METHODVS INTEGRANDI
FORMVLAS DIFFERENTIALES RATIONALES VNICAM VARIABILEM INVOLVENTES.

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§ 1.

Omnes formulae differentiales, quarum integrationem hic sum traditurus, continentur in hac forma generali Xdx, vbi X denotat functionem quamcunque rationalem ipsius x. Cum igitur omnis functio rationalis sit vel integra vel fracta, tractatio nostra esset bipartita constituenda, nisi integratio illis casibus, quibus X est functio integra, nulla laboraret difficultate. Si enim X huiusmodi est functio, denominatore, qui quidem variabilem x complectatur, destituta, semper ad hanc formam reuocabitur, vt sit:

$$X = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 \text{ etc.}$$

hocque casu erit

$$\int X dx = \Delta + Ax + \tfrac{1}{2}Bx^2 + \tfrac{1}{3}Cx^3 + \tfrac{1}{4}Dx^4 + \tfrac{1}{5}Ex^5 + \text{ etc.}$$

vbi Δ constantem quamcunque denotat. Latius autem ratio huius integrationis patet, atque ad exponentes ipsius x non solum integros affirmatiuos sed etiam negatiuos et fractos extenditur. Ita si m, n, p, q etc. exponant numeros quoscunque siue integros siue fractos, siue positiuos siue negatiuos, fueritque

$$X = Ax^m + Bx^n + Cx^p + Dx^q \text{ etc.}$$

erit, vti sponte patet

$$\int X dx = \frac{1}{m+1} Ax^{m+1} + \frac{1}{n+1} Bx^{n+1} + \frac{1}{p+1} Cx^{p+1} + \frac{1}{q+1} Dx^{q+1} \text{ etc.}$$

Quae cum sint iam fere triuialia, huic priori generi, quo X est functio ipsius x integra, amplius non immoror, sed ad functiones, quae forma exprimuntur fracta, progredior.

§. 2. Sit igitur X functio quaecunque fracta ipsius x, numeratore ac denominatore contenta, atque semper in huiusmodi forma latissime patente continebitur:

$$X = \frac{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{ etc.}}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 + \text{etc.}}$$

De qua primum obserua, si x in numeratore tot vel plures habeat dimensiones quam in denominatore, formulam ad aliam reuocari posse, in qua summa ipsius x dimensio in numeratore minor sit quam in denominatore: quae reducta vti constat diuisione absoluitur, si enim sit

$$X = \frac{A + Bx + Cx^2 + Dx^3 + Ex^4}{\alpha + \beta x + \gamma x^2 + \delta x^3} \text{ fiet}$$

$$X = \frac{E}{\delta}x + \frac{A + (B - \frac{E\alpha}{\delta})x + (C - \frac{E\beta}{\delta})x^2 + (D - \frac{E\gamma}{\delta})x}{\alpha + \beta x + \gamma x^2 + \delta x^3}$$

atque vlterius resoluendo

$$X = \frac{E}{\delta}x(\frac{D}{\delta} - \frac{E\gamma}{\delta\delta}) + A - \frac{D\alpha}{\delta} + \frac{E\alpha\gamma}{\delta\delta}) + (B - \frac{E\alpha}{\delta} - \frac{D\beta}{\delta} + \frac{E\beta\gamma}{\delta\delta})x$$
$$\frac{+ (C - \frac{E\beta}{\delta} - \frac{D\gamma}{\delta} + \frac{E\gamma\gamma}{\delta\delta})xx}{\alpha + \beta x + \gamma x^2 + \delta x^3}$$

Cum iam prioris partis $\frac{E}{\delta}x + \frac{D}{\delta} - \frac{E\gamma}{\delta\delta}$, si per dx multiplitiplicetur, integratio ſit obuia, tota difficultas ad integrationem partis poſterioris, quae eſt vera forma fracta, reducitur. Ideoque cardo rei verſatur in integratione huiusmodi formae $X\,dx$, ſi fuerit:

$$X = \frac{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 + \text{etc.}}$$

vbi quidem ſumma poteſtas ipſius x in numeratore minor ſit, quam ſumma poteſtas ipſius x in denominatore. Ex quo regulas ſequentes tantum ad huiusmodi formulas ſum relaturus.

§. 3. Si in denominatore terminus primus α vel aliquot termini initiales deſint ſeu euaneſcant, integratio multum leuari atque ad caſum faciliorem, quem poſtmodum tractabimus, reduci poteſt. Reductio autem in hoc conſtat, quod denominatur tum habeat vnum factorem cognitum, qui erit vel x, vel x^2, vel x^3, etc. prout vnus pluresue termini initiales denominatoris euaneſcant, ideoque poterit fractio propoſita in duas alias fractiones reſolvi, quarum altera, cum habeat poteſtatem ipſius x ſimplicem pro denominatore, nullo negotio integratur, ita vt tantum altera remaneat, cuius integrale quaeratur. Sic ſi primus tantum terminus denominatoris deſit, erit formula differentialis propoſita huiusmodi.

$$\frac{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}}{x(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.})}\,dx$$

quae aequalis eſt his duabus iunctim ſumtis

$$\frac{A}{\alpha x}\,dx + \frac{(B - \frac{A\beta}{\alpha}) + (C - \frac{A\gamma}{\alpha})x + (D - \frac{A\delta}{\alpha})x^2 + (E - \frac{A\varepsilon}{\alpha})x^3 \text{ etc.}}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.}}\,dx$$

 quarum

quarum prioris partis $\frac{A\,dx}{\alpha x}$ integrale est $\frac{A}{\alpha}\, l\, x$. posterioris vero partis integrale methodo deinceps tradenda reperiri debet.

§. 4. Si bini termini initiales denominatoris euanescant, formula differentialis erit huiusmodi.

$$\frac{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}}{x^2(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.}} \cdot dx$$

quae vt resoluatur in duas fractiones factores hos denominatoris pro denominatoribus habentes, ponatur ea aequalis his duabus fractionibus:

$$\frac{\mathfrak{a} + \mathfrak{b}x}{x^2}\, dx + \frac{\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \text{etc.}}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.}}\, dx$$

Addantur hae fractiones more consueto, ac denominator summae quidem sponte aequalis fiet denominatori fractionis propositae; numerator autem erit:

$$\left.\begin{array}{l} \mathfrak{a}\alpha + \mathfrak{a}\beta x + \mathfrak{a}\gamma x^2 + \mathfrak{a}\delta x^3 + \mathfrak{a}\varepsilon x^4 + \text{etc.} \\ \quad + \mathfrak{b}\alpha x + \mathfrak{b}\beta x^2 + \mathfrak{b}\gamma x^3 + \mathfrak{b}\delta x^4 + \text{etc.} \\ \qquad\quad + \mathfrak{A}x^2 + \mathfrak{B}x^3 + \mathfrak{C}x^4 + \text{etc.} \end{array}\right\} dx$$

qui vt numeratori proposito aequalis fiat, termini singuli homologi aequentur vnde elicietur.

$$\begin{aligned} \mathfrak{a} &= \tfrac{A}{\alpha} \\ \mathfrak{b} &= \tfrac{B}{\alpha} - \tfrac{\mathfrak{a}\beta}{\alpha} = \tfrac{B}{\alpha} - \tfrac{A\beta}{\alpha^2} \\ \mathfrak{A} &= C - \tfrac{A\gamma}{\alpha} - \tfrac{B\beta}{\alpha} + \tfrac{A\beta^2}{\alpha^2} \\ \mathfrak{B} &= D - \tfrac{A\delta}{\alpha} - \tfrac{B\gamma}{\alpha} + \tfrac{A\beta\gamma}{\alpha^2} \\ \mathfrak{C} &= E - \tfrac{A\varepsilon}{\alpha} - \tfrac{B\delta}{\alpha} + \tfrac{A\beta\delta}{\alpha^2} \\ &\text{etc.} \end{aligned}$$

Hi

His coefficientibus initio assumtis determinatis innotescent binae fractiones simpliciores, in quas proposita resoluitur; ac prioris quidem $\frac{\mathfrak{a}+\mathfrak{b}x}{xx}dx$ integrale est $= -\frac{\mathfrak{a}}{x} + \mathfrak{b}lx$; ita vt integratio formulae propositae iam ad integrationem partis posterioris reducatur. Ceterum ex ipsa terminorum comparatione intelligitur non opus fuisse, vt numeratori fractionis prioris $\mathfrak{a}+\mathfrak{b}x$ plures quam duos terminos tribueremus; cum litterarum determinandarum numerus hoc modo cum numero aequationum congruat: vnicus autem terminus $\mathfrak{a}$ ad hoc non suffecisset, eo quod si posuissemus $\mathfrak{b}=0$, secundae aequationi $B=\mathfrak{a}\beta$ ob $\mathfrak{a}$ iam determinatum satisfieri non potuisset.

§. 5. Ponamus iam tres terminos initiales denominatoris formulae initio propositae abesse; ac formula differentialis integranda erit huiusmodi

$$\frac{A+Bx+Cx^2+Dx^3+Ex^4+\text{etc.}}{x^3(\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4+\text{etc.})}\cdot dx$$

quae in duas fractiones huius formae resolui poterit:

$$\frac{\mathfrak{a}+\mathfrak{b}x+\mathfrak{c}x^2}{x^3}dx+\frac{\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3+\text{etc.}}{\alpha+\beta x+\gamma x^2+\delta x^3+\text{etc.}}dx$$

quarum summae denominator cum denominatore proposito congruit, numerator vero erit

$$\begin{array}{l}\mathfrak{a}\alpha+\mathfrak{a}\beta x+\mathfrak{a}\gamma x^2+\mathfrak{a}\delta x^3+\mathfrak{a}\varepsilon x^4+\text{etc.}\\ \quad+\mathfrak{b}\alpha x+\mathfrak{b}\beta x^2+\mathfrak{b}\gamma x^3+\mathfrak{b}\delta x^4+\text{etc.}\\ \qquad\quad+\mathfrak{c}\alpha x^2+\mathfrak{c}\beta x^3+\mathfrak{c}\gamma x^4+\text{etc.}\\ \qquad\qquad\quad+\mathfrak{A}x^3+\mathfrak{B}x^4+\text{etc.}\end{array}$$

qui vt aequalis reddatur numeratori proposito debebit esse

$\mathfrak{a} =$

$$\mathfrak{a} = \frac{A}{\alpha}$$
$$\mathfrak{b} = \frac{B}{\alpha} - \frac{\mathfrak{a}\beta}{\alpha} = \frac{B}{\alpha} - \frac{A\beta}{\alpha^2}$$
$$\mathfrak{c} = \frac{C}{\alpha} - \frac{\mathfrak{a}\gamma}{\alpha} - \frac{\mathfrak{b}\beta}{\alpha} = \frac{C}{\alpha} - \frac{A\gamma}{\alpha^2} - \frac{B\beta}{\alpha^2} + \frac{A\beta^2}{\alpha^3}$$

atque

$$\mathfrak{A} = D - \frac{A\delta}{\alpha} - \frac{B\gamma}{\alpha} + \frac{2A\beta\gamma}{\alpha^2} - \frac{C\mathfrak{b}}{\alpha} + \frac{B\beta^2}{\alpha^2} - \frac{A\beta^3}{\alpha^3}$$
$$\mathfrak{B} = E - \frac{A\varepsilon}{\alpha} - \frac{B\delta}{\alpha} + \frac{A\beta\delta}{\alpha^2} - \frac{C\gamma}{\alpha} + \frac{A\gamma\gamma}{\alpha^2} + \frac{B\beta\gamma}{\alpha^2} - \frac{A\beta^2\gamma}{\alpha^3}$$

etc.

Apparet igitur nec plures nec pauciores terminos pro numeratore fractionis prioris accipi oportuiſſe, quam tres; atque ex natura rei generaliter intelligitur pro numeratore prioris fractionis tot terminos aſſumi debere, quoad perveniatur ad exponentem ipſius x vnitate minorem, quam continet exponens denominatoris, ſeu quod eodem redit, numerator tot terminos habere debet, quot vnitates continet exponens denominatoris.

§. 6. Ex his iam ſatis patet modus, quemadmodum fractio differentialis, cuius denominator factorem habeat, qui ſit poteſtas ipſius x, cuiusmodi eſt

$$\frac{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc}}{x^n(\alpha + \beta x + \gamma x^2 + \delta x^3 + \text{etc.})}\, dx$$

reſolui debeat in binas alias fractiones, quarum denominatores ſint hi bini factores ſeorſim ſumti. Scilicet ea transmutabitur in huiusmodi binas formulas:

$$\frac{\mathfrak{a} + \mathfrak{b}x + \mathfrak{c}x^2 + \mathfrak{d}x^3 \ldots\ldots + \mathfrak{n}x^{n-1}}{x^n}\, dx$$
$$+ \frac{\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \text{etc.}}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.}}\, dx$$

habe-

habebuntque coefficientes assumti hos valores:

$$\mathfrak{a} = \frac{A}{\alpha}$$
$$\mathfrak{b} = \frac{B}{\alpha} - \frac{\mathfrak{a}\beta}{\alpha}$$
$$\mathfrak{c} = \frac{C}{\alpha} - \frac{\mathfrak{a}\gamma}{\alpha} - \frac{\mathfrak{b}\beta}{\alpha}$$
$$\mathfrak{d} = \frac{D}{\alpha} - \frac{\mathfrak{a}\delta}{\alpha} - \frac{\mathfrak{b}\gamma}{\alpha} - \frac{\mathfrak{c}\beta}{\alpha}$$
etc.

$$\mathfrak{A} = P - \mathfrak{n}\beta - \mathfrak{m}\gamma - \mathfrak{l}\delta - \mathfrak{k}\varepsilon - \text{etc.}$$
$$\mathfrak{B} = Q - \mathfrak{n}\gamma - \mathfrak{m}\delta - \mathfrak{l}\varepsilon - \mathfrak{k}\zeta - \text{etc.}$$
$$\mathfrak{C} = R - \mathfrak{n}\delta - \mathfrak{m}\varepsilon - \mathfrak{l}\zeta - \mathfrak{k}\eta - \text{etc.}$$
etc.

vbi in numeratore proposito $A + Bx + Cx^2 +$ etc. denotat P coefficientem potestatis x^n, et Q coefficientem potestatis x^{n+1}, et R coefficientem potestatis x^{n+2}, et ita porro. Hac ergo facta resolutione prioris fractionis integrale est in promtu per §. 2, ita vt ad plenam integrationem supersit modus integrandi fractionem posteriorem. Hanc obrem tota difficultas huc redit, vt modus tradatur integrandi huiusmodi formulam

$$\frac{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.}} \cdot dx$$

in cuius denominatore primus terminus α non sit $= 0$.

§. 7. Casus simplicissimus, qui in hac forma continetur, erit, si in numeratore omnes termini, praeter primum, in denominatore vero omnes praeter duos primos euanescant, ita vt haec habeatur formula integranda:

$$\frac{A}{\alpha + \mathfrak{b}x} dx.$$

Ponatur ea $= dy$; vt ſit $dy = \frac{A\,dx}{\alpha + \mathfrak{b}x}$ erit $\frac{\mathfrak{b}dy}{A} = \frac{\mathfrak{b}dx}{\alpha + \mathfrak{b}x}$ quod cum ſit differentiale ipſius $l(\alpha + \mathfrak{b}x)$ erit $\frac{\mathfrak{b}y}{A} = l(\alpha + \mathfrak{b}x)$, ideoque integrale quaeſitum

$$\int \frac{A}{\alpha + \mathfrak{b}x}\,dx = y = \frac{A}{\mathfrak{b}}\,l(\alpha + \mathfrak{b}x)$$

ſeu adiiciendo conſtantem $\int \frac{A}{\alpha + \mathfrak{b}x} = \frac{A}{\mathfrak{b}}\,l\,\frac{\alpha + \mathfrak{b}x}{\alpha}$. ſimili modo ſi numerator totus per dx multiplicatus ſit differentiale denominatoris, integratio facile per logarithmos expeditur. Si enim formula integranda ſit:

$$\frac{\mathfrak{b} + 2\gamma x + 3\delta x^2 + 4\varepsilon x^3}{\alpha + \mathfrak{b}x + \gamma x^2 + \delta x^3 + \varepsilon x^4}\,dx$$

integrale erit logarithmus denominatoris, ſcilicet $l(\alpha + \mathfrak{b}x + \gamma x^2 + \delta x^3 + \varepsilon x^4)$. Quodſi autem formula differentialis ſit huiusmodi, vt numerator in dx ductus ſit multiplum quodpiam denominatoris, nempe:

$$dy = \frac{n\mathfrak{b} + 2n\gamma x + 3n\delta x^2 + 4n\varepsilon x^3}{\alpha + \mathfrak{b}x + \gamma x^2 + \delta x^3 + \varepsilon x^4}\,dx$$

erit pariter per logarithmos integrale quaeſitum

$$y = n\,l(\alpha + \mathfrak{b}x + \gamma x^2 + \delta x^3 + \varepsilon x^4).$$

§. 8. Deinde etiam alius caſus eſt obuius, ſi denominator ſit quaepiam poteſtas, ac numerator in dx ductus ſit differentiale radicis denominatoris, vel eius multiplum veluti ſi fuerit

$$dy = \frac{n\mathfrak{b} + 2n\gamma x + 3n\delta x^2 + 4n\varepsilon x^3}{(\alpha + \mathfrak{b}x + \gamma x^2 + \delta x^3 + \varepsilon x^4)^m}\,dx$$

Ponatur breuitatis ergo denominatoris radix

$$\alpha + \mathfrak{b}x + \gamma x^2 + \delta x^3 + \varepsilon x^4 = z$$

erit $(\mathfrak{b} + 2\gamma x + 3\delta x^2 + 4\varepsilon x^3)dx = dz$

hincque

hincque habebitur $dy = \frac{n\,dz}{z^m}$, cuius integrale erit $y = -\frac{n}{(m-1)z^{m-1}}$ atque valore ipsius z restituto prodibit integrale quaesitum

$$y = -\frac{n}{(m-1)(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4)^{m-1}}$$

cui insuper pro arbitrio quantitatem constantem adiicere licet. Praeterea vero etiam vsu venire potest, vt integrale sit quantitas algebraica, etiamsi numerator non sit ita comparatus, vti in hoc casu assumsimus. Omnes autem hi casus vna formula comprehendi poterunt, si in genere huiusmodi functio

$$\frac{\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \text{etc.}}{(\alpha + \mathfrak{b}x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.})^{m-1}}$$

differentietur; cum enim differentiale huiusmodi habiturum sit formam:

$$\frac{A + Bx + Cx^2 + Dx^3 + \text{etc.}}{(\alpha + \mathfrak{b}x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.})^m}\,dx$$

huius formulae vicissim integrale habebitur.

§. 9. His casibus exceptis, nulla alia via ad huiusmodi formulas differentiales fractas integrandas patet, nisi vt denominator $\alpha + \mathfrak{b}x + \gamma x^2 +$ etc. in suos factores simplices resoluatur, vbi quidem, cum non sit $\alpha = 0$, pro α vnitas scribi potest, quod opus autem saepenumero maximis difficultatibus est obnoxium, quas tollere huius non est loci. Quanquam enim radicum inuestigatio, cum qua resolutio in factores congruit, adhuc non vltra aequationes quatuor dimensionum generatim est perducta, tamen

 in

in integrationum negotio merito nobis resolutionem aequationum quotcunque dimensionum concedi postulamus. Atque is formulae seu aequationis differentialis integrationem perfecte dedisse censendus est, qui eam ad resolutionem seu constructionem aequationis algebraicae reuocauerit. Quamobrem assumamus denominatoris propositi:

$$1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \text{etc.}$$

factores simplices, in quibus x vnicam habeat dimensionem, esse hos:

$$(1 + px)(1 + qx)(1 + rx)(1 + sx)(1 + tx) \text{ etc.}$$

quorum factorum numerus, vti constat, aequalis est maximae dimensioni ipsius x, quam habet in denominatore proposito. Praeterea vero manifestum est coefficientes p, q, r, s, etc. cum coefficientibus cognitis α, β, γ, δ, etc. ita esse connexos, vt sit

$\alpha = p + q + r + s +$ etc. $=$ summae singulorum
$\beta = pq + pr + ps + qr +$ etc. $=$ sum. factor. ex binis
$\gamma = pqr + pqs + qrs +$ etc. $=$ sum. factor. ex ternis
$\delta = pqrs + pqrt +$ etc. $=$ sum. factor. ex quat.
$\varepsilon = pqrst +$ etc. $=$ sum. factor. ex quinis

et ita porro.

Quamobrem has quantitates p, q, r, s etc. tanquam datas ac determinatas per cognitas α, β, γ, δ, etc. iure accipere licet.

§. 10. Hac posita denominatoris in factores resolutione, formula differentialis

$$\frac{A + Bx + Cx^2 + Dx^3 + \text{etc.}}{1 + \alpha x + \beta x^2 + \gamma x^3 + \text{etc.}}\, dx$$

abibit in hanc formam:

A +

$$\frac{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}}{(1+px)(1+qx)(1+rx)(1+sx)\ \text{etc.}} \cdot dx$$

quae porro resolui poterit in fractiones totidem simplices, quot denominator continet factores, sintque haec fractiones simplices differentiales hae:

$$\frac{Pdx}{1+px} + \frac{Qdx}{1+qx} + \frac{Rdx}{1+rx} + \frac{Sdx}{1+sx} + \text{etc.}$$

Patet enim his fractionibus addendis expressionem esse prodituram superiori similem; namque denominator summae sponte aequalis fiet denominatori oblato: numerator quidem illi non congruens orietur, verum tamen tot non prodibunt dimensiones ipsius x in numeratore quam in denominatore; quamobrem litterae adhuc ignotae P, Q, R, S, etc. ita determinari poterunt vt numerator ipsi proposito congruat. Tot enim sunt litterae P, Q, R, etc. quot x habet dimensiones in denominatore, totidem vero numerator continet terminos, ita vt haec operatio sufficiat ad omnes literas P, Q, R, etc. determinandas. Ex hocque patet ratio, cur x in numeratore pauciores habere debeat quam in denominatore; si enim totidem haberet vel plures, litterae assumtae P, Q, R, etc. non sufficerent ad numeratorem propositum producendum.

§. 11. Si ad valores litterarum P, Q, R, S etc. inueniendos, omnes fractiones simplices actu addere, ac numeratorem resultantem cum numeratore proposito congruentem reddere velimus, poterimus quidem valores illarum litterarum P, Q, R, etc. omnium assignare, verum si numerus fractionum simplicium fuerit modicus tantum, labor fere fit insuperabilis. Eaedem autem aequationes multo facilius eruentur; si singulae fractiones simplices reiicien-

reiiciendo dx per diuiſionem in ſeries conuertantur, quo facto prodibit ſumma omnium per ſeriem expreſſa:

$$+P-Ppx+Pp^2x^2-Pp^3x^3+Pp^4x^4-Pp^5x^5+\text{etc.}$$
$$+Q-Qqx+Qq^2x^2-Qq^3x^3+Qq^4x^4-Qq^5x^5+\text{etc.}$$
$$+R-Rrx+Rr^2x^2-Rr^3x^3+Rr^4x^4-Rr^5x^5+\text{etc.}$$
$$+S-Ssx+Ss^2x^2-Ss^3x^3+Ss^4x^4-Ss^5x^5+\text{etc.}$$
$$\text{etc.}$$

Quarum ſumma cum aequalis eſſe debeat fractioni propoſitae reiecto pariter factore differentiali dx;

$$\frac{A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\text{etc.}}{1+\alpha x+\beta x^2+\gamma x^3+\delta x^4+\varepsilon x^5+\text{etc.}}$$

conuertatur haec pariter per diuiſionem in ſeriem infinitam, quae erit ſeries recurrens;

$$\begin{array}{llll}
A & -A\alpha x & +A\alpha^2x^2 & -A\alpha^3x^3+ \\
 & +Bx & -B\alpha x^2 & +B\alpha^2x^3- \\
 & & -A\beta x^2 & +2A\alpha\beta x^3- \\
 & & +Cx^2 & -B\beta x^3+ \qquad \text{etc.} \\
 & & & -A\gamma x^3+ \\
 & & & -C\alpha x^3+ \\
 & & & +Dx^3-
\end{array}$$

Quod ſi iam termini homologi inter ſe comparentur, atque inter ſe aequales reddantur, obtinebuntur ſequentes aequationes:

$$P+Q+R+S+\text{etc.}=A$$
$$Pp+Qq+Rr+Ss+\text{etc.}=A\alpha-B$$
$$Pp^2+Qq^2+Rr^2+Ss^2+\text{etc.}=A(\alpha^2-\beta)-B\alpha+C$$
$$Pp^3+Qq^3+Rr^3+Ss^3+\text{etc.}=A(\alpha^3-2\alpha\beta+\gamma)-B(\alpha^2\beta)+C\alpha-D$$
$$\text{etc.}$$

Harum

Harum aequationum, quarum numerus quidem eſt infinitus, capiantur tot, quot habentur litterae determinandae P, Q, R, etc. ex iisque earum valores more conſueto definiantur.

§. 12. Calculus hac methodo inſtituendus fit autem admodum prolixus, ſi plures habeantur litterae determinandae: attamen ſi a ſimplicioribus ad magis compoſita progrediamur, per inductionem certam non difficulter deprehendetur, valores quaeſitos ſequenti modo expreſſum iri, vt ſit:

$$P = \frac{A - \frac{1}{p}B + \frac{1}{p^2}C - \frac{1}{p^3}D + \text{ etc.}}{(1-\frac{q}{p})(1-\frac{r}{p})(1-\frac{s}{p})(1-\frac{t}{p}) \text{ etc.}}$$

$$Q = \frac{A - \frac{1}{q}B + \frac{1}{q^2}C - \frac{1}{q^3}D + \text{ etc.}}{(1-\frac{p}{q})(1-\frac{r}{q})(1-\frac{s}{q})(1-\frac{t}{q}) \text{ etc.}}$$

$$R = \frac{A - \frac{1}{r}B + \frac{1}{r^2}C - \frac{1}{r^3}D + \text{ etc.}}{(1-\frac{p}{r})(1-\frac{q}{r})(1-\frac{s}{r})(1-\frac{t}{r}) \text{ etc.}}$$

$$S = \frac{A - \frac{1}{s}B + \frac{1}{s^2}C - \frac{1}{s^3}D + \text{ etc.}}{(1-\frac{p}{s})(1-\frac{q}{s})(1-\frac{r}{s})(1-\frac{t}{s}) \text{ etc.}}$$

etc.

Inductio haec, qua valores litterarum P, Q, R, S etc. eruimus, etſi eſt certiſſima, tamen non ſine ingenti moleſtia, atque loco α, β, γ, etc. ſuos valores per p, q, r, s etc. expreſſos ſubſtituendo reperitur, quare cum non cuique liceat hunc calculum repetere alium modum faciliorem idem efficiendi proponamus, cuius ſimul in ſequentibus amplior ſit vſus.

