2}{n} \int \frac{1}{n} d. \frac{ndq}{dn} = -1. n^{-1} - \frac{1.1}{2.2} 1 n^{-2} - \frac{1.1.1.3}{2.2.4.4} 3 n^{-3} - \&c.$$

Multiplicetur per $\frac{dn}{n^2}$ & integrando prodibit,

$$\frac{2}{n} \int \frac{dn}{n} \int \frac{1}{n} d. \frac{ndq}{dn} = \frac{1}{n} - \frac{1 \cdot 1}{2 \cdot 2} n - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} n^3 - \text{etc.}$$

que series cum sit ipsa proposita, per n divisa erit

$$\frac{2}{n} \int \frac{dn}{n} \int \frac{1}{n} d. \frac{ndq}{dn} = \frac{2q}{n} \text{ seu } \int \frac{dn}{n} \int \frac{1}{n} d. \frac{ndq}{dn} = \frac{q}{n}$$

§. XXXV. Sumamus nunc differentialis, habebiturque

$$\frac{dn}{n^2} \int \frac{1}{n} d. \frac{ndq}{dn} = \frac{ndq - qdn}{nn} \text{ seu}$$

$$\int \frac{1}{n} d. \frac{ndq}{dn} = \frac{vndq}{dn} - nq.$$

porroque differentiando

$$\frac{1}{n} d. \frac{ndq}{dn} = n d. \frac{ndq}{dn} + ndq - ndq - qdn$$

$$\text{seu } (1 - nn) d. \frac{ndq}{dn} + qdn = 0$$

Jam ob $1 - nn = pp$ erit $ndn = -pdp$ & $\frac{dn}{n} = -\frac{pdp}{1 - pp}$

$$\text{unde fit } -ppd. \frac{(1 - pp) dq}{pdp} - pqdp = 0 \text{ seu}$$

$$d. \frac{(1 - pp) dq}{pdp} + \frac{qdp}{p} = 0. \text{ Sumatur jam } dp \text{ constans erit}$$

$$\frac{(1 - pp) dq}{pdp} - \frac{dq(1 + pp)}{pp} + \frac{qdp}{p} = 0 \text{ seu}$$

$$p(1 - pp) ddq - dpdq(1 + pp) + pqdp = 0.$$

§. XXXVI.

§. XXXVI. En igitur æquationem differentialem secun-
di gradus pro curva proposita

$$p(1 - pp) ddq - dpdq(1 + pp) + pqdp^2 = 0$$

ex qua potestas illa ipsius p in æquatione $q = 1 + Ap^m$ effici de-
bet, si abscissa p valde parva statuat. Cum igitur fiat $dq = m$

$$\left. \begin{aligned} Ap^{m-1} dp & \text{ \& } ddq = 0 \quad (m-1) Ap^{m-2} dp \text{ orietur} \\ m(m-1) Ap^{m-2} & - m Ap^{m-1} + p \\ -m(m-1) Ap^{m+1} & - m Ap^{m+1} + Ap^{m+1} \end{aligned} \right\} = 0$$

$$\text{scu } m(m-2) Ap^{m-1} - m Ap^{m+1} + p = 0$$

Deberet ergo esse $m = 2$, ut terminus Ap^m cum p comparari
posset, sed tum iterum obtinetur $A = 0$: præterea vero hinc
perspicitur exponentem m nullo modo numerum fractum esse posse,
ita ut hinc difficultas supra memorata augeri potius quam tolli
videatur.

§. XXXVII. Quod si regulis consuetis uti velimus ad æqua-
tionem inventam in seriem evolvendam, quæ secundum potesta-
tes ipsius p procedat, quoniam novimus primum seriei terminum
esse $= 1$, nullam aliam formam inde colligere licet nisi hæc:

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + \&c.$$

unde fit

$$\frac{dq}{dp} = 2Ap + 4Bp^3 + 6Cp^5 + 8Dp^7 + \&c.$$

$$\&c. \frac{ddq}{dp^2} = 2A + 12Bp^2 + 30Cp^4 + 56Dp^6 + \&c.$$

qui valores in æquatione substituti præbebunt:

$$\left. \begin{array}{l} + 2Ap + 12Bp^3 + 30Cp^5 + 56Dp^7 + \&c. \\ - 2A - 12B - 30C - \&c. \\ - 2Ap - 4B - 6C - 8D - \&c. \\ - 2A - 4B - 6C - \&c. \\ + p + A + B + C + \&c. \end{array} \right\} = 0$$

unde omnes coefficientes A, B, C, &c. prodeunt infiniti.

