



OBSERVATIONES
ANALYTICAE VARIAE
DE COMBINATIONIBVS

Auctore

L. Euler.

§. I.

Proposita nobis sit series quantitatum quarumcunque si-
ve finita siue in infinitum excurrens haec :

a, b, c, d, e, f, g, h , etc.

quae litterae denotent quantitates quascunque siue inter se
aequales siue inaequales. Interim tamen quantitates, quae
diuersis litteris indicantur, inter se inaequales vocabo,
etiamsi in exemplis earum loco numeros aequales substitui-
re liceat.

§. 2. Nunc primo ex his quantitatibus formentur
potestatis sumendis nouae series, quarum summae de-
signentur litteris maiusculis A, B, C, D etc. ut sequitur :
sit scilicet

$$A = a + b + c + d + e + \text{etc.}$$

$$B = a^2 + b^2 + c^2 + d^2 + e^2 + \text{etc.}$$

$$C = a^3 + b^3 + c^3 + d^3 + e^3 + \text{etc.}$$

$$D = a^4 + b^4 + c^4 + d^4 + e^4 + \text{etc.}$$

$$E = a^5 + b^5 + c^5 + d^5 + e^5 + \text{etc.}$$

etc.

quae series singulae erunt infinitae, si numerus quantita-
tum a, b, c, d , etc. assumtarum fuerit infinitus; si au-
tem numerus harum quantitatum sit finitus ac determi-
natus



natus puta $= n$, tum singulae istae series totidem terminos complectentur.

§. 3. Deinde sequenti modo ex quantitatibus assumentis a, b, c, d , etc. productis inaequalium sumendis formantur series. Primo scilicet colligantur quantitates singulae, tum facta ex binis inaequalibus; tertio ex ternis inaequalibus; quarto ex quaternis inaequalibus et ita porro; atque hae series litteris graecis $\alpha, \beta, \gamma, \delta$, etc. indicentur ut sequitur.

$$\alpha = a + b + c + d + \text{etc.}$$

$$\beta = ab + ac + ad + bc + bd + \text{etc.}$$

$$\gamma = abc + abd + a be + bcd + \text{etc.}$$

$$\delta = abcd + abce + bcde + \text{etc.}$$

$$\epsilon = abcde + \text{etc.}$$

etc.

quae series si quantitatum assumtarum a, b, c, d , etc. numerus fuerit infinitus, non solum omnes in infinitum excurrent, sed etiam ipsarum serierum hoc modo formandarum numerus erit infinitus. Quodsi autem numerus quantitatum a, b, c, d , etc. fuerit finitus puta $= n$, tum series α continebit n terminos, secunda series β constabit ex $\frac{n(n-1)}{1 \cdot 2}$ terminis, tertia γ ex $\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$ terminis, quarta δ ex $\frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}$ terminis, et ita porro, donec tandem ad seriem perueniatur ex vnico termino constantem, quam sequentes omnes evanescunt terminis omnino carentes. Perspicuum autem est serierum, quae hoc modo generantur, numerum fore $= n$, eorumque ultimam vnico constare termino, qui sit pro-

ductum ex omnibus quantitatibus assumtis a, b, c, d, e , etc.

§. 4. Quemadmodum autem hic producta ex quantitatibus inaequalibus tantum assumsimus, ex iisque series expositas formauimus; ita iisdem quantitatibus in productis quoties fieri poterit repetendis, nanciscemur nouas productorum ex singulis, binis, ternis, quaternis, etc. series, in quibus factores aequales non, vt ante exclusantur; haec ergo series ita se habebunt.

$$\mathfrak{A} = a + b + c + d + e + \text{etc.}$$

$$\mathfrak{B} = a^2 + ab + b^2 + ac + bc + c^2 + \text{etc.}$$

$$\mathfrak{C} = a^3 + a^2b + ab^2 + b^3 + a^2c + abc + \text{etc.}$$

$$\mathfrak{D} = a^4 + a^3b + a^2b^2 + a^2bc + abcd + \text{etc.}$$

$$\mathfrak{E} = a^5 + a^4b + a^3b^2 + a^3bc + a^2bcd + \text{etc.}$$

etc.

in his nempe seriebus omnes continentur quantitates, quae per multiplicationem ex quantitatibus assumtis a, b, c, d , etc. produci possunt. Ceterum notandum est si numerus quantitatum a, b, c, d , etc. fuerit finitus $= n$, tum seriem primam \mathfrak{A} esse habituram n terminos, secunda autem \mathfrak{B} habebit $\frac{n(n+1)}{1+2}$ terminos, tercia \mathfrak{C} habebit $\frac{n(n+1)(n+2)}{1+2+3}$ terminos, quarta \mathfrak{D} vero $\frac{n(n+1)(n+2)(n+3)}{1+2+3+4}$ terminos, et ita porro.

§. 5. Tres hi serierum, quas ex quantitatibus assumtis a, b, c, d , etc. triplici modo composuimus, ordines multifariam inter se connectuntur, ita vt uno serierum ordine cognito, bini reliqui ordines inde possint determinari. Atque in hoc negotio ad connexionis legem et rationem inuestigandam obseruatio atque induc*tion* plu

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plurimum adhiberi solet; hocque pacto primum quidem certissime constat esse $A - \alpha = \mathfrak{A}$; ac de reliquis compertum est esse:

$$\begin{aligned}\alpha &= A \\ \mathfrak{B} &= \frac{\alpha A - B}{2} \\ \gamma &= \frac{\epsilon A - \alpha B + C}{3} \\ \delta &= \frac{\gamma A - \mathfrak{B} B + \alpha C - D}{4} \\ \epsilon &= \frac{\delta A - \gamma B + \mathfrak{C} C - \alpha D + E}{5}\end{aligned}$$

etc. item

$$\begin{aligned}\mathfrak{A} &= A \\ \mathfrak{B} &= \frac{\mathfrak{A} A + B}{2} \\ \mathfrak{C} &= \frac{\mathfrak{B} A + \mathfrak{A} B + C}{3} \\ \mathfrak{D} &= \frac{\mathfrak{C} A + \mathfrak{B} B + \mathfrak{A} C + D}{4} \\ \mathfrak{E} &= \frac{\mathfrak{D} A + \mathfrak{C} B + \mathfrak{B} C + \mathfrak{A} D + E}{5}\end{aligned}$$

etc.
praetereaque)

$$\begin{aligned}\mathfrak{A} &= \alpha \\ \mathfrak{B} &= \alpha \mathfrak{A} - \mathfrak{B} \\ \mathfrak{C} &= \alpha \mathfrak{B} - \mathfrak{C} \mathfrak{A} + \gamma \\ \mathfrak{D} &= \alpha \mathfrak{C} - \mathfrak{C} \mathfrak{B} + \gamma \mathfrak{A} - \delta \\ \mathfrak{E} &= \alpha \mathfrak{D} - \mathfrak{C} \mathfrak{C} + \gamma \mathfrak{B} - \delta \mathfrak{A} + \epsilon\end{aligned}$$

etc.

Harumque relationum ope ex datis summis serierum cuiuscunque classis definiri poterunt summae serierum, quae in duabus reliquis classibus continentur.