§. 13. In hac methodo tantum ad vnicum factorem $1 + px$ tanquam cognitum respicimus, atque sine respectu ad reliquos factores simplices determinabimus valorem litterae P pro fractione simplici vna $\frac{P\,dx}{1+px}$. Pari deinceps ratione, qua vna fractio simplex est inuenta reperientur reliquae omnes $\frac{Q\,dx}{1+qx}$, $\frac{R\,dx}{1+rx}$ etc. quarum omnium summa aequetur formulae differentiali propositae. Discerpamus igitur formulam differentialem propositam

$$\frac{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}}{1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \text{etc.}}\, dx$$

in cuius numeratore pauciores inesse ponimus dimensiones ipsius x quam in denominatore; discerpamus inquam hanc formulam in binas partes quarum altera sit $= \frac{P\,dx}{1+px}$, alterius vero denominator erit quotus qui resultat, si ille denominator formulae propositae per $1 + px$ diuidatur, id quod vtique fieri potest, cum $1 + px$ sit factor illius denominatoris. Ponamus quotum ex hac diuisione oriundum esse $1 + \mathfrak{a}x + \mathfrak{b}x^2 + \mathfrak{c}x^3 + \mathfrak{d}x^4 +$ etc. in quo ergo maximus dimensionum numerus ipsius x vnitate deficit ab illo, quem habet in denominatore primo

$$1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \text{etc:}$$

Sit igitur altera pars praeter $\frac{P\,dx}{1+px}$, in quam formula proposita resoluitur haec

$$\frac{\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \text{etc.}}{1 + \mathfrak{a}x + \mathfrak{b}x^2 + \mathfrak{c}x^3 + \mathfrak{d}x^4 + \text{etc.}}\, dx$$

vbi ob eandem rationem in numeratore x pauciores habere debet dimensiones quam in denominatore.

§. 14.

§. 14. Cum igitur ſumma harum duarum formularum

$$\frac{P\,dx}{1+px}+\frac{\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3+\text{etc.}}{1+\mathfrak{a}x+\mathfrak{b}x^2+\mathfrak{c}x^3+\mathfrak{d}x^4\ \text{etc.}}\,dx$$

aequalis eſſe debeat formulae propoſitae

$$\frac{A+Bx+Cx^2+Dx^3+Ex^4+\text{etc.}}{1+\alpha x+\beta x^2+\gamma x^3+\delta x^4+\varepsilon x^5+\text{etc.}}\,dx$$

primum ob denominatores aequales habebitur

$$\begin{array}{ll} \alpha=\mathfrak{a}+p & \mathfrak{a}=\alpha-p \\ \beta=\mathfrak{b}+\mathfrak{a}p & \mathfrak{b}=\beta-\alpha p+p^2 \\ & \text{etc.} \\ \gamma=\mathfrak{c}+\mathfrak{b}p & \mathfrak{c}=\gamma-\beta p+\alpha p^2-p^3 \\ \delta=\mathfrak{d}+\mathfrak{c}p & \mathfrak{d}=\delta-\gamma p+\beta p^2-\alpha p^3+p^4 \\ \text{etc.} & \text{etc.} \end{array}$$

Vel cum formulae $1+\mathfrak{a}x+\mathfrak{b}x^2+\mathfrak{c}x^3+\mathfrak{d}x^4+$ etc. factores ſint $(1+qx)(1+rx)(1+sx)$ etc. fiet $\mathfrak{a}$ quantitatum p, r, s, etc. ſumma, $\mathfrak{b}$ ſumma factorum ex binis, $\mathfrak{c}$ ſumma factorum ex ternis, $\mathfrak{d}$ ex quaternis et ita porro. Quamobrem valores litterarum $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, etc. duplici modo cognoſcuntur, primo ſcilicet ex coefficientibus α, β, γ, δ, etc. ac deinde etiam ex factoribus $(1+px)(1+qx)(1+rx)$ etc. in quos denominator $1+\alpha x+\beta x^2+\gamma x^3+$ etc. reſoluitur. Quodſi ergo praeter primum factorem $1+px$ quem hic ſolum contemplamur, alii reliqui fuerint incogniti, priori modo, quo litteras $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, etc. determinauimus, vtendum erit.

§. 15. Cum iam per additionem more ſolito abſoluendam denominator ſummae congruens prodeat cum denominatore formulae propositae, superest vt numeratores identicos reddamus. Fiet itaque

$$\begin{array}{l|l}
A = \mathfrak{A} + P & \mathfrak{A} = A - P \\
B = \mathfrak{A}p + \mathfrak{B} + \mathfrak{a}P & \mathfrak{B} = B - Ap + P(p - \mathfrak{a}) \\
C = \mathfrak{B}p + \mathfrak{C} + \mathfrak{b}P & \mathfrak{C} = C - Bp + Ap^2 - P(p^2 - \mathfrak{a}p + \mathfrak{b}) \\
D = \mathfrak{C}p + \mathfrak{D} + \mathfrak{c}P & \mathfrak{D} = D - Cp + Bp^2 - Ap^3 + P(p^3 - \mathfrak{a}p^2 + \mathfrak{b}p - \mathfrak{c}) \\
E = \mathfrak{D}p + \mathfrak{E} + \mathfrak{d}P & \mathfrak{E} = E - Dp + Cp^2 - Bp^3 + Ap^4 \\
 & \qquad - P(p^4 - \mathfrak{a}p^3 + \mathfrak{b}p^2 - \mathfrak{c}p + \mathfrak{d}) \\
\text{etc.} & \text{etc.}
\end{array}$$

Quoniam vero termini numeratoris $\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 +$ etc. non in infinitum progrediuntur, sed ibi terminantur vbi exponens ipsius x est vnitate minor, quam maximus exponens in denominatore, in litteris $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. tandem peruenietur ad euanescentem, postquam sequentes omnes euanescent; ac tum valorem ipsius P definire licebit. Quo igitur valor ipsius P generatim determinetur, ponamus successiue numeratorem $\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 +$ etc. nullo tum vnico, post duobus, tribus, quatuor etc. terminis tantum constare, eritque si

$$\begin{array}{l}
\mathfrak{A} = 0;\ P = A \\
\mathfrak{B} = 0;\ P = \frac{Ap - B}{p - \mathfrak{a}} \\
\mathfrak{C} = 0;\ P = \frac{Ap^2 - Bp + C}{p^2 - \mathfrak{a}p + \mathfrak{b}} \\
\mathfrak{D} = 0;\ P = \frac{Ap^3 - Bp^2 + Cp - D}{p^3 - \mathfrak{a}p^2 + \mathfrak{b}p - \mathfrak{c}} \\
\qquad \text{etc.}
\end{array}$$

Hinc facile concluditur fore generaliter quotcunque affuerint dimensiones ipsius x;

$$P = \frac{A - \frac{1}{p}B + \frac{1}{p^2}C - \frac{1}{p^3}D + \frac{1}{p^4}E - \text{etc.}}{1 - \frac{1}{p}\mathfrak{a} + \frac{1}{p^2}\mathfrak{b} - \frac{1}{p^3}\mathfrak{c} + \frac{1}{p^4}\mathfrak{d} - \text{etc.}}$$

quae

quae expreſſio perpetuo terminatur, ſi formula differentialis propoſita finito terminorum numero conſtet.

§. 16. Numeratorem quidem huius fractionis, quam pro valore ipſius P inuenimus apprime conuenit cum numeratore fractionis praecedenti modo §. 12. pro eadem quantitate P inventae. At denominatores a ſe inuicem diſcrepare videntur: re autem propius perpenſa apparebit ſummum inter vtrosque eſſe conſenſum. Sumamus enim denominatorem priorem: $(1-\frac{q}{p})(1-\frac{r}{p})(1-\frac{s}{p})(1-\frac{t}{p})$ etc. atque patebit actuali multiplicatione eiusmodi prodituram eſſe expreſſionem:

$1-\frac{\mathfrak{P}}{p}+\frac{\mathfrak{Q}}{p^2}-\frac{\mathfrak{R}}{p^3}+\frac{\mathfrak{S}}{p^4}-\frac{\mathfrak{T}}{p^5}+$ etc.

in qua ſit $\mathfrak{P}$ = ſummae quantitatum q, r, s, t etc.

et $\mathfrak{Q}$ = ſummae factorum ex binis

$\mathfrak{R}$ = ſummae factorum ex ternis

$\mathfrak{S}$ = ſummae factorum ex quaternis

etc.

Cum igitur expreſſio $1+\mathfrak{a}x+\mathfrak{b}x^2+\mathfrak{c}x^3+\mathfrak{d}x^4+$ etc aequalis ſit producto $(1+qx)(1+rx)(1+sx)(1+tx)$ etc. (§. 14.) erit ob eandem rationem

$\mathfrak{a}$ = ſummae quantitatum q, r, s, t, etc.

$\mathfrak{b}$ = ſummae factorum ex binis

$\mathfrak{c}$ = ſummae factorum ex ternis

$\mathfrak{d}$ = ſummae factorum ex quaternis

etc.

Conſequenter erit $\mathfrak{P}=\mathfrak{a}$; $\mathfrak{Q}=\mathfrak{b}$; $\mathfrak{R}=\mathfrak{c}$; $\mathfrak{S}=\mathfrak{d}$; etc. ideoque denominator prius inuentus

$(1-\frac{q}{p})(1-\frac{r}{p})(1-\frac{s}{p})(1-\frac{t}{p})$ etc.

transmutabitur in ſequentem,

$1-\frac{1}{p}\mathfrak{a}+\frac{1}{p^2}\mathfrak{b}-\frac{1}{p^3}\mathfrak{c}+\frac{1}{p^4}\mathfrak{d}-$ etc.

 qui

qui est ipse denominator modo posteriori erutus, qui adeo priori est aequalis.

§. 17. Denominatorem hunc etiam poterimus exprimere per coëfficientes α, β, γ, δ, etc. qui continentur in denominatore formulae differentialis propositae, eo quod supra §. 14 per hos coëfficentes valores literarum $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{e}$ etc. determinauimus. In hoc autem negotio nosse oportebit, ex quot omnino terminis constet expressio $1 - \frac{1}{p}\mathfrak{a} + \frac{1}{p^2}\mathfrak{b} - \frac{1}{p^3}\mathfrak{c} +$ etc. Ponamus ergo eam constare ex

terminis	erit valor ipsius expressionis
1	1
2	$2 - \frac{1}{p}\alpha$
3	$3 - \frac{2}{p}\alpha + \frac{1}{p^2}\beta$
4	$4 - \frac{3}{p}\alpha + \frac{2}{p^2}\beta - \frac{1}{p^3}\gamma$
5	$5 - \frac{4}{p}\alpha + \frac{3}{p^2}\beta - \frac{2}{p^3}\gamma + \frac{1}{p^4}\delta$
:	
:	
n	$n - \frac{(n-1)\alpha}{p} + \frac{(n-2)\beta}{p^2} - \frac{(n-3)\gamma}{p^3} + \frac{(n-4)\delta}{p^4} -$ etc.

Vbi numerus n indicat quot sint termini in formula $1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 +$ etc. seu si forte qui termini desint, $n-1$ dat maximum ipsius x exponentem in denominatore formulae differentialis propositae. Cum autem $1 + px$ sit factor huius expressionis, ea si loco x ponatur $-\frac{1}{p}$ euadet $= 0$, hoc est, erit

$$0 = 1 - \frac{\alpha}{p} + \frac{\beta}{p^2} - \frac{\gamma}{p^3} + \frac{\delta}{p^4} - \frac{\varepsilon}{p^5} + \text{etc.}$$

quae ab illa n vicibus subtracta relinquit hanc expressionem

$$\frac{\alpha}{p} - \frac{2\beta}{p^2} + \frac{3\gamma}{p^3} - \frac{4\delta}{p^4} + \frac{5\varepsilon}{p^5} - \text{etc.}$$

quae

quae ergo eſt tertia expreſſio eandem denominatorem pro fractione ipſi P aequali ſuppeditans.

§. 18. Si ergo propoſita fuerit formula differentialis

$$\frac{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}}{1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.}}\, dx$$

in cuius numeratore x pauciores habeat dimenſiones, quam in denominatore; tum ea in tot formulas differentiales ſimplices logarithmicas reſolui poterit, quot vnitates contineat maximus ipſius x dimenſionum numerus in denominatore. Ad quas inueniendas ponamus denominatorem eſſe productum ex his factoribus

$$(1 + px)(1 + qx)(1 + rx)(1 + sx) \text{ etc.}$$

atque vnusquisque factor vnam ſuppeditabit fractionem ſimplicem; nempe ex factore $1 + px$ orietur formula differentialis $\frac{P\, dx}{1 + px}$, eritque

$$P = \frac{A - \frac{1}{p}B + \frac{1}{p^2}C - \frac{1}{p^3}D + \frac{1}{p^4}E - \text{etc.}}{(-\frac{q}{p})(1 - \frac{r}{p})(1 - \frac{s}{p})(1 - \frac{t}{p}) \text{ etc.}}$$

vel quod eodem redit

$$P = \frac{A - \frac{1}{p}B + \frac{1}{p^2}C - \frac{1}{p^3}D + \frac{1}{p^4}E - \text{etc.}}{\frac{1}{p}\alpha - \frac{2}{p^2}\beta + \frac{3}{p^3}\gamma - \frac{4}{p^4}\delta + \frac{5}{p^5}\varepsilon - \text{etc.}}$$

Simili autem modo, quo hic ex factore $1 + px$ formulam differentialem $\frac{P\, dx}{1 + px}$ inuenimus, ex reliquis omnibus factoribus totidem formulae differentiales $\frac{Q\, dx}{1 + qx}$, $\frac{R\, dx}{1 + rx}$, $\frac{S\, dx}{1 + sx}$ etc. reperientur. Quibus omnibus inuentis erit formulae propoſitae differentialis integrale quaeſitum $=$

$$\frac{P}{p} l(1 + px) + \frac{Q}{q} l(1 + qx) + \frac{R}{r} l(1 + rx) + \frac{S}{s} l(1 + sx) \text{ etc.}$$

 tot

tot conſtans membris longarithmicis, quot x in denominatore formulae propoſitae habet dimenſiones.

§. 19. Fieri autem nequit, vt horum membrorum vllum euaneſcat, ſeu vt vnquam fiat $P = 0$, niſi in ipſa formula differentiali propoſita communis diuiſor numeratoris ac denominatoris exiſtat. Quod vt clarius appareat ponamus numeratorem fractionis valorem ipſius P exhibentis $= v$ hoc eſt $A - \frac{1}{p} B + \frac{1}{p^2} C - \frac{1}{p^3} D +$ etc. $= 0$. Haec autem expreſſio reſultat ex numeratore formulae differentialis

$$A + Bx + Cx^2 + Dx^3 + \text{etc.}$$

ponendo $- \frac{1}{p}$ loco x; quare cum haec poſtrema expreſſio fiat $= 0$ poſito $- \frac{1}{p}$ loco x, ſequitur $x + \frac{1}{p}$ ſeu $1 + px$ eius diuiſorem eſſe; hocque caſu quo $P = 0$ neceſſe eſt, vt numerator et denominator formulae differentialis propoſitae communem habeant diuiſorem. Contra autem facile euenire poteſt, vt valor ipſius P in infinitum excreſcat euaneſcente denominatore

$$(1 - \frac{q}{p})(1 - \frac{r}{p})(1 - \frac{s}{p})(1 - \frac{t}{p}) \text{ etc.}$$

quod eueniet ſi inter reliquas litteras q, r, s, t, etc. vna pluresue reperiantur ipſi p aequales. Ponamus eſſe $p = q$, ſeu denominatorem $1 + \alpha x + \beta x^2 +$ etc.

duos habere factores aequales, tum in vtraque fractione $\frac{P dx}{1 + px}$ et $\frac{Q dx}{1 + qx}$ numerator in infinitum excreſcet. Interim tamen integrale ipſum non erit infinitum, ob bina iſta infinita ſe deſtruentia, ſed finitum, atque adeo ad quantitatem algebraicam reducetur quantitas alias perpetuo a logarithmis pendens. Tradamus igitur modum illam integralis partem, quae a duobus factoribus aequalibus oritur defi-

definiendi, cum ea ex praecedentibus formulis infinitis difficulter colligi queat.

§. 20. Si igitur duo pluresue denominatoris factores interse fuerint aequales, eos a se inuicem disiungi non conuenit, sed integralis membrum, quod ex illis coniunctim nascitur, peculiari modo est inuestigandum. Sint igitur duo denominatoris factores $(1 + px)^2$ aequales atque ponamus formulam differentialem propositam

$$\frac{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}}{1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.}}\, dx$$

resolui in has duas partes

$$\frac{\mathfrak{P}dx + \mathfrak{Q}xdx}{1 + 2px + ppxx} + \frac{\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \text{etc.}}{1 + \mathfrak{a}x + \mathfrak{b}x^2 + \mathfrak{c}x^3 + \text{etc.}}\, dx$$

erit primo vt per additionem denominator propositus proveniat

$\alpha = \mathfrak{a} + 2p$	$\mathfrak{a} = \alpha - 2p$
$\beta = \mathfrak{b} + 2p\mathfrak{a} + pp$	$\mathfrak{b} = \beta - 2\alpha p + 3pp$
$\gamma = \mathfrak{c} + 2p\mathfrak{b} + pp\mathfrak{a}$	$\mathfrak{c} = \gamma - 2\beta p + 3\alpha pp - 4p^3$
$\delta = \mathfrak{d} + 2p\mathfrak{c} + pp\mathfrak{b}$	$\mathfrak{d} = \delta - 2\gamma p + 3\beta pp - 4\alpha p^3 + 5p^4$

Deinde vt numerator propositus producatur esse oportebit

$$A = \mathfrak{A} + \mathfrak{P}$$
$$B = \mathfrak{B} + 2\mathfrak{A}p + \mathfrak{Q} + \mathfrak{P}\mathfrak{a}$$
$$C = \mathfrak{C} + 2\mathfrak{B}p + \mathfrak{A}pp + \mathfrak{Q}\mathfrak{a} + \mathfrak{P}\mathfrak{b}$$
$$D = \mathfrak{D} + 2\mathfrak{C}p + \mathfrak{B}pp + \mathfrak{Q}\mathfrak{b} + \mathfrak{P}\mathfrak{c}$$
$$E = \mathfrak{E} + 2\mathfrak{D}p + \mathfrak{C}pp + \mathfrak{Q}\mathfrak{c} + \mathfrak{P}\mathfrak{d}$$
etc.

Ex his aequationibus vicissim elicientur valores litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. sequentes.

$\mathfrak{A} =$

$\mathfrak{A} = A - \mathfrak{P}$
$\mathfrak{B} = B - 2Ap + \mathfrak{P}(2p - \mathfrak{a}) - \mathfrak{Q}$
$\mathfrak{C} = C - 2Bp + 3App - \mathfrak{P}(3p^2 - 2\mathfrak{a}p + \mathfrak{b}) + \mathfrak{Q}(2p - \mathfrak{a})$
$\mathfrak{D} = D - 2Cp + 3Bpp - 4Ap^3 + \mathfrak{P}(4p^3 - 3\mathfrak{a}p^2 + 2\mathfrak{b}p - \mathfrak{c}) - \mathfrak{Q}(3pp - 2\mathfrak{a}p + \mathfrak{b})$

etc.

Hae aequationes eousque ſunt continuandae, donec in expreſſione $\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 +$ etc. quae finito conſtat terminorum numero, ad finem perueniatur; tum enim ob ſequentes valores in ſerie litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. euaneſcentes ſtatim occurrent duae aequationes ex quibus coefficientes $\mathfrak{P}$ et $\mathfrak{Q}$ determinari poterunt.

§. 21. Si ordine progrediamur, ac primo $\mathfrak{A}$ et $\mathfrak{B}$, deinde $\mathfrak{B}$ et $\mathfrak{C}$, tum $\mathfrak{C}$ et $\mathfrak{D}$ etc. euaneſcentes ponamus, tum pro caſibus particularibus valores litterarum $\mathfrak{P}$ et $\mathfrak{Q}$ inuenimus, ex earum autem formis difficulter generales expreſſiones colligentur. Interim tamen alio modo ſatis concinne valores ipſarum $\mathfrak{P}$ et $\mathfrak{Q}$ determinari poterunt: Ponatur

$$\frac{A - \frac{1}{p}B + \frac{1}{p^2}C - \frac{1}{p^3}D + \text{etc.}}{\frac{1}{p} - \frac{1}{p^2}\mathfrak{a} + \frac{1}{p^3}\mathfrak{b} - \frac{1}{p^4}\mathfrak{c} + \text{etc.}} = V$$

erit V functio ipſius p et quantitatum cognitarum A, B, C, etc. et $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, etc. Quodſi ergo quantitas haec V differentietur ponendo tantum p variabili fiet $\frac{dV}{dp}$ quantitas algebraica eaque cognita. Iam dico valores ipſarum $\mathfrak{P}$ et $\mathfrak{Q}$ ita definiri vt ſit

$$\mathfrak{P} = \frac{dV}{dp} \text{ et } \mathfrak{Q} = \frac{pp}{dp}\, d.\frac{V}{p} = \frac{p\,dV}{dp} - V$$

Ad quas expreſſiones demonſtrandas, notari debet cum in ſerie litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. ad euaneſcentes lit-

teras

teras fuerit peruentum, tum binas eiusmodi prodituras esse aequationes.

$$nAp^{n-1}-(n-1)Bp^{n-2}+(n-2)Cp^{n-3}-(n-3)Dp^{n-4}+\text{ etc.}$$
$$=\mathfrak{P}(np^{n-1}-(n-1)\mathfrak{a}p^{n-2}+(n-2)\mathfrak{b}p^{n-3}-(n-3)\mathfrak{c}p^{n-4}+\text{ etc.})$$
$$-\mathfrak{Q}((n-1)p^{n-2}-(n-2)p^{n-3}+(n-3)\mathfrak{b}p^{n-4}-(n-4)\mathfrak{c}p^{n-5}+\text{ etc.})$$

et

$$(n+1)Ap^{n}-nBp^{n-1}+(n-1)Cp^{n-2}-(n-2)Dp^{n-3}+\text{ etc.}$$
$$=\mathfrak{P}((n+1)p^{n}-n\mathfrak{a}p^{n-1}+(n-1)\mathfrak{b}p^{n-2}-(n-2)\mathfrak{c}p^{n-3}+\text{ etc.})$$
$$-\mathfrak{Q}(np^{n-1}-(n-1)\mathfrak{a}p^{n-2}+(n-2)\mathfrak{b}p^{n-3}-(n-3)\mathfrak{c}p^{n-4}+\text{ etc.})$$

Ponatur iam:

$$Ap^{n}-Bp^{n-1}+Cp^{n-2}-Dp^{n-3}+\text{ etc.}=M$$
$$p^{n-1}-\mathfrak{a}p^{n-2}+\mathfrak{b}p^{n-3}-2p^{n-4}+\text{ etc.}=N$$

atque binae illae aequationes transibunt in has

$$\frac{dM}{dp}=\frac{\mathfrak{P}d.Np}{dp}-\frac{\mathfrak{Q}dN}{dp}$$
$$\frac{d.Mp}{ep}=\frac{\mathfrak{P}d.Np^2}{dp}-\frac{\mathfrak{Q}d.Np}{dp}$$

ex quibus elicitur:

$$\mathfrak{Q}=\frac{\mathfrak{P}Ndp+\mathfrak{P}pdN-dM}{dN}\text{ et}$$
$$\mathfrak{Q}=\frac{2\mathfrak{P}Npdp+\mathfrak{P}p^2dN-Mdp-pdM}{Ndp+pdN}$$

Atque ex harum comparatione oritur

$$\mathfrak{P}=\frac{NdM-MdN}{N^2dp}=\frac{1}{dp}d.\frac{M}{N}$$
$$\mathfrak{Q}=-\frac{M}{N}+\frac{p(NdM-MdN)}{N^2dp}=-\frac{M}{N}+\frac{p}{dp}d.\frac{M}{N}$$

Ponatur iam $\frac{M}{N}=V$ vt sit

$$V=\frac{Ap^{n}-Bp^{n-1}+Cp^{n-2}-Dp^{n-3}+\text{ etc.}}{p^{n-1}-\mathfrak{a}p^{n-2}+\mathfrak{b}p^{n-3}-\mathfrak{c}p^{n-4}+\text{ etc.}}$$

erit si numerator et denominator per p^n diuidatur, prorsus vt ante assumsimus

$$V = \frac{A - \frac{1}{p}B + \frac{1}{p^2}C - \frac{1}{p^3}D + \text{ etc.}}{\frac{1}{p} - \frac{1}{p^2}\mathfrak{a} + \frac{1}{p^3}\mathfrak{b} - \frac{1}{p^4}\mathfrak{c} + \text{ etc.}}$$

Hocque adeo valore ipsius V assumto habebimus:

$$\mathfrak{P} = \frac{dV}{dp} \text{ et } \mathfrak{Q} = \frac{pdV}{dp} - V \text{ seu } \mathfrak{P} = \frac{p}{dp} \cdot \frac{dV}{p}\ \mathfrak{Q} = \frac{pp}{dp} d \cdot \frac{V}{p}.$$

§. 22. Simili modo si formulae propositae

$$\frac{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}}{1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.}}\, dx$$

denominator habeat tres factores simplices aequales, ita vt sit diuisibilis per $(1 + px)^3$, tum resolutio in duas istiusmodi partes fieri debet

$$\frac{\mathfrak{P}dx + \mathfrak{Q}xdx + \mathfrak{R}x^2dx}{1 + 3px + 3p^2x^2 + p^3x^3} + \frac{\mathfrak{A} + \mathfrak{B}x + \text{etc.}}{1 + \mathfrak{a}x + \mathfrak{b}x^2 + \text{etc.}}\, dx$$

vt iam summae denominator cum proposito congruat, oportebit esse

$$\mathfrak{a} = \alpha - 3p$$
$$\mathfrak{b} = \beta - 3\alpha p + 6pp$$
$$\mathfrak{c} = \gamma - 3\beta p + 6\alpha pp - 10p^3$$
$$\mathfrak{d} = \delta - 3\gamma p + 6\beta pp - 10\alpha p^3 + 15p^4$$
etc.

Numeratorum autem identitas dabit has aequationes:

$$A = \mathfrak{A} + \mathfrak{P}$$
$$B = \mathfrak{B} + 3\mathfrak{A}p + \mathfrak{Q} + \mathfrak{P}\mathfrak{a}$$
$$C = \mathfrak{C} + 3\mathfrak{B}p + 3\mathfrak{A}p^2 + \mathfrak{R} + \mathfrak{Q}\mathfrak{a} + \mathfrak{P}\mathfrak{b}$$
$$D = \mathfrak{D} + 3\mathfrak{C}p + 3\mathfrak{B}p^2 + \mathfrak{A}p^3 + \mathfrak{R}\mathfrak{a} + \mathfrak{Q}\mathfrak{b} + \mathfrak{P}\mathfrak{c}$$
$$E = \mathfrak{E} + 3\mathfrak{D}p + 3\mathfrak{C}p^2 + \mathfrak{B}p^3 + \mathfrak{R}\mathfrak{b} + \mathfrak{Q}\mathfrak{c} + \mathfrak{P}\mathfrak{d}$$
etc.