§. XXXVIII. Hinc igitur videmus regulas ordinarias, secundum quas vulgo forma seriei, in quam æquatio differentialis transmutanda sit, dijudicari solet, non esse sufficientes, cum hoc casu nullam afferant utilitatem: unde nostra æquatio eo majorem meretur attentionem. Sequenti tamen modo ex ea natura curvæ prope punctum A colligi poterit, ex quo simul intelligetur, quemadmodum quoque in aliis casibus defectus iste regularum usu receptarum suppleri, eæque ad praxin accommodari debeant. Quia enim abscissam p hic pro infinite parva habemus, in æquatione pro $1 - pp$ & $1 + pp$ ponere licebit 1, & quia novimus esse hoc casu proxime $q = 1$, pro quantitate finita q unitatem scribamus: quo facto æquatio differentio-differentialis inventa pro casu, quo abscissa p est minima sequentem induet formam;

$$pddq - dydq + psp^2 = 0.$$

§. XXXIX. Hujus jam æquationis resolutio est facilis, cum e-
nim dp sit constans, ponatur $dq = rdp$, erit $ddq = drdp$, habebiturque:

$$pdr - rrp + pdp = 0$$

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$$\text{sive } \frac{pdr - rdp}{pp} + \frac{dp}{p} = 0$$

cujus integrale est: $\frac{r}{p} + lp = C$, unde fit

$$r = Cp - plp$$

ideoque $dq = Cpdp - pdp lp$

Hæc jam æquatio integrata dabit:

$$q = 1 + \frac{1}{2} Cp^2 - fp lp + \frac{1}{2} pp$$

in qua cum terminus pp incomparabiliter sit minor quam $fpld$, erit pro curvæ initio A:

$$q = 1 - \frac{1}{2} pp lp$$

§. XL. Nunc igitur naturam curvæ prope initium A æquatione simplici definire possumus: si enim vocemus $AR = x$ & $RQ = y$, ob $p = y$ & $q = 1 + x$, orietur hæc $x = -\frac{1}{2} yyly$, ad quam æquatio generalis pro curva revocatur, si coordinatæ x & y sint quam minimæ. Patet igitur ne minimum quidem arcum circa A tanquam portiunculam curvæ algebraicæ spectari posse, sed ejus naturam logarithmos implicare. Et quoniam æquatio logarithmica in exponentialem transformari potest, initium curvæ nostræ A commune erit cum linea transcendente, cujus æquatio

est $e^{-2x} = y$, sumto e pro numero cujus logarithmus hyperbolicus est $= 1$.

§. XLI. Æquatione hac $x = -\frac{1}{2} yyly$ confirmantur quoque ea, quæ supra jam de affectionibus hujus curvæ in puncto A notavimus. Primo enim patet si sit $y = 0$, fore quoque $yyly$ ac proinde $x = 0$, et si hoc casu sit $ly = -\infty$. Deinde cum sit $dy = -y ly ly - \frac{1}{2} y ly$, quia y incomparabiliter est minus quam $yyly$, erit

erit $dx = -y dy / ly$, ac propterea $\frac{dy}{dx} = \frac{-1}{y/ly} = \infty$ posito $y = 0$; unde patet tangentem curvæ in A ad abscissam AR esse perpendicularem. Porro cum sit subnormalis $\frac{y dy}{dx} = \frac{-1}{ly}$, hocque casu subnormalis radio evolutæ æquetur, ob $ly = \infty$ si $y = 0$, manifestum est radium osculi curvæ in A esse infinite parvum.

§. XLII. Maxime autem differt hæc curva a curvis algebraicis, quæ in initio A quoque habent radium osculi evanescentem. Curvarum enim algebraicarum, quæ hac indole gaudent,

naturæ circa initium A hujusmodi formula exprimitur $x = ay^m$ existente $m < 2$, at tamen $m > 1$. Sit igitur $m = 2 - \epsilon$ existente

fractione unitate minore, ut sit $x = ay^{2-\epsilon}$, erit $dx = a(2-\epsilon)$

$y^{1-\epsilon} dy$, ideoque $\frac{dy}{dx} = \frac{1}{a(2-\epsilon)y^{1-\epsilon}}$ $= \infty$ ob $y^{1-\epsilon} = 0$: ac

radius osculi, qui subnormali $\frac{y dy}{dx}$ æqualis est, erit $= \frac{y}{a(2-\epsilon)}$

$= 0$. Pro nostra vero curva radius osculi inventus est $= \frac{-1}{ly}$,

unde radius osculi evanescens in curva algebraica quacunque erit

ad radium osculi in nostræ curvæ puncto A ut $-y / ly$ ad $a(2-\epsilon)$ hoc est ut 0 ad 1; quantumvis enim exiguus sit exponent ϵ , casu

$y = 0$ semper est $y / ly = 0$, etiam si sit $ly = -\infty$. Quare in nostra quidem curva radius osculi in A est infinite parvus, sed tamen

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men infinities major est, quam radius osculi evanescens in omni curva algebraica.