§. 6. Ad naturam atque indolem harum serierum diligentius attendenti facile quidem per obseruationem et inductionem veritas istius mutuae relationis patebit. Verum tamen quo magis de veritate huius nexus conuincamur, expediet sequenti modo totum hoc negotium considerare; quo simul aliae insuper proprietates nobis offerentur, ad quas sola inductio non tam facile viam aperit. Assuntis scilicet pro libitu quantitatibus

$$a, b, c, d; e; \text{etc.}$$

ex iisque formatis trium classium seriebus supra memoratis, contempleremur hanc expressionem:

$$P = \frac{az}{1-az} + \frac{bz}{1-bz} + \frac{cz}{1-cz} + \frac{dz}{1-dz} + \frac{ez}{1-ez} + \text{etc.}$$

cuius singuli termini in progressiones geometricas resoluti more solito, dabunt

$$\begin{aligned} P = & +z(a+b+c+d+e+\text{etc.}) \\ & + z^2(a^2+b^2+c^2+d^2+e^2+\text{etc.}) \\ & + z^3(a^3+b^3+c^3+d^3+e^3+\text{etc.}) \\ & + z^4(a^4+b^4+c^4+d^4+e^4+\text{etc.}) \end{aligned}$$

quae series omnes in prima classe continentur. Quare si earum loco summae supra (z) positae scribantur fiet:

$$P = Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \text{etc.}$$

cuius ideo seriei summa erit, uti sumimus

$$P = \frac{az}{1-az} + \frac{bz}{1-bz} + \frac{cz}{1-cz} + \frac{dz}{1-dz} + \text{etc.}$$

Simili autem modo si fuerit:

$$Q = \frac{az}{1+az} + \frac{bz}{1+bz} + \frac{cz}{1+cz} + \frac{dz}{1+dz} + \text{etc.}$$

erit per series primae classis:

$$Q = Az - Bz^2 + Cz^3 - Dz^4 + Ez^5 - \text{etc.}$$

§. 7. Consideremus porro hanc expressionem:

$R = (1 + az)(1 + bz)(1 + cz)(1 + dz)(1 + ez)$ etc.
 cuius factores, si actu in se multiplicentur; ac termini secundum exponentes ipsius z disponantur fiet coëfficiens ipsius z aequalis summae quantitatum assumtarum a, b, c, d , etc. Coëfficiens ipsius z^2 erit aggregatum omnium productorum ex binis inaequalibus; coëfficiens ipsius z^3 erit aggregatum omnium productorum ex ternis inaequalibus et ita porro: ex quibus sequitur fore

$$R = 1 + az + \mathfrak{c}z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc.}$$

secundum definitiones supra (§. 3) datas.

Quodsi autem ponatur:

$S = (1 - az)(1 - bz)(1 - cz)(1 - dz)(1 - ez)$ etc.
 erit faciendo tantum z negatiuo

$$S = 1 - az + \mathfrak{c}z^2 - \gamma z^3 + \delta z^4 - \varepsilon z^5 + \text{etc.}$$

§. 8. Ut series hae R et S cum praecedentibus P et Q comparentur, notandum est esse

$$\frac{dR}{dz} = l(1 + az) + l(1 + bz) + l(1 + cz) + l(1 + dz) + \text{etc.}$$

vnde sumendis differentialibus erit:

$$\frac{dR}{dz} = \frac{a}{1 + az} + \frac{b}{1 + bz} + \frac{c}{1 + cz} + \frac{d}{1 + dz} + \text{etc.}$$

quae per z multiplicata dat illam ipsam expressionem quam supra Q vocauimus, ita vt sit: $Q = \frac{z dR}{R dz}$. Simili autem modo erit $\frac{dS}{dz} = \frac{-a}{1 - az} - \frac{b}{1 - bz} - \frac{c}{1 - cz} - \text{etc.}$

$$\text{vnde habebitur } P = \frac{-z dS}{S dz}.$$

§. 9. Cum nunc sit $R = 1 + az + \mathfrak{c}z^2 + \gamma z^3 + \text{etc.}$ erit $\frac{z dR}{dz} = az + 2\mathfrak{c}z^2 + 3\gamma z^3 + 4\delta z^4 + 5\varepsilon z^5 + \text{etc.}$ ideoque $Q = Az - Bz^2 + Cz^3 - Dz^4 + Ez^5 - \text{etc.}$

$$= \frac{\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc.}}{1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc.}}$$

At ex aequalitate harum expressionum sequuntur sequentes relationes inter litteras A, B, C, D, E, etc. et α , β , γ , δ , ε , etc.

$$A = \alpha$$

$$\alpha A - B = 2 \beta$$

$$\beta A - \alpha B + C = 3 \gamma$$

$$\gamma A - \beta B + \alpha C - D = 4 \delta$$

$$\delta A - \gamma B + \beta C - \alpha D + E = 5 \varepsilon$$

etc.

Simili vero modo ex altera aequatione $P = \frac{-z ds}{s dz}$ sequitur $P = A z + B z^2 + C z^3 + D z^4 + E z^5 + \text{etc.}$

$$= \frac{\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc.}}{1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc.}}$$

quae pariter easdem praebet determinationes, quas supra §. 5. tradidimus.

§. 10. Praeterea autem ex aequatione $Q = \frac{z d R}{R dz}$ consequimur integrando $\int \frac{Q dz}{z} = l R$. Quoniam vero est $Q = A z - B z^2 + C z^3 - D z^4 + \text{etc.}$ erit $\int \frac{Q dz}{z} = A z - \frac{B z^2}{2} + \frac{C z^3}{3} - \frac{D z^4}{4} + \text{etc.}$ cuius seriei valor itaque exprimet logarithmum huius seriei $R = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}$

Quemadmodum igitur est :

$$l(1 + \alpha z + \beta z^2 + \gamma z^3 + \text{etc.}) = Az - \frac{1}{2}Bz^2 + \frac{1}{3}Cz^3 - \frac{1}{4}Dz^4 + \text{etc.}$$

ita etiam ex aequatione $\int \frac{P dz}{z} = -l S$ erit

$$l(1 - \alpha z + \beta z^2 - \gamma z^3 + \text{etc.}) = -Az - \frac{1}{2}Bz^2 - \frac{1}{3}Cz^3 - \frac{1}{4}Dz^4 - \text{etc.}$$

Quare si k scribatur pro numero, cuius logarithmus hyperbolicus est $= 1$, habebitur :

$$1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.} = k^{Az - \frac{1}{2}Bz^2 + \frac{1}{3}Cz^3 - \frac{1}{4}Dz^4 + \text{etc.}}$$

et

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et

$$x - az + bz^2 - cz^3 + dz^4 - \text{etc.} = k - Ax - \frac{1}{2}Bz^2 - \frac{1}{3}Cz^3 - \frac{1}{4}Dz^4 - \text{etc.}$$

§. 11. Notatu praeterea dignae sunt expressiones harum R et S reciprocae nempe $\frac{1}{R}$ et $\frac{1}{S}$. Est vero $\frac{1}{S} = \frac{1}{(1 - az)(1 - bz)(1 - cz)(1 - dz) \text{etc.}}$ ad cuius fractionis valorem per seriem, cuius termini secundum potestates ipsius z progrediantur, exprimendum, perspicuum est in se inuicem multiplicari oportere cunctas has progressiones geometricas

$$\frac{1}{1 - az} = 1 + az + a^2 z^2 + a^3 z^3 + a^4 z^4 + \text{etc.}$$

$$\frac{1}{1 - bz} = 1 + bz + b^2 z^2 + b^3 z^3 + b^4 z^4 + \text{etc.}$$

$$\frac{1}{1 - cz} = 1 + cz + c^2 z^2 + c^3 z^3 + c^4 z^4 + \text{etc.}$$

$$\frac{1}{1 - dz} = 1 + dz + d^2 z^2 + d^3 z^3 + d^4 z^4 + \text{etc.}$$

In Producto autem post primum terminum 1 coëfficiens ipsius z erit summa quantitatum $a + b + c + d + \text{etc.}$ coëfficiens ipsius z^2 erit summa factorum ex binis non excipiendo factores aequales in eodem facto; coëfficiens ipsius z^3 erit summa factorum ex ternis, et ita porro quas productorum summas supra (§. 4) litteris alphabeti geometrici $A, B, C, D, E, \text{etc.}$ designauimus.

His itaque litteris introductis habebimus:

$$S = 1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \text{etc.}$$

atque simili modo valorem ipsius R tractando erit:

$$R = 1 - Az + Bz^2 - Cz^3 + Dz^4 - Ez^5 + \text{etc.}$$

§. 12. Hae ergo series reciprocae sunt earum, quas supra sub litteris R et S (§. 7) protulimus. Atque hanc ob causam erit:

$$x =$$

$$1 = (1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}) (1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.}) \quad \text{pariterque}$$

$$1 = (1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \text{etc.}) (1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.})$$

Ex utraque autem sequitur una eademque relatio inter valores litterum $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. et $\alpha, \beta, \gamma, \delta$, erit scilicet :

$$\mathfrak{A} - \alpha = 0$$

$$\mathfrak{B} - \alpha \mathfrak{A} + \beta = 0$$

$$\mathfrak{C} - \alpha \mathfrak{B} + \beta \mathfrak{A} - \gamma = 0$$

$$\mathfrak{D} - \alpha \mathfrak{C} + \beta \mathfrak{B} - \gamma \mathfrak{A} + \delta = 0$$

etc.

quam eandem relationem iam supra (§. 5) tradidimus.