Hinc-

Hincque viciſſim ſequentes valores pro litteris $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, etc. eliciuntur.

$$\mathfrak{A} = A - \mathfrak{P}$$
$$\mathfrak{B} = B - 3Ap + \mathfrak{P}(3p - \mathfrak{a}) - \mathfrak{Q}$$
$$\mathfrak{C} = C - 3Bp + 6Ap^2 - \mathfrak{P}(6pp - 3\mathfrak{a}p + \mathfrak{b}) + \mathfrak{Q}(3p - \mathfrak{a}) - \mathfrak{R}$$
$$\mathfrak{D} = D - 3Cp + 6Bp^2 - 10Ap^3 + \mathfrak{P}(10p^3 - 6\mathfrak{a}p^2 + 3\mathfrak{b}p - \mathfrak{c}) - \mathfrak{Q}(6pp - 3\mathfrak{a}p + \mathfrak{b}) + \mathfrak{R}(3p - \mathfrak{a})$$
etc.

Ponatur iam vt ante fecimus:

$$Ap^n - Bp^{n-1} + Cp^{n-2} - Dp^{n-3} + \text{etc.} = M$$
et
$$p^{n-2} - \mathfrak{a}p^{n-3} + \mathfrak{b}p^{n-4} - \mathfrak{c}p^{n-5} + \text{etc.} = N$$

erit vbi ad valores euaneſcentes peruenitur

$$ddM - \mathfrak{P}\,dd.Np^2 + \mathfrak{Q}\,dd.Np - \mathfrak{R}\,ddN = 0$$
$$dd.Mp - \mathfrak{P}\,dd.Np^3 + \mathfrak{Q}\,dd.Np^2 - \mathfrak{R}\,dd.Np = 0$$
$$dd.Mp^2 - \mathfrak{P}\,dd.Np^4 + \mathfrak{Q}\,dd.Np^3 - \mathfrak{R}\,dd.Np^2 = 0$$

poſito in duplici differentiatione dp conſtante.

Quod ſi iam ponatur $\frac{M}{N} = V$ ita vt ſit

$$V = \frac{A - \frac{1}{p}B + \frac{C}{p^2}C - \frac{1}{p^3}D + \text{etc.}}{\frac{1}{pp} - \frac{\mathfrak{a}}{p^3} + \frac{\mathfrak{b}}{p^4} - \frac{\mathfrak{c}}{p^5} + \text{etc.}}$$

reperientur ſequentes valores pro $\mathfrak{P}$, $\mathfrak{Q}$ et $\mathfrak{R}$.

$$\mathfrak{P} = \frac{ddV}{2dp^2} = \frac{p}{2dp^2} \cdot \frac{ddV}{p}$$
$$\mathfrak{Q} = \frac{pddV}{dp^2} - \frac{dV}{dp} = \frac{pp}{dp^2} \,.\, d \,.\, \frac{dV}{p}$$
$$\mathfrak{R} = \frac{ppddV}{2dp^2} - \frac{pdV}{dp} + V = \frac{p^3}{2dp^2} \,.\, dd \,.\, \frac{V}{p}.$$

§. 23. Haec iam ſufficiunt ad legem cognoſcendam cuius beneficio reſolutio formulae propoſitae in duas partes

 abſolui

abſolui debeat, ſi denominator quotcunque habeat factores ſimplices aequales. Quae lex vt facilius perſpiciatur repetamus breuitur, quae hactenus ſunt tradita. Sitque propoſita haec formula

$$\frac{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}}{1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.}}\, dx$$

cuius integrale requiritur in cuius numeratore x pauciores habeat dimenſiones quam in denominatore; ſitque denominatoris factor aliquis $1 + px$. Quod ſi iam hic factor $1 + px$ alium non habeat ſui aequalem, ex ipſo integralis quaeſiti pars reperietur $\int \frac{\mathfrak{P}\, dx}{1 + px}$, in quo coefficiens $\mathfrak{P}$ ita determinabitur. Diuidatur primo denominator $1 + \alpha x + \beta x^2 + \gamma x^3 + \text{etc.}$ per $1 + px$ ſitque quotus $= 1 + \mathfrak{a} x + \mathfrak{b} x^2 + \mathfrak{c} x^3 + \text{etc.}$ Tum ponatur

$$\frac{A - \frac{1}{p}B + \frac{1}{p^2}C - \frac{1}{p^3}D + \text{ etc.}}{1 - \frac{1}{p}\mathfrak{a} + \frac{1}{p^2}\mathfrak{b} - \frac{1}{p^3}\mathfrak{c} + \text{etc.}} = V$$

eritque $\mathfrak{P} = p \cdot \frac{V}{p}$, atque integrale ex factore $1 + px$ ortum erit

$$p \cdot \frac{V}{p} \int \frac{dx}{1 + px} = \frac{V}{p}\, l\,(1 + px)$$

hocque modo ex ſingulis denominatoris factoribus ſimplicibus reſpondentes integralis partes eruantur, quae ſimul ſumtae totum integrale quaeſitum praebebunt.

§. 24. Quod ſi autem denominator duos factores ſimplices habeat aequales, ſeu $(1 + px)^2$ factor fuerit denominatoris, tum ex hoc factore quadrato integralis pars inueniri debet, quae erit huiusmodi $\int \frac{\mathfrak{P}\, dx + \mathfrak{Q}\, x\, dx}{(1 + px)^2}$; in qua coefficientes $\mathfrak{P}$ et $\mathfrak{Q}$ hoc modo reperientur. Diuidatur deno-

denominator $1 + \alpha x + \beta x^2 +$ etc. per $(1+px)^2$ fitque quotus $= 1 + \mathfrak{a}x + \mathfrak{b}x^2 + \mathfrak{c}x^3 +$ etc. Tum ponatur

$$\frac{A - \frac{1}{p}B + \frac{1}{p^2}C - \frac{1}{p^3}D + \text{ etc.}}{\frac{1}{p} - \frac{1}{p^2}\mathfrak{a} + \frac{1}{p^3}\mathfrak{b} - \frac{1}{p^4}\mathfrak{c} + \text{ etc.}} = V$$

erit V functio ipfius p quae differentietur more confueto ponendo tantum p variabile, fietque

$$\mathfrak{P} = \frac{p}{1dp} \cdot \frac{dV}{p}$$

$$\mathfrak{Q} = \frac{pp}{1dp} \cdot d \cdot \frac{V}{p}$$

Atque integrale ex factore $(1+px)^2$ oriundum erit

$$\frac{p}{1dp} d\frac{V}{p} \int \frac{dx}{1+px} + p \cdot \frac{V}{pp} \int \frac{dx}{(1+px)^2}$$

feu $\frac{1}{1dp} d \cdot \frac{V}{p} l(1+px) - \frac{V}{pp} \cdot \frac{1}{1+px}$.

§. 25. Habeat denominator tres factores fimplices aequales, feu fit $(1+px)^3$ eius diuifor; tum ex hoc toto factore quaeratur integralis pars, quae fit

$$\int \frac{\mathfrak{P}dx + \mathfrak{Q}xdx + \mathfrak{R}x^2dx}{(1+px)^3}.$$

Coefficientes autem $\mathfrak{P}$, $\mathfrak{Q}$, et $\mathfrak{R}$ hoc modo definientur. Diuidatur denominator $1 + \alpha x + \beta x^2 + \gamma x^3 +$ etc. per $(1+px)^3$ fitque quotus $1 + \mathfrak{a}x + \mathfrak{b}x^2 + \mathfrak{c}x^3 +$ etc.

Tum ponatur $\frac{A - \frac{1}{p}B + \frac{1}{pp}C - \frac{1}{p^3}D + \text{ etc.}}{\frac{1}{p^2} - \frac{1}{p^3}\mathfrak{a} + \frac{1}{p^4}\mathfrak{b} - \frac{1}{p^5}\mathfrak{c} + \text{ etc.}} = V$

ex hoc ipfius V valore cognito erit

$$\mathfrak{P} = \frac{p}{1 \cdot 2dp^2} \cdot \frac{ddV}{p}$$

$$\mathfrak{Q} = \frac{2p^2}{1 \cdot 2dp^2} \cdot d \cdot \frac{dV}{p}$$

$$\mathfrak{R} = \frac{p^3}{1 \cdot 2dp^2} \cdot dd \cdot \frac{V}{p}$$

Atque integralis quaesiti membrum ex factore hoc $(1+px)^3$ oriundum erit:

$$\frac{p}{1.2dp^2}dd.\frac{V}{p}\int\frac{dx}{1+px}+\frac{p}{1dp}.d.\frac{V}{p^2}\int\frac{dx}{(1+px)^2}+p.\frac{V}{p^3}\int\frac{dx}{(1+px)^3}$$

seu

$$\frac{dd.\frac{V}{p}}{1.2dp^2}l(1+px)-\frac{d.\frac{V}{p^2}}{1dp}.\frac{1}{1+px}-\frac{V}{p^3}.\frac{1}{2(1+px)^2}$$

§. 26. Habeat denominator formulae differentialis propositae quatuor factores simplices inter se aequales, ita vt $(1+px)^4$ sit eius diuisor, tum ex toto hoc factore quaeratur integralis pars

$$\int\frac{\mathfrak{P}dx+\mathfrak{Q}xdx+\mathfrak{R}x^2dx+\mathfrak{S}x^3dx}{(1+px)^4}$$

vbi coefficientes $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$, et $\mathfrak{S}$ hoc modo determinabuntur. Diuidatur denominator propositus $1+\alpha x+\beta x^2+\gamma x^3+$ etc. per $(1+px)^4$ sitque quotus $1+\mathfrak{a}x+\mathfrak{b}x^2+\mathfrak{c}x^3+$ etc. Tum ponatur

$$\frac{A-\frac{1}{p}B+\frac{1}{p^2}C-\frac{1}{p^3}D+\text{ etc.}}{\frac{1}{p^3}-\frac{1}{p^4}\mathfrak{a}+\frac{1}{p^5}\mathfrak{b}-\frac{1}{p^6}\mathfrak{c}+\text{ etc.}}=V$$

Atque ex hoc ipsius V valore cognito erit

$$\mathfrak{P}=\frac{p}{1.2.3dp^3}.\frac{d^3V}{p}$$
$$\mathfrak{Q}=\frac{3p^2}{1.2.3dp^3}d.\frac{ddV}{p}$$
$$\mathfrak{R}=\frac{3p^3}{1.2.3dp^3}dd.\frac{dV}{p}$$
$$\mathfrak{S}=\frac{p^4}{1.2.3dp^3}d^3.\frac{V}{p}$$

Hinc integralis membrum ex factore $(1+px)^4$ oriundum erit:

$$\frac{p}{1.2.3dp^3}d^3.\frac{V}{p}\int\frac{dx}{1+px}+\frac{p}{1.2dp^2}d^2.\frac{V}{pp}\int\frac{dx}{(1+px)^2}$$
$$+\frac{p}{1dp}d.\frac{V}{p^3}.\int\frac{dx}{(1+px)^3}+p.\frac{V}{p^4}\int\frac{dx}{(1+px)^4}.$$

§. 27.

§. 27. Perſpicitur hinc duplex lex, altera quam valores coefficientium aſſumtorum $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$, etc. inter ſe tenent, altera vero, quam formulae integrales ipſae obſeruant; atque ex vtraque integralis quaeſiti membrum id poterit definiri quod oritur ex poteſtate quacunque factoris cuiuspiam ſimplicis $1+px$. Sufficit autem alteram tantum legem notaſſe, cum altera ex altera ſequatur; ac poſterior quidem non ſolum facilior videtur, verum etiam maiorem praeſtat vtilitatem, cum ex ea ſtatim integrale formari queat. Probe autem cauendum eſt ne vera variabilis x cum variabili aſſumtitia p confundatur; nam in formula differentiali propoſita vnica ineſt variabilis x, praeter quam omnes reliquae quantitates non excepta p ſunt conſtantes et in integratione qua tales tractantur: quando vero coefficientes per ſe quidem conſtantes inueſtigantur, tum vera variabilis x non amplius in computum ducitur; ſed inueſtigatio per meras conſtantes abſoluitur. In hoc autem opere ingens nanciſcimur ſubſidium ad coefficientes determinandos ope differentiationis, vbi quantitatem V tanquam functionem variabilis p conſideramus, eamque differentiamus ſemel, bis pluriesue ponendo dp conſtans. Eſt adeo haec differentiatio tantum operatio ſubſidiaria, quae ad coefficientes indagandos ſuſcipitur; qui, quam primum fuerint inuenti, tum rurſus quantitas p tanquam conſtans tractatur; et integrale more conſueto expreſſum exhibetur.

§. 28. Vnicam difficultatem, quae ſaepenumero maximam moleſtiam parere poſſet, hic adhuc remouere poſſumus, quo ipſo calculus mirum in modum contrahetur. Haec

Haec autem difficultas verſatur in inuentione ſeriei $1 + \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^2 + \mathfrak{D}x^3 +$ etc. quae reſultat ſi denominator $1 + \alpha x + \beta x^2 + \gamma x^3 +$ etc. vel per $1 + px$ vel per $(1 + px)^2$ vel per $(1 + px)^3$ vel etc. diuidatur. Facilior igitur calculus reddetur ſi pro quouis caſu in valore ipſius V loco litterarum $\mathfrak{A}, \mathfrak{B}, \mathfrak{C},$ etc. earum valores per $\alpha, \beta, \gamma,$ etc. ſubſtituamus, qui cum ſint ſponte cogniti inuentione litterarum $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D},$ etc. ſuperſedere poterimus. Caſu igitur primo §. 23. pertractato, quo denominator $1 + \alpha x + \beta x^2 + \gamma x^3 +$ etc. ſemel tantum diuiſorem $1 + px$ admittit, erit:

$$V = \frac{A - \frac{1}{p}B + \frac{1}{p^2}C - \frac{1}{p^3}D + \text{etc.}}{\frac{\alpha}{p} - \frac{2\beta}{p^2} - \frac{3\gamma}{p^3} + \frac{4\delta}{p^4} - \text{etc.}} \quad (\S.\ 18.)$$

quae operatio, vt adhuc breuius abſolui queat, ponatur in formula differentiali propoſita

$$\frac{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}}{1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.}}\, dx$$

numerator $A + Bx + Cx^2 + Dx^3 +$ etc. $= P$

et denomin: $1 + \alpha x + \beta x^2 + \gamma x^3 +$ etc. $= Q$

erit $V = \frac{Ppdx}{dQ}$ poſito $-\frac{1}{p}$ loco x; quo facto obtinebit V eum ipſum valorem per p expreſſum, qui ipſi conuenit pro caſu, quo $1 + px$ eſt factor denominatoris Q; ex hocque factore integralis quaeſiti pars oritur haec

$$\frac{V}{p}\int\frac{pdx}{1+px}$$

§. 29. Si denominator Q factorem habeat $(1 + px)^2$ tum ſumatur $V = \frac{1.2\,Pp^3dx^2}{ddQ}$ ponendo in differentiatione ipſius Q x variabile et dx conſtans, tumque loco x ſcribendo $-\frac{1}{p}$, quo facto V fiet functio ipſius p, quae proin

poſita

posita hac p variabili differentiari poterit. Erit autem integralis quaesiti membrum ex factore $(1+px)^2$ oriundum:

$$\frac{d.\frac{V}{p}}{1dp}\int\frac{pdx}{1+px}+\frac{V}{p^2}\int\frac{pdx}{(1+px)^2}.$$

Sin denominator Q factorem habeat $(1+px)^3$ tum sumatur $V=\frac{1.2.3Pp^5dx^3}{d^3Q}$; vbi primo in differentiatione ipsius $Q=1+\alpha x+\beta x^2+\gamma x^3+$ etc. ponatur x variabile et in sequentibus differentiationibus dx constans; tum fiat $x=-\frac{1}{p}$, vt prodeat V functio ipsius p deinceps differentianda posito p variabili. Atque integralis quaesiti pars ex factore $(1+px)^3$ oriunda erit

$$\frac{dd.\frac{V}{p}}{1.2dp^2}\int\frac{pdx}{1+px}+\frac{d.\frac{V}{p^2}}{1dp}\int\frac{pdx}{(1+px)^2}+\frac{V}{p^3}\int\frac{pdx}{(1+px)^3}.$$

Si denominator Q factorem habeat $(1+px)^4$ tum sumatur $V=\frac{1.2.3.4Pp^7dx^4}{d^4Q}$ et seruatis iisdem circa differentiationes legibus erit integralis portio ex denominatoris factore $(1+px)^4$ oriunda $=$

$$\frac{d^3.\frac{V}{p}}{1.2.3dp^3}\int\frac{pdx}{1+px}+\frac{d^2.\frac{V}{pp}}{1.2dp^2}\int\frac{pdx}{(1+px)^2}+\frac{d.\frac{V}{p^3}}{1dp}\int\frac{pdx}{(1+px)^3}+\frac{V}{p^4}\int\frac{pdx}{(1+px)^4}$$

sicque vlterius, quousque libuerit, progredi licebit.

§. 30. Generatim igitur, quae hactenus sunt tradita huc redeunt. Si proposita sit formula differentialis rationalis

$$\frac{A+Bx+Cx^2+Dx^3+Ex^4+\text{etc.}}{1+\alpha x+\beta x^2+\gamma x^3+\delta x^4+\varepsilon x^5+\text{etc.}}\,dx$$

in qua variabilis x pauciores habeat dimensiones in numeratore quam in denominatore huiusque integrale quaeratur: dico integrale ex tot compositum fore partibus, quot denominator contineat factores simplices a se inuicem diversos. Atque ex quolibet factore denominatoris simplici eiusque potentia quacunque pars integralis quaesiti sequenti modo inuestigatur. Sit

$(1+px)^n$ factor seu diuisor denominatoris

Atque ponatur breuitatis gratia:

numerator $A+Bx+Cx^2+Dx^3+$ etc. $=P$

denominat. $1+\alpha x+\beta x^2+\gamma x^3+$ etc. $=Q$

Tum continuo differentiando denominatorem Q, posito x variabili et dx constanti, capiatur

$$V=\frac{Pp^{2n-1}dx^n}{d^nQ} \text{ vel quodidem est } V=\frac{Pp^{n-1}(1+px)^n}{1.2.3\ldots n.Q}$$

ac fiat postea $x=-\frac{1}{p}$, vt exeat ex hac expressione x, maneatque V functio ipsius p et constantium; quae deinceps differentietur ponendo p variabile et dp constans. Quo facto ex factore $(1+px)^n$ nascetur integralis quaesiti ista pars:

$$\frac{n\,d^{n-1}.\frac{V}{p}}{dp^{n-1}}\int\frac{pdx}{1+px}+\frac{n(n-1)d^{n-2}.\frac{V}{p^2}}{dp^{n-2}}\int\frac{pdx}{(1+px)^2}+\frac{n(n-1)(n-2)d^{n-3}.\frac{V}{p^3}}{dp^{n-3}}\int\frac{pdx}{(1+px)^3}$$
$$+\frac{n(n-1)(n-2)(n-3)d^{n-4}.\frac{V}{p^4}}{dp^{n-4}}\int\frac{pdx}{(1+px)^4}+\text{etc.}$$

quae tot constabit membris, quot n continet vnitates.

§. 31. Habemus hic valorem ipsius V duplici modo expressum, quorum eo, qui commodior visus fuerit, vti conueniet. Si quidem denominator Q iam fuerit in suos factores

factores resolutus, tum expediet vti modo posteriori quo est

$$V = \frac{P p^{n-1}(1+px)^n}{1.2.3\ldots\ldots nQ}$$

tum enim statim fiet $\frac{Q}{(1+px)^n}$ quantitas integra, eo, quod denominatorem Q diuisibilem esse ponimus per $(1+px)^n$. Quod autem diximus inuento hoc modo valore ipsius V loco x poni debere $-\frac{1}{p}$; hic probe cauendum est, ne loco p numerum determinatum scribendo contra institutum peccemus, ac talem pro V expressionem obtineamus, quae differentiari nequeat. Quamobrem etsi reuera p eam quantitatem determinatam denotat, quae reddat $(1+px)^n$ diuisorem denominatoris Q, tamen loco $-\frac{1}{x}$ non illa quantitas determinata, sed potius character p, quasi adhuc esset incognitus, indeterminate substitui debet, quem etiam tamdiu retinebit donec per differentiationem singuli integralis coefficientes fuerint reperti. Quod vt clarius ob oculos ponatur, sit integranda ista formula differentialis Exemplum

$$\frac{x\,dx}{(1-x)^3(1+x)^2(1+2x)}$$

Erit ergo $P = x$ et $Q = (1-x)^3(1+x)^2(1+2x)$ atque integrale ex tribus constabit partibus, quae ex tribus factoribus $(1-x)^3$, $(1+x)^2$ et $1+2x$ reperientur. Sumatur primum factor $(1-x)^3$, ex quo est $p = -1$ quem autem valorem tum demum loco p substituemus, cum omnes coefficientes fuerint determinati.

Erit ergo $n = 3$ et $\frac{Q}{(1+px)^3} = (1+x)^2(1+2x)$, ex quo fit $V = \frac{p^2x}{6(1+x)^2(1+2x)} = \frac{-p^4}{6(p-1)^2(p-2)}$. Hinc erit $\frac{V}{p^3} = \frac{-p}{6(p-1)^2(p-2)}$ et $\frac{6V}{p^3} = \frac{-p}{p^3-4pp+5p-2}$; $\frac{6d.\frac{V}{pp}}{dp} = \frac{p^3+pp-4p}{(p-1)^3(p-2)^2}$ itemque $\frac{3V}{p} = \frac{-p^3}{2(p-1)^2(p-2)}$ vnde erit $\frac{3}{dp}d.\frac{V}{p} = \frac{2p^3-3pp}{(p-1)^3(p-2)^2}$; atque $\frac{3}{dp^2}d^2.\frac{V}{p} = \frac{-4p^4+7p^3+6pp-12p}{(p-1)^4(p-2)^3}$. Cum nunc fit $p = -1$ erit

$$\frac{3}{dp^2}d^2.\frac{V}{p} = \frac{7}{16.27}$$
$$\frac{6}{dp}d.\frac{V}{p^2} = \frac{-4}{8.9}$$
$$6\frac{V}{p^3} = \frac{-1}{4.3}$$

atque integrale ex factore $(1-x)^3$ oriundum erit

$$\frac{7}{16.27}\int\frac{dx}{1-x} + \frac{4}{8.9}\int\frac{dx}{(1-x)^2} + \frac{1}{4.3}\int\frac{dx}{(1-x)^3} =$$
$$\frac{7}{16.27}l\frac{1}{1-x} + \frac{4}{8.9}.\frac{1}{1-x} + \frac{1}{8.3(1-x)^2}$$

Porro sumatur factor $(1+x)^2$ erit $n = 2$, et $p = 1$ atque $V = \frac{px}{2(1-x)^3(1+2x)} = \frac{-p^4}{2(p+1)^3(p-2)}$. Hinc est $2.\frac{V}{pp} = \frac{-pp}{(p+1)^3(p-2)} = \frac{1}{8}$ posito $p = 1$: ac $\frac{2}{dp}d.\frac{V}{p} = \frac{p^4-2p^3+6pp}{(p+1)^4(p-2)^2} = \frac{5}{16}$ posito $p = 1$. Ergo ex denominatoris factore $(1+x)^2$ nascitur integralis pars haec.

$$\frac{5}{16}\int\frac{dx}{1+x} + \frac{1}{8}\int\frac{dx}{(1+x)^2} = \frac{5}{16}\ l(1+x) - \frac{1}{8(1+x)}$$

Denique ex factore $1+2x$, fit $n = 1$ et $p = 2$ oriturque $V = \frac{x}{(1-x)^3(1+x)^2} = \frac{-p^4}{(p+1)^3(p-1)^2}$ et quia nulla differentiatione opus est ponatur $p = 2$ fiet $\frac{V}{p} = \frac{-8}{27}$ et integralis pars ex factore $1+2x$ oriunda erit $= \frac{-8}{27}\int\frac{2dx}{1+2x} = \frac{-8}{27}\ l(1+2x)$

Ex

Ex his itaque formulae differentialis huius

$$\frac{x\,dx}{(1-x)^3(1+x)^2\cdot(1+2x)}$$

integrale completum colligitur esse:

$$\frac{7}{16\cdot 27}\,l\frac{1}{1-x}+\frac{5}{16}\,l(1+x)+\frac{8}{27}\,l\frac{1}{1+2x}+\frac{1}{18(1-x)}+\frac{1}{24(1-x)^2}-\frac{1}{8(1+x)}+\text{Const.}$$

§. 32. Ex his igitur dilucide perspicitur, quibus operationibus cuiuscunque formulae differentialis, dummodo sit rationalis, integrale inueniri oporteat. Primum enim si denominator formulae propositae fuerit diuisibilis per x, vel eius potestatem quamcunque, tum docuimus formulam in duas partes discerpere, quarum altera integrationem sponte admittat, habens pro denominatore illam ipsam potestatem ipsius x, altera pars autem habeat denominatorem non amplius diuisibilem per x; quae adeo reducetur ad hanc formam:

$$\frac{A+Bx+Cx^2+Dx^3+Ex^4+\text{etc.}}{1+\alpha x+\beta x^2+\gamma x^3+\delta x^4+\text{etc.}}\,dx$$

in cuius integrale inquiri oporteat. Secundo videndum est vtrum in hac forma variabilis x in numeratore tot vel plures habeat dimensiones quam in denominatore, an pauciores. Nam si tot habeat vel plures dimensiones, tum iterum formula differentialis in duas partes discerpi poterit, quarum altera sit nullo negotio integrabilis, altera autem habitura sit in numeratore pauciores dimensiones ipsius x, quam in denominatore; quae ideo erit istius modi:

$$\frac{A+Bx+Cx^2+Dx^3+Ex^4+\text{etc.}}{1+\alpha x+\beta x^2+\gamma x^3+\delta x^4+\varepsilon x^5+\text{etc.}}$$

in qua fractione variabilis x in denominatore plures habeat dimensiones quam in numeratore. Tertio ergo tota integrandi difficultas reducitur ad integrationem istiusmodi formularum; ad quam absoluendam oportet denominatorem

$$1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.}$$

in suos factores simplices singulos huius formae $1 + px$ resoluere, quorum vnusquisque modo exposito tractatus dabit vnam integralis quaesiti partem, ita vt singulis his partibus, quae ex singulis denominatoris factoribus oriuntur, colligendis obtineatur integrale quaesitum. In hoc autem negotio molestiam facessit, si duo pluresue istorum factorum simplicium fuerint inter se aequales, huic vero incommodo remedium attulimus, dum docuimus, quomodo ex factoribus aequalibus coniunctis integralis pars ex ipsis oriunda inueniri debeat. Vnicum autem incommodum adhuc saepenumero accedit, quod in hoc consistit, vt quoties denominator habeat factores imaginarios, toties integralis partes ex iis oriundae fiant quoque imaginariae: quae etsi coniunctim sumtae quantitatem realem praebent, tamen ea ex imaginariis non tam liquido appareat. Quamobrem operam dabimus, vt integrale ab imaginariis omnino liberam ac solis quantitatibus realibus expressum exhibeamus.