§. XLIII. Cognito jam initio seriei, qua valor applicatæ $PQ = q$ per abscissam $CP = p$ exprimitur, scilicet,

$$q = 1 - \frac{1}{2} p p' p + A p p'$$

non difficile erit hinc formam totius seriei colligere. Cum enim ex æquatione differentio-differentiali intelligatur sequentium terminorum potestates ipsius p binario crescere, valor ipsius q generatim gemina serie infinita exprimetur, eritque

$$q = 1 + A p^2 + B p^4 + C p^6 + D p^8 + \&c.
- a p p' p - c p^3 p' p - e p^5 p' p - g p^7 p' p - \&c.$$

in qua quidem nunc jam novimus esse $\alpha = \frac{1}{2}$.

§. XLIV. Cum igitur verus valor ipsius q duplici serie contineatur, ut utramque seorsim eliciamus, ponamus

$$q = r - s p \text{ eritque differentiendo}$$

$$dq = dr - \frac{sdp}{p} - ds p$$

$$ddq = ddr - \frac{2dp ds}{p} + \frac{sdp^2}{pp} - dds p$$

Hi valores in nostra æquatione differentiali

$$p(1 - pp) ddq - dpdq(1 + pp) + p q dp^2 = 0$$

Substituantur, ac termini per sp affecti seorsim nihilo æquantur, hoc modo duæ obtinebuntur æquationes:

I. q

I. $p(1 - pp) ddr - (1 + pp) dpdr + prdp^2 = 0$

II. $p(1 - pp) ddr - (1 + pp) dpdr + prdp^2 -$

$2(1 - pp) dpdr + \frac{2r dp}{p} = 0$

§. XLV. Ad has æquationes resolvendas ponatur

$r = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + \&c.$

$s = ap^2 + \beta p^4 + \gamma p^6 + \delta p^8 + \epsilon p^{10} + \&c.$

eritque differentialibus sumendis

$\frac{dr}{dp} = 2Ap + 4Bp^3 + 6Cp^5 + 8Dp^7 + \&c.$

$\frac{ddr}{dp^2} = 2A + 12Bp^2 + 30Cp^4 + 56Dp^6 + \&c.$

$\frac{ds}{dp} = 2ap + 4\beta p^3 + 6\gamma p^5 + 8\delta p^7 + \&c.$

$\frac{dds}{dp^2} = 2a + 12\beta p^2 + 30\gamma p^4 + 56\delta p^6 + \&c.$

His valoribus substitutis prima æquatio abibit in hanc:

| | | |
|---|---|-----|
| $2ap + 12\beta p^3 + 30\gamma p^5 + 56\delta p^7 + 90\epsilon p^9 + \&c.$ | } | = 0 |
| $- 2a - 12\beta - 30\gamma - 56\delta - \&c.$ | | |
| $- 2a - 4\beta - 6\gamma - 8\delta - 10\epsilon - \&c.$ | | |
| $- 2a - 4\beta - 6\gamma - 8\delta - \&c.$ | | |
| $+ a + \beta + \gamma + \delta + \&c.$ | | |

§. XLVI.

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Euleri

§. XLVI. Si jam singularum potestatum ipsius p coefficientes nihilo aequales ponantur, erit:

$$2\alpha - 2\alpha = 0; \quad \alpha \text{ manet indeterminatum}$$

$$8\beta - 3\alpha = 0; \quad \beta = \frac{1 \cdot 3}{2 \cdot 4} \alpha$$

$$24\gamma - 15\beta = 0; \quad \gamma = \frac{3 \cdot 5}{4 \cdot 6} \beta = \frac{1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 4 \cdot 6} \alpha$$

$$48\delta - 35\gamma = 0; \quad \delta = \frac{5 \cdot 7}{6 \cdot 8} \gamma = \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8} \alpha$$

$$80\varepsilon - 63\delta = 0; \quad \varepsilon = \frac{7 \cdot 9}{8 \cdot 10} \delta = \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10} \alpha$$

&c.

&c.

Si igitur valor coefficientis primi α constaret, quem quidem jam vidimus esse $= 1$, omnes sequentes coefficientes β, γ, δ , &c. forent cogniti. Verum resolutio alterius aequationis quoque hunc nobis valorem ipsius α patefaciet.

§. XLVII. Substitutis enim seriebus antè traditis in altera aequatione proveniet;

$$\begin{aligned} & 2Ap + 12Bp^2 + 30Cp^3 + 56Dp^4 + 90Ep^5 + \&c. \\ & \quad - 2A - 12B - 30C - 56D - \&c. \\ -2A & - 4B - 6C - 8D - 10E - \&c. \\ & \quad - 2A - 4B - 6C - 8D - \&c. \\ + 1 & + A + B + C + D + \&c. \\ -4\alpha & - 8\beta - 12\gamma - 16\delta - 20\varepsilon - \&c. \\ & \quad + 4\alpha + 8\beta + 12\gamma + 16\delta + \&c. \\ + 2\alpha & + 2\beta + 2\gamma + 2\delta + 2\varepsilon + \&c. \end{aligned}$$

Euleri Opuscula Tom. II.