§. 13. Quodsi ponamus $\frac{1}{R} = T$ et $\frac{1}{S} = V$, vt sit

$$T = 1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \mathfrak{D}z^4 - \text{etc.}$$

$$\text{et } V = 1 - \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.}$$

erit $\frac{dR}{R} = -\frac{dT}{T}$ et $\frac{dS}{S} = -\frac{dV}{V}$; hincque

fiet $P = \frac{zdV}{vdz}$ et $Q = -\frac{zdT}{Tdz}$. Quare cum

$$\text{sit } \frac{zdV}{dz} = \mathfrak{A}z + 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 + 4\mathfrak{D}z^4 + \text{etc.}$$

$$\text{et } -\frac{zdT}{dz} = \mathfrak{A}z - 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 - 4\mathfrak{D}z^4 + \text{etc.}$$

habebimus loco P et Q valores debitos ex (§. 6.) scribendo has aequationes

$$Az + Bz^2 + Cz^3 + Dz^4 + \text{etc.} = \frac{\mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.}}{1 - \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.}}$$

et

$$Az - Bz^2 + Cz^3 - Dz^4 + \text{etc.} = \frac{\mathfrak{A}z - 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 - 4\mathfrak{D}z^4 + \text{etc.}}{1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.}}$$

ex quibus eadem sequitur relatio inter litteras A, B, C, D, etc. et $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, etc. quam supra (§. 5.) dedimus. Erit scilicet

\mathfrak{A}

$$\begin{aligned}
 1 & \mathfrak{A} = A \\
 2 & \mathfrak{B} = \mathfrak{A}A + \mathfrak{B} \\
 3 & \mathfrak{C} = \mathfrak{B}A + \mathfrak{A}B + C \\
 4 & \mathfrak{D} = \mathfrak{C}A + \mathfrak{B}B + \mathfrak{A}C + D \\
 5 & \mathfrak{E} = \mathfrak{D}A + \mathfrak{C}B + \mathfrak{B}C + AD + E \\
 & \quad \text{etc.}
 \end{aligned}$$

§. 14. Ex aequationibus (§. 12.) datis sequitur fore
 $(1 + az + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \text{etc.}) = -(1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \text{etc.})$
 et

$$(1 - az + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \text{etc.}) = -(1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \text{etc.})$$

His igitur ad §. 10. accommodatis erit:

$$(1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \text{etc.}) = Az + \frac{1}{2}Bz^2 - \frac{1}{3}Cz^3 + \frac{1}{4}Dz^4 - \text{etc.}$$

et

$$(1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \text{etc.}) = Az + \frac{1}{2}Bz^2 + \frac{1}{3}Cz^3 + \frac{1}{4}Dz^4 + \text{etc.}$$

Hincque sumto k pro numero, cuius logarithmus = 1 erit.

$$1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \text{etc.} = -Az + \frac{1}{2}Bz^2 - \frac{1}{3}Cz^3 + \frac{1}{4}Dz^4 - \text{etc.}$$

atque

$$1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \text{etc.} = k^{Az + \frac{1}{2}Bz^2 + \frac{1}{3}Cz^3 + \frac{1}{4}Dz^4 + \text{etc.}}$$

§. 15. Si iam litterae R et S retineant valores supra assumtos (§. 7.) erit

$$1 + az + \mathfrak{B}z^2 + \gamma z^3 + \delta z^4 + \text{etc.} = R$$

$$1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \mathfrak{D}z^4 - \text{etc.} = \frac{1}{R}$$

et

$$1 - az + \mathfrak{B}z^2 - \gamma z^3 + \delta z^4 - \text{etc.} = S$$

$$1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.} = \frac{1}{S}$$

Ex quibus deducuntur sequentia conjectaria.

$$1 + \mathfrak{B}z^2 + \delta z^4 + \zeta z^6 + \theta z^8 + \text{etc.} = \frac{R+S}{2}$$

$$az + \gamma z^3 + \varepsilon z^5 + \eta z^7 + \nu z^9 + \text{etc.} = \frac{R-S}{2}$$

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$$I + Bz^2 + Dz^4 + Fz^6 + Hz^8 + \text{etc.} = \frac{R+S}{2RS}$$

$$A z + Cz^3 + Ez^5 + Gz^7 + Jz^9 + \text{etc.} = \frac{R-S}{2RS}$$

hincque colligitur ista proportio :

$$I + Bz^2 + Dz^4 + Fz^6 + \text{etc.} : az + \gamma z^3 + \varepsilon z^5 + \eta z^7 + \text{etc.} = R - I$$

$$I + Bz^2 + Dz^4 + Fz^6 + \text{etc.} : Az + Cz^3 + Ez^5 + Gz^7 + \text{etc.} = I - \frac{1}{R}$$

Cum praeterea sit :

$$R - I = az + Bz^2 + \gamma z^3 + \delta z^4 + \text{etc.}$$

$$I - \frac{1}{R} = Az - Bz^2 + Cz^3 + Dz^4 + \text{etc.}$$

erit

$$R = \frac{az + Bz^2 + \gamma z^3 + \delta z^4 + \text{etc.}}{Az - Bz^2 + Cz^3 + Dz^4 + \text{etc.}}$$

similique modo propter

$$I - S = az - Bz^2 + \gamma z^3 - \delta z^4 + \text{etc.}$$

$$\varepsilon - I = Az + Bz^2 + Cz^3 + Dz^4 + \text{etc.}$$

erit

$$S = \frac{az - Bz^2 + \gamma z^3 - \delta z^4 + \text{etc.}}{az + Bz^2 + Cz^3 + Dz^4 + \text{etc.}}$$

§. 16. Deinde vero si ut supra (§. 6.) ponamus

$$P = Az + Bz^2 + Cz^3 + Dz^4 + \text{etc.}$$

$$Q = Az - Bz^2 + Cz^3 - Dz^4 + \text{etc.}$$

erit ex paragr. 9.

$$az + 2Bz^2 + 3\gamma z^3 + 4\delta z^4 + \text{etc.} = QR$$

$$az - 2Bz^2 + 3\gamma z^3 - 4\delta z^4 + \text{etc.} = PS$$

similique modo ex paragr. 13 habebitur

$$Az + 2Bz^2 + 3Cz^3 + 4Dz^4 + \text{etc.} = \frac{P}{S}$$

$$Az - 2Bz^2 + 3Cz^3 + 4Dz^4 + \text{etc.} = \frac{Q}{R}$$

Ex quibus sequentia corollaria facile derivantur

$$\frac{az - Bz^2 + \gamma z^3 + \delta z^4 + \text{etc.}}{Az + Bz^2 + Cz^3 + Dz^4 + \text{etc.}} = S = \frac{Az + Bz^2 + Cz^3 + Dz^4 + \text{etc.}}{Az - Bz^2 + Cz^3 + Dz^4 + \text{etc.}}$$

$$\frac{az + Bz^2 + \gamma z^3 + \delta z^4 + \text{etc.}}{Az - Bz^2 + Cz^3 - Dz^4 + \text{etc.}} = R = \frac{Az - Bz^2 + Cz^3 - Dz^4 + \text{etc.}}{Az + Bz^2 + Cz^3 - Dz^4 + \text{etc.}}$$

Pro

Pro litteris igitur R et S habemus quintuplices valores hos :

$$R = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}$$

$$R = \frac{1 - \mathfrak{A}z - \mathfrak{B}z^2 - \mathfrak{C}z^3 - \mathfrak{D}z^4 - \text{etc.}}{\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}}$$

$$R = \frac{\mathfrak{A}z - \mathfrak{B}z^2 - \mathfrak{C}z^3 - \mathfrak{D}z^4 + \text{etc.}}{\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}}$$

$$R = \frac{\alpha z - \beta z^2 + \gamma z^3 - \delta z^4 + \text{etc.}}{\mathfrak{A}z - \mathfrak{B}z^2 + \mathfrak{C}z^3 - \mathfrak{D}z^4 + \text{etc.}}$$

$$R = \frac{\mathfrak{A}z - \mathfrak{B}z^2 + \mathfrak{C}z^3 - \mathfrak{D}z^4 + \text{etc.}}{\alpha z - \beta z^2 + \gamma z^3 - \delta z^4 + \text{etc.}}$$

qui posito $-z$ loco z totidem praebent valores pro S. Atque ex horum quinque valorum multipliciti combinacione quam plurimae proprietates elici possunt, quas terni litterarum nostrarum ordines, scilicet: A, B, C, D etc. α , β , γ , δ , etc. \mathfrak{A} , \mathfrak{B} , \mathfrak{C} , \mathfrak{D} , etc. inter se tenent, quibus autem euoluendis hic supersedemus.