§. 33. Dubium enim est nullum, quin formulae differentialis realis integrale pariter sit reale; namque integrale nil aliud est nisi summa omnium valorum differentialium formulae propositae a minimo ipsius x valore ad maximum continuo progredientium, qui cum sint omnes reales, necesse est, vt etiam eorum summa, hoc est integrale sit, quantitas realis. Quocirca etiam

etiam si denominator formulae differentialis habeat factores imaginarios, qui proinde integralis partes producant imaginarias, tamen totum integrale speciem tantum imaginarii prae se feret, atque re ipsa erit quantitas realis. Ita vidimus integrale huius formulae $\frac{dx}{1+xx}$, si denominator $1+xx$ in suos factores simplices imaginarios $1+x\sqrt{-1}$ et $1-x\sqrt{-1}$ resoluatur, integrale componi ex duobus logarithmis imaginariis $\frac{1}{2\sqrt{-1}}l(1+x\sqrt{-1})-\frac{1}{2\sqrt{-1}}l(1-x\sqrt{-1})$ qui autem simul sumti reducuntur ad quadraturam circuli, ita vt integrale sit arcus circuli, cuius tangens est $=x$ posito radio $=1$. Cum igitur quilibet factor simplex imaginarius in integrale inferat logarithmum imaginarium, iure concludimus cunctos hos logarithmos imaginarios simul sumtos ad quadraturam circuli aliusue curuae redire, quae eorum loco substituta integrali formam realem inducat. Demonstrabimus autem omnes logarithmos imaginarios, cuiuscunque demum sint formae, dummodo quantitatem realem referant, ad quadraturam circuli reuocari posse, ita vt nullam aliam quadraturam introducere sit opus. Hincque patebit omnis formulae differentialis rationalis, vt cunque fuerit composita, integrale per logarithmos et quadraturam circuli perpetuo exhiberi posse, neque ad hoc vllam aliam quadraturam requiri.

§. 34. Dico autem quotcunque denominator formulae differentialis propositae habeat factores simplices imaginarios, tum eorum numerum semper esse parem, binosque ex iis semper ita esse comparatos, vt eorum productum fiat reale. Quod ad numerum parem factorum imaginariorum attinet, id quidem iam pridem constat, atque fir-

missimis

missimis argumentis est confirmatum. Factores enim simplices expressionis algebraicae formantur ex radicibus eiusdem expressionis nihilo aequalis positae, sic si aequationis huius

$$z^n + \alpha z^{n-1} + \beta z^{n-2} + \gamma z^{n-3} + \text{etc.} = 0$$

radices fuerint $z = p$; $z = q$; $z = r$; etc. tum huius expressionis algebraicae:

$$z^n + \alpha z^{n-1} + \beta z^{n-2} + \gamma z^{n-3} + \text{etc.}$$

factores simplices erunt: $z - p$; $z - q$; $z - r$; etc. itaque inuentio factorum simplicium absoluitur inuentione radicum aequationis algebraicae: atque radices reales praebebunt factores simplices reales imaginariae vero imaginarios. Demonstratum autem est, si maximus ipsius z exponens n sit numerus impar; tum aequationem vel vnam habere radicem realem, vel tres, vel quinque, vel septem, etc. ex quo numerus radicum non realium seu imaginariarum erit par, eo quod numerus radicum omnium aequetur numero n qui est impar. Deinde etiam demonstratum est si maximus incognitae z exponens n fuerit numerus par, tum aequationem vel nullam habere radicem realem, vel duas, vel quatuor, vel sex, etc. vnde etiam hoc casu numerus radicum imaginariarum erit par. Ex quibus colligitur, si quaecunque expressio algebraica habuerit factores simplices imaginarios, tum eorum numerum perpetuo esse parem, ideoque factores imaginarios habebit vel nullum, vel duos, vel quatuor, vel sex, etc. secundum numeros pares.

§. 35. Si iam haec ad denominatorem formulae nostrae differentialis propositae accommodemus, is vel nullum habebit factorem simplicem imaginarium, quo casu vtique

vtique integrale per methodum traditam in forma reali reperitur; vel habebit duos factores simplices imaginarios, vel quatuor, vel sex, vel octo etc. Ponamus denominatorem duos tantum habere factores simplices imaginarios, reliquos vero omnes reales: ac diuidamus eum per productum ex omnibus factoribus realibus, quod vtique erit quantitas realis; manifestum est quotum fore quantitatem realem. At quotus erit productum ex binis illis factoribus imaginariis; ideoque horum productum erit quantitas realis. Hinc si denominator $1 + \alpha x + \beta x^2$ $\gamma x^3 + \delta x^4 +$ etc. duos tantum habeat factores imaginarios, eorum productum erit huiusmodi $1 + px + qx^2$, in quo ceofficientes p et q sint reales; ideoque iste denominator loco duorum factorum simplicium imaginariorum habebit vnum factorem trinomialem $1 + px + qx^2$ realem, cuius factores cum sint imaginarii erit $q > \frac{pp}{4}$. Cum igitur sit $\frac{pp}{4}$ quantitas positiua, erit q etiam quantitas positiua atque maior quam quadratum semissis ipsius p. Quodsi iam in producto omnium factorum simplicium realium ponamus in coefficientibus eiusmodi mutationem fieri, vt vnus factor fiat imaginarius, simul alium imaginarium fieri oportebit, horumque duorum productum per idem ratiocinium fiet reale. Vnde colligitur loco omnium factorum simplicium imaginariorum, quorum numerus sit $= 2m$, factores trinomiales huius formae $1 + px + qxx$ reales, quorum numerus sit $= m$ substitui posse.

§. 36. Si igitur formulae differentialis propositae

$$\frac{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}}{1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.}}\, dx$$

 deno-

denominator habeat quotcunque factores simplices imaginarios, eorum loco poterimus adhibere factores trinomiales $1 + px + qxx$ reales; hocque modo resoluemus denominatorem in factores meros reales. Scilicet tot simplices quot habebit reales, ac pro imaginariis factores trinomiales reales introducemus. Quare cum iam ostenderimus, quomodo ex singulis factoribus simplicibus partes integralis respondentes inueniri oporteat; superest vt modum tradamus inueniendi integralis partes ex factoribus trinomialibus oriundas. Sit igitur factor denominatoris trinomialis $1 + px + qxx$, atque formula differentialis proposita in has duas partes discerpi concipiatur:

$$\frac{\mathfrak{P}dx + \mathfrak{Q}xdx}{1 + px + qxx} + \frac{\mathfrak{A}dx + \mathfrak{B}xdx + \mathfrak{C}x^2dx + \text{ect.}}{1 + \mathfrak{a}x + \mathfrak{b}x^2 + \mathfrak{c}x^3 + \text{etc.}}$$

Atque ex parte $\frac{\mathfrak{P}dx + \mathfrak{Q}xdx}{1 + px + qxx}$ nascetur integrale

$$\frac{\mathfrak{Q}}{2q}l(1 + px + qxx) + \frac{2\mathfrak{P}q - \mathfrak{Q}p}{q\sqrt{(4q - pp)}}\text{A tang.}\frac{x\sqrt{(4q - pp)}}{2 + px}$$

erit enim ob duos factores simplices imaginarios in $1 + px + qxx$ contentos $4q > pp$. Quodsi autem bini factores essent reales, et $4q < pp$, tum integrale formulae $\frac{\mathfrak{P}dx + \mathfrak{Q}xdx}{1 + px + qxx}$ a solis logarithmis penderet foretque

$$\frac{\mathfrak{Q}}{2q}l(1 + px + qxx + \frac{2\mathfrak{P}q - \mathfrak{Q}p}{2q\sqrt{(pp - 4q)}}l\frac{2qx + p - \sqrt{(pp - 4q)}}{2qx + p + \sqrt{(pp - 4q)}}$$

§. 37. Valores autem coefficientium $\mathfrak{P}$ et $\mathfrak{Q}$ ex his aequationibus definientur:

A =

$A = \mathfrak{A} + \mathfrak{P}$
$B = \mathfrak{B} + \mathfrak{A}p + \mathfrak{Q} + \mathfrak{P}\mathfrak{a}$
$C = \mathfrak{C} + \mathfrak{B}p + \mathfrak{A}q + \mathfrak{Q}\mathfrak{a} + \mathfrak{P}\mathfrak{b}$
$D = \mathfrak{D} + \mathfrak{C}p + \mathfrak{B}q + \mathfrak{Q}\mathfrak{b} + \mathfrak{P}\mathfrak{c}$
etc.

ex quibus eliciuntur fequentes valores:

$\mathfrak{A} = A - \mathfrak{P}$
$\mathfrak{B} = B - Ap + \mathfrak{P}(p-\mathfrak{a}) - \mathfrak{Q}$
$\mathfrak{C} = C - Bp + A(pp-q) - \mathfrak{P}(pp-q-\mathfrak{a}p+\mathfrak{b}) + \mathfrak{Q}(p-\mathfrak{a})$
$\mathfrak{D} = D - Cp + B(pp-q) - A(p^3-2pq) +$
$\mathfrak{P}(p^3-2pq-\mathfrak{a}(p^2-q)+\mathfrak{b}p-2) - \mathfrak{Q}(pp-q-\mathfrak{a}p+\mathfrak{b})$
etc.

vbi cum in ferie litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. tandem ad euanefcentes perueniatur, prodibunt aequationes, ex quibus $\mathfrak{P}$ et $\mathfrak{Q}$ determinari poterunt. His autem aequationibus implicabitur haec feries

$1;\ p;\ p^2-q;\ p^3-2pq;\ p^4-3p^2q+qq;$ etc.

cuius terminus generalis feu indici n refpondens eft $= \frac{(\frac{1}{2}p+\sqrt{(\frac{1}{4}pp-q)})^n - (\frac{1}{2}p-\sqrt{(\frac{1}{4}pp-q)})^n}{\sqrt{(pp-4q)}}$. Ponatur breuitatis gratia $\frac{1}{2}p+\sqrt{(\frac{1}{4}pp-q)} = r$ et $\frac{1}{2}p-\sqrt{(\frac{1}{4}pp-q)} = s$ atque cum ad coefficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. euanefcentes fuerit peruentum, habebitur ifta aequatio indefinita

$+\mathfrak{P}(r^n \mp \mathfrak{a}r^{n-1} \pm \mathfrak{b}r^{n-2} \mp 2r^{n-3}$ etc. $- \mathfrak{P}(s^n - \mathfrak{a}s^{n-1} \pm \mathfrak{b}s^{n-2} \mp$ etc.$)$
$-\mathfrak{Q}(r^{n-1} \mp \mathfrak{a}r^{n-2} \pm \mathfrak{b}r^{n-3} \mp$ etc.$) + \mathfrak{Q}(s^{n-1} \mp \mathfrak{a}s^{n-2} \pm \mathfrak{b}s^{n-3} \mp$ etc.$)$
$= Ar^n \mp Br^{n-1} \pm Cr^{n-2} \mp$ etc. $- As^n + Bs^{n-1} \mp Cs^{n-2} \pm$ etc.

cuius duo cafus fufficient ad valores ipforum $\mathfrak{P}$ et $\mathfrak{Q}$ determinandos.

§. 38. Ponamus breuitatis gratia:

$$A r^{n} - B r^{n-1} + C r^{n-2} - D r^{n-3} + \text{etc.} = R$$

$$A s^{n} - B s^{n-1} + C s^{n-2} - D s^{n-3} + \text{etc.} = S$$

$$r^{n-1} - \mathfrak{a} r^{n-2} + \mathfrak{b} r^{n-3} - \mathfrak{c} r^{n-4} + \text{etc.} = P$$

$$s^{n-1} - \mathfrak{a} r^{n-2} + \mathfrak{b} r^{n-3} - \mathfrak{c} s^{n-4} + \text{etc.} = Q$$

atque obtinebimus hanc aequationem:

$$\mathfrak{P} P r - \mathfrak{P} Q s - \mathfrak{Q} P + \mathfrak{Q} Q = R - S$$

hancque pari modo ſequetur iſta in loco $n+1$

$$\mathfrak{P} P r^{2} - \mathfrak{P} Q s^{2} - \mathfrak{Q} P r + \mathfrak{Q} Q s = R r - S s$$

ex his eliminando $\mathfrak{Q}$ reperitur:

$$\mathfrak{Q} = \frac{\mathfrak{P}(Pr - Qs) - R + S}{P - Q} = \frac{\mathfrak{P}(Pr^{2} - Qs^{2}) - Rr + Ss}{Sr - Qs}$$

hincque obtinebuntur ſequentes determinationes.

$$\mathfrak{P} = \frac{QR - PS}{(r-s)PQ} = \frac{QR - PS}{PQ\sqrt{(pp-4q)}} = \frac{1}{r-s}\left(\frac{R}{P} - \frac{S}{Q}\right)$$

$$\mathfrak{Q} = \frac{QRs - PSr}{(r-s)PQ} = \frac{QRs - PSr}{PQ\sqrt{(pp-4q)}} = \frac{1}{r-s}\left(\frac{Rs}{P} - \frac{Sr}{Q}\right)$$

at in his valoribus eſt independenter ab n;

$$\frac{R}{P} = \frac{A - \frac{1}{r}B + \frac{1}{r^{2}}C - \frac{1}{r^{3}}D + \text{etc.}}{\frac{1}{r} - \frac{\mathfrak{a}}{r^{2}} + \frac{\mathfrak{b}}{r^{3}} - \frac{\mathfrak{c}}{r^{4}} + \text{etc.}}$$

$$\frac{S}{Q} = \frac{A - \frac{1}{s}B + \frac{1}{s^{2}}C - \frac{1}{s}D + \text{etc.}}{\frac{1}{s} - \frac{\mathfrak{a}}{s^{2}} + \frac{\mathfrak{b}}{s^{3}} - \frac{\mathfrak{c}}{s^{4}} + \text{etc.}}$$

Denotant autem $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, etc. ſequentes valores

$$\mathfrak{a} = \alpha - p$$

$$\mathfrak{b} = \beta - \alpha p + pp - q$$

$$\mathfrak{c} = \gamma - \beta p + \alpha(pp - q) - (p^{3} - 2pq)$$

$$\mathfrak{d} = \delta - \gamma p + \beta(pp - q) - \alpha(p^{3} - 2pq) + (p^{4} - 3ppq + qq)$$

etc.

vel

vel cum fit $p = r + s$; et $q = rs$ erit vt fequitur

$$-\mathfrak{a} = r + s - \alpha$$

$$+\mathfrak{b} = rr + rs + ss - \alpha(r+s) + \beta$$

$$-\mathfrak{c} = r^3 + r^2 s + rss + s^3 - \alpha(rr + rs + ss) + \beta(r+s) - \gamma$$

His valoribus fubftitutis prodibit denominator

$$\frac{1}{r} - \frac{\mathfrak{a}}{rr} + \frac{\mathfrak{b}}{r^3} - \frac{\mathfrak{c}}{r^4} + \frac{\mathfrak{d}}{r^5} - \text{etc.}$$

aequalis huic expreffioni pro n dimenfionibus

$$\frac{n}{r} + \frac{(n-1)s}{rr} + \frac{(n-2)ss}{r^3} + \frac{(n-3)s^3}{r^4} + \frac{(n-4)s^4}{r^5} + \text{etc.}$$

$$-\frac{\alpha}{r}\left(\frac{n-1}{r} + \frac{(n-2)s}{r^2} + \frac{(n-3)s^2}{r^3} + \frac{(n-4)s^3}{r^4} + \text{etc.}\right.$$

$$+\frac{\beta}{r^2}\left(\frac{n-2}{r} + \frac{(n-3)s}{r^2} + \frac{(n-4)s^2}{r^3} + \text{etc.}\right)$$

$$\frac{\gamma}{r^3}\left(\frac{n-3}{r} + \frac{(n-4)s}{rr} + \text{etc.}\right.$$

etc.

quae feries, cum fint omnes fummabiles, habebitur

$$\frac{1}{(r-s)^2}\left(nr-(n+1)s+\frac{s^{n+1}}{r^n} - \frac{\alpha}{r}\left((n-1)r-ns+\frac{s^n}{r^{n-1}}\right) + \frac{\beta}{r^2}\left((n-2)r-(n-1)s+\frac{s^{n-1}}{r^{n-2}}\right) - \text{etc.}\right)$$

§. 39. Quoniam autem $1 + px + qxx$ euanefcit pofito loco x vel $-\frac{1}{r}$ vel $-\frac{1}{s}$, eaque ipfa quantitas eft diuifor denominatoris $1 + \alpha x + \beta x^2 + \gamma x^3 +$ etc. erit quoque pofito pro x vel $-\frac{1}{r}$ vel $-\frac{1}{s}$

$$0 = 1 - \frac{\alpha}{r} + \frac{\beta}{rr} - \frac{\gamma}{r^3} + \frac{\delta}{r^4} - \text{etc.}$$

$$0 = 1 - \frac{\alpha}{s} + \frac{\beta}{ss} - \frac{\gamma}{s^3} + \frac{\delta}{s^4} - \text{etc.}$$

Hinc itaque erit

$$\frac{s^{n+1}}{r^n} - \frac{\alpha}{r} \cdot \frac{s^n}{r^{n-1}} + \frac{\beta}{rr} \cdot \frac{s^{n-1}}{r^{n-2}} - \frac{\gamma}{r^3} \cdot \frac{s^{n-2}}{r^{n-3}} + \text{etc.} = 0$$

atque

$$nr-(n+1)s - \frac{\alpha}{r}(nr-(n+1)s) + \frac{\beta}{r^2}(nr-(n+1)s) - \text{etc.} = 0$$

quae duae expressiones si coniunctim subtrahantur a superiori, transmutabitur ista expressio

$\frac{1}{r} - \frac{\mathfrak{a}}{r^2} + \frac{\mathfrak{b}}{r^3} - \frac{\mathfrak{c}}{r^4} + \text{etc.}$ in hanc

$$\frac{1}{(r-s)^2}\left(\frac{\alpha(r-s)}{r} - \frac{2\beta(r-s)}{r^2} + \frac{3\gamma(r-s)}{r^3} - \frac{4\delta(r-s)}{r^4} + \text{etc.}\right)$$

quae cum diuidi queat per $r-s$ emergit:

$$\frac{1}{r-s}\left(\frac{\alpha}{r} - \frac{2\beta}{r^2} + \frac{3\gamma}{r^3} - \frac{4\delta}{r^4} + \text{etc.}\right)$$

simili modo ista expressio;

$$\frac{1}{s} - \frac{\mathfrak{a}}{s^2} + \frac{\mathfrak{b}}{s^3} - \frac{\mathfrak{c}}{s^4} + \frac{\mathfrak{d}}{s^5} - \text{etc.}$$

factis substitutionibus transibit in hanc:

$$\frac{1}{s-r}\left(\frac{\alpha}{s} - \frac{2\beta}{s^2} + \frac{3\gamma}{s^3} - \frac{4\delta}{s^4} + \text{etc.}\right)$$

Quare tandem pro §. 38. habebitur;

$$\frac{R}{(r-s)P} = \frac{A - \frac{1}{r}B + \frac{1}{r^2}C - \frac{1}{r^3}D + \text{etc.}}{\frac{\alpha}{r} - \frac{2\beta}{r^2} + \frac{3\gamma}{r^3} - \frac{4\delta}{r^4} + \text{etc.}}$$

$$\frac{S}{(r-s)Q} = \frac{A - \frac{1}{s}B + \frac{1}{s^2}C - \frac{1}{s^3}D + \text{etc.}}{\frac{\alpha}{s} - \frac{2\beta}{s^2} + \frac{3\gamma}{s^3} - \frac{4\delta}{s^4} + \text{etc.}}$$

Hisque inuentis est $\mathfrak{P} = \frac{R}{(r-s)P} - \frac{S}{(r-s)Q}$ atque

$\mathfrak{Q} = \frac{Rs}{(r-s)P} - \frac{Sr}{(r-s)Q}$; et $2\mathfrak{P}q - \mathfrak{Q}p = \frac{Rs}{P} + \frac{Sr}{Q}$.

§. 40. Etsi hic quantitates r et s habeant valores imaginarios, tamen in valoribus pro $\mathfrak{P}$ et $\mathfrak{Q}$ imaginaria se destruunt, atque orientur valores reales. Ponatur enim primum productum amborum denominatorum.

$(\frac{\alpha}{r}$

$$\left(\frac{\alpha}{r}-\frac{2\beta}{r^2}+\frac{3\gamma}{r^3}-\text{etc.}\right)\left(\frac{\alpha}{s}-\frac{2\beta}{s^2}+\frac{3\gamma}{s^3}-\text{etc.}\right)=W$$

erit ob $rs=q$ et $r+s=p$ multiplicatione peracta

$$W=\begin{cases}+\frac{\alpha^2}{q}+\frac{4\beta^2}{qq}+\frac{9\gamma^2}{q^3}+\frac{16\delta^2}{q^4}+\frac{25\varepsilon^2}{q^5}+\text{etc.}\\ -p\left(\frac{2\alpha\beta}{qq}+\frac{6\beta\gamma}{q^3}+\frac{12\gamma\delta}{q^4}+\frac{20\delta\varepsilon}{q^5}+\text{etc.}\right)\\ +(pp-2q)\left(\frac{3\alpha\gamma}{q^3}+\frac{8\beta\delta}{q^4}+\frac{15\gamma\varepsilon}{q^5}+\frac{24\delta\zeta}{q^6}+\text{etc.}\right)\\ -(p^3-3pq)\left(\frac{4\alpha\delta}{q^4}+\frac{10\beta\varepsilon}{q^5}+\frac{18\gamma\zeta}{q^6}+\text{etc.}\right)\\ \text{etc.}\end{cases}$$

Ex his inuenitur $\mathfrak{P}$ et $\mathfrak{Q}$ vti sequitur

$$\mathfrak{P}=\begin{cases}+\frac{A}{W}\left(\frac{\alpha p}{q}-\frac{2\beta(pp-2q)}{qq}+\frac{3\gamma(p^3-3pq)}{q^3}-\frac{4\delta(p^4-4p^2q+2qq)}{q^4}+\text{etc.}\right)\\ -\frac{B}{W}\left(\frac{2\alpha}{q}-\frac{2\beta p}{qq}+\frac{3\gamma(pp-2q)}{q^3}-\frac{4\delta(p^3-3pq)}{q^4}+\text{etc.}\right)\\ +\frac{C}{W}\left(\frac{\alpha p}{qq}-\frac{4\beta}{qq}+\frac{3\gamma p}{q^3}-\frac{4\delta(pp-2q)}{q^4}+\text{etc.}\right)\\ -\frac{D}{W}\left(\frac{\alpha(pp-2q)}{q^3}-\frac{2\beta p}{q^3}+\frac{6\gamma}{q^3}-\frac{4\delta p}{q^4}+\text{etc.}\right)\\ \text{etc.}\end{cases}$$

$$\mathfrak{Q}=\begin{cases}+\frac{A}{W}\left(2\alpha-\frac{2\beta p}{q}+\frac{3\gamma(pp-2q)}{qq}-\frac{4\delta(p^3-3pq)}{q^3}+\text{etc.}\right)\\ -\frac{B}{W}\left(\frac{\alpha p}{q}-\frac{4\beta}{q}+\frac{3\gamma p}{qq}-\frac{4\delta(pp-2q)}{q^3}+\text{etc.}\right)\\ +\frac{C}{W}\left(\frac{\alpha(pp-2q)}{qq}-\frac{2\beta p}{qq}+\frac{6\gamma}{qq}-\frac{4\delta p}{q^3}+\text{etc.}\right)\\ -\frac{D}{W}\left(\frac{\alpha(p^3-3pq)}{q^3}-\frac{2\beta(pp-2q)}{q^3}+\frac{3\gamma p}{q^3}-\frac{8\delta}{q^3}+\text{etc.}\right)\\ \text{etc.}\end{cases}$$

Inuentis ergo hoc modo $\mathfrak{P}$ et $\mathfrak{Q}$ reperietur integrale, quod ex denominatoris factore $1+px+qxx$ oritur, quippe quod est

$$\frac{\mathfrak{Q}}{2q}\,l(1+px+qxx)+\frac{2\mathfrak{P}q-\mathfrak{Q}p}{q\sqrt{(4q-pp)}}\,A\,\text{tang.}\,\frac{x\sqrt{(4q-pp)}}{2+px}$$

In casibus autem particularibus saepenumero praestat litteras r et s retinere, donec valores pro $\mathfrak{P}$ et $\mathfrak{Q}$ fuerint

inuenti;

inuenti; etsi enim hi valores sunt imaginarii, tamen ordo progressionis, quo in formulas $\mathfrak{P}$ et $\mathfrak{Q}$ ingrediuntur, facilius apparet, simulque sponte imaginaria se tollunt. Huius adeo methodi beneficio omnis formulae differentialis rationalis, vtcunque factoribus imaginariis scateat, integrale reale ope logarithmorum et arcuum circularium poterit exhiberi.

§. 41. Quae hic non mediocri labore pro factore trinomiali inuenimus, ea multo facilius directe ex iis quae de factoribus simplicibus attulimus, deriuari possunt. Sit enim in formula differentiali proposita

$$\frac{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}}{1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.}}$$

vbi x vti ponimus iam pauciores habeat dimensiones in numeratore quam in denominatore. Sit inquam $1 + px + qxx$ factor trinomialis, isque realis denominatoris $1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 +$ etc. cuiusmodi factores vtique dantur; bini enim factores simplices in $1 + px + qxx$ contenti sunt vel reales vel imaginarii, atque vtroque casu eorum productum est reale. Sint igitur $1 + rx$ et $1 + sx$ bini factores simplices siue reales siue imaginarii, quorum productum sit $= 1 + px + qxx$ ita vt sit $r = \frac{p + \sqrt{(pp - 4q)}}{2}$ et $s = \frac{p - \sqrt{(pp - 4q)}}{2}$; et quaerantur integralis partes, quae ex vtroque factore simplici oriuntur. Pro primo quidem factore si ponatur

$$R = \frac{A - \frac{1}{r}B + \frac{1}{r^2}C - \frac{1}{r^3}D + \frac{1}{r^4}E - \text{etc.}}{\frac{\alpha}{r} - \frac{2\beta}{r^2} + \frac{3\gamma}{r^3} - \frac{4\delta}{r^4} + \frac{5\varepsilon}{r^5} - \text{etc.}}$$

erit integralis pars inde oriunda $= \int \frac{R\,dx}{1 + rx}$.