U

Unde

Unde simili modo elicitur:

$$2A - 2A + 1 - 2a = 0; \text{ hinc fit } a = \frac{1}{2}$$

$$8B - 3A - 6\beta + 4a = 0; 2.4 B - 1.3A + 2(2 - \frac{1.3.3}{2.4})a = 0$$

$$24C - 15B - 10\gamma + 8\beta = 0; 4.6 C - 3.5B + 2(4 - \frac{3.5.5}{4.6})\beta = 0$$

$$48D - 35C - 14\delta + 12\gamma = 0; 6.8 D - 5.7C + 2(6 - \frac{5.7.7}{6.8})\gamma = 0$$

$$80E - 63D - 18\epsilon + 16\delta = 0; 8.10E - 7.9D + 2(8 - \frac{7.9.9}{8.10})\delta = 0$$

§. XLVIII. Cognito igitur valore ipsius $a = \frac{1}{2}$ altera series s , quae logarithmum ipsius p involvit, tota innotescit, erit enim:

$$a = \frac{1}{2}$$

$$\beta = \frac{1.1.3}{2.2.4}$$

$$\gamma = \frac{1.1.3.3.5}{2.2.4.4.6}$$

$$\delta = \frac{1.1.3.3.5.5.7}{2.2.4.4.6.6.8}$$

$$\epsilon = \frac{1.1.3.3.5.5.7.7.9}{2.2.4.4.6.6.8.8.10}$$

&c.

sicque hinc $s = a p p + \beta p^2 + \gamma p^3 + \delta p^4 + \epsilon p^5 + \dots$

§. XLIX. Quod autem ad alteram seriem attinet

$$r = r + Ap^2 + Bp^3 + Cp^4 + Dp^5 + Ep^6 + \dots$$

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primus coefficientis A hinc manet indeterminatus, cujus rei ratio est, quod has series ex equatione differentiali secundi gradus ellicuimus, quae duplici determinatione indiget, ut ad nostrum casum accommodetur. Quare valorem hujus coefficientis A ex ipsa curvae natura definiri oportet, eo autem invento, reliqui immutescent ex his formulis, ad quas superiores redeunt.

$$B = \frac{1.3}{2.4} A - \frac{1}{2} \alpha \left(\frac{3}{2.2} + \frac{1}{1.1} \right)$$

$$C = \frac{3.5}{2.4} B - \frac{1}{2} \beta \left(\frac{3}{3.3} + \frac{1}{2.2} \right)$$

$$D = \frac{5.7}{6.8} C - \frac{1}{2} \gamma \left(\frac{3}{4.4} + \frac{1}{4.4} \right)$$

$$E = \frac{7.9}{8.10} D - \frac{1}{2} \delta \left(\frac{3}{5.5} + \frac{1}{4.4} \right)$$

§. L. His autem omnibus coefficientibus inventis ad datam quamvis abscissam CP = p, valor respondens applicatae PQ = q ita definitur, ut sit

$$q = 1 + Ap^2 + Bp^3 + Cp^4 + Dp^5 + \&c.$$

$$= \alpha p^2 + \beta p^3 + \gamma p^4 + \delta p^5 + \&c.$$

quae series si abscissa p fuerit unitate multo minor, satis prompte convergit, ut inde valor ipsius q cognosci queat. Hinc vero etiam applicatae, quae abscissis multo majoribus unitate respondent, definiri poterunt, quia abscissa $\frac{1}{p}$ respondet applicata $\frac{q}{p}$. Quare si abscissa unitate multo major ponatur = P eique respondens applicata = Q ob $p = \frac{1}{P}$ & $q = p Q = \frac{Q}{P}$ erit

U 2

Q =

$$Q = P + AP^{-1} + BP^{-2} + CP^{-3} + DP^{-4} + \&c. \\ + aP^{-1} / P + \beta P^{-2} / P + \gamma P^{-3} / P + \delta P^{-4} / P + \&c.$$

Hinc si abscissa P fiat infinita erit

$$Q = P + \frac{a/P}{P} \text{ seu } Q - P = \frac{a/P}{P}$$

unde natura remi Dq in infinitum extensi & ad asymptotam CV appropinquantis colligitur.