§. 17. His, quae latissime ptent, paraemissis atque expositis ad magis particularia descendamus, ac primo quidem pro serie litterarum a, b, c, d , etc. accipiatur progressio geometrica infinita haec: $n, n^2, n^3, n^4, n^5, n^6$, etc. qua in formulas superiores successiue introducta habebimus:

$$A = n + n^2 + n^3 + n^4 + n^5 + \text{etc.} = \frac{n}{1-n}$$

$$B = n^2 + n^4 + n^6 + n^8 + n^{10} + \text{etc.} = \frac{n^2}{1-n^2}$$

$$C = n^3 + n^6 + n^9 + n^{12} + n^{15} + \text{etc.} = \frac{n^3}{1-n^3}$$

$$D = n^4 + n^8 + n^{12} + n^{16} + n^{20} + \text{etc.} = \frac{n^4}{1-n^4}$$

etc.

Iam ex §. 6 duplices pro litteris P et Q nanciscimur va- res, qui erunt:

$$P = \frac{nz}{1-nz} + \frac{n^2z}{1-n^2z} + \frac{n^3z}{1-n^3z} + \frac{n^4z}{1-n^4z} + \text{etc.}$$

K 2

$$Q =$$

$$Q = \frac{nz}{1-nz} + \frac{n^2 z^2}{1-n^2 z} + \frac{n^3 z^3}{1-n^3 z} + \frac{n^4 z^4}{1-n^4 z} + \text{etc.}$$

hincque ex inuentis litterarum A, B, C, D, etc. valoribus nascentur hi alteri :

$$P = \frac{nz}{1-n} + \frac{n n z^2}{1-n n} + \frac{n^3 z^3}{1-n^3} + \frac{n^4 z^4}{1-n^4} + \text{etc.}$$

$$Q = \frac{nz}{1-n} - \frac{n^2 z^2}{1-n n} + \frac{n^3 z^3}{1-n^3} - \frac{n^4 z^4}{1-n^4} + \text{etc.}$$

§. 18. Ex paragr. porro 7 habebimus pro R et S sequentes expressiones :

$$R = (1+nz)(1+n^2 z)(1+n^3 z)(1+n^4 z) \text{ etc.}$$

$$S = (1-nz)(1-n^2 z)(1-n^3 z)(1-n^4 z) \text{ etc.}$$

qui factores actu in se multiplicati, et producta secundum dimensiones ipsius z ordinata praebebunt pro R et S has series :

$$R = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}$$

$$S = 1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \text{etc.}$$

ubi litterae $\alpha, \beta, \gamma, \delta$, etc ex serie assumta

$$n, n^2, n^3, n^4, n^5, n^6, n^7, \text{ etc.}$$

ita determinabuntur, vt sit :

I. $\alpha =$ summae singulorum terminorum ; vnde erit :

$$\alpha = n + n^2 + n^3 + n^4 + n^5 + n^6 + n^7 + \text{etc.}$$

quae est ipsa progressio geometrica assumta in qua quaevis potestas ipsius n occurrit, atque coëfficientem habet $+1$.

II. $\beta =$ summae factorum ex binis terminis : vnde erit :

$$\beta = n^3 + n^4 + 2n^5 + 2n^6 + 3n^7 + 3n^8 + 4n^9 + 4n^{10} + \text{etc.}$$

in qua serie post potestatem tertiam omnes sequentes ipsius n potestates occurruunt : quaelibet autem potestas toties occurrit, quoties ex multiplicatione binorum terminorum seriei α oriri potest. Cum autem multiplicatio potestatum consistat in exponentium additione, coëfficiens cuiusque potestatis ipsius n in serie β ostendet, quot va-

riis

riis modis exponens ipsius n possit in duas partes inaequales distribui, seu quoties iste exponens ex additione duorum numerorum integrorum inaequalium produci queat. Sic potestatis decimae n^{10} coëfficiens est 4, quia 10 quatuor modis in duas partes inaequales distribui potest nempe $10 = 1 + 9$; $10 = 3 + 7$.
 $10 = 2 + 8$; $10 = 4 + 6$.

III. $\gamma =$ summae factorum ex ternis terminis seriei α inaequalibus; vnde erit:

$\gamma = n^6 + n^7 + 2n^8 + 3n^9 + 4n^{10} + 5n^{11} + 7n^{12} + 8n^{13} + \text{etc.}$
in qua post potestatem sextam omnes sequentes ipsius n potestates occurunt. Cuiuslibet autem potestatis coëfficiens indicat, quot variis modis exponens distribui possit in tres partes inaequales, seu quoties idem exponens produci queat ex additione trium numerorum integrorum inter se inaequalium. Sic potestas n^{12} coëfficientem habet 7, quia exponens 12 septem modis in tres partes inaequales partiri potest: vt

$$\begin{aligned} 12 &= 1 + 2 + 9; 12 = 2 + 3 + 7 \\ 12 &= 1 + 3 + 8; 12 = 2 + 4 + 6 \\ 12 &= 1 + 4 + 7; 12 = 3 + 4 + 5 \\ 12 &= 1 + 5 + 6; \end{aligned}$$

IV. $\delta =$ summae factorum ex quatuor terminis seriei α inaequalibus inter se; vnde erit

$\delta = n^{10} + n^{11} + 2n^{12} + 3n^{13} + 5n^{14} + 6n^{15} + 9n^{16} + \text{etc.}$
cuius prima potestas est n^{10} , quippe cuius exponens est $1+2+3+4$ seu numerus trigonalis quartus. Sequentium potestatum quaelibet toties adest, quoties eius exponens oriri potest ex additione quatuor numerorum integrorum inter se inaequalium. Sic potestas sexta deci-

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ma n^{16} coëfficientem habet 9; quia 16 nouem modis in quatuor partes inter se inaequales dispertiri potest, quae nouem partitiones sunt:

$$\begin{aligned} 16 &= 1 + 2 + 3 + 10; \quad 16 = 1 + 3 + 4 + 8 \\ 16 &= 1 + 2 + 4 + 9; \quad 16 = 1 + 3 + 5 + 7 \\ 16 &= 1 + 2 + 5 + 8; \quad 16 = 1 + 4 + 5 + 6 \\ 16 &= 1 + 2 + 6 + 7; \quad 16 = 2 + 3 + 4 + 7 \\ 16 &= 2 + 3 + 5 + 6 \end{aligned}$$

Simili modo res se habet in sequentium litterarum ϵ , ζ , η , etc. valoribus qui erunt

$$\begin{aligned} \epsilon &= n^{15} + n^{16} + 2n^{17} + 3n^{18} + 5n^{19} + 7n^{20} + 10n^{21} + \text{etc.} \\ \zeta &= n^{21} + n^{22} + 2n^{23} + 3n^{24} + 5n^{25} + 7n^{26} + 11n^{27} + \text{etc.} \\ \eta &= n^{28} + n^{29} + 2n^{30} + 3n^{31} + 5n^{32} + 7n^{33} + 11n^{34} + \text{etc.} \end{aligned}$$

in quibus seriebus omnibus cuiusvis ipsius n potestatis coëfficiens indicat, quot variis modis exponens ipsius n possit resoluti in tot partes inaequales, quota series est a principio numerata. Seu coëfficiens cuiusque termini declarat, quoties exponens ipsius n oriri queat ex additione tot numerorum integrorum inter se inaequalium quota ipsa series, ex qua terminus desumitur, est, numerando a prima α . Sic in serie septima coëfficiens potestatis n^{34} est 11, quia numerus 34 undecim modis distribui potest in septem partes inaequales, quae distributiones sunt:

$$\begin{aligned} 34 &= 1 + 2 + 3 + 4 + 5 + 6 + 13 \\ 34 &= 1 + 2 + 3 + 4 + 5 + 7 + 12 \\ 34 &= 1 + 2 + 3 + 4 + 5 + 8 + 11 \\ 34 &= 1 + 2 + 3 + 4 + 5 + 9 + 10 \\ 34 &= 1 + 2 + 3 + 4 + 6 + 7 + 11 \\ 34 &= 1 + 2 + 3 + 4 + 6 + 8 + 10 \end{aligned}$$

34 =

$$34 = 1 + 2 + 3 + 4 + 7 + 8 + 9$$

$$34 = 1 + 2 + 3 + 5 + 6 + 7 + 10$$

$$34 = 1 + 2 + 3 + 5 + 6 + 8 + 9$$

$$34 = 1 + 2 + 4 + 5 + 6 + 7 + 9$$

$$34 = 1 + 3 + 4 + 5 + 6 + 7 + 8$$

Atque ex his natura serierum, quae hoc pacto pro litteris α , β , γ , δ , etc. prodeunt, facile perspicitur.