At

At pro altero factore $1+sx$, si ponatur:

$$S = \frac{A - \frac{1}{s}B + \frac{1}{s^2}C - \frac{1}{s^3}D + \frac{1}{s^4}E - \text{etc.}}{\frac{\alpha}{s} - \frac{2\beta}{s^2} + \frac{3\gamma}{s^3} - \frac{4\delta}{s^4} + \frac{5\varepsilon}{s^5} - \text{etc.}}$$

erit integralis pars inde oriunda $= \int \frac{S\,dx}{1+sx}$.
Quamobrem ex vtroque factore coniunctim hoc est ex factore trinomiali $1+px+qxx$ orietur integralis pars haec $\int \frac{R\,dx}{1+rx} + \int \frac{S\,dx}{1+sx} = \int \frac{(R+S)dx + (Rs+Sr)x\,dx}{1+px+qxx}$ vbi tam $R+S$ quam $Rs+Sr$ erunt quantitates reales; atque hoc integrale vel a solis logarithmis vel simul a quadratura circuli pendebit, prout r et s fuerint vel quantitates reales, vel imaginariae: hocque apprime congruit cum eo, quod ante inuenimus.

§. 42. Simili modo si denominator fuerit diuisibilis per $(1+px+qxx)^2$, atque factores simplices ipsius $1+px+qxx$, qui sint $1+rx$ et $1+sx$ fuerint imaginarii, integralis portio inde oriunda in forma reali dabitur. Cum enim ipsius $(1+px+qxx)^2$ factores sint $(1+rx)^2$ et $(1+sx)^2$, tractetur vterque modo supra exposito. Scilicet pro factore $(1+rx)^2$ ponatur

$$R = \frac{A - \frac{1}{r}B + \frac{1}{r^2}C - \frac{1}{r^3}D + \text{etc.}}{\frac{1}{r^3}\left(1\cdot2\beta - \frac{2\cdot3\gamma}{r} + \frac{3\cdot4\delta}{r^2} - \text{etc.}\right)}$$

eritque integrale hinc oriundum:

$$\frac{2\,d\cdot\frac{R}{r}}{dr}\int\frac{r\,dx}{1+rx} + 2\cdot1\cdot\frac{R}{r^2}\int\frac{r\,dx}{(1+rx)^2}$$

Simili modo pro factore $(1+sx)^2$ ponatur:

$$S = \frac{A - \frac{1}{s}B + \frac{1}{s^2}C - \frac{1}{s^3}D + \text{etc.}}{\frac{1}{s^3}\left(1\cdot2\beta - \frac{2\cdot3\gamma}{s} + \frac{3\cdot4\delta}{s^2} - \text{etc.}\right)}$$

 atque

atque hinc orietur integralis portio:

$$\frac{2d.\frac{\mathfrak{S}}{s}}{ds}\int\frac{s\,dx}{1+sx}+2.1\frac{\mathfrak{S}}{s^2}\int\frac{s\,dx}{(1+sx)^2}$$

quae duae formae ſi r et s fuerint quantitates imaginariae inuicem addantur, prodibitque formula differentialis integranda realis huius formae

$$\frac{\mathfrak{P}dx+\mathfrak{Q}x\,dx+\mathfrak{R}x^2dx+\mathfrak{S}x^3dx}{(1+px+qxx)^2}$$

cuius integrale reſoluitur in has duas partes

$$\frac{\mathfrak{P}p-2\mathfrak{Q}+\mathfrak{R}\frac{p}{q}-\mathfrak{S}\left(\frac{pp-2q}{qq}\right)+\left(2\mathfrak{P}q-\mathfrak{Q}p+\mathfrak{R}\left(\frac{pp-2q}{q}\right)-\mathfrak{S}\left(\frac{p^3-3pq}{qq}\right)\right)x}{(4q-pp)\,(1+px+qxx)}$$

$$+\int\frac{(2\mathfrak{P}q-\mathfrak{Q}p+2\mathfrak{R}-\mathfrak{S}\frac{p}{q})\,dx-\mathfrak{S}\left(\frac{pp-4q}{q}\right)x\,dx}{(4q-pp)\,(1+px+qxx)}$$

quae ergo integratio ope logarithmorum et arcuum circularium abſolui poteſt. Pari autem modo erit procedendum, ſi denominatoris factor fuerit $(1+px+qxx)^3$ vel alia altior poteſtas quaecunque.

§. 43. Ex his igitur intelligitur, quomodo formulae cuiuscunque differentialis rationalis integrale inueniri atque adeo in forma reali exhiberi oporteat; poſtquam enim differentiale per modos prius expoſitos ad formam poſterius pertractatam fuerit perductum, tum totius negotii cardo vertetur in inuentione factorum realium denominatoris, qui, vti oſtendimus, ſunt vel ſimplices binomiales vel trinomiales; atque quilibet factor dabit partem integralis quaeſiti; quare ſi methodo praeſcripta ex ſingulis factoribus integralis partes reſpondentes fuerint inuentae, tum omnium harum partium aggregatum erit integrale quaeſitum

tum. Ex his porro patet veritas non exigui momenti, quod omnis formulae differentialis rationalis perpetuo concessis logarithmis et arcubus circularibus exhiberi queat; ita vt integratio si algebraice absolui nequeat alias quantitates transcendentes non requirat praeter logarithmos et arcus circulares. Cum igitur modus, quo ad integrale perueniendum est, satis sit expositus; nil aliud super est, nisi vt vsum eius in aliquot exemplis alias difficilioribus monstremus; quae partim iam sint ab aliis tractata, partim hic de nouo producta vel saltem fusius exposita. In hoc autem negotio, quia praecipuum opus in inuentione factorum versatur, alia exempla proferre non licet, nisi in quibus factores denominatoris actu exhiberi queant; hanc obrem in subsidium vocabimus praecipua illa artificia a Celeb. Moivraeo aliisque excogitata, quorum beneficio factores propositae cuiuspiam expressionis assignari possunt, sicque plura occurrent ad algebrae finitae incrementum facientia, quae etsi huc minus pertinent, tamen fusius prosequemur.

Problema 1.

§. 44. Inuenire integrale huius formulae differentialis $\frac{x^m\,dx}{1+x^{2n}}$ existente m numero integro minore, quam $2n$.

Solutio.

Si exponens m maior esset quam $2n$ tum per diuisionem formula ad casum praesentem perduceretur; perspicuum autem est denominatorem nullum omnino factorem simplicem realem habere, ex quo factores trinomiales quaeri debebunt reales quorum numerus erit n. Sit huiusmodi

 factor

factor $= 1 + px + qxx$ realis, qui ſit productum ex his imaginariis $(1 + rx)(1 + sx)$, exiſtente $r + s = p$ et $rs = q$. His ad paragr. 41. reuocatis erit $R = (-\frac{1}{r})^m : -2n(-\frac{1}{r})^{2n} = \frac{-(-r)^{2n-m}}{2n}$ in ſubſidium vocatis quae §. 28. ſunt tradita. Simili modo eſt $S = \frac{-(-s)^{2n-m}}{2n}$; vnde ex factore $1 + px + qxx$ orietur iſta integralis pars

$$\int \frac{-((-r)^{2n-m} + (-s)^{2n-m})\,dx + rs\,((-r)^{2n-m-1} + (-s)^{2n-m-1})\,x\,dx}{2n(1 + px + qxx)}$$

eſt vero $r = \frac{p + \sqrt{(pp-4q)}}{2}$ et $s\ \frac{p - \sqrt{(pp-4q)}}{2}$

Formetur hinc ſeries, cuius terminus generalis ſit $= (-r)^k + (-s)^k$, eiusque termini ita progredientur

$$\begin{matrix} 1 & 2 & 3 & 4 \\ -p; & +(pp-2q); & -(p^3 - 3pq); & +(p^4 - 4p^2q + 2qq); \text{ etc.} \end{matrix}$$

ponatur huius feriei terminus, cuius index eſt $2n - m - 1 = M$ et terminus ſequens cuius index eſt $= 2n - m$ ſit $= N$, habebiturque iſtud integrale

$$\int \frac{-N\,dx + Mqx\,dx}{2n(1 + px + qxx)}$$

quod per logarithmos et arcus circulares dat:

$$+ \frac{M}{4n}\,l(1 + px + qxx) - \frac{2N - Mp}{2n\sqrt{(4q-pp)}}\,\text{A tang.}\,\frac{x\sqrt{(4q-pp)}}{2 + px}$$

at ex natura ſerierum recurrentium M et N ita a ſe inuicem pendent, vt ſit $N^2 + MNp + M^2q = -q^{2n-m-1}(pp-4q)$ ſeu $N = -\frac{Mp}{2} + (4q-pp)(q^{2n-m-1} - \frac{M^2}{4})$

Cum ergo ſit $2N + Mp = \sqrt{(4q-pp)(4q^{2n-m-1} - M^2)}$ erit integrale formulae $\int \frac{-Ndx + Mqxdx}{2n(1 + px + qxx)} =$

$$+ \frac{M}{4n}\,l(1 + px + qxx) - \frac{\sqrt{(4q^{2n-m-1} - M^2)}}{2n}\,\text{A tang.}\,\frac{x\sqrt{(4q-pp}}{2 + px}$$

eſtque $M = \pm \left(\frac{p + \sqrt{(pp-4q)}}{2}\right)^{2n-m-1} \pm \left(\frac{p - \sqrt{(pp-4q)}}{2}\right)^{2n-m-1}$

vbi

vbi ſigna ſuperiora $+$ valent ſi $2-m-1$ fuerit numerus par, ſin ſit impar, ſigna inferiora $-$ ſunt capienda. Haec expreſſio M autem commode per multiplicationem arcuum circularium exprimi poteſt, ſi quidem eſt $4q > pp$, vti ponimus. Nam ſi arcus cuiuspiam circuli Φ coſinus fuerit $= u$, ita vt ſit $\Phi = \text{A cof.}\, u$ poſito ſinu toto $= 1$. erit coſinus arcus $k\,\Phi = \frac{(u+\sqrt{(uu-1)})^k + (u-\sqrt{(uu-1)})^k}{2}$. Quodſi iam ponamus $u = \frac{p}{2\sqrt{q}}$, ita vt ſit $\Phi = \text{A cof.}\, \frac{p}{2\sqrt{q}}$, orietur: $M = \mp 2\, q^{\frac{2n-m-1}{2}} \text{cof. A.}\, (2n-m-1)\,\Phi$; vbi ſignum $+$ habet locum ſi fuerit $2n-m-1$ numerus par, contrarium autem $-$, ſi $2n-m-1$ ſit numerus impar. Hinc fiet $\sqrt{(4q^{2n-m-1} - MM)} = \mp 2 q^{\frac{2n-m-1}{2}} \text{ſin. A.}\, (2n-m-1)\,\Phi$; ideoque integrale ex denominatoris factore $1+px+qxx$ oriundum erit $= \pm \frac{q^{\frac{2n-m-1}{2}} \text{cof. A.}\, (2n-m-1)\Phi}{2n} l(1+px+qxx) \mp \frac{q^{\frac{2n-m-1}{2}} \text{ſin. A.}\, (2n-m-1)\Phi}{n} \text{A tang.}\, \frac{x\sqrt{(4q-pp)}}{2+px}$ exiſtente Φ arcu circuli, cuius coſinus eſt $= \frac{p}{2\sqrt{q}}$, in circulo cuius radius $= 1$. Valent autem ſigna ſuperiora ſi $2n-m-1$ fuerit numerus par, inferiora autem ſi ſit numerus impar. Cum igitur ſinguli factores denominatoris $1+x^{2n}$ huius formae $1+px+qxx$ praebeant partes integralis, quae ex cognitis coefficientibus p et q modo praeſcripto aſſignari queant, hos ipſos factores trinomiales denominatoris $1+x^{2n}$ eruere debemus. At ex theoremate quodam Moivreano ſi habeatur expreſſio huiusmodi $1+ax+bx^2+cx^3+\ldots\ldots$

 $+m$

$+ mx^{n} + \ldots\ldots + cx^{2n-3} + bx^{2n-2} + ax^{2n-1} + x^{2n}$
cuius coefficientes finales ordine retrogrado congrunt cum coefficientibus initialibus; ea resoluitur in factores trinomiales $1 + \alpha x + xx$; $1 + \beta x + xx$; $1 + \gamma x + xx$; $1 + \delta x + xx$; etc. quorum numerus est n eruntque coefficientes $\alpha, \beta, \gamma, \delta$, etc. radices huius aequationis n dimensionum.

$$Z^{n} - Az^{n-1} + Bz^{n-2} - Cz^{n-3} + Dz^{n-4} - \text{etc.} = 0$$

cuius coefficientes sequentem tenent legem:

$$A = a$$
$$B = b - n$$
$$C = c - (n-1)a$$
$$D = d - (n-2)b + \frac{n(n-3)}{1\cdot 2}$$
$$E = e - (n-3)c + \frac{(n-1)(n-4)}{1\cdot 2}a$$
$$F = f - (n-4)d + \frac{(n-2)(n-5)}{1\cdot 2}b - \frac{n(n-4)(n-5)}{1\cdot 2\cdot 3}$$
$$G = g - (n-5)e + \frac{(n-3)(n-6)}{1\cdot 2}c - \frac{(n-1)(n-5)(n-6)}{1\cdot 2\cdot 3}a$$
$$H = h - (n-6)f + \frac{(n-4)(n-7}{1\cdot 2}d - \frac{(n-2)(n-6)(n-7)}{1\cdot 2\cdot 3}b + \frac{n(n-5)(n-6)(n-7)}{1\cdot 2\cdot 3\cdot 4}$$

etc.

His valoribus substitutis habebimus hanc aequationem

$$\left.\begin{array}{l} z^{n} - nz^{n-2} + \frac{n(n-3)}{1\cdot 2}z^{n-4} - \frac{n(n-4)(n-5)}{1\cdot 2\cdot 3}z^{n-6} + \text{etc.}) \\ - a(z^{n-1} - (n-1)z^{n-3} + \frac{(n-1)(n-4)}{1\cdot 2}z^{n-5} - \frac{(n-1)(n-5)(n-6)}{1\cdot 2\cdot 3}z^{n-7} + \text{etc.}) \\ + b(z^{n-2} - (n-2)z^{n-4} + \frac{(n-2((n-5)}{1\cdot 2}z^{n-6} - \frac{(n-2)(n-6)(n-7)}{1\cdot 2\cdot 3}z^{n-8} + \text{etc.}) \\ - c(z^{n-3} - (n-3)z^{n-5} + \frac{(n-3)(n-6}{1\cdot 2}z^{n-7} - \frac{(n-3)(n-7)(n-8)}{1\cdot 2\cdot 3})z^{n-9} + \text{etc.}) \\ \qquad\qquad \text{etc.} \end{array}\right\} = 0$$

Ad

Ad expreſſionem hanc diſtinctius cognoſcendam ſit ψ arcus circuli cuius coſinus $= \frac{z}{2}$, erit

2 coſ. A. $k\psi = z^k - kz^{k-2} + \frac{k(k-3)}{1\cdot 2} z^{k-4} - \frac{k(k-4)(k-5)}{1\cdot 2\cdot 3} z^{k-6} +$ etc.

vnde ſuperior aequatio pro incognita z transmutabitur in ſequentem;

coſ. A. $n\psi - a$ coſ. A $(n-1)\psi + b$ coſ. A $(n-2)\psi - c$ coſ. A $(n-3)\psi + \ldots + \frac{m}{2} = 0$

Ex hoc aequatione reperientur n diuerſi valores pro ψ; quorum coſinus bis ſumti dabunt totidem valores quaeſitos pro $\alpha, \beta, \gamma, \delta$, etc. Ex hac generali reductione theorematis Moivreani pro noſtro caſu, quo omnes litterae a, b, c, etc. euaneſcunt, obtinemus hanc ſimplicem aequationem coſ. A. $n\psi = 0$. Quaeramus ergo omnes arcus, quorum coſinus ſunt $= 0$, qui ſunt

$\frac{\pi}{2}; \frac{3\pi}{2}; \frac{5\pi}{2}; \frac{7\pi}{2}; \frac{9\pi}{2}; \frac{11\pi}{2}$; etc.

denotante $1 : \pi$ rationem diametri ad peripheriam ex his valoribus capiantur n, qui erunt totidem valores pro $n\psi$, vnde ipſius ψ valores erunt ſequentes arcus in circulo cuius radius $= 1$

$\frac{\pi}{2n}; \frac{3\pi}{2n}; \frac{5\pi}{2n}; \frac{7\pi}{2n}; \frac{9\pi}{2n}; \frac{11\pi}{2n}; \ldots\ldots \frac{(2n-1)\pi}{2n}$.

Ex his valores pro coefficientibus $\alpha, \beta, \gamma, \delta$, etc. erunt 2 coſ. A. $\frac{\pi}{2n}$; 2 coſ. A. $\frac{3\pi}{2n}$; 2 coſ. A. $\frac{5\pi}{2n}$; 2 coſ. A. $\frac{7\pi}{2n}$; ...; 2 coſ. A $\frac{(2n-1)\pi}{2n}$. At eſt coſ. A. $\frac{(2n-1)\pi}{2n} = -$ coſ. A. $\frac{\pi}{2n}$; coſ. A $\frac{(2n-3)\pi}{2n} = -$ coſ. A. $\frac{3\pi}{2n}$ vnde ſi n fuerit numerus par, valores pro $\alpha, \beta, \gamma, \delta$, etc. erunt.

± 2 coſ. A. $\frac{\pi}{2n}$; ± 2 coſ. A. $\frac{3\pi}{2n}$; ± 2 coſ. A. $\frac{5\pi}{2n}$; .... ± 2 coſ. A. $\frac{(n-1)\pi}{2n}$ ſin autem n ſit numerus impar, tum valores litte-

litterarum $\alpha, \beta, \gamma, \delta,$ etc. erunt sequentes

$\pm 2 \operatorname{cof.A.} \frac{\pi}{2n}; \pm 2 \operatorname{cof.A.} \frac{3\pi}{2n}; \pm 2 \operatorname{cof.A.} \frac{5\pi}{2n}; \ldots \pm 2 \operatorname{cof.} \operatorname{A.} \frac{(n-2)\pi}{2n} + 2 \operatorname{cof.A.} \frac{\pi}{2}.$

Ex quo patet casus, quibus n vel est numerus par vel impar, probe a se inuicem esse distinguendos in nostro instituto.

Denominatoris ergo nostri $1 + x^{2n}$ factores trinomiales omnes continentur in hac forma generali

$$1 + 2x \operatorname{cof.A.} \frac{k\pi}{2n} + xx$$

denotante k omnes numeros impares minores quam $2n$. Comparetur forma assumta $1 + px + qxx$ cum hac inventa, erit $q = 1$, et $p = + 2 \operatorname{cof.A.} \frac{k\pi}{2n}$: hincque $\frac{p}{2\sqrt{q}} = \operatorname{cof.A.} \frac{k\pi}{2n}$. Cum iam ϕ sit arcus circuli cuius cosinus est $\frac{p}{2\sqrt{q}}$, erit $\phi = \frac{k\pi}{2n}$. Ex isto igitur factore generali reperietur pars integralis inde oriunda haec:

$$\pm \frac{1}{2n} \operatorname{cof.A.} \frac{(2n-m-1)k\pi}{2n} l\left(1 + 2x \operatorname{cof.A.} \frac{k\pi}{2n} + xx\right) \mp \frac{1}{n} \operatorname{fin.A.} \frac{(2n-m-1)k\pi}{2n} \operatorname{A tang.} \frac{x \operatorname{fin.A.} \frac{k\pi}{2n}}{1 + x \operatorname{cof.A.} \frac{k\pi}{2n}}$$

vbi signorum ambiguorum ante coefficientes superiora valent, si $2n-m-1$ sit numerus par, inferiora si impar. In casu quo $k = n$, quo occurrit si n est numerus impar, tum fit $\operatorname{cof.A.} \frac{k\pi}{2n} = 0$, et denominatoris diuisor erit $1 + xx$ ex quo nascitur integrale

$$\pm \frac{1}{2n} \operatorname{cof.A.} \frac{(2n-m-1)\pi}{2} l(1 + xx) \mp \frac{1}{n} \operatorname{fin.A.} \frac{(2n-m-1)\pi}{2} \operatorname{A tang.} x$$

Quod si iam loco k successiue substituantur omnes numeri impares minores quam $2n$, et omnes expressiones in

vnam

vnam summam colligantur, habebitur integrale quaesitum formulae differentialis propositae

$$\frac{x^m dx}{1 + x^{2n}}$$

si quidem m fuerit numerus minor quam $2n$.

Q. E. I.

Exemplum. 1.

§. 45. Formulae differentialis $\frac{dx}{1+xx}$ integrale inuenire: Hic fit $m = 0$; $n = 1$; $2n - m - 1 = 1$ numero impari, valent ergo signa inferiora, habebiturque

$$-\tfrac{1}{2}\,\text{cos. A.}\,\tfrac{k\pi}{2}\,l\left(1 + 2x\,\text{cos. A.}\,\tfrac{k\pi}{2} + xx\right) + \text{sin. A.}\,\tfrac{k\pi}{2}.\ \text{A tang.}\,\frac{x\,\text{sin. A.}\,\frac{k\pi}{2}}{1 + x\,\text{cos. A.}\,\frac{k\pi}{2}}$$

Ob $2n = 2$ habebit k vnicum valorem nempe $k = 1$ ex quo propter cos. A. $\frac{\pi}{2} = 0$, et sin. A. $\frac{\pi}{2} = 1$, reperietur integrale quaesitum $=$ A tang. x.

Exemplum. 2.

§. 46. Formulae differentialis $\frac{dx}{1+x^4}$ integrale inuenire: Hic fit $m = 0$; $n = 2$; $2n - m - 1 = 3$ numero impari, ita vt signa inferiora valeant: habetur ergo

$$-\tfrac{1}{4}\,\text{cos. A.}\,\tfrac{3k\pi}{4}\,l\left(1 + 2x\,\text{cos. A.}\,\tfrac{k\pi}{4} + xx\right) + \tfrac{1}{2}\,\text{sin. A.}\,\tfrac{3k\pi}{4}.\ \text{A tang.}\,\frac{x\,\text{sin. A.}\,\frac{k\pi}{4}}{1 + x\,\text{cos. A.}\,\frac{k\pi}{4}}$$

ob $2n = 4$ tribuendi sunt ipsi k duo valores 1 et 3 successiue, ex quo integrale quaesitum erit:

$-\frac{1}{4}\operatorname{cos}.A.\frac{3\pi}{4}.\,l(1+2x\operatorname{cos}.A.\frac{\pi}{4}+xx)+\frac{1}{2}\operatorname{sin}.A.\frac{3\pi}{4}$

$$\text{A tang.}\frac{x\operatorname{sin}.A.\frac{\pi}{4}}{1+x\operatorname{cos}.A.\frac{\pi}{4}}$$

$-\frac{1}{4}\operatorname{cos}.A.\frac{9\pi}{4}.\,l(1+2x\operatorname{cos}.A.\frac{3\pi}{4}+xx)+\frac{1}{2}\operatorname{sin}.A.\frac{9\pi}{4}.$

$$\text{A tang.}\frac{x\operatorname{sin}.A.\frac{3\pi}{4}}{1+x\operatorname{cos}.A.\frac{3\pi}{4}}$$

At eſt $\operatorname{cof}.A.\frac{3\pi}{4}=-\operatorname{cof}.A.\frac{\pi}{4}$; et $\operatorname{cof}.A.\frac{9\pi}{4}=\operatorname{cof}.A\frac{\pi}{4}$; itemque, $\operatorname{fin}.A.\frac{3\pi}{4}=\operatorname{fin}.A.\frac{\pi}{4}$ et $\operatorname{fin}.A.\frac{9\pi}{4}=\operatorname{fin}.A.\frac{\pi}{4}$ vnde tandem formulae propoſitae integrale prodit:

$+\frac{1}{4}\operatorname{cof}.A.\frac{\pi}{4}.\,l(1+2x\operatorname{cof}.A.\frac{\pi}{4}+xx)+\frac{1}{2}\operatorname{fin}.A.\frac{\pi}{4}.$

$$\text{A. tang.}\frac{x\operatorname{fin}.A.\frac{\pi}{4}}{1+x\operatorname{cof}.A.\frac{\pi}{4}}$$

$-\frac{1}{4}\operatorname{cof}.A.\frac{\pi}{4}\,l(1-2x\operatorname{cof}.A.\frac{\pi}{4}+xx)+\frac{1}{2}\operatorname{fin}.A.\frac{\pi}{4}.$

$$\text{A tang.}\frac{x\operatorname{fin}.A.\frac{\pi}{4}}{1-x\operatorname{cof}.A.\frac{\pi}{4}}$$

quod ad formam ſimpliciorem reductum ob $\operatorname{fin}.A.\frac{\pi}{4}=\operatorname{cof}.A.\frac{\pi}{4}=\frac{1}{\sqrt{2}}$ dabit

$$\frac{1}{4\sqrt{2}}\,l\,\frac{1+x\sqrt{2}+xx}{1-x\sqrt{2}+xx}+\frac{1}{2\sqrt{2}}\text{ A tang.}\frac{x\sqrt{2}}{1-xx}.$$

Exemplum. 3.

§. 47. Formulae differentialis $\frac{dx}{1+x^6}$ integrale inuenire. Hic eſt $m=0$, $n=3$, $2n-m-1=5$ numero impari, vnde haec habetur partium integralis forma

$-\frac{1}{3}\operatorname{cof}.A.\frac{5k\pi}{6}\,l(1+2x\operatorname{cof}.A.\frac{k\pi}{6}+xx)+\frac{1}{3}\operatorname{fin}.A.\frac{5k\pi}{6}$

$$\text{A tang.}\frac{x\operatorname{fin}.A.\frac{k\pi}{6}}{1+x\operatorname{cof}.A.\frac{k\pi}{6}}$$

vbi loco k ſucceſſiue ſubſtitui debent numeri 1, 3, 5, at eſt

cof.

$\text{cos. A.}\frac{5\pi}{6} = -\text{cos. A.}\frac{\pi}{6}, \text{sin. A.}\frac{5\pi}{6} = \text{sin. A.}\frac{\pi}{6}$
$\text{cos. A.}\frac{15\pi}{6} = \text{cos. A.}\frac{\pi}{2}; \text{sin. A.}\frac{15\pi}{6} = \text{sin. A.}\frac{\pi}{2}$
$\text{cos. A.}\frac{25\pi}{6} = \text{cos. A.}\frac{\pi}{6}; \text{sin. A.}\frac{25\pi}{6} = \text{sin. A}\frac{\pi}{6}$
Ex quibus ob $\text{cos. A.}\frac{\pi}{2} = 0$ et $\text{sin. A.}\frac{\pi}{2} = 1$ colligitur integrale quaesitum:

$$+\tfrac{1}{6}\text{cos. A.}\tfrac{\pi}{6}\, l\,(1 + 2x\,\text{cos. A.}\tfrac{\pi}{6} + xx) + \tfrac{1}{3}\text{sin. A.}\tfrac{\pi}{6}\ \text{A tang.}\frac{x\,\text{sin. A.}\frac{\pi}{6}}{1 + x\,\text{cos. A.}\frac{\pi}{6}}$$
$$-\tfrac{1}{6}\text{cos. A.}\tfrac{\pi}{6}\, l\,(1 - 2x\,\text{cos. A.}\tfrac{\pi}{6} + xx) + \tfrac{1}{3}\text{A tang.}\,x$$
$$+\tfrac{1}{3}\text{sin. A.}\tfrac{\pi}{6}.\ \text{A tang.}\frac{x\,\text{sin. A.}\frac{5\pi}{6}}{1 + x\,\text{cos. A.}\frac{5\pi}{6}}$$

Cum iam sit $\text{cos. A.}\frac{\pi}{6} = \frac{\sqrt{3}}{2}$; $\text{sin. A.}\frac{\pi}{6} = \frac{1}{2}$; erit integrale
$$\frac{\sqrt{3}}{12}\, l\,\frac{1 + x\sqrt{3} + xx}{1 - x\sqrt{3} + xx} + \tfrac{1}{3}\text{A tang.}\,x + \tfrac{1}{6}\text{A tang.}\frac{x}{2 + x\sqrt{3}} + \tfrac{1}{6}\text{A tang.}\frac{x}{2 - x\sqrt{3}}.$$

Exemplum. 4.