§. LI. Quia porro novimus, si $p = 1$ fore $q = \frac{r}{2}$ pro

hoc casu æquatio inventa hanc formam ob $1/1 =$ inducet

$$\frac{r}{2} = 1 + A + B + C + D + E + \&c.$$

Cum igitur valor A nondum sit definitus, reliqui vero $B, C, D \&c.$ ab eo pendent, hæc æquatio conditionem continet, qua valor ipsius A determinatur. Ita scilicet valorem ipsius A comparatum esse oportet, ut summa seriei infinitæ $1 + A + B + C + \&c.$

fiat $\frac{r}{2}$. Verum si valores reliquarum litterarum $B, C, D \&c.$ qui

ab A pendunt, evolvantur, tam complicatæ resultant expressiones, ut hinc valor ipsius A nequaquam erui possit.

§. LII. Ad hanc constantem A determinandam alia patet via, si datæ cujuscumque ellipsis perimeter ex altera formula in numeris fuerit inventa. Quæ methodus cum requirat, ut omnes coefficientes in fractionibus decimalibus evolvantur, computo peracto reperietur:

$\alpha = 0,5000000000$ $A = \text{quaritur}$
 $\beta = 0,1875000000$; $B = 0,3750000000$ $A = 0,1093750000$
 $\gamma = 0,1171875000$; $C = 0,2343750000$ $A = 0,0820312500$
 $\delta = 0,0854492188$; $D = 0,1708984375$ $A = 0,0641886393$
 $\epsilon = 0,0672912198$; $E = 0,1345825195$ $A = 0,0524978638$
 $\zeta = 0,0555152893$; $F = 0,1110305786$ $A = 0,0443481445$
 $\eta = 0,0472540815$; $G = 0,0945081711$ $A = 0,0383663416$
 $\theta = 0,0411363691$; $H = 0,082272738$ $A = 0,0337966962$
 $i = 0,0364228268$; $I = 0,0728456536$ $A = 0,0301949487$
 $\kappa = 0,0326793696$; $K = 0,0653587392$ $A = 0,0272843726$
 &c.

Hisque valoribus inventis, si abscissa sit $CP = p$, valor applicatae q ita definietur ut sit:

$$\begin{aligned}
 q = & 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + Fp^{12} + Gp^{14} + Hp^{16} \\
 & + Ip^{18} + Kp^{20} + \text{\&c.} \\
 & - pp^2p.(a + \beta p^2 + \gamma p^4 + \delta p^6 + \epsilon p^8 + \zeta p^{10} + \eta p^{12} + \theta p^{14} + i p^{16} \\
 & + \kappa p^{18} + \text{\&c.})
 \end{aligned}$$

§. LIII. Deinde vero supra ejusdem applicatae q valorem ita invenimus expressum ut sit:

$$\begin{aligned}
 q = & \frac{1}{2} \left(1 - \frac{1 \cdot 1}{2 \cdot 2} (1 - pp) - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} (1 - pp)^2 - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} \right. \\
 & \left. (1 - pp)^3 - \text{\&c.} \right)
 \end{aligned}$$

Nunc igitur ex utraque formula pro eodem quopiam valore

re ipsius p erimus valorem ipsius q , ut deinceps ex æqualitate horum duorum elicere queamus valorem coefficientis A . Pro p vero non nimis exiguum fractionem substitui conveniet, ne expressio posterior nimis lente convergat, tam parvum tamen assumamus ut coefficientes pro superiore forma computati valori q ad 10 figuras inveniendo sufficiant.

$A p^0$
 $B p^1$
 $C p^2$
 $D p^3$
 $E p^4$
 $F p^5$
 $G p^6$

§. LIV. Ponamus ergo ad commodum calculi $p = \frac{2}{5}$. erit in logarithmis hyperbolicis:

$— \log p = 1,60943791243$

Jam vero est

| | | |
|----------------|-----|----------------|
| αp^0 | $=$ | 0,020000000000 |
| βp^1 | $=$ | 0,000300000000 |
| γp^2 | $=$ | 0,000007500000 |
| δp^3 | $=$ | 0,00000021875 |
| ϵp^4 | $=$ | 0,00000000689 |
| ζp^5 | $=$ | 0,00000000023 |
| ηp^6 | $=$ | 0,00000000007 |

$q =$

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 $q =$

0,02030772588

coefficientis ipsius $— \log p$

1,60943791243

0,03268402394

productum.

Verum
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Deinde est

$A p$

| | |
|---------------------------|-------------------|
| $A p^0 = 0,04000000000 A$ | |
| $B p^1 = 0,00060000000 A$ | $- 0,00017500000$ |
| $C p^2 = 0,00001500000 A$ | $- 0,00000525000$ |
| $D p^3 = 0,00000043750 A$ | $- 0,00000016432$ |
| $E p^4 = 0,00000001378 A$ | $- 0,00000000538$ |
| $F p^5 = 0,00000000045 A$ | $- 0,00000000016$ |
| $G p^6 = 0,00000000002 A$ | $- 0,00000000001$ |