§. 19. Inuestigando igitur, quot variis modis quisque numerus in partes inaequales numero datas distribui possit, series istae litteris α , β , γ , δ , etc. signatae formari poterunt: quod autem opus foret summopere molestum. Vicissim autem ex his seriebus aliunde cognitis et formatis resoluti poterit problema hoc non inelegans, quod mihi a Viro Clar. Naudaeo propositum ita se habet; „Definire, quot variis modis datus numerus produci queat ex additione aliquot numerorum integrorum inter se inaequalium; quorum numerus detur.

Sic Clariss Propositore quaerit, quot variis modis numerus 50 oriri possit ex additione septem numerorum integrorum inaequalium. Ad quam quaestione resolendum manifestum est in subsidium vocari debere seriem η in qua coëfficiens cuiusque termini indicat, quot variis modis exponens ipsius n resoluti possit in 7 partes inaequales. Quare series illa

$$\eta = n^{28} + n^{29} + 2n^{30} + 3n^{31} + 5n^{32} + 7n^{33} + 11n^{34} + \text{etc.}$$

cotinuari debet usque ad terminum, in quo potestas quinquagesima ipsius n continetur, cuius coëfficiens qui erit 522 ostendet numerum 50 omnino 522 modis diversis ex additione septem numerorum integrorum inter se inaequalium produci posse. Ex quo perspicuum est,

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si modus habeatur commodus et facilis formandi illas series $\alpha, \beta, \gamma, \delta, \text{ etc.}$ eo ipso problema istud Naudaeanum perfectissime solutum iri.

§. 20. Cum igitur supra §. §. 5 et 9. modus traditus sit inueniendi valores litterarum $\alpha, \beta, \gamma, \delta, \text{ etc.}$ ex cognitis valoribus litterarum A, B, C, D, etc. in praesenti negotio resolutionem facile expedire poterimus propterea quod ex §. 17 cognitos habemus valores A, B, C, D, etc. atque praeterea est, ut sequitur

$$\begin{aligned}\alpha &= A \\ \beta &= \frac{\alpha A - B}{2} \\ \gamma &= \frac{\beta A - \alpha B + C}{3} \\ \delta &= \frac{\gamma A - \beta B + \alpha C - D}{4} \\ \varepsilon &= \frac{\delta A - \gamma B + \beta C - \alpha D + E}{5} \\ &\vdots \end{aligned}$$

etc.

Ex his igitur obtinebimus :

$$\begin{aligned}\alpha &= \frac{n}{1-n} \\ 2\beta &= \frac{\alpha n}{1-n} = \frac{n^2}{1-n^2} \\ 3\gamma &= \frac{\beta n}{1-n} = \frac{\alpha n^2}{1-n^2} + \frac{n^3}{1-n^4} \\ 4\delta &= \frac{\gamma n}{1-n} = \frac{\beta n^2}{1-n^2} + \frac{\alpha n^3}{1-n^4} = \frac{n^4}{1-n^4} \\ &\vdots \end{aligned}$$

etc.

Quod si autem loco $\alpha, \beta, \gamma, \delta, \text{ etc.}$ successive substituantur valores ante reperti, prodibunt :

$$\begin{aligned}\alpha &= \frac{n}{1-n} \\ \beta &= \frac{n^2}{(1-n)(1-n^2)} \\ \gamma &= \frac{n^3}{(1-n)(1-n^2)(1-n^3)} \\ \delta &= \frac{n^4}{(1-n)(1-n^2)(1-n^3)(1-n^4)} \\ \varepsilon &= \frac{n^5}{(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5)} \\ &\vdots \end{aligned}$$

etc.

Ex

Ex his itaque intelligitur esse in hoc casu

$$\alpha = A$$

$$\beta = A B$$

$$\gamma = A B C$$

$$\delta = A B C D$$

$$\epsilon = A B C D E$$

etc.

§. 21. Lex haec, qua valores litterarum $\alpha, \beta, \gamma, \delta$, etc. progredi sunt inuenti, compluribus formulis evolutis obseruatur, eiusque veritas nisi per inductionem adhuc non constat. Quo igitur haec veritas firmius confirmetur, conueniet eandem progressionis legem alio modo planissimo, in quo inductioni nullus locis relinquatur, elicere. Cum itaque nobis propositum sit valores litterarum $\alpha, \beta, \gamma, \delta$, etc. indagare, quas fortiuntur in serie

$$R = 1 + az + \beta z^2 + \gamma z^3 + \delta z^4 + \epsilon z^5 + \text{etc.}$$

Si fuerit uti initio assumimus.

$$R = (1 + nz)(1 + n^2z)(1 + n^3z)(1 + n^4z) \dots$$

notandum est si loco z scribatur $n z$, expressionem, cum modo R erat aequalis, mutari in hanc formam

$$(1 + n^2z)(1 + n^3z)(1 + n^4z)(1 + n^5z) \dots$$

quae multiplicata per $1 + nz$ ipsam priorem expressionem producit. Quamobrem recte concludimus, si in serie

$$1 + az + \beta z^2 + \gamma z^3 + \delta z^4 + \epsilon z^5 + \text{etc.}$$

locu z scribamus $n z$ ut habeamus

$$1 + anz + \beta n^2z^2 + \gamma n^3z^3 + \delta n^4z^4 + \epsilon n^5z^5 + \text{etc.}$$

hancque expressionem per $1 + nz$ multiplicemus, tum productum, quod erit

$$1 + anz + \beta n^2z^2 + \gamma n^3z^3 + \delta n^4z^4 + \epsilon n^5z^5 + \text{etc.}$$

$$+ nz + an^2z^2 + \beta n^3z^3 + \gamma n^4z^4 + \delta n^5z^5 + \text{etc.}$$

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aequale esse debere illi ipsi priori seriei

$$1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc.}$$

Quodsi ergo actu coëfficientes terminorum homologorum coëquemus, nanciscemur sequentes pro α , β , γ , etc determinationes.

$$\begin{aligned}\alpha &= \frac{n}{1-n} = \frac{n}{n^3} \\ \beta &= \frac{\alpha n^2}{1-n^2} = \frac{(1-n)(1-n^2)}{n^5} \\ \gamma &= \frac{\beta n^3}{1-n^3} = \frac{(1-n)(1-n^2)(1-n^3)}{n^{10}} \\ \delta &= \frac{\gamma n^4}{1-n^4} = \frac{(1-n)(1-n^2)(1-n^3)(1-n^4)}{\text{etc.}}\end{aligned}$$

§. 22. Hoc igitur modo inuenimus summas serierum illarum α , β , γ , δ , etc. satis commode expressas ex quibus vicissim ipsae illae series formari poterunt. Nam cum illae series secundum potestates ipsius n progrediantur eae prodire debebunt, si istae expressiones summarum per diuisionem more confueto euoluantur atque in series infinitas secundum potestates ipsius n procedentes conuentantur. Quae operatio cum diuisione absoluatur manifestum est omnes illas series α , β , γ , δ , etc. ad id genus pertinere, quod nomine serierum recurrentium indicari solet; atque adeo quilibet terminus ex aliquot praecedentibus determinabitur. Ut autem pateat, quomodo in singulis his seriebus quisque terminus ex praecedentibus sit formandus, denominatores illarum expressionum pro litteris α , β , γ , δ , etc. inuentarum per multiplicationem actu euolui debent, quo facto habebitur:

$$\begin{aligned}\alpha &= \frac{n}{1-n} n^3 \\ \beta &= \frac{\alpha n^2}{1-n-n^2+n^3} n^5 \\ \gamma &= \frac{\beta n^3}{1-n-n^2+n^4+n^5-n^6} n^{10} \\ \delta &= \frac{\gamma n^4}{1-n-n^2+n^6-n^8-n^9+n^{10}} \text{ etc.}\end{aligned}$$

$$\begin{aligned}\epsilon &= \frac{n^{15}}{1-n-n^2+n^5+n^6+n^7-n^8-n^9+n^{10}+n^{13}+n^{14}-n^{15}} \\ \gamma &= \frac{n^{21}}{1-n-n^2+n^5+n^6+n^7-n^9-n^{10}-n^{11}-n^{12}+n^{14}+n^{16}-n^{18}-n^{19}-n^{20}+n^{21}}\end{aligned}$$

etc.