§. 48. Formulae differentialis $\frac{dx}{1+x^8}$ integrale inuenire.
Hic est $m = 0$, $n = 4$; et $2n - m - 1 = 7$, ex quo erit partium integralis forma:

$$-\tfrac{1}{8}\text{cos. A.}\tfrac{7k\pi}{8}\, l\,(1 + 2x\,\text{cos. A.}\tfrac{k\pi}{8} + xx) + \tfrac{1}{4}\text{sin. A.}\tfrac{7k\pi}{8}\ \text{A tang.}\frac{x\,\text{sin. A.}\frac{k\pi}{8}}{1 + x\,\text{cos. A.}\frac{k\pi}{8}}$$

At quia k est numerus impar quippe 1, 3, 5, 7 successiue est $-\text{cos. A.}\frac{7k\pi}{8} = \text{cos. A.}\frac{k\pi}{8}$; et $\text{sin. A.}\frac{7k\pi}{8} = \text{sin. A.}\frac{k\pi}{8}$ vnde substituendo loco k valores 1, 3, 5, 7, erit integrale

$$+\tfrac{1}{8}\text{cos. A.}\tfrac{\pi}{8}\, l\,\frac{1 + 2x\,\text{cos. A.}\frac{\pi}{8} + xx}{1 - 2x\,\text{cos. A.}\frac{\pi}{8} + xx} + \tfrac{1}{4}\text{sin. A.}\tfrac{\pi}{8}\ \text{A tang.}\frac{2x\,\text{sin. A.}\frac{\pi}{8}}{1 - xx}$$

 +

$$+\frac{1}{8}\operatorname{cos}.A.\frac{3\pi}{8}\, l\, \frac{1+2x\operatorname{cos}.A.\frac{3\pi}{8}+xx}{1-2x\operatorname{cos}.A.\frac{3\pi}{8}+xx}+\frac{1}{4}\operatorname{fin}.A.\frac{3\pi}{8}\ \text{A tang.}\frac{2x\operatorname{fin}.A.\frac{3\pi}{8}}{1-xx}$$

Exemplum 5.

§. 49. Formulae differentialis $\frac{dx}{1+x^{2n}}$ integrale invenire.

Si $2n$ fuerit numerus pariter par, seu n numerus par, tum integrale erit

$$+\frac{1}{2n}\operatorname{cos}.A.\frac{\pi}{2n}\, l\, \frac{1+2x\operatorname{cos}.A.\frac{\pi}{2n}+xx}{1-2x\operatorname{cos}.A.\frac{\pi}{2n}+xx}+\frac{1}{n}\operatorname{fin}.A.\frac{\pi}{2n}\ \text{A tang.}\frac{2x\operatorname{fin}.A.\frac{\pi}{2n}}{1-xx}$$

$$+\frac{1}{2n}\operatorname{cos}.A.\frac{3\pi}{2n}\, l\, \frac{1+2x\operatorname{cos}.A.\frac{3\pi}{2n}+xx}{1-2x\operatorname{cos}.A.\frac{3\pi}{2n}+xx}+\frac{1}{n}\operatorname{fin}.A.\frac{3\pi}{2n}\ \text{A tang}\frac{2x\operatorname{fin}.A.\frac{3\pi}{2n}}{1-xx}$$

$$+\frac{1}{2n}\operatorname{cos}.A.\frac{5\pi}{2n}\, l\, \frac{1+2x\operatorname{cos}.A.\frac{5\pi}{2n}+xx}{1-2x\operatorname{cos}.A.\frac{5\pi}{2n}+xx}+\frac{1}{n}\operatorname{fin}.A.\frac{5\pi}{2n}\ \text{A tang}\frac{2x\operatorname{fin}.A.\frac{5\pi}{2n}}{1-xx}$$

$$\vdots$$

$$+\frac{1}{2n}\operatorname{cos}.A.\frac{(n-1)\pi}{2n}\, l\, \frac{1+2x\operatorname{cos}.A.\frac{(n-1)\pi}{2n}+xx}{1-2x\operatorname{cos}.A.\frac{(n-1)\pi}{2n}+xx}+\frac{1}{n}\operatorname{fin}.A.\frac{(n-1)\pi}{2n}\ \text{A tang.}\frac{2x\operatorname{fin}.A.\frac{(n-1)\pi}{2n}}{1-xx}$$

Sin autem $2n$ fuerit numerus impariter par, seu n numerus impar, tum erit integrale

+

$$+\tfrac{1}{2n}\operatorname{cof.A}.\tfrac{\pi}{2n}\, l\frac{1+2x\operatorname{cof.A}.\frac{\pi}{2n}+xx}{1-2x\operatorname{cof.A}.\frac{\pi}{2n}+xx}+\tfrac{1}{n}\operatorname{fin.A}.\tfrac{\pi}{2n}.\operatorname{A tang.}\frac{2x\operatorname{fin.A}.\frac{\pi}{2n}}{1-xx}$$

$$+\tfrac{1}{2n}\operatorname{cof.A}.\tfrac{3\pi}{2n}\, l\frac{1+2x\operatorname{cof.A}.\frac{3\pi}{2n}+xx}{1-x2\operatorname{cof.A}.\frac{3\pi}{2n}+xx}+\tfrac{1}{n}\operatorname{fin.A}.\tfrac{3\pi}{2n}.\operatorname{A tang.}\frac{2x\operatorname{fin.A}.\frac{3\pi}{2n}}{1-xx}$$

$$+\tfrac{1}{2n}\operatorname{cof.A}.\tfrac{5\pi}{2n}\, l\frac{1+2x\operatorname{cof.A}.\frac{5\pi}{2n}+xx}{1-2x\operatorname{cof.A}.\frac{5\pi}{2n}+xx}+\tfrac{1}{n}\operatorname{fin.A}.\tfrac{5\pi}{2n}.\operatorname{A tang.}\frac{2x\operatorname{fin.A}.\frac{5\pi}{2n}}{1-xx}$$

$$\vdots$$

$$+\tfrac{1}{2n}\operatorname{cof.A}\tfrac{(n-2)\pi}{2n}\, l\frac{1+2x\operatorname{cof.A}\frac{(n-2)\pi}{2n}+xx}{1-2x\operatorname{cof.A}\frac{(n-2)\pi}{2n}+xx}+\tfrac{1}{n}\operatorname{fin.A}.\tfrac{(n-2)\pi}{2n}\operatorname{A tang.}\frac{2x\operatorname{fin.A}\frac{(n-2)\pi}{2n}}{1-xx}+\tfrac{1}{n}\operatorname{A tang.}x$$

Exemplum 6.

§. 50. Formulae differentialis $\frac{x^m dx}{1+x^{2n}}$ integrale invenire exiſtente m numero pari.

Quoniam m eſt numerus par erit $2n-m-1$ numerus impar, ideoque ſigna inferiora valent; Porro autem erit $\operatorname{cof.A}\frac{(2n-m-1)k\pi}{2n}=-\operatorname{cof.A}\frac{(m+1)k\pi}{2n}$ ob k numerum imparem et $\operatorname{fin.A}\frac{(2n-m-1)k\pi}{2n}=\operatorname{fin.A}\frac{(m+1)k\pi}{2n}$; tum etiam habetur

betur $\cos. A \frac{(m+1)(2n-k)\pi}{2n} = -\cos. A \frac{(m+1)k\pi}{2n}$ atque $\sin. A. \frac{(m+1)(2n-k)\pi}{2n} = \sin. A. \frac{(m+1)k\pi}{2n}$.

His positis distinguendi sunt casus, quibus n est vel numerus par vel impar.

Ac primo quidem si n est numerus par erit integrale

$$+\frac{1}{2n}\cos. A. \frac{(m+1)\pi}{2n}\, l\, \frac{1+2x\cos A.\frac{\pi}{2n}+xx}{1-2x\cos. A.\frac{\pi}{2n}+xx} + \frac{1}{n}\sin. A. \frac{(m+1)\pi}{2n}\, A\,\text{tang.}\frac{2x\sin. A.\frac{\pi}{2n}}{1-xx}$$

$$+\frac{1}{2n}\cos. A. \frac{3(m+1)\pi}{2n}\, l\, \frac{1+2x\cos A.\frac{3\pi}{2n}+xx}{1-2x\cos. A.\frac{3\pi}{2n}+xx} + \frac{1}{n}\sin. A. \frac{3(m+1)\pi}{2n}\, A\,\text{tang.}\frac{2x\sin. A.\frac{3\pi}{2n}}{1-xx}$$

$$+\frac{1}{2n}\cos. A. \frac{5(m+1)\pi}{2n}\, l\, \frac{1+2x\cos. A.\frac{5\pi}{2n}+xx}{1-2x\cos. A.\frac{5\pi}{2n}+xx} + \frac{1}{n}\sin. A. \frac{5(m+1)\pi}{2n}\, A\,\text{tang.}\frac{2x\sin. A.\frac{5\pi}{2n}}{1-xx}$$

$$\vdots$$

$$+\frac{1}{2n}\cos. A. \frac{(n-1)(m+1)\pi}{2n}\, l\, \frac{1+2x\cos. A.\frac{(n-1)\pi}{2n}+xx}{1-2x\cos. A.\frac{(n-1)\pi}{2n}+xx} + \frac{1}{n}\sin. A. \frac{(n-1)(m+1)}{2n}\, A\,\text{tang.}\frac{2x\sin. A\frac{(n-1)\pi}{2n}}{1-xx}$$

Deinde si n sit numerus impar erit integrale;

$$+\frac{1}{2n}\cos. A. \frac{(m+1)\pi}{2n}\, l\, \frac{1+2x\cos. A.\frac{\pi}{2n}+xx}{1-2x\cos. A.\frac{\pi}{2n}+xx} + \frac{1}{n}\sin. A. \frac{(m+1)\pi}{2n}\, A\,\text{tang.}\frac{2x\sin. A.\frac{\pi}{2n}}{1-xx}$$

+

$$+\frac{1}{2n}\operatorname{cof.A.}\frac{3(m+1)\pi}{2n}\, l\, \frac{1+2x\operatorname{cof.A.}\frac{3\pi}{2n}+xx}{1-2x\operatorname{cof.A.}\frac{3\pi}{2n}+xx}+\frac{1}{n}\operatorname{fin.A.}\frac{3(m+1)\pi}{2n}\operatorname{A\,tang.}\frac{2x\operatorname{fin.A.}\frac{3\pi}{2n}}{1-xx}$$

$$+\frac{1}{2n}\operatorname{cof.A.}\frac{5(m+1)\pi}{2n}\, l\, \frac{1+2x\operatorname{cof.A.}\frac{5\pi}{2n}+xx}{1-2x\operatorname{cof.A.}\frac{5\pi}{2n}+xx}+\frac{1}{n}\operatorname{fin.A.}\frac{5(m+1)\pi}{2n}\operatorname{A.tang.}\frac{2x\operatorname{fin.A.}\frac{5\pi}{2n}}{1-xx}$$

$$\vdots$$

$$+\frac{1}{2n}\operatorname{cof.A.}\frac{(n-2)(m+1)\pi}{2n}\, l\, \frac{1+2x\operatorname{cof.A}\frac{(n-2)\pi}{2n}+xx}{1-2x\operatorname{cof.A}\frac{(n-2)\pi}{2n}+xx}+\frac{1}{n}\operatorname{fin.A.}\frac{(n-2)(m+1)\pi}{2n}\operatorname{A\,tang.}\frac{2x\operatorname{fin.A.}\frac{(n-2)\pi}{2n}}{1-xx}$$

$$+\frac{1}{n}\operatorname{fin.A.}\frac{(m+1)\pi}{2n}\operatorname{A\,tang.}x.$$

vbi si m est numerus pariter par, erit $\operatorname{fin.A.}\frac{(m+1)\pi}{2}=1$ at si m est impariter par, erit $\operatorname{fin.A.}\frac{(m+1)\pi}{2}=-1.$

Exemplum. 7.

§. 51. Formulae differentialis $\frac{x^m dx}{1+x^{2n}}$ integrale inuenire, si m fit numerus impar.

Quoniam $2n-m-1$ est numerus par, signa superiora valent, eritque integralis pars quaeuis huius formae

$$-\tfrac{1}{2n}\,\text{cof. A.}\,\tfrac{(m+1)k\pi}{2n}\,l\,(1+2x\,\text{cof. A.}\,\tfrac{k\pi}{2n}+xx)-\tfrac{1}{n}\,\text{fin. A.}\,\tfrac{(m+1)k\pi}{2n}\,\text{A. tang.}\,\frac{x\,\text{fin. A.}\,\frac{k\pi}{2n}}{1+x\,\text{cof. A.}\,\frac{k\pi}{2n}}$$

vbi loco k omnes numeri impares vsque ad $2n-1$ fubftitui debent. Si pro k ponamus $2n-k$, habebitur haec forma

$$-\tfrac{1}{2n}\,\text{cof. A.}\,\tfrac{(m+1)k\pi}{2n}\,l\,(1-2x\,\text{cof. A.}\,\tfrac{k\pi}{2n}+xx)+\tfrac{1}{n}\,\text{fin. A.}\,\tfrac{(m+1)k\pi}{2n}\,.\,\text{A. tang.}\,\frac{x\,\text{fin. A.}\,\frac{k\pi}{2n}}{1-x\,\text{cof. A.}\,\frac{k\pi}{2n}}$$

quae duae formae coniunctim fumtae dabunt:

$$-\tfrac{1}{2n}\,\text{cof. A.}\,\tfrac{(m+1)k\pi}{2n}\,l\,(1-2xx\,\text{cof. A.}\,\tfrac{k\pi}{n}+x^4)+\tfrac{1}{n}\,\text{fin. A.}\,\tfrac{(m+1)k\pi}{2n}\,\text{A tang.}\,\frac{xx\,\text{fin. A.}\,\frac{k\pi}{n}}{1-xx\,\text{cof. A.}\,\frac{k\pi}{n}}$$

Hoc modo bini valores ipfius k coniunguntur, vnde fi n fit numerus par integrale reperitur fi loco k fucceffiui omnes numeri impares vsque ad $n-1$ fubftituantur, erit ergo integrale:

$$-\tfrac{1}{2n}\,\text{cof. A}\,\tfrac{(m+1)\pi}{2n}\,l\,(1-2xx\,\text{cof. A}\,\tfrac{\pi}{n}+x^4)+\tfrac{1}{n}\,\text{fin. A.}\,\tfrac{(m+1)\pi}{2n}\,\text{A tang.}\,\frac{xx\,\text{fin. A.}\,\frac{\pi}{n}}{1-xx\,\text{cof. A.}\,\frac{\pi}{n}}$$

$$-\tfrac{1}{2n}\,\text{cof. A.}\,\tfrac{3(m+1)\pi}{2n}\,l\,(1-2xx\,\text{cof. A.}\,\tfrac{3\pi}{n}+x^4)+\tfrac{1}{n}\,\text{fin. A.}\,\tfrac{3(m+1)\pi}{2n}\,\text{A tang.}\,\frac{xx\,\text{fin. A.}\,\frac{3\pi}{n}}{1-xx\,\text{cof. A.}\,\frac{3\pi}{n}}$$

$$-\tfrac{1}{2n}\,\text{cof. A.}\,\tfrac{5(m+1)\pi}{2n}\,l\,(1-2xx\,\text{cof. A.}\,\tfrac{5\pi}{n}+x^4)+\tfrac{1}{n}\,\text{fin. A.}\,\tfrac{5(m+1)\pi}{2n}\,\text{A tang.}\,\frac{xx\,\text{fin. A.}\,\frac{5\pi}{n}}{1-xx\,\text{cof. A.}\,\frac{5\pi}{n}}$$

$\vdots$

$$-\frac{1}{2n}\operatorname{cof.} A. \frac{(n-1)(m+1)\pi}{2n}\, l\left(1-2xx\operatorname{cof.} A \frac{(n-1)\pi}{n} + x^4\right) + \frac{1}{n}$$
$$\operatorname{fin.} A. \frac{(n-1)(m+1)\pi}{2n} A \operatorname{tang.} \frac{xx \operatorname{fin.} A. \frac{(n-1)\pi}{n}}{1-xx\operatorname{cof.} A \frac{(n-1)\pi}{n}}$$

Quodſi autem n ſit numerus impar, tum loco k ſubſtitui debent omnes numeri impares vsque ad $n-2$, et numerus impar medius n erit ſolitarius, vnde ſequens prodibit integrale.

$$-\frac{1}{2n}\operatorname{cof.} A. \frac{(m+1)\pi}{2n}\, l\left(1-2xx\operatorname{cof.} A. \frac{\pi}{n} + x^4\right) + \frac{1}{n}\operatorname{fin.} A.$$
$$\frac{(m+1)\pi}{2n} A \operatorname{tang.} \frac{xx \operatorname{fin.} A. \frac{\pi}{n}}{1-xx\operatorname{cof.} A. \frac{\pi}{n}}$$

$$-\frac{1}{2n}\operatorname{cof.} A. \frac{3(m+1)\pi}{2n}\, l\left(1-2xx\operatorname{cof.} A. \frac{3\pi}{n} + x^4\right) + \frac{1}{n}\operatorname{fin.} A.$$
$$\frac{3(m+1)\pi}{2n} A \operatorname{tang.} \frac{xx \operatorname{fin.} A. \frac{3\pi}{n}}{1-xx\operatorname{cof.} A. \frac{3\pi}{n}}$$

$$-\frac{1}{2n}\operatorname{cof.} A. \frac{5(m+1)\pi}{2n}\, l\left(1-2xx\operatorname{cof.} A. \frac{5\pi}{n} + x^4\right) + \frac{1}{n}\operatorname{fin.} A.$$
$$\frac{5(m+1)\pi}{2n} A \operatorname{tang.} \frac{xx \operatorname{fin} A. \frac{5\pi}{n}}{1-xx\operatorname{cof.} A. \frac{5\pi}{n}}$$

$\vdots$

$$-\frac{1}{2n}\operatorname{cof.} A \frac{(n-2)(m+1)\pi}{2n}\, l\left(1-2xx\operatorname{cof.} A \frac{(n-2)\pi}{n} + x^4\right) + \frac{1}{n}\operatorname{fin.} A$$
$$\frac{(n-2)(m+1)\pi}{2n} A \operatorname{tang.} \frac{xx \operatorname{fin.} A. \frac{(n-2)\pi}{n}}{1-xx\operatorname{cof.} A. \frac{(n-2)\pi}{n}}$$
$$-\frac{1}{2n}\operatorname{cof.} A \frac{(m+1)\pi}{2}\, l\,(1+xx)$$

 vbi

vbi cof. A $\frac{(m+1)\pi}{2}$ erit vel $+1$ vel -1, prout $m+1$ fuerit vel numerus pariter par vel impariter par.

Problema 2.

§. 52. Inuenire integrale huius formulae differentialis $\frac{x^m dx}{1+x^{2n+1}}$, exiſtente m numero integro minore quam $2n+1$.

Solutio.

Huius formulae denominator $1+x^{2n+1}$ vnum habet factorem realem $1+x$, reliqui factores ſimplices omnes ſunt imaginarii; eorumque adeo loco factores trinomiales accipi debent. Quod ad factorem ſimplicem $1+x$ attinet; intelligitur ſi pro eo ponatur $1+rx$, inde hanc integralis partem eſſe orituram $\frac{-(-r)^{2n-m+1}}{(2n+1}\int\frac{dx}{1+rx}$. Sit ergo $r=1$, atque ex denominatoris factore $1+x$ naſcetur integralis pars $\frac{-(-1)^{2n-m+1}}{(2n+1)}\ l(1+x)=\frac{(-1)^{2n-m}}{(2n+1)}$ $l(1+x)$ quae adeo erit $\pm\frac{1}{2n+1}\ l(1+x)$ vbi ſignum $+$ valet, ſi $2n-m$ fuerit numerus par, ſeu ſi m ſit numerus par, alterum vero ſignum $-$ valet, ſi m ſit numerus impar.

Quod iam ad factores trinomiales attinet, ſit eorum quilibet $1+px+qxx$; atque ex ſolutione praecedentis problematis ponendo $2n+1$ loco $2n$ colligitur integralis pars ex hoc factore oriunda $=$

$$\pm q^{\frac{2n-m}{2}}\frac{\text{cof.A}(2n-m)\Phi}{2n+1}\,l(1+px+qxx)\mp 2q^{\frac{2n-m}{2}}\frac{\text{fin.A}(2n-m)\Phi}{2n+1}\,\text{A tang.}\frac{x\surd(4q-pp)}{2+px}$$

de-

denotante ϕ arcum circuli, cuius cofinus $= \frac{p}{2\sqrt{q}}$; valent autem fignorum ambiguorum fuperiora, fi $2n-m$ fuerit numerus par, hoc eft fi m fuerit numerus par, fin autem m fit numerus impar, valebunt figna inferiora.

Iam ad factores trinomiales inueniendos diuidatur denominator $1+x^{2n+1}$ per factorem fimplicem $1+x$, prodibitque quotus:

$$1-x+x^2-x^3+\ldots\ldots-x^{2n-3}+x^{2n-2}-x^{2n-1}+x^{2n}$$

cuius factores trinomiales quaeri oportet, id quod fiet ope theorematis in folutione praecedente adhibiti. Erunt fcilicet factores trinomiales huiusmodi $1+\alpha x+xx$; $1+\beta x+xx$; $1+\gamma x+xx$ etc. quorum numerus erit n, et cum fit $a=-1$, $b=+1$, $c=-1$, etc. formanda eft aequatio

$$\text{cof. A}\,n\psi+\text{cof. A}\,(n-1)\,\psi+\text{cof. A}\,(n-2)\,\psi+\ldots\ldots+\text{cof. A}\,\psi+\tfrac{1}{2}=0$$

ex qua aequatione n valores pro arcu ψ orientur, quorum cofinus bis fumti in loca litterarum α, β, γ, etc. fubftitui debent. Aequatio autem haec refolui poterit ex ifto principio, quod cofinus arcuum in arithmetica progreffione crefcentium tenent feriem recurrentem: eft nempe cof. A. $n\psi = z$ cof. A $(n-1)\,\psi$ − cof. A $(n-2)\,\psi$ exiftente cof. A $\psi = \frac{z}{2}$. Cum iam fit

$$+\text{cof.A.}\,n\psi+\text{cof.A.}\,(n-1)\psi+\ldots\ldots+\text{cof.A.}\,2\psi+\text{cof.A.}\,\psi+1=\tfrac{1}{2}$$

erit
$$+z\,\text{cof.A}\,n\psi+z\,\text{cof.A}(n-1)\psi+z\,\text{cof.A}(n-2)\psi+\ldots\ldots+z\,\text{cof.A}\,\psi+z\;=\tfrac{z}{2}$$
$$-\text{cof.A}\,n\psi-\text{cof.A}(n-1)\psi-\text{cof.A}(n-2)\psi-\text{cof.A}(n-3)\psi-\ldots\ldots-1\;=-\tfrac{1}{2}$$

fubtrahantur inferiores aequationes a fuperiore, orietur

$$(1-z)\,\text{cof.A}\,n\psi+\text{cof.A}\,(n-1)\,\psi+\text{cof.A.}\,\psi+1-z=1-\tfrac{z}{2}$$

 quae

quae ob $\text{cof. A}\,\psi = \frac{z}{2}$ transmutatur in hanc

$(1-z)\,\text{cof. A}.\,n\psi + \text{cof. A}\,(n-1)\,\psi = 0$ seu

$\text{cof. A}.\,n\psi - 2\,\text{cof. A}.\,\psi.\,\text{cof. A}\,n\psi + \text{cof. A}\,(n-1)\,\psi = 0$

at eft $\text{cof. A}(n-1)\psi = \text{cof. A}.\,\psi.\,\text{cof. A}.\,n\psi + \text{fin. A}\,\psi\,\text{fin. A}.\,n\psi$

erit ergo

$(1 - \text{cof. A}.\,\psi)\,\text{cof. A}.\,n\psi + \text{fin. A}.\,\psi.\,\text{fin. A}\,n\psi = 0$; vnde concluditur fore

$$\text{tang. A}.\,\tfrac{\psi}{2} + \text{tang. A}.\,n\psi = 0.$$

At ex natura tangentium conftat effe:

$$\text{tang. A}.\,\tfrac{\psi}{2} + \text{tang. A}\,(k\pi - \tfrac{\psi}{2}) = 0$$

denotante k numerum quemcunque integrum; ex quo erit $n\psi = k\pi - \frac{\psi}{2}$, hincque $\psi = \frac{2k\pi}{2n+1}$. Subftituendo ergo pro k fucceffiue numeros $1,\ 2,\ 3,\ 4 \ldots\ldots n$, oriuntur n valores pro arcu ψ, quorum cofinus bis fumti dabunt coefficientes α, β, γ, δ, etc. in factoribus $1 + \alpha x + xx$; $1 + \beta x + xx$; $1 + \gamma x + xx$; etc. Quilibet ergo factor continetur in hac forma $1 + 2x\,\text{cof. A}.\,\frac{2k\pi}{2n+1} + xx$. Quare cum pro factore generali affumferimus hanc formam $1 + px + qxx$ erit $q = 1$ et $p = 2\,\text{cof. A}\,\frac{2k\pi}{2n+1}$ atque $\frac{p}{2\sqrt{q}} = \text{cof. A}.\,\frac{2k\pi}{2n+1}$; quia nunc Φ eft arcus, cuius cofinus $= \frac{p}{2\sqrt{q}}$ erit $\Phi = \frac{2k\pi}{2n+1}$; et hanc obrem ex factore $1 + 2x\,\text{cof. A}.\,\frac{2k\pi}{2n+1} + xx$ orietur integralis pars

$$\pm \frac{1}{2n+1}\,\text{cof. A}\,\frac{2k(2n-m)\pi}{2n+1}\; l\left(1 + 2x\,\text{cof. A}.\,\frac{2k\pi}{2n+1} + xx\right)$$

$$\mp \frac{2}{2n+1}\,\text{fin. A}.\,\frac{2k(2n-m)\pi}{2n+1}\;\text{A tang.}\,\frac{x\,\text{fin. A}.\,\frac{2k\pi}{2n+1}}{1 + x\,\text{cof. A}.\,\frac{2k\pi}{2n+1}}$$

vbi

vbi signa superiora valent si m fuerit numerus par, inferiora autem si m numerus impar. At est

$\cos. A. \frac{2k(2n-m)\pi}{2n+1} = \cos. A. \frac{2k(m+1)\pi}{2n+1}$ atque

$\sin. A. \frac{2k(2n-m)\pi}{2n+1} = -\sin. A. \frac{2k(m+1)\pi}{2n+1}$, vnde cuiusque partis integralis ex factore denominatoris trinomiali oriunda est

$$\pm \frac{1}{2n+1} \cos. A. \frac{2k(m+1)\pi}{2n+1}\, l\left(1+2x\cos. A. \frac{2k\pi}{2n+1}+xx\right) \pm \frac{2}{2n+1} \sin. A. \frac{2k(m+1)}{2n+1} \text{ A tang.} \frac{x \sin. A. \frac{2k\pi}{2n+1}}{1+x\cos. A. \frac{2k\pi}{2n+1}}$$

successiue scilicet loco k scribantur omnes numeri integri $1, 2, 3, \ldots\ldots n$; addanturque formae resultantes, atque ad summam addantur insuper integrale ex factore simplici $1+x$ oriundum $\pm \frac{1}{2n+1}\, l(1+x)$, quo facto habebitur formulae differentialis propositae $\frac{x^m dx}{1+x^{2n+1}}$ integrale quaesitum hoc:

$$\pm \frac{1}{2n+1}\, l(1+x)$$

$$\pm \frac{1}{2n+1} \cos. A. \frac{2(m+1)\pi}{2n+1}\, l\left(1+2x\cos. A. \frac{2\pi}{2n+1}+xx\right) \pm \frac{2}{2n+1} \sin. A. \frac{2(m+1)\pi}{2n+1} \text{ A tang.} \frac{x \sin. A. \frac{2\pi}{2n+1}}{1+x\cos. A. \frac{2\pi}{2n+1}}$$

$$\pm \frac{1}{2n+1} \cos. A. \frac{4(m+1)\pi}{2n+1}\, l\left(1+2x\cos. A. \frac{4\pi}{2n+1}+xx\right) \pm \frac{2}{2n+1} \sin. A. \frac{4(m+1)\pi}{2n+1} \text{ A tang.} \frac{x \sin. A. \frac{4\pi}{2n+1}}{1+x\cos. A. \frac{4\pi}{2n+1}}$$

$$\pm \frac{1}{2n+1} \text{cos. A.} \frac{6(m+1)\pi}{2n+1}\, l\left(1+2x \text{ cos. A.} \frac{6\pi}{2n+1} + xx\right) \pm \frac{2}{2n+1}$$

$$\text{sin. A.} \frac{6(m+1)\pi}{2n+1} \text{ A tang.} \frac{x \text{ sin. A.} \frac{6\pi}{2n+1}}{1 + x \text{ cos. A.} \frac{6\pi}{2n+1}}$$

$$\vdots$$

$$\pm \frac{1}{2n+1} \text{cos. A.} \frac{2n(m+1)\pi}{2n+1}\, l\left(1+2x \text{ cos. A.} \frac{2n\pi}{2n+1} + xx\right) \pm \frac{2}{2n+1}$$

$$\text{sin. A.} \frac{2n(m+1)\pi}{2n+1} \text{ A tang.} \frac{x \text{ sin. A.} \frac{2n\pi}{2n+1}}{1 + x \text{ cos. A.} \frac{2n\pi}{2n+1}}$$

vbi ſignorum ambiguorum ſuperiora valent, ſi m fuerit numerus par, inferiora autem ſunt capienda, ſi m ſit numerus impar. Q. E. I.