$$0,04061545175 A - 0,00018041987$$

Ex his conficitur

$$q = 0,04061545175 A + 1,032503250360407$$

§. LV. Nunc eundem valorem ipsius q ex altera æquatione quaeramus, & cum sit $p = \frac{1}{5}$, erit $1 - pp = \frac{24}{25}$ sic $pn = \frac{24}{25}$ erit

$$q = \frac{r}{1} \left(1 - \frac{1 \cdot 1}{2 \cdot 2} pn - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} p^4 - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} p^6 - \&c. \right)$$

ponatur ad abbreviandum

$$q = \frac{r}{2} - A_n^2 - B_n^4 - C_n^6 - D_n^8 - E_n^{10} - \&c.$$

Verum hoc casu ob $pn = \frac{24}{25}$ series ista nimis lente convergit, quam ut hinc valor ipsius q satis exacte elici queat, quare ut utrinque parem convergentiam obtineamus ponamus $p = \frac{1}{\sqrt{5}}$, ut sit tam $pp = \frac{1}{5}$ quam $pn = \frac{4}{5}$; calculum vero tantum ad 6 figuras expediamus: eritque

APP

$$\begin{aligned}
 App &= 0,500000 A \\
 Bp^6 &= 0,093730 A - 0,027344 \\
 Cp^6 &= 0,019297 A - 0,010254 \\
 Dp^6 &= 0,010681 A - 0,004012 \\
 Ep^6 &= 0,004206 A - 0,001640 \\
 Fp^6 &= 0,001735 A - 0,000693 \\
 Gp^6 &= 0,000738 A - 0,000300 \\
 Hp^6 &= 0,000321 A - 0,000132 \\
 Ip^6 &= 0,000142 A - 0,000059 \\
 Kp^6 &= 0,000064 A - 0,000026 \\
 \text{Summa reliq: } &60 A - 24
 \end{aligned}$$

$$\text{Si. om. } 0,640994 A - 0,044484 + 0,320497. \frac{1^2}{p} + 1$$

$$\text{ergo } q = 1,066392 + 0,640994 A$$

at altera expressio dat $q = 1,350647$, unde fit

$$A = \frac{284055}{640994} = 0,443147$$

§. LVI. Quoniam hic valor non ultra 6 figuras extenditur, tamen casui non tribuendum videtur, quod iste numerus inventus 0,443147 a logarithmo binarii 0,69314718 unitatis quadrate 0,25 precise deficiat. Quae conjectura si veritati esset consentanea, valorem litterae A ad plurimas figuras exhibere liceret, cum enim sit

$$12 = 0,6931471805599453094172321$$

$$\text{foret } A = 12 - \frac{1}{2} \text{ ideoque}$$

$$A = 0,4431471805599453094172321.$$

Quod

Quod autem valor coefficientis hujus A sit reversus $= 12 = \frac{1}{2}$, sequenti modo demonstro, hancque conjecturam confirmo.

§. LVII. Comparo scilicet arcum ellipticum AY , cujus Fig. 2. semiaxes $AC = 1$, $CP = p$ cum arcu parabolico AZS super eodem axe AC descripto, qui in A cum ellipsi communem habeat curvaturam. Sumta abscissa communi $AX = x$, sit applicata ellipsis $XY = y$ & parabolæ $XZ = z$, erit $y = p \sqrt{(2x - xx)}$ & $z = p \sqrt{2x}$, ideoque $dy = \frac{p dx (1-x)}{\sqrt{(2x - xx)}}$ & $dz = \frac{p dx}{\sqrt{2x}}$; unde sit

$$\text{arcus ellipticus } AY = \int dx \sqrt{1 + \frac{pp(1-x)^2}{2x - xx}}$$

$$\text{arcus parabolicus } AZ = \int dx \sqrt{1 + \frac{pp}{2x}}. \text{ Constat autem}$$

$$\text{erit } AZ = x \sqrt{1 + \frac{pp}{2x}} + \frac{1}{2} pp \frac{\sqrt{1 + \frac{pp}{2x}} + 1}{\sqrt{1 + \frac{pp}{2x}} - 1}; \text{ Hinc si ponatur}$$

$$x = 1, \text{ erit arcus parabolæ } AZS = \sqrt{1 + \frac{1}{2} pp} + \frac{1}{2} pp \frac{\sqrt{1 + \frac{1}{2} pp} + 1}{\sqrt{1 + \frac{1}{2} pp} - 1}$$

At in formulis integralibus erit

$$\sqrt{1 + \frac{pp(1-x)^2}{2x - xx}} = \sqrt{1 + \frac{pp}{2x} - \frac{pp(3-2x)}{4-2x}}$$

Quia autem comparisonem non ad altiores ipsius p potestates extendere opus est quam ad secundam: coefficientes enim altiorum ipsius p potestatum ex minoribus jam definitimus, rejectis terminis, qui continent p^3 & altiores potestates, erit

Euleri Opera Tom. II.