Atque ex his denominatoribus intelligitur, quomodo in singulis seriebus quisque terminus ex praecedentibus componi debeat, si praecpta, quae de formatione serierum recurrentium habentur, in subsidium vocentur.

§. 23. At ex forma expressionum pro litteris α , β , γ , δ , etc. inuentarum, qua quaelibet est productum ex praecedente in nouum quempiam factorem, alitis deducitur modus sat is idoneus ex quavis serie iam inuenta seriem sequentem inueniendi. Sic, cum series $\alpha = \frac{n}{1-n}$ sit progressio geometrica

$$\alpha = n + n^2 + n^3 + n^4 + n^5 + n^6 + n^7 + \text{etc.}$$

ex hac reperietur series β , si ea multiplicetur per $\frac{n^2}{1-n^2}$, vel si multiplicetur per hanc progressionem geometricam: $n^2 + n^4 + n^6 + n^8 + n^{10} + n^{12} + n^{14} + \text{etc.}$

Ex serie porro β hoc pacto inuenta, si ea multiplicetur per $\frac{n^3}{1-n^3} = n^3 + n^6 + n^9 + n^{12} + n^{15} + n^{18} + \text{etc.}$ producetur series γ . Haecque multiplicata per

$$\frac{n^4}{1-n^4} = n^4 + n^8 + n^{12} + n^{16} + n^{20} + n^{24} + \text{etc.}$$

producet seriem δ . Atque ita porro seriem cuiusque ordinis multiplicando per certam quamdam progressionem geometricam reperietur series sequens. Hocque pacto non difficulter has series quoisque libuerit, continuare licebit: atque sic problema supra memoratum a Clar. Naudaeo propositum resoluetur.

§. 24. Facilius autem quaelibet series ex se ipsa ope praecedentis poterit continuari, si ad modum respiciamus

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quo valor cuiusque litterarum $\alpha, \beta, \gamma, \delta, \text{etc.}$ ex praecedente determinatur. Sic cum sit $\beta = \frac{\alpha n^2}{1-n^2}$ erit $\beta = \beta nn + \alpha nn$; quare si ad seriem β per nn multiplicatam addatur series α per nn multiplicata, ipsa series β oriri debebit. Cum igitur constet seriei β primum terminum esse n ponamus; $\beta = \alpha n + \beta n^2 + \gamma n^3 + \delta n^4 + \epsilon n^5 + \zeta n^6 + \eta n^7 + \text{etc.}$ eritque

$$\begin{array}{l} \beta n^2 = \dots + \alpha n^3 + \beta n^4 + \gamma n^5 + \delta n^6 + \epsilon n^7 + \text{etc.} \\ \alpha n^2 = n^3 + n^4 + n^5 + n^6 + n^7 + n^8 + n^9 + \text{etc.} \\ \text{Aequatis iam terminis propter } \beta = \beta nn + \alpha nn \text{ habebimus:} \\ \alpha = 1 & | \quad \epsilon = \epsilon + 1 = 3 \\ \beta = 1 & | \quad f = \delta + 1 = 3 \\ \epsilon = \alpha + 1 = 2 & | \quad g = e + 1 = 4 \\ \delta = \beta + 1 = 2 & | \quad h = f + 1 = 4 \\ & \quad \text{etc.} \end{array}$$

Simili modo cum sit $\gamma = \frac{\epsilon n^3}{1-n^2}$ seu $\gamma = \gamma n^3 + \beta n^4$, ex serie β formabitur series γ , atque porro ex serie γ operacionis $\delta = \delta n^4 + \gamma n^5$ producetur series δ ; pariterque sequentes omnes conficiuntur.

§. 25. Quoniam in expressione

$$R = 1 + az + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}$$

valores litterarum $a, \beta, \gamma, \delta, \text{etc.}$ inueniamus, sitque

$$R = (1+nz)(1+n^2z)(1+n^3z)(1+n^4z) \dots$$

conueretur productum hoc ex infinitis factoribus constans:

$$(1+nz)(1+n^2z)(1+n^3z)(1+n^4z) \dots$$

in seriem, hanc secundum potestates ipsius z procedentem

$$1 + \frac{nz}{1-n} + \frac{n^2z^2}{(1-n)(1-n^2)} + \frac{n^3z^3}{(1-n)(1-n^2)(1-n^3)} + \frac{n^4z^4}{(1-n)(1-n^2)(1-n^3)(1-n^4)} + \text{etc.}$$

Atque summae huius seriei logarithmus hyperbolicus ex §. 10 erit

$$= \frac{nz}{1-n} - \frac{n^2z^2}{2(1-n^2)} + \frac{n^3z^3}{3(1-n^3)} - \frac{n^4z^4}{4(1-n^4)} + \text{etc.}$$

Vel si k scribatur pro numero, cuius logar. = 1, erit

k

$$\frac{n^2}{2^{1-n}} - \frac{n^2 z^2}{2(1-n^2)} + \frac{n^3 z^3}{3(1-n^3)} - \frac{n^4 z^4}{4(1-n^4)} + \text{etc.} = R$$

seu ista expressio exponentialis est aequalis summae illius seriei, in quam valorem ipsius R transmutarimus.

§. 26. Verum vt ad propositum problema reuertamur quo definiendum sit, quot variis modis datus numerus m , partiri queat in μ partes inaequales inter se et integras; indicemus hunc modorum numerum, quem quaerimus, huiusmodi scriptione $m^{(\mu)}$

qua nobis perpetuo numerus modorum indicetur, quibus numerus m per additionem produci queat ex μ numeris integris inter se inaequalibus; atque ad hanc partium inaequalitatem denotandam supra litteram i adiunximus; quae omittetur si quaestio formabitur, de numero modorum inveniendo, quibus datus numerus m omnino in μ partes tam aequales quam inaequales distribui queat. Quod problema postea pari facilitate solutum exhibebitur.

§. 27. Iste ergo modorum numerus $m^{(\mu)}$ erit coëficiens potestatis n^m in illa serierum $\alpha, \beta, \gamma, \delta, \epsilon, \text{etc.}$ quae a prima α numerata in ordine est tota quot μ continet unitates. Huius autem seriei summa est =

$$\frac{\mu(\mu+1)}{n^{1-\mu}}$$

$$(1-n)(1-n^2)(1-n^3)(1-n^4) \dots \dots (1-n^\mu)$$

ideoque seriei, quae ex hac forma nascitur terminus generalis est $= m^{(\mu)} n^m$. Seriei autem quae nascitur ex hac forma

$$\frac{\mu(\mu-1)}{n^{1-\mu}}$$

$$(1-n)(1-n^2)(1-n^3)(1-n^4) \dots \dots (1-n^\mu)$$

terminus generalis erit $= m^{(\mu)} n^{m-\mu}$, seu pro eadem ipius n potestate erit terminus generalis $= (m+\mu)^{(\mu)} n^m$. Subtrahatur prior expressio a posteriore, atque residuae expressionis.