Exemplum 1.

§. 53. Huius formulae differentialis $\frac{dx}{1+x^3}$ integrale invenire.

Hic eſt $m = 0$, et $n = 1$, atque $2n + 1 = 3$. Deinde habemus

$$\text{cos. A.} \frac{2(m+1)\pi}{2n+1} = \text{cos. A.} \frac{2\pi}{3} = \text{cos. A. } 120° = -\frac{1}{2}$$
$$\text{sin. A.} \frac{2(m+1)\pi}{2n+1} = \text{sin. A.} \frac{2\pi}{3} = \text{sin. A. } 120° = \frac{\sqrt{3}}{2}$$

Quoniam igitur ob m numerum parem ſigna ſuperiora valent, erit formulae propoſitae integrale

$$+ \frac{1}{3}\, l(1+x)$$
$$- \frac{1}{6}\, l(1-x+xx) + \frac{1}{\sqrt{3}} \text{ A tang.} \frac{x\sqrt{3}}{2-x}$$

vbi conſtantem non adiicimus, quia hoc integrale iam euaneſcit poſito $x = 0$.

Exem-

Exemplum. 2.

§. 54. Huius formulae differentialis $\frac{x\,dx}{1+x^3}$ integrale invenire.

Hic eſt $m = 1$ et $n = 1$, ex quo ſigna inferiora valebunt. Erit autem pro hoc caſu:

$$\text{coſ. A. } \frac{2(m+1)\pi}{2n+1} = \text{coſ. A. } \frac{4\pi}{3} = -\text{coſ. A. } \frac{\pi}{3} = -\frac{1}{2}$$

$$\text{coſ. A. } \frac{2\pi}{2n+1} = \text{coſ. A. } \frac{2\pi}{3} = -\text{coſ. A. } \frac{\pi}{3} = -\frac{1}{2}$$

$$\text{ſin. A. } \frac{2(m+1)\pi}{2n+1} = \text{ſin. A. } \frac{4\pi}{3} = -\text{ſin. A. } \frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$\text{ſin. A. } \frac{2\pi}{2n+1} = \text{ſin. A. } \frac{2\pi}{3} = \text{ſin. A. } \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

vnde integrale quaeſitum erit hoc

$$-\frac{1}{3}\,l(1+x)$$

$$+\frac{1}{6}\,l(1-x+xx) + \frac{1}{\sqrt{3}}\,\text{A tang. } \frac{x\sqrt{3}}{2-x}.$$

Corollarium.

§. 55. Huius ergo formulae differentialis $\frac{(1-x)\,dx}{1+x^3}$ integrale erit $\frac{2}{3}\,l(1+x) - \frac{1}{3}\,l(1-x+xx) = \frac{1}{3}\,l\,\frac{1+2x+xx}{1-x+xx}$

Exemplum. 3.

§. 56. Huius formulae differentialis $\frac{dx}{1+x^5}$ integrale invenire.

Hic eſt $m = 0$, ideoque ſigna ſuperiora valent, et $n = 2$, vnde integrale quaeſitum erit:

$$+\frac{1}{5}\,l(1+x)$$

$$+\frac{1}{5}\,\text{coſ. A. } \frac{2\pi}{5}\,l\left(1+2x\,\text{coſ. A. } \frac{2\pi}{5}+xx\right) + \frac{2}{5}\,\text{ſin. A. } \frac{2\pi}{5}$$

$$\text{A tang. } \frac{x\,\text{ſin. A. } \frac{2\pi}{5}}{1+x\,\text{coſ. A. } \frac{2\pi}{5}}$$

— $\frac{1}{5}$

$- \frac{2}{5} \cos. A. \frac{\pi}{5} \, l (1 - 2x \cos. A. \frac{\pi}{5} + xx) + \frac{2}{5} \sin. A. \frac{\pi}{5}$

$$\text{A tang.} \frac{x \sin. A. \frac{\pi}{5}}{1 - x \cos. A. \frac{\pi}{5}}$$

At eſt $\cos. A. \frac{\pi}{5} = \frac{1 + \sqrt{5}}{4}$; $\sin. A. \frac{\pi}{5} = \frac{\sqrt{(10 - 2\sqrt{5})}}{4}$; atque $\cos. A. \frac{2\pi}{5} = \frac{-1 + \sqrt{5}}{4}$ et $\sin. A. \frac{2\pi}{5} = \frac{\sqrt{(10 + 2\sqrt{5})}}{4}$

Exemplum. 4.

§. 57. Huius formulae differentialis $\frac{xdx}{1 + x^5}$ integrale invenire

Hic eſt $n = 1$, ideoque ſigna inferiora valent, et $n = 2$; ex quo integrale quaeſitum erit.

$$- \frac{1}{5} \, l (1 + x)$$

$+ \frac{2}{5} \cos. A. \frac{\pi}{5} \, l (1 + 2x \cos. A. \frac{2\pi}{5} + xx) \frac{2}{5} \sin. A. \frac{\pi}{5}$

$$\text{A tang.} \frac{x \sin. A. \frac{2\pi}{5}}{1 + x \cos. A \frac{2\pi}{5}}$$

$- \frac{1}{5} \cos. A. \frac{2\pi}{5} \, l (1 - 2x \cos. A. \frac{\pi}{5} + xx) + \frac{2}{5} \sin. A. \frac{2\pi}{5}$

$$\text{A tang.} \frac{x \sin. A. \frac{\pi}{5}}{1 - x \cos. A. \frac{\pi}{5}}$$

Exemplum. 5.

§. 58. Huius formulae differentialis $\frac{xxdx}{1 + x^5}$ integrale invenire.

Hic eſt $n = 2$ et $m = 2$, vnde ſigna ſuperiora valent: ex quo integrale quaeſitum erit

$$+ \frac{1}{5} \, l (1 + x)$$

$- \frac{2}{5} \cos. A. \frac{\pi}{5} \, l (1 + 2x \cos. A. \frac{2\pi}{5} + xx) - \frac{2}{5} \sin. A. \frac{\pi}{5}.$

$$\text{A tang.} \frac{x \sin. A. \frac{2\pi}{5}}{1 + x \cos. A \frac{2\pi}{5}}$$

+

$+ \frac{2}{5} \cos. A. \frac{2\pi}{5}\, l\, (1 - 2x \cos. A. \frac{\pi}{5} + xx) + \frac{2}{5} \sin. A. \frac{2\pi}{5}.$

$$\text{A tang.} \frac{x \sin. A. \frac{\pi}{5}}{1 - x \cos. A. \frac{\pi}{5}}$$

Exemplum. 6.

§. 59. Huius formulae differentialis $\frac{x^3 dx}{1 + x^5}$ integrale invenire.

Hic est $n = 2$ et $m = 3$, vnde signa inferiora valent, ex quo integrale quaesitum erit:

$$-\frac{1}{5} l (1 + x)$$

$$- \frac{2}{5} \cos. A. \frac{2\pi}{5}\, l\, (1 + 2x \cos. A \frac{2\pi}{5} + xx) + \frac{2}{5} \sin. A. \frac{2\pi}{5}$$

$$\text{A tang.} \frac{x \sin. A. \frac{2\pi}{5}}{1 + x \cos. A. \frac{2\pi}{5}}$$

$$+ \frac{2}{5} \cos. A. \frac{\pi}{5}\, l\, (1 - 2x \cos. A. \frac{\pi}{5} + xx) + \frac{2}{5} \sin. A. \frac{\pi}{5}$$

$$\text{A tang,} \frac{x \sin. A. \frac{\pi}{5}}{1 - x \cos. A. \frac{\pi}{5}}$$

Problema. 3.

§. 60. Inuenire integrale huius formulae differentialis $\frac{x^m dx}{1 - x^{2n+1}}$ existente m numero integro minore, quam $2n + 1$.

Solutio.

Quia denominator $1 - x^{2n+1}$ hic habet vnum factorem simplicem realem $1 - x$, per quem, diuisione peracta, resultat quotus $1 + x + x^2 + x^3 + \ldots\ldots + x^{2n}$ huius factores trinomiales perinde ac in praecedente problemate poterunt inueniri, ex iisque integrale quaesitum determinari. At si rem probe perpendamus, solutio praecedentis problematis simul nobis suppeditabit solutionem praesentis; nam si hic ponamus $x = -y$, habebimus hanc formulam $\frac{-(-y)^m dy}{1+y^{2n+1}}$ seu $\frac{\mp y^m dy}{1+y^{2n+1}}$; vbi signum superius valet, si m fuerit numerus par, inferius vero, si m numerus impar. Huius autem formulae integrale iam inuenimus in solutione praecedentis problematis, vbi simul hoc commode accedit, vt signa illa ambigua tollantur, et vbique idem signum $-$ locum habeat: tum vero in illa solutione loco x poni oportet $-x$; hincque formulae nostrae propositae integrale erit:

$$-\tfrac{1}{2n+1}\, l(1-x)$$

$$-\tfrac{1}{2n+1}\cos. A. \tfrac{2(m+1)\pi}{2n+1}\, l\left(1-2x\cos. A. \tfrac{2\pi}{2n+1}+xx\right)+\tfrac{2}{2n+1}\sin. A. \tfrac{2(m+1)\pi}{2n+1}\ A\ tang. \frac{x\sin. A. \frac{2\pi}{2n+1}}{1-x\cos. A. \frac{2\pi}{2n+1}}$$

$$-\tfrac{1}{2n+1}\cos. A. \tfrac{4(m+1)\pi}{2n+1}\, l\left(1-2x\cos. A. \tfrac{4\pi}{2n+1}+xx\right)+\tfrac{2}{2n+1}\sin. A. \tfrac{4(m+1)\pi}{2n+1}\ A\ tang. \frac{x\sin. A. \frac{\pi}{2n+1}}{1-x\cos. A. \frac{4\pi}{2n+1}}$$

$$-\tfrac{1}{2n+1}\cos. A. \tfrac{6(m+1)\pi}{2n+1}\, l\left(1-2x\cos. A. \tfrac{6\pi}{2n+1}+xx\right)+\tfrac{2}{2n+1}\sin. A. \tfrac{6(m+1)\pi}{2n+1}\ A\ tang. \frac{x\sin. A. \frac{6\pi}{2n+1}}{1-x\cos. A. \frac{6\pi}{2n+1}}$$

$$\vdots$$

$$-\tfrac{1}{2n+1}\cos. A. \tfrac{2n(m+1)\pi}{2n+1}\, l\left(1-2x\cos. A. \tfrac{2n\pi}{2n+1}+xx\right)+\tfrac{2}{2n+1}\sin. A. \tfrac{2n(m+1)\pi}{2n+1}\ A\ tang. \frac{x\sin. A. \frac{2n\pi}{2n+1}}{1-x\cos. A. \frac{2n\pi}{2n+1}}$$

Q. E. I.

Proble-

Problema. 4.

§. 61. Inuenire integrale huius formulae differentialis $\frac{x^m dx}{1-x^{2n+2}}$, existente m numero integro minore, quam $2n+2$.

Solutio.

Denominator $1-x^{2n+2}$ duos habet factores reales, nempe $1+x$ et $1-x$, reliqui factores simplices omnes sunt imaginarii. Sit ergo $1+rx$ factor simplex. Ex eo si consulatur §. 41., orietur $R = \frac{(-r)^{2n-m+2}}{2n+2}$, atque integralis pars ex factore $1+rx$ oriunda erit $= \frac{1}{2n+2} \int \frac{(-r)^{2n-m+2} dx}{1+rx}$.

Sit iam primo $r = +1$, ac factor $1+x$ in integrale dabit hanc partem

$$\frac{1}{2n+2} \int \frac{\pm dx}{1+x} = \pm \frac{1}{2n+2} l(1+x)$$

vbi signum superius valet, si m fuerit numerus par, inferius si m impar. Sit nunc $r = -1$; ac factor $1-x$ in integrale inducet hoc:

$$\frac{1}{2n+2} \int \frac{dx}{1-x} = -\frac{1}{2n+2} l(1-x).$$

Pro factoribus trinomialibus sit nobis propositus iste: $1+px+qxx = (1+rx)(1+sx)$, ita vt sit $r+s=p$ et $rs=pq$. Quare cum ex factore simplici $1+rx$ oriatur integralis pars haec: $\frac{1}{2n+2} \int \frac{(-r)^{2n-m+2} dx}{1+rx}$; ex factore composito $1+px+qxx = (1+rx)(1+sx)$ orietur pro integrali

$$\int \frac{((-r)^{2n-m+2}+(-s)^{2n-m+2})dx - rs((-r)^{2n-m+1}+(-s)^{2n-m+1})xdx}{(2n+2)(1+px+qxx)}$$

quae formula cum ea, quam in ſolutione problematis 1. habuimus, ita congruit, vt ſi ibi loco $2n$ ponamus $2n+2$, prodeat haec noſtra negatiue ſumta. His conſideratis, ſi ſit Φ arcus circuli, cuius coſinus eſt $= \frac{p}{2\sqrt{q}}$, ex factore trinomiali $1+px+qxx$ orietur iſta integralis pars:

$$\pm \frac{q^{\frac{2n-m+1}{2}} \operatorname{cof.} A\,(2n-m+1)\Phi}{2n+2}\, l(1+px+qxx) \mp \frac{q^{\frac{2n-m+1}{2}} \operatorname{fin.} A\,(2n-m+1)\Phi}{n+1}\, A\ \operatorname{tang.} \frac{x\sqrt{(4q-pp)}}{2+px}$$

vbi ſigna ſuperiora valent, ſi m ſit numerus par; inferiora vero, ſi m ſit numerus impar. Supereſt igitur, vt in factores trinomiales denominatoris $1-x^{2n+2}$ inquiramus, ex quibus ob $1-xx$ factorem iam in computum ductum conſtet productum:

$$1+x^2+x^4+x^6+\ldots\ldots\ldots+x^{2n}$$

Haec forma, ſi cum theoremate in ſolutione primi problematis allegato comparetur, erit alternatim $a=0$, $b=1$, $c=0$, $d=1$, etc. At quod ad terminum medium attinet, quem poſuimus mx^n, erit vtique $m=1$, ſi n ſit numerus par; at erit $m=0$ ſi n ſit numerus impar. Quare duo caſus ſunt tractandi, alter quo n eſt numerus par, qui dat hanc aequationem:

$$\operatorname{cof.} A.n\psi + \operatorname{cof.} A(n-2)\psi + \operatorname{cof.} A(n-4)\psi + \ldots\ldots + \operatorname{cof.} A\,2\psi + \tfrac{1}{2} = 0$$

alter caſus, quo n eſt numerus impar, dat hanc aequationem:

cof. A.

$\text{cos. A} . n\psi + \text{cos. A}(n-2)\psi + \text{cos. A}(n-4)\psi + \ldots\ldots + \text{cos. A}\,\psi = 0$

ex quibus n diuersi arcus ψ eruuntur, quorum cosinus bis sumti praebebunt valores pro p substituendos in factore generali $1 + px + qxx$, et q semper est $= 1$ ita vt factor quisque trinomialis sit futurus

$$1 + 2x\,\text{cos. A}\,\psi + xx, \text{ estque } \Phi = \psi$$

Sit primo n numerus par, atque aequatio

$\text{cos. A}\, n\psi + \text{cos. A}(n-2)\psi + \text{cos. A}(n-4)\psi + \ldots\ldots\ldots + \text{cos. A}\, 2\psi + \frac{1}{2} = 0$

quae eodem modo, quo in solutione probl. 2. tractata tandem dabit $\psi = \frac{k\pi}{n+1}$, atque loco k substituendo successiue numeros $1, 2, 3$, etc. .... n, prodibunt n diuersi valores pro ψ simulque pro Φ. Quamobrem casu quo n est numerus par, formulae propositae differentialis $\frac{x^m dx}{1 - x^{2n+2}}$ integrale erit

$$\pm \frac{1}{2(n+1)}\, l(1+x) - \frac{1}{2(n+1)}\, l(1-x)$$

$$\pm \frac{1}{2(n+1)}\, \text{cos. A.} \frac{(m+1)\pi}{n+1}\, l(1 + 2x\,\text{cos. A.} \frac{\pi}{n+1} + xx) \pm \frac{1}{n+1}\, \text{sin. A} \frac{(m+1)\pi}{n+1}\, \text{A tang.} \frac{x\,\text{sin. A.} \frac{\pi}{n+1}}{1 + x\,\text{cos. A} \frac{\pi}{n+1}}$$

$$\pm \frac{1}{2(n+1)}\, \text{cos. A.} \frac{2(m+1)\pi}{n+1}\, l(1 + 2x\,\text{cos. A} \frac{2\pi}{n+1} + xx) \pm \frac{1}{n+1}\, \text{sin. A} \frac{2(m+1)\pi}{n+1}\, \text{A tang.} \frac{x\,\text{sin. A.} \frac{2\pi}{n+1}}{1 + x\,\text{cos. A} \frac{2\pi}{n+1}}$$

$$\pm \frac{1}{2(n+1)} \operatorname{cos}.A.\frac{3(m+1)\pi}{n+1}\, l(1+2x\operatorname{cos}.A.\frac{3\pi}{n+1}+xx) \pm \frac{1}{n+1}$$

$$\operatorname{sin}.A\frac{3(m+1)\pi}{n+1} A\operatorname{tang}.\frac{x \operatorname{sin}.A\frac{3\pi}{n+1}}{1+x\operatorname{cos}.A\frac{3\pi}{n+1}}$$

⋮

$$\pm \frac{1}{2(n+1)} \operatorname{cos}.A.\frac{n(m+1)\pi}{n+1}\, l(1+2x\operatorname{cos}.A.\frac{n\pi}{n+1}+xx) \pm \frac{1}{n+1}$$

$$\operatorname{sin}.A\frac{n(m+1)\pi}{n+1} A\operatorname{tang}.\frac{x \operatorname{sin}.A\frac{n\pi}{n+1}}{1+x\operatorname{cos}.A\frac{n\pi}{n+1}}$$

vbi signorum ambiguorum superiora valent, si m est numerus par, inferiora vero si m numerus impar.

Ponamus iam n esse numerum imparem, atque ad arcuum ψ vel Φ valores inueniendos, resolui oportet hanc aequationem.

$$\operatorname{cos}.A\,n\psi + \operatorname{cos}.A(n-2)\psi + \operatorname{cos}.A(n-4)\psi + \ldots\ldots + \operatorname{cos}.A.3\psi + \operatorname{cos}.A\psi = 0$$

Quorum arcuum in progressione arithmetica progredientium cum sit differentia $= 2\psi$, erit $\operatorname{cos}.A\,n\psi = 2\operatorname{cos}.A.2\psi.\operatorname{cos}.A(n-2)\psi - \operatorname{cos}.A(n-4)\psi$. Formemus ergo has aequationes:

$$+\operatorname{cos}.A\,n\psi + \ldots\ldots\ldots + \operatorname{cos}.A.5\psi + \operatorname{cos}.A.3\psi + \operatorname{cos}.A.\psi = 0$$

$$-2\operatorname{cos}.A2\psi\operatorname{cos}.A\,n\psi - 2\operatorname{cos}.A.2\psi.\operatorname{cos}.A(n-2)\psi - \ldots - 2\operatorname{cos}.A2\psi\operatorname{cos}.A.3\psi - 2\operatorname{cos}.A.2\psi\operatorname{cos}.A\psi = 0$$

$$+\operatorname{cos}.A\,n\psi + \operatorname{cos}.A(n-2)\psi + \operatorname{cos}.A(n-4)\psi + \ldots - \operatorname{cos}.A\,\psi = 0$$

quarum summa dabit hanc aequationem:

$$(1-2\operatorname{cos}.A.2\psi)\operatorname{cos}.A.n\psi + \operatorname{cos}.A(n-2)\psi + \operatorname{cos}.A.3\psi + (1-2\operatorname{cos}.A.2\psi)\operatorname{cos}.A\psi = 0$$

At est $\operatorname{cos}.A.3\psi = \operatorname{cos}.A\psi\operatorname{cos}.A.2\psi - \operatorname{sin}.A\psi.\operatorname{sin}.A.2\psi$ et $\operatorname{cos}.A\,3\psi - 2\operatorname{cos}.A\psi\operatorname{cos}.A2\psi = -\operatorname{cos}.A\psi\operatorname{cos}.A2\psi - \operatorname{sin}.A\psi\operatorname{sin}.A2\psi = -\operatorname{cos}.A.\psi$ ex quo erit $\operatorname{cos}.A.3\psi + (1-2\operatorname{cos}.A.2\psi)\operatorname{cos}.A\psi = 0$. Deinde est $\operatorname{cos}.A(n-2)\psi = \operatorname{sin}.A.2\psi\operatorname{sin}.A\,n\psi +$

cos.

cof. A 2ψ. cof. A $n\psi$. Quibus fubftitutis habetur haec aequatio:

$$\text{cof.}\,A.n\psi + \text{fin.}\,A.2\psi.\,\text{fin.}\,A n\psi - \text{cof.}\,A.2\psi.\,\text{cof.}\,A.n\psi = 0$$

feu

$$\text{cof.}\,A.\,n\,\psi = \text{cof.}\,A\,(n+2)\,\psi$$

At eft generaliter cof. A. $n\psi =$ cof. A $(2k\pi - n\psi)$ denotante k numerum quemcunque integrum: vnde fit $2k\pi - n\psi = (n+2)\psi$, atque $\psi = \frac{k\pi}{n+1}$, qui valor quia congruit cum eo, quem cafu praecedente, quo n eft numerus par, inuenimus. Patet quoque ifto cafu idem proditurum effe integrale, quod in cafu praecedente. Quocirca fiue n fit numerus par fiue impar idem prodit integrale, hocque integrale iam cafu praedente exhibuimus: ita vt problemati ex affe fit fatisfactum. Q. E. I.

Scholion I.