X

$\sqrt{1}$

$$\sqrt{\left(1 + \frac{pp(1-x)^2}{2x-xx}\right)} = \sqrt{\left(1 + \frac{pp}{2x} - \frac{pp(3-2x)}{2(2-x)}\right)} \text{ ideoque}$$

$$AY = \int dx \sqrt{\left(1 + \frac{pp}{2x}\right)} - \frac{1}{2} pp \int \frac{dx(3-2x)}{2-x}, \text{ Integralibusque}$$

acta sumis

$$AY = x \sqrt{\left(1 + \frac{pp}{2x}\right)} + \frac{1}{2} pp \int \frac{V(1+\frac{pp}{2x})+1}{V(1+\frac{pp}{2x})-1} \frac{2x}{2x} - \frac{1}{2} pp x - \frac{1}{2} pp \int \frac{2-x}{2}$$

Ponitur jam $x = 1$, ut prodeat arcus $AYP = q$, erit

$$q = V\left(1 + \frac{1}{2} pp\right) + \frac{1}{2} pp \int \left(V\left(1 + \frac{1}{2} pp\right) + 1\right) - \frac{1}{2} pp \int \left(V\left(1 + \frac{1}{2} pp\right) - 1\right) - \frac{1}{2} pp + \frac{1}{2} pp \int 2.$$

§. LVIII. Jam quoniam ad altiores ipsius p potestates non respicimus, erit $\sqrt{\left(1 + \frac{1}{2} pp\right)} = 1 + \frac{1}{4} pp$, unde fiet

$$q = 1 + \frac{1}{4} pp + \frac{1}{4} pp \int \left(2 + \frac{1}{4} pp\right) - \frac{1}{4} pp \int \frac{1}{4} pp$$

$$- \frac{1}{2} pp + \frac{1}{4} pp \int 2$$

ubi pro $\int \left(2 + \frac{1}{4} pp\right) = \int 2 + \frac{1}{8} pp$ scribere licet $\int 2$, ita ut

$$\text{fit } q = 1 - \frac{1}{4} pp + \frac{1}{2} pp \int 2 - \frac{1}{2} pp \int p + \frac{1}{2} pp \int 2$$

$$\text{seu } q = 1 - \frac{1}{2} pp \int p + pp \left(\int 2 - \frac{1}{4}\right)$$

unde perspicitur coefficientem ipsius pp , quem ante littera A indica-
vimus

vimus esse $\approx f^2 - \frac{1}{4}$, omnino uti ex cursu ante computato conjectura sumus consecuti.

§. LIX. Pro curva igitur intello proposita AQD γ , si su- Fig. I. erit abscissa CP $\approx p$ & applicata PQ $\approx q$, erit

$$q \approx 1 + A p^2 + B p^4 + C p^6 + D p^8 + E p^{10} + \&c.$$

$-(a p^2 + \beta p^4 + \gamma p^6 + \delta p^8 + \epsilon p^{10} + \&c.)$ sp
ubi coefficientes ita determinantur:

$$\begin{array}{l|l} A \approx \frac{1}{2} - \frac{1}{4} & a \approx \frac{1}{2} \\ B \approx \frac{1.3}{2.4} A - \frac{1}{2}(a - \beta) + \frac{1}{2} \cdot \frac{5}{2} & \beta \approx \frac{1.3}{2.4} a \\ C \approx \frac{3.5}{4.6} B - \frac{1}{2}(\beta - \gamma) + \frac{1}{4} \cdot \frac{7}{3} & \gamma \approx \frac{3.5}{4.6} \beta \\ D \approx \frac{5.7}{6.8} C - \frac{1}{4}(\gamma - \delta) + \frac{1}{6} \cdot \frac{8}{4} & \delta \approx \frac{5.7}{6.8} \gamma \\ E \approx \frac{7.9}{8.10} D - \frac{1}{5}(\delta - \epsilon) + \frac{1}{8} \cdot \frac{9}{5} & \epsilon \approx \frac{7.9}{8.10} \delta \\ F \approx \frac{9.11}{10.12} E - \frac{1}{6}(\epsilon - \zeta) + \frac{1}{10} \cdot \frac{10}{6} & \zeta \approx \frac{9.11}{10.12} \epsilon \end{array}$$

&c.

series hæc valde convergit, si abscissa p fuerit fractio valde parva, sin autem sit unitate multo major, iisdem manentibus coefficientibus erit

X 2

q =

$$q = p + \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p^3} + \frac{D}{p^4} + \frac{E}{p^5} + \dots$$

$$+ \left(\frac{a}{p} + \frac{b}{p^2} + \frac{c}{p^3} + \frac{d}{p^4} + \frac{e}{p^5} + \dots \right) p$$