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$\frac{\mu(\mu-i)}{n^{i+2}}$

$$(1-n)(1-n^2)(1-n^3)(1-n^4) \dots \dots (1-n^{\mu-1})$$

terminus generalis erit $= n^m((m+\mu)^{(\mu)i} - m^{(\mu)i})$ huius autem eiusdem seriei terminus generalis est $m^{(\mu-i)i} n^m$, quocirca habebimus:

$$m^{(\mu-i)i} = (m+\mu)^{(\mu)i} - m^{(\mu)i}$$

vnde hanc adipiscimur regulam, vt sit

$$(m+\mu)^{(\mu)i} = m^{(\mu)i} + m^{(\mu-i)i}$$

cuius ope, si constiterit, quod varies modis numerus m distribui possit cum in μ partes, tum in $\mu - 1$ partes inaequales, hos binos modorum numeros addendo reperietur, quot variis modis numerus maior $m + \mu$ distribui possit in μ partes inaequales. Atque ita resolutio casum difficiliorum ad simpliciores reducitur, atque tandem ad simplicissimos per se notos; quippe constat, si fuerit $m < \frac{\mu \mu + \mu}{2}$ tum fore $m^{(\mu)i} = 0$, et si fuerit $m = \frac{\mu \mu + \mu}{2}$, tum erit $m^{(\mu)i} = 1$.

§. 28. Cum formula $m^{(\mu)i}$ sit terminus generalis huius expressionis

$$\frac{\mu(\mu-i)}{n^{i+2}}$$

$$(1-n)(1-n^2)(1-n^3) \dots \dots (1-n^{\mu})$$

Videamus qualem seriem praebat ista expressio

$$\frac{1}{(1-n)(1-n^2)(1-n^3) \dots \dots (1-n^{\mu})}$$

si enolurat, atque secundum dimensiones ipsius n disponatur. Ponamus autem prodire hanc seriem

$$1 + an + \frac{b}{2} n^2 + 2 n^3 + n^4 + \dots + n^5 + \text{etc.}$$

ex cuius generatione perspicitur, coefficientem cuiusque potesta-

poteſtatis ipſius n monſtare quoſ variis modiſ exponenſ ipſius n per additionem produci queat ex hiſ datiſ numeriſ

$$1, 2, 3, 4, 5, 6, \dots, \mu$$

hiſque nec certus partium numeriſ praefribitur, ex quibus componatur, nec iſta conditio ponitur, vt partes ſint inter ſe inaequales. Hanc itaque ob cauſam expreſſio $m^{(\mu)}$ ſimul indicabit, quoſ variis omnino modiſ numeriſ $m - \frac{\mu(\mu+1)}{2}$ per additionem prodici queat ex numeriſ $1, 2, 3, 4, 5, \dots, \mu$. Sic fi quaeratur quoſ variis modiſ numeriſ 50 diſtribui poſſit in 7 partes inaequales, propter $m = 50$ et $\mu = 7$, quaefatio eo reducitur, vt inueſtigetur quoſ variis modiſ numeriſ $50 - 28$ ſeu 22 oriri queat per additionem ex hiſ ſeptem numeriſ: $1, 2, 3, 4, 5, 6, 7$. Hoc ergo pacto dupliſ generiſ quaefioneſ vna eademque opera reſol- vuntur.

§. 29. Definitiſ hoc pacto littoris $a, b, c, d,$ etc. pro caſu, quo loco litterarum $a, b, c, d,$ etc. progressionem geometricam n, n^2, n^3, n^4, n^5 , etc. infinitam affumſimus, ordo poſtulat, vt etiam in valoress ter- tii ordinis A, B, C, D, E , etc. inquiramus. Ad- hibuiſimus autem has litteras A, B, C, D, E , etc. in- ſeriębus, hiſ valoribus $\frac{1}{R}$, et $\frac{1}{S}$ aequalibus, ſumſimus enim ſupra §. 11. eſſe

$$\frac{1}{S} = 1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \text{etc.}$$

et

$$\frac{1}{R} = 1 - Az + Bz^2 - Cz^3 + Dz^4 - Ez^5 + \text{etc.}$$

obtinentibus R et S valoress primū affumtoſ quibus erat:

R

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$$R = (1 + nz)(1 + n^2 z)(1 + n^3 z)(1 + n^4 z) \text{ etc.}$$

$$S = (1 - nz)(1 - n^2 z)(1 - n^3 z)(1 - n^4 z) \text{ etc.}$$

Intelligitur autem hinc seriem

$$\frac{1}{1-nz} = 1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.}$$

oriri, si innumerabiles istae progressiones geometricae in se inuicem multiplicentur.

$$\frac{1}{1-nz} = 1 + nz + n^2 z^2 + n^3 z^3 + n^4 z^4 + \text{etc.}$$

$$\frac{1}{1-n^2 z} = 1 + n^2 z + n^4 z^2 + n^6 z^3 + n^8 z^4 + \text{etc.}$$

$$\frac{1}{1-n^3 z} = 1 + n^3 z + n^6 z^2 + n^9 z^3 + n^{12} z^4 + \text{etc.}$$

$$\frac{1}{1-n^4 z} = 1 + n^4 z + n^8 z^2 + n^{12} z^3 + n^{16} z^4 + \text{etc.}$$

etc.

Posito autem $-z$ loco z prodit simili modo series $\frac{1}{1-nz}$

§. 30. Ex ista harum serierum generatione manifestum est esse:

$$\text{I. } \mathfrak{A} = n + n^2 + n^3 + n^4 + n^5 + \text{etc.}$$

quae est progressio geometrica omnes ipsius n potestates complectens singulas per coëfficientem $+1$ multiplicatas.

II. $\mathfrak{B} = n^2 + n^3 + 2n^4 + 2n^5 + 3n^6 + 3n^7 + 4n^8 + 4n^9 + \text{etc.}$
in qua coëfficiens cuiusque ipsius n potestatis tot continet unitates, quot variis modis exponens ipsius n in duas partes siue aequales siue inaequales partiri potest.
Sic potestatis n^3 coëfficiens est 4 quia 8 quatuor modis in 2 partes partitur

$$8 = 1 + 7; \quad 8 = 3 + 5$$

$$8 = 2 + 6; \quad 8 = 4 + 4$$

III. $\mathfrak{C} = n^3 + n^4 + 2n^5 + 3n^6 + 4n^7 + 5n^8 + 7n^9 \text{ etc.}$
in qua cuiusque potestatis ipsius n coëfficiens tot continet unitates, quot variis modis exponens ipsius n in tres

tres partes siue aequales siue inaequales distribui potest, sic n^o coëfficientem habet 7, quia 7 modis 9 in tres partes dispergiri se patitur.

$$\begin{aligned} 9 &= 1 + 1 + 7; & 9 &= 2 + 2 + 5 \\ 9 &= 1 + 2 + 6; & 9 &= 2 + 3 + 4 \\ 9 &= 1 + 3 + 5; & 9 &= 3 + 3 + 3 \\ 9 &= 1 + 4 + 4; \end{aligned}$$

IV. $\mathfrak{D} = n^4 + n^5 + 2n^6 + 3n^7 + 5n^8 + 6n^9 + 9n^{10} + \text{etc.}$
vbi cuiusque potestatis ipsius n coëfficiens tot continet vñitates, quot variis modis exponens ipsius n in quatuor partes siue aequales siue inaequales resolvi potest. Atque similis est ratio sequentium serierum; quae pro litteris \mathfrak{E} , \mathfrak{F} , \mathfrak{G} , etc. reperiuntur.

§. 31. Harum ergo serierum ope alterum problema, quod simul cum praecedente Vir Cl. Naudaeus mihi proposuit, resolvi potest, quot ita se habet.

„Inuenire quot variis modis datus numerus m partiri possit in μ partes tam aequales quam inaequales:
„Siue inueniri quot variis modis datus numerus m
„per additionem μ numerorum integrorum siue aequalium siue inaequalium produci queat

Quod problema a praecedente eo tantum discrepat, quod in praecedente partitio ad partes tantum inter se inaequales sit restricta, haec autem partes quoque aequales admittat. Ad numerum autem omnium modorum in hoc problemate quaesitum signo exprimendum vertamus hac forma:

$$m^{(\mu)}$$

quae scilicet declarat, quot variis modis numerus m partiiri queat in μ partes integras, partium aliquot aequa-

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litate non exclusa: quamobrem in signo supra affixo (μ) ante adnexa littera i , qua inaequalitas partium indicatur, hic est praetermissa.

§. 32. Solutio ergo huius problematis ad formationem serierum A , B , C , D , E , etc. reducitur, at supra iam ostendimus (§. 5) quomodo harum litterarum valores ex valoribus litterarum a , b , c , d , e , etc. iam cognitis definiantur. Quanquam autem iste modus est generalis et ex rei natura petitus, tamen non satis dilucide legem, qua hi valores progrediuntur, ob oculos ponit. Quamobrem valores harum litterarum A , B , C , D , E , etc. via huic casui propria inuestigabo, simili ei, qua supra (§. 21) usus sum.

Quoniam est.

$$\frac{1}{s} = \frac{1}{(1-nz)(1-n^2z)(1-n^3z)(1-n^4z)\dots}$$

perspicuum est, si in hac forma loco z scribatur $n z$, tum prodituram esse hanc formam

$$\frac{1}{(1-n^2z)(1-n^3z)(1-n^4z)(1-n^5z)\dots}$$

Ad ipsam autem hanc formam prior $\frac{1}{s}$ perducitur, si ea multiplicetur per $1 - nz$. Hancobrem cum assumserimus esse

$$s = 1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \dots$$

ponamus in hac $n z$ loco z , habebimusque

$$1 + Anz + Bn^2z^2 + Cn^3z^3 + Dn^4z^4 + \dots$$

Iam priorem seriem $\frac{1}{s}$ multiplicemus per $1 - nz$

$$1 + Az + Bz^2 + Cz^3 + Dz^4 + \dots$$

$$- n z - Anz^2 - Bn^2z^3 - Cn^3z^4 + \dots$$

Quae forma cum illi esse debeat aequalis, erit

$A =$

$$\begin{aligned}\mathfrak{A} &= \frac{n}{1-n} = \frac{n}{1-n} \\ \mathfrak{B} &= \frac{\mathfrak{A} n}{1-n^2} = \frac{n^2}{(1-n)(1-n^2)} \\ \mathfrak{C} &= \frac{\mathfrak{B} n}{1-n^3} = \frac{n^3}{(1-n)(1-n^2)(1-n^3)} \\ \mathfrak{D} &= \frac{\mathfrak{C} n}{1-n^4} = \frac{n^4}{(1-n)(1-n^2)(1-n^3)(1-n^4)}\end{aligned}$$

etc.

§. 33. Hinc igitur noua percipitur relatio inter valores litterarum $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, etc. et litterarum $\alpha, \mathfrak{c}, \gamma, \delta$, etc. quae eo magis est notatu digna quo minus hi valores a se inuicem discrepant Collato enim (§. 21) intelligitur esse :

$$\begin{aligned}\alpha &= \mathfrak{A} \\ \mathfrak{c} &= n \mathfrak{B} \\ \gamma &= n^3 \mathfrak{C} \\ \delta &= n^6 \mathfrak{D} \\ \varepsilon &= n^1 \mathfrak{E}\end{aligned}$$

etc.

Manifestum ergo est ratione coëfficientium series $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, etc. omnino cum seriebus $\alpha, \mathfrak{c}, \gamma, \delta$, etc. congruere, totumque discrimen in exponentibus ipsius n situm esse. In serie quidem \mathfrak{A} , exponentes quoque aequales sunt exponentibus in serie α ; at in serie \mathfrak{B} exponentes unitate deficiunt ab exponentibus seriei \mathfrak{c} : in serie \mathfrak{C} exponentes ternario deficiunt ab exponentibus seriei γ : et ita porro defectus secundum numeros trigonales progrediuntur.

§. 34. Ex seriebus ergo $\alpha, \mathfrak{c}, \gamma, \delta$, etc. quas supra formare docuimus, et quibus prius problema Naudaeanum resoluitur, simul hoc posterius problema a Naudaeo propositum ita resolui potest, vt eius solutio reducatur ad

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solutionem prioris. Erit nempe

$$m^{(1)} = m^{(1)i}$$

$$m^{(2)} = (m+1)^{(2)i}$$

$$m^{(3)} = (m+2)^{(3)i}$$

$$m^{(4)} = (m+3)^{(4)i}$$

et generaliter

$$m^{(\mu)} = \left(m + \frac{\mu(\mu-1)}{2}\right)^{(\mu)i}$$

et vicissim

$$m^{(\mu)i} = \left(m - \frac{\mu(\mu-1)}{2}\right)^{(\mu)}$$

Quoniam autem porro inuenimus esse :

$$(m+\mu)^{(\mu)i} = m^{(\mu)i} + m^{(\mu-1)i}$$

erit reductione ad casum praesentem facta :

$$\left(m - \frac{\mu(\mu-1)}{2}\right)^{(\mu)} = \left(m - \frac{\mu(\mu-1)}{2}\right)^{(\mu)} + \left(m - \frac{\mu(\mu-1)(\mu-2)}{2}\right)^{(\mu-1)}$$

seu commodius

$$m^{(\mu)} = (m-\mu)^{(\mu)} + (m-1)^{(\mu-1)}$$

ex qua proprietate etiam facile series litterarum **A**, **B**, **C**, etc. formabuntur, sicque hoc alterum problema resoluetur.

§. 35. Ad exemplum huius problematis quaestioneum Vir Clar. affert, ut determinetur, quot variis modis numerus 50 in septem omnino partes siue aequales siue inaequales dispertiri queat. Haec ergo quaestio ad prius problema reducetur, ob $m=50$ et $\mu=7$, si quaeratur quot variis modis numerus $50+21$, seu numerus 71, in septem partes inaequales partiri queat. Vtrumque autem fieri posse 8946 modis diuersis. Praeterea vero hic idem numerus 8946 indicat (§. 28), quot variis modis $71-28=43$ per additionem produci queat ex his numeris 1, 2, 3, 4, 5, 6, 7. Atque generaliter numerus modorum $m^{(\mu)}$, quibus numerus m in μ partes siue aequales siue in-

inaequales resoluitur, simul ostendit, quot variis modis numerus $m-\mu$ produci queat per additionem ex his numeris definitis

$$1, 2, 3, 4, 5, \dots \mu.$$

§. 36. Finem huic dissertationi faciat obseruatio notatu digna, quam quidem rigore geometrico demonstrare mihi nondum licuit. Obseruaui scilicet hoc infinitorum factorum productum

$$(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5) \text{ etc.}$$

si per multiplicationem actu euoluatur, praebere hanc seriem:
 $1-n-n^2+n^3+n^7-n^{12}-n^{15}+n^{22}+n^{26}-n^{35}-n^{40}+n^{51}+$ etc.
 vbi eae tantum ipsius n potestates occurrent, quarum exponentes continentur hac forma: $\frac{xxx+x}{2}$. Ac si x sit numerus impar, potestates ipsius n , quae sunt $\frac{xxx+x}{2}$ coëfficientem habent -1; si autem x sit numerus par, tum potestates $\frac{xxx+x}{2}$ coëfficientem habent +1.

§. 37. Praeterea notari meretur series huius reciproca, quae oritur ex euolutione huius fractionis

$$(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5) \text{ etc.}$$

prodibit scilicet ista series recurrens:

$$1+1n+2n^2+3n^3+5n^4+7n^5+11n^6+15n^7+22n^8+\text{etc.}$$

quippe quae per seriem superiorem

$$1-n-n^2+n^5+n^7-n^{12}-n^{15}+n^{22}+n^{26}-\text{etc.}$$

multiplicata producit unitatem. In illa autem serie coëficiens cuiusque potestatis ipsius n tot continet unitates, quot variis modis exponens ipsius n in partes dispertiri potest; sic 5 septem modis in partes resolui potest, vti

$$\begin{array}{c|c|c} \frac{s}{s} \equiv \frac{s}{s} & \frac{s}{s} \equiv \frac{s}{s} & \frac{s}{s} \equiv \frac{s}{s} \\ \frac{s}{s} \equiv \frac{s}{s} + 1 & \frac{s}{s} \equiv \frac{s}{s} + 1 & \frac{s}{s} \equiv \frac{s}{s} + 1 \\ \hline & s \equiv 1 + 1 + 1 + 1 + 1 & \end{array}$$

nec numerus scilicet partium hic praescribitur nec inaequalitas.