§. LX. Verum si abscissa p non multum ab unitate discrepet, uti conveniet hac serie supra §. XXVI. inventa

$$q = 1 + pp \left\{ \begin{aligned} & \left(\frac{\pi}{2} - 1 \right) + \left(\frac{1.3}{2.2} \cdot \frac{\pi}{2} - 1 \right) (1 - pp) + \\ & \left(\frac{1.3.3.5}{2.2.4.4} \cdot \frac{\pi}{2} - 1 \right) (1 - pp)^2 + \left(\frac{1.3.3.5.5.7}{2.2.4.4.6.6} \right. \\ & \left. \frac{\pi}{2} - 1 \right) (1 - pp)^3 + \dots \end{aligned} \right.$$

quae etiam ex natura ellipsis in hanc convertitur

$$q = p + \frac{1}{p} \left\{ \begin{aligned} & \left(\frac{\pi}{2} - 1 \right) - \left(\frac{1.3}{2.2} \cdot \frac{\pi}{2} - 1 \right) \frac{(1 - pp)}{pp} + \\ & \left(\frac{1.3.3.5}{2.2.4.4} \cdot \frac{\pi}{2} - 1 \right) \frac{(1 - pp)^2}{p^2} - \left(\frac{1.3.3.5.5.7}{2.2.4.4.6.6} \right. \\ & \left. \frac{\pi}{2} - 1 \right) \frac{(1 - pp)^3}{p^3} + \dots \end{aligned} \right.$$

unde prout fuerit vel $p > 1$ vel $p < 1$ eam eligere licet, cujus termini vel iisdem signis procedant, vel alternantibus. Plerumque autem praestat ad summam proximam definiendam signa eligere alternantia.

Pro-

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L. 9
II. 9
III. 4
IV. 4

Problema.

§. LXI. *Datis axibus conjugatis ellipsis, in numeris praxiam exhibere ejus perimetrum.*

Solutio.

Sint semiaxes ellipsis a & b , & quadrans perimetri $\equiv q$, atque per formulas inventas valor ipsius q in numeris definiti poterit, dummodo ea eligatur, quibus terminis maxima converget. Quatuor autem adeo firmis formulis quae sunt:

$$I. q = 1 + A p^2 + B p^4 + C p^6 + D p^8 + E p^{10} + F p^{12} \text{ \&c.}$$

$$= (a p^2 + B p^4 + C p^6 + D p^8 + E p^{10} + F p^{12} \text{ \&c.}) / p$$

$$II. q = p + A \frac{1}{p} + B \frac{1}{p^3} + C \frac{1}{p^5} + D \frac{1}{p^7} + E \frac{1}{p^9} + F \frac{1}{p^{11}} + \text{\&c.}$$

$$+ \left(\frac{a}{p} + \frac{B}{p^3} + \frac{C}{p^5} + \frac{D}{p^7} + \frac{E}{p^9} + \frac{F}{p^{11}} + \text{\&c.} \right) p$$

$$III. q = 1 + p p (A + B (1 - p p) + C (1 - p p)^2 + D (1 - p p)^3 + E (1 - p p)^4 + \text{\&c.})$$

$$IV. q = p + \frac{1}{p} (A - B \frac{(1 - p p)}{p} + C \frac{(1 - p p)^2}{p^2} - D \frac{(1 - p p)^3}{p^3} + E \frac{(1 - p p)^4}{p^4} - \text{\&c.})$$

Horvitz

Horum autem tergenitorum coefficientium valores sunt
 In numeris:

| | | |
|-------------------|-------------------|---------------------------|
| A = 0,44314718056 | a = 0,5000000000 | q = 0,57079632679 |
| B = 0,05680519271 | β = 0,1875000000 | q' = 0,17809724510 |
| C = 0,02183137044 | γ = 0,11718750000 | Q = 0,10446616728 |
| D = 0,01154452143 | δ = 0,08544921875 | D = 0,07378655152 |
| E = 0,00714200029 | ε = 0,06729125977 | Q' = 0,05700863665 |
| F = 0,00485474337 | ζ = 0,0511527031 | Q'' = 0,04643855029 |
| G = 0,00351468795 | η = 0,0425408554 | Q''' = 0,03917161592 |
| H = 0,00266223578 | θ = 0,04113636911 | Q'''' = 0,03386971991 |
| I = 0,00208639732 | ι = 0,03642282682 | Q''''' = 0,02983116632 |
| K = 0,00167916842 | κ = 0,03267936962 | Q'''''' = 0,02665267507 |
| | | Q'''''''' = 0,02408604339 |

Hinc pro quavis ellipsis specie habebitur series convergens,
 unde ejus perimenter definiti poterit, veluti

si ponatur $p = \frac{1}{10}$ erit $q = 1,015993545021$

si sit $p = \frac{1}{5}$ erit $q = 1,05050222700$

si sit $p = \frac{1}{\sqrt{2}}$ erit $q = 1,35001200$ (1 - 1)