§. 62. Quod ambae aequationes, quas pro arcu ψ determinando inuenimus, cum cafu, quo n eft numerus par tum quo eft impar, eosdem plane valores arcus ψ praebeant, etiamfi ipfae aequationes omnino difcrepant, mirum videri poteft. Sin autem rem curatius infpiciamus, reperiemus binas illas aequationes in hac vna contineri:

$$0 = \text{cof.}\,A\,n\psi + \text{cof.}\,A(n-2)\psi + \text{cof.}\,A(n-4)\psi + \ldots + \text{cof.}\,A-(n-4)\psi + \text{cof.}\,A-(n-2)\psi + \text{cof.}\,A-n\psi$$

cum enim cofinus arcuum negatiuorum aequentur cofinibus eorundem arcuum affirmatiue fumtorum, termini extremi inter fe funt aequales, ideoque eundem terminum duplicatum dabunt, et fi n fit numerus vnus, terminus in medio: cof. A. $0\,\psi = 1$ folitarius relinquetur. Quare cum refolutio huius aequationis pro vtroque cafu valeat, neceffe

eft

eſt, vt eadem reperiatur expreſſio pro arcu ψ, ſiue n ſit numerus par ſiue impar. Si enim ad modum ſerierum recurrentium ſummam omnium terminorum inueſtigemus, proueniet $o = (1 - 2 \operatorname{cof.} A. 2\psi) \operatorname{cof.} A n\psi + \operatorname{cof.} A(n-2)\psi + (1 - 2\operatorname{cof.} A. 2\psi)\operatorname{cof.} A. -n\psi + \operatorname{cof.} A. -(n-2)\psi$ hoc eſt ob $\operatorname{cof.} A. -n\psi = \operatorname{cof.} A. n\psi$ et $\operatorname{cof.} A. -(n-2)\psi = \operatorname{cof.} A. (n-2)\psi$ erit $o = \operatorname{cof.} A(n-2)\psi - 2 \operatorname{cof.} A. 2\psi \operatorname{cof.} A. n\psi + \operatorname{cof.} A. n\psi$, atque ex lege progreſſionis ob $\operatorname{cof.} A(n+2)\psi = 2 \operatorname{cof.} A. 2\psi. \operatorname{cof.} A n\psi - \operatorname{cof.} A(n-2)\psi$ erit $\operatorname{cof.} A(n+2)\psi = \operatorname{cof.} A n\psi$. At eſt generaliter $\operatorname{cof.} A. n\psi = \operatorname{cof.} A(2k\pi - n\psi)$, vnde oritur $2k\pi - n\psi = (n+2)\psi$, hincque $\psi = \frac{k\pi}{n+1}$; ſiue n ſit numerus affirmatiuus ſiue negatiuus. Adnotari hic conuenit arcum ψ eſſe eiusmodi, vt $2n+2$ vicibus ſumtus det peripheriam totam aliquoties ſumtam, ex quo erit $\operatorname{cof.} A 2(n+1)\psi = 1$. Quamobrem ſi expreſſionis $1 - x^{2n+2}$ factor ſeu diuiſor fuerit $1 + 2x \operatorname{cof.} A\psi + xx$, arcus ψ ita erit comparatus, vt ſit $1 - \operatorname{cof.} A(2n+2)\psi = o$, quo ipſo ingens analogia cum expreſſione $1 - x^{2n+2}$ perſpicitur in reliquis caſibus confirmanda. Iſte autem factor $1 + 2x \operatorname{cof.} A\psi + xx$ non ſolum factores trinomiales formae $1 - x^{2n+2}$ in ſe complectitur, verum etiam ipſos factores ſimplices reales eiusdem formulae nempe $1 + x$ et $1 - x$ indicat: namque vi determinationis eſſe poteſt $\psi = \pi$ et $\psi = 2\pi$; priori caſu fit $\operatorname{cof.} A\psi = -1$, altero $\operatorname{cof.} A. \psi = 1$, vnde oriuntur hi factores $1 + 2x + xx$ et $1 - 2x + xx$, qui ſunt quadrata factorum ſimplicium $1 + x$ et $1 - x$. Neque vero diſcrimen inter quadrata et radices ſcrupulum mouere poteſt, cum in logarithmis, ad quos totum negotium refertur, totum diſcrimen in coefficientes

cadat,

cadat, quos hic non respicimus. Haec vero obseruatio confirmatur in reliquis formulis adhuc tractatis; nam si formae $1 + x^{2n}$ factor sit $1 + 2x \cos. A\psi + xx$, erit $\psi = \frac{k\pi}{2n}$, denotante k numerum quemcunque imparem; erit ergo $2n\psi = k\pi$ et $1 + \cos. A. 2n\psi = 0$. In problemate secundo vidimus, si formulae $1 + x^{2n+1}$ factor seu diuisor sit $1 + 2x \cos. A. \psi + xx$, fore $\psi = \frac{k\pi}{2n+1}$ denotante k numerum parem, ex quo erit $1 + \cos. A\ (2n+1)\psi = 0$. Atque ex solutione problematis 3. colligitur, si formae $1 - x^{2n+1}$ factor fuerit $1 - 2x \cos. A\psi + xx$ fore $1 - \cos. A\ (2n+1)\psi = 0$. Haecque omnia huc redeunt, vt si expressionis $1 \pm x^k$ diuisor fuerit $1 \pm 2x \cos. A\psi + xx$ fore $1 \pm \cos. A.\ k\psi = 0$. In casu ergo signi superioris $+$ arcus ψ valores sunt $\frac{\pi}{k}$; $\frac{3\pi}{k}$; $\frac{5\pi}{k}$; etc. pro signo autem inferiore sunt $\frac{0\pi}{k}$, $\frac{2\pi}{k}$; $\frac{4\pi}{k}$; $\frac{6\pi}{k}$; etc. Si pro ψ tot capiantur termini, quot k continet vnitates, quilibet factor $1 \pm 2x \cos. A. \psi + xx$ bis occurrunt, exceptis aliquot casibus, quibus est $\cos. A\psi$ vel $+1$ vel -1. Ex quo sequitur $1 \pm x^k$ esse productum ex k factoribus huius formae $\sqrt{(1 \pm 2x \cos. A. \psi + xx)}$, tribuendo ipsi ψ successiue valores, huius progressionis

$$\frac{\pi}{k};\ \frac{3\pi}{k};\ \frac{5\pi}{k};\ \frac{7\pi}{k};\ \ldots\ \frac{(2k-1)\pi}{k}$$

si signum $+$ valeat, at pro signo $-$, hos

$$\frac{0\pi}{k};\ \frac{2\pi}{k};\ \frac{3\pi}{k};\ \frac{6\pi}{k};\ \ldots\ \frac{(2k-2)\pi}{k}.$$

Huius igitur theorematis ope per diuisionem circuli factores tam simplices quam trinomiales formulae $1 \pm x^k$ exhiberi possunt: hocque theorema elegantissimum Cotesio debetur. Est vero $\sqrt{(1 \pm 2x \cos. A\psi + xx)} =$

$\sqrt{((x \pm \text{cos. A}\,\psi)^2 + (\text{fin. A}\,\psi)^2)}$; vnde fatis illa concinna conftructio geometrica fponte fequitur.

Scholion. 2.

§. 63. Inueniri hinc poffunt per circuli diuifionem omnes radices huius aequationis $x^k \pm 1 = 0$: hoc eft omnes numeri fiue reales fiue imaginarii, quorum poteftates exponentis k faciunt vel -1 vel $+1$. Ac primo quidem aequationis $x^k - 1 = 0$ radices inuenientur ex aequatione $xx - 2x\,\text{cof. A}\,\psi + 1 = 0$ fubftituendo loco ψ fucceffiue hos numero k arcus:

$\frac{0\pi}{k}$; $\frac{2\pi}{k}$; $\frac{4\pi}{k}$; . . . . . . $\frac{(2k-2)\pi}{k}$

eritque $x = \text{cof. A.}\,\psi + \sqrt{-1}$, fin. A ψ. Ex quo omnes radices, quarum numerus eft k, huius aequationis

$$x^k - 1 = 0$$

erunt fequentes:

$$x = \text{cof. A}.\tfrac{0\pi}{k} - \tfrac{1}{\sqrt{-1}}.\,\text{fin. A}.\tfrac{0\pi}{k} = 1$$
$$x = \text{cof. A}.\tfrac{2\pi}{k} - \tfrac{1}{\sqrt{-1}}.\,\text{fin. A}.\tfrac{2\pi}{k}$$
$$x = \text{cof. A}.\tfrac{4\pi}{k} - \tfrac{1}{\sqrt{-1}}.\,\text{fin. A}.\tfrac{4\pi}{k}$$

:

:

:

$$x = \text{cof. A}.\tfrac{(2k-2)\pi}{k} - \tfrac{1}{\sqrt{-1}}\,\text{fin. A}.\tfrac{(2k-2)\pi}{k}$$

Harum ergo expreffionum omnium poteftates, quarum exponens eft $= k$, faciunt vnitatem.

Deinde

Deinde aequationis $x^k + 1 = 0$ radices omnes inueniuntur ex aequatione $xx + 2x \operatorname{cof.} A . \psi + 1 = 0$ substituendo loco ψ successiue hos arcus numero k; qui sunt:

$$\frac{\pi}{k};\ \frac{3\pi}{k};\ \frac{5\pi}{k};\ \frac{7\pi}{k} \ldots\ldots \frac{(2k-1)\pi}{k}$$

eritque adeo $x = -\operatorname{cof.} A . \psi + \frac{1}{\sqrt{-1}} . \operatorname{fin.} A . \psi.$
Hanc obrem omnes radices huius aequationis

$$x^k + 1 = 0$$

quarum numerus est k erunt sequentes

$$x = -\operatorname{cof.} A . \frac{\pi}{k} + \frac{1}{\sqrt{-1}} \operatorname{fin.} A . \frac{\pi}{k}$$
$$x = -\operatorname{cof.} A . \frac{3\pi}{k} + \frac{1}{\sqrt{-1}} \operatorname{fin.} A . \frac{3\pi}{k}$$
$$x = -\operatorname{cof.} A . \frac{5\pi}{k} + \frac{1}{\sqrt{-1}} \operatorname{fin.} A . \frac{5\pi}{k}$$
$$\vdots$$
$$x = -\operatorname{cof.} A \frac{(2k-1)\pi}{k} + \frac{1}{\sqrt{-1}} \operatorname{fin.} A \frac{(2k-1)\pi}{k}$$

harumque expressionum omnium potestates exponentis k faciunt -1.

Problema 5.

§. 64. Inuenire integrale huius formulae differentialis $\frac{x^m dx}{1 + 2bx^n + x^{2n}}$ existente m numero integro minore quam $2n$, et $bb < 1$.

Solutio.

Quia est $bb < 1$ denominator $1 + 2bx^n + x^{2n}$ factorem simplicem realem non habebit: quare is in factores trinomiales resolui debet. Sit factor trinomialis $1 + px + qxx$ qui sit productum ex his duobus simplicibus imagi-

 nariis

nariis $(1+rx)(1+sx)$. Quaeratur ergo integralis pars ex vtroque factore simplici $1+rx$ et $1+sx$ oriund secundum praecepta §. 28. Hunc in finem erit numerator $P = x^m$ et denominator $Q = 1 + 2bx^n + x^{2n}$, vnde $\frac{dQ}{dx} = 2nbx^{n-1} + 2nx^{2n-1}$. Ex his fit propter $p = r$ vel s illo loco:

$$V = \frac{x^m p}{2n(bx^{n-1} + x^{2n-1})} = \frac{r\left(-\frac{1}{r}\right)^m}{2n\left(b\left(-\frac{1}{r}\right)^{n-1} + \left(-\frac{1}{r}\right)^{2n-1}\right)}$$

seu $V = \dfrac{-(-r)^{2n-m}}{2n + 2nb(-r)^n}$. Atque ex factore $1 + px + qxx = (1+rx)(1+sx)$ nascitur integralis pars haec

$$\frac{-(-r)^{2n-m}}{2n(1+b(-r)^n)}\int\frac{dx}{1+rx} \quad \frac{-(-s)^{2n-m}}{2n(1+b(-s)^n)}\int\frac{dx}{1+sx}$$

seu

$$\int\frac{-\left((-r)^{2n-m} \pm (-s)^{2n-m} \pm bq^n(-r)^{n-m} \pm bq^n(-s)^{n-m}\right)dx + \left(q(-r)^{2n-m-1} \pm q(-s)^{2n-m-1} \pm bq^{n+1}(-r)^{n-m-1} \pm bq^{n+1}(-s)^{n-m-1}\right)xdx}{2n(1+b(-r)^n + b(-s)^n + bbq^n)(1+px+qxx)}$$

At est $(-r)^k + (-s)^k = \pm r^k \pm s^k$ vbi signa superiora valent, si sit k numerus par, inferiora, si sit impar. Hinc ad eundem modum, quo in solutione problematis primi, posito Φ arcu circuli, cuius cosinus $= \frac{p}{2\sqrt{q}}$; erit $(-r)^k + (-s)^k = \pm 2q^{\frac{k}{2}} \cos. A.k\Phi = 2q^{\frac{k}{2}} \cos. A\, k(\pi - \Phi)$.

Hinc facta substitutione erit integrale ex factore $1 + px + qxx$ oriundum

$\int(-$

$$\frac{\int\left(-2q^{\frac{2n-m}{2}}\operatorname{cos}. A(2n-m)(\pi-\Phi)-2bq^{\frac{3n-m}{2}}\operatorname{cos}. A(n-m)(\pi-\Phi)\right)dx + \int\left(2q^{\frac{2n-m+1}{2}}\operatorname{cos}. A(2n-m-1)(\pi-\Phi)+2bq^{\frac{3n-m+1}{2}}\operatorname{cos}. A(n-m-1)(\pi-\Phi)\right)x\,dx}{2n\left(1+2bq^{\frac{n}{2}}\operatorname{cos}. A\,n(\pi-\Phi)+bbq^{n}\right)(1+px+qxx)}$$

cuius integrale est

$$+\frac{q^{\frac{2n-m-1}{2}}\operatorname{cos}. A(2n-m-1)(\pi-\Phi)+bq^{\frac{3n-m-1}{2}}\operatorname{cos}. A(n-m-1)(\pi-\Phi)}{2n\left(1+2bq^{\frac{n}{2}}\operatorname{cos}. A.\,n(\pi-\Phi)+bbq^{n}\right)}\,l(1+px+qxx)$$

$$+\frac{q^{\frac{2n-m-1}{2}}\operatorname{sin}. A(2n-m-1)(\pi-\Phi)+bq^{\frac{3n-m-1}{2}}\operatorname{sin}. A(n-m-1)(\pi-\Phi)}{n\left(1+2bq^{\frac{n}{2}}\operatorname{cos}. A\,n(\pi-\Phi)+bbq^{n}\right)}\,A\operatorname{tang}.\frac{x\sqrt{(4q-pp)}}{2+pq}$$

superest, vt singulos factores trinomiales denominatoris investigemus: in quem finem theorema in solutione primi problematis adhibitum huc transferamus; eritque $m=2b$; et obtinebimus hanc aequationem $\operatorname{cos}. A.\,n\psi \pm b = 0$, signum $+$ valet si n sit numerus par, signum $-$ vero si n numerus impar. Sit ω arcus cuius cosinus $= \mp b$, nempe $-b$, si n sit numerus par, et $+b$; si n sit impar; eritque $\operatorname{cos}. A.\,n\psi = \operatorname{cos}. A.\,\omega = \operatorname{cos}. A(2k\pi-\omega)$, vnde nascitur $\psi = \frac{2k\pi-\omega}{n}$; cuius n sunt valores differentes ponendo loco k successiue numeros $0, 1, 2, 3, \ldots (n-1)$ Quilibet ergo factor trinomialis denominatoris continetur in hac forma

$$1+2x\operatorname{cos}. A.\frac{2k\pi-\omega}{n}+xx$$

et huiusmodi factorum numerus erit $= n$, quare, cum hactenus $1 + px + qxx$ pro factore generali assumserimus, erit $q = 1$ et $p = 2$ cos. A. $\frac{2k\pi - \omega}{n}$, hincque $\Phi = \frac{2k\pi - \omega}{n}$. Integralis ergo quaesitae pars ex vnoquoque denominatoris factore trinomiali oriunda erit

$$\frac{+ \text{cos. A.} (2n-m-1)\left(\frac{(n-2k)\pi + \omega}{n}\right) + b \text{ cos. A} (n-m-1)\left(\frac{(n-2k)\pi + \omega}{n}\right)}{2n(1 + 2b \text{ cos. A} (n\pi + \omega) + bb)}$$

$$l\left(1 + 2x \text{ cos. A.} \tfrac{2k\pi - \omega}{n} + xx\right.$$

$$\frac{+ \text{fin. A} (2n-m-1)\left(\frac{(n-2k)\pi + \omega}{n}\right) + b \text{ fin. A} (n-m-1)\left(\frac{(n-2k)\pi + \omega}{n}\right)}{n(1 + 2b \text{ cof. A} (n\pi + \omega) + bb)}$$

$$\text{A tang.} \frac{x \text{ fin. A.} \frac{2k\pi - \omega}{n}}{1 + x \text{ cof. A.} \frac{2k\pi - \omega}{n}}$$

Completum ergo integrale obtinebitur, si loco k successiue numeri $0, 1, 2, 3, \ldots\ldots (n-1)$ substituantur, atque omnes valores resultantes in vnam summam colligantur: existente $\omega =$ A cos. $\mp b$. Scilicet si n est numerus par, erit $\omega =$ A cos. $-b$, et si n est numerus impar, erit $\omega =$ A cos. $+ b$. Q. E. I.

Exemplum 1.

§. 65. Huius formulae differentialis $\frac{dx}{1 + 2bx^2 + x^4}$ integrale inuenire, existente $bb < 1$.

Hic est $m = 0$, et $n = 2$, vnde ω erit arcus cuius cosinus $= -b$; seu, si arcus, cuius cosinus $= + b$ sit ϱ erit $\omega = \pi - \varrho$. Cognito ergo arcu ω, erunt bini denominatoris factores $1 + 2x$ cos. A. $\frac{\omega}{2} + xx$ et $1 - 2x$ cos. A. $\frac{\omega}{2} + xx$, ex quibus nascetur integrale quaesitum

$=$ cos.

$$\frac{-\operatorname{cos}.A.\frac{3\omega}{2}-b\operatorname{cos}.A.\frac{\omega}{2}}{4(1+2b\operatorname{cos}.A.\omega+bb)}l(1+2x\operatorname{cos}.A\tfrac{\omega}{2}+xx)+\frac{\operatorname{fin}A.\frac{3\omega}{2}+b\operatorname{fin}A.\frac{\omega}{2}}{2(1+2b\operatorname{cos}A\omega+bb)}$$

$$A\,\mathrm{tang}.\frac{x\operatorname{fin}.A.\frac{\omega}{2}}{1+x\operatorname{cos}.A.\frac{\omega}{2}}$$

$$\frac{+\operatorname{cos}.A.\frac{3\omega}{2}+b\operatorname{cos}.A.\frac{\omega}{2}}{4(1+2b\operatorname{cos}.A\omega+bb)}l(1-2x\operatorname{cos}.A\tfrac{\omega}{2}+xx)+\frac{\operatorname{fin}.A.\frac{3\omega}{2}+b\operatorname{fin}.A.\frac{\omega}{2}}{2(1+2b\operatorname{cos}A\omega+bb)}$$

$$A\,\mathrm{tang}.\frac{x\operatorname{fin}.A.\frac{\omega}{2}}{1-x\operatorname{cos}.A.\frac{\omega}{2}}$$

At cum fit $\operatorname{cof}.A\omega = -b$ erit $1+2b\operatorname{cof}.A\omega+bb = 1-bb$, et $\operatorname{cof}.A.\frac{3\omega}{2}+b\operatorname{cof}.A.\frac{\omega}{2} = -\operatorname{fin}.A\omega.\operatorname{fin}.A\frac{\omega}{2}$, vnde erit integrale quaefitum

$$\frac{1}{8\operatorname{cof}.A.\frac{\omega}{2}}\,l\,\frac{1+2x\operatorname{cof}.A\frac{\omega}{2}+xx}{1-2x\operatorname{cof}.A\frac{\omega}{2}+xx}+\frac{1}{4\operatorname{fin}.A\frac{\omega}{2}}A\,\mathrm{tang}.\frac{2x\operatorname{fin}.A\frac{\omega}{2}}{1-xx}$$

Scholion.

§. 66. Ex hoc exemplo videmus generaliter effe $1+2b\operatorname{cof}.A(n\pi+\omega)+bb = 1-bb = \operatorname{fin}.A\omega\operatorname{fin}.A\omega$ nam fi n fit numerus par, erit $\operatorname{cof}.A(n\pi+\omega) = \operatorname{cof}.A\omega = -b$; et, fi n fit numerus impar, erit $\operatorname{cof}.A(n\pi+\omega) = -\operatorname{cof}.A\omega = -b$. Deinde etiam numeratores in genere compendiofius exprimere poterimus. Si enim n fit numerus par, quo cafu eft $b = -\operatorname{cof}.A\omega$ erit $\operatorname{cof}.A\,n\left(\frac{(n-2k)\pi+\omega}{n}\right) = \operatorname{cof}.A\omega$, et $\operatorname{cof}.A(2n-m-1)\left(\frac{(n-2k)\pi+\omega}{n}\right) = \operatorname{cof}.A\omega\operatorname{cof}.A(n-m-1)\left(\frac{(n-2k)\pi+\omega}{n}\right) - \operatorname{fin}.A\omega.\operatorname{fin}.A(n-m-1)\left(\frac{(n-2k)\pi+\omega}{n}\right)$ atque $\operatorname{fin}.A(2n-m-1)\left(\frac{(n-2k)\pi+\omega}{n}\right) = \operatorname{fin}.A\omega.\operatorname{cof}.A(n-m-1)\left(\frac{(n-2k)\pi+\omega}{n}\right) + \operatorname{cof}.A\omega.\operatorname{fin}.A(n-m-1)\left(\frac{(n-2k)\pi+\omega}{n}\right)$.

Cafu

Casu ergo, quo n est numerus par, erit forma integralis:

$$\frac{-\text{sin. A}(n-m-1)\left(\frac{(n-2k)\pi+\omega}{n}\right)}{2n\,\text{sin. A}.\omega}\,l\left(1+2x\,\text{cos A}.\tfrac{2k\pi-\omega}{n}+xx\right)+\frac{\text{cos A}(n-m-1)\left(\frac{(n-2k)\pi+\omega}{n}\right)}{n\,\text{sin. A}\,\omega}$$

$$\text{A tang.}\frac{x\,\text{sin. A}.\frac{2k\pi-\omega}{n}}{1+x\,\text{cos. A}\frac{2k\pi-\omega}{n}}$$

Sit iam n numerus impar, erit $b=\text{cos. A}\,\omega$, et $\text{cos. A}((n-2k)\pi+\omega)=-\text{cos. A}.\omega$; $\text{sin. A}((n-2k\,\pi+\omega)=-\text{sin A}.\omega$ ex his oritur $\text{cos. A}(2n-m-1)\left(\frac{(n-2k)\pi+\omega}{n}\right)=-b\,\text{cos. A}(n-m-1)\left(\frac{(n-2k)\pi+\omega}{n}\right)+\text{sin. A}\,\omega.\,\text{sin. A}(n-m-1)\left(\frac{(n-2k)\pi+\omega}{n}\right)$; parique modo $\text{sin. A}(2n-m-1)\left(\frac{(n-2k)\pi+\omega}{n}\right)=-\text{sin A}\,\omega.\,\text{cos. A}(n-m-1)\left(\frac{(n-2k)\pi+\omega}{n}\right)-\text{cos. A}\,\omega.\,\text{sin. A}(n-m-1)\left(\frac{(n-2k)\pi+\omega}{n}\right)$, ex quo casu quo n est numerus impar, erit integralis forma

$$\frac{+\text{sin A}(n-m-1)\left(\frac{(n-2k)\pi+\omega}{n}\right)}{2n\,\text{sin. A}.\omega}\,l\left(1+2x\,\text{cos A}\tfrac{2k\pi-\omega}{n}+xx\right)-\frac{\text{cos A}(n-m-1)\left(\frac{(n-2k)\pi+\omega}{n}\right)}{n\,\text{sin. A}.\omega}$$

$$\text{A tang.}\frac{x\,\text{sin. A}.\frac{2k\pi-\omega}{n}}{1+x\,\text{cos. A}\frac{2k\pi-\omega}{n}}$$

quae duae expressiones vtique multo sunt simpliciores ea, quae in solutione prodiit.

Exemplum 2.

§. 67. Huius formulae differentialis $\frac{dx}{1+2bx^3+x^6}$ integrale inuenire, existente $bb<1$.

Hic est $m=0$, $n=3$, ideoque forma scholii posterior valet, et erit $\omega=\text{A cos.}\,b$, hinc erit integrale quaesitum.

$$\frac{+\text{sin. A}.\frac{2}{3}\omega}{6\,\text{sin. A}.\omega}\,l\left(1+2x\,\text{cos. A}.\tfrac{\omega}{3}+xx\right)+\frac{\text{cos. A}.\frac{2}{3}\omega}{3\,\text{sin. A}.\omega}\,\text{A tang}\frac{x\,\text{sin. A}.\frac{\omega}{3}}{1+x\,\text{cos A}\frac{\omega}{3}}$$

+ sin.

$$+\frac{\text{fin.}A.\frac{2}{3}(\pi+\omega)}{6\,\text{fin.}A.\omega}\,l(1+2x\,\text{cof.}A\tfrac{2\pi-\omega}{3}+xx)-\frac{\text{cof.}A.\frac{2}{3}(\pi+\omega)}{3\,\text{fin.}A\,\omega}$$

$$\text{A tang.}\frac{x\,\text{fin.}A.\frac{2\pi-\omega}{3}}{1+x\,\text{cof.}A.\frac{2\pi-\omega}{3}}$$

$$-\frac{\text{fin.}A.\frac{2}{3}(\pi-\omega)}{6\,\text{fin.}A.\omega}\,l(1+2x\,\text{cof.}A\tfrac{2\pi+\omega}{3}+xx)+\frac{\text{cof.}A.\frac{2}{3}(\pi-\omega)}{3\,\text{fin.}A\,\omega}$$

$$\text{A tang.}\frac{x\,\text{fin.}A\frac{2\pi+\omega}{3}}{1+x\,\text{cof.}A.\frac{2\pi+\omega}{3}}$$

Scholion. 2.

§. 68. Formulae illae integrales adhuc commodius exprimi poſſunt, ita vt nunquam arcus negatiui occurrant. Primo nimirum, ſi n ſit numerus par, quo caſu eſt cof. $A.\omega = -b$, erit cuiusuis partis integralis haec forma, poſito $n-m-1=i$

$$-\frac{\text{fin.}A.\frac{i}{n}((n+2k)\pi+\omega)}{2n\,\text{fin.}A\,\omega}\,l(1+2x\,\text{cof.}A.\tfrac{2k\pi+\omega}{n}+xx)-$$

$$\frac{\text{cof.}A\frac{i}{n}((n+2k)\pi+\omega)}{n\,\text{fin.}A\,\omega}\,\text{A tang.}\frac{x\,\text{fin.}A.\frac{2k\pi+\omega}{n}}{1+x\,\text{cof.}A.\frac{2k\pi+\omega}{n}}$$

Altero autem caſu quo eſt n numerus impar et cof. $A.\omega = +b$, poſito iterum $n-m-1=i$ erit integralis portio quaecunque:

$$+\frac{\text{fin.}A.\frac{i}{n}((n+2k)\pi+\omega)}{2n\,\text{fin.}A.\omega}\,l(1+2x\,\text{cof.}A.\tfrac{2k\pi+\omega}{n}+xx)$$

$$+\frac{\text{cof.}A.\frac{i}{n}((n+2k)\pi+\omega)}{n\,\text{fin.}A.\omega}\,\text{A tang.}\frac{x\,\text{fin.}A\frac{2k\pi+\omega}{n}}{1+x\,\text{cof.}A.\frac{2k\pi+\omega}{n}}$$

In vtroque caſu integrale conſtabit ex n huiusmodi partibus, quae obtinentur, ſi loco k ſucceſſiue ſubſtituantur nu-

 meri,

meri, 0, 1, 2, 3, ... $(n-1)$. Praeterea hic notandum est, si sit i numerus par, fore

$\sin. A . \frac{i}{n}((n+2k)\pi+\omega) = \sin. A . \frac{i}{n}(2k\pi+\omega)$ et

$\cos. A . \frac{i}{n}((n+2k)\pi+\omega) = \cos. A . \frac{i}{n}(2k\pi+\omega)$.

Quodsi autem fuerit i numerus impar, erit

$\sin. A . \frac{i}{n}((n+2k)\pi+\omega) = -\sin. A . \frac{i}{n}(2k\pi+\omega)$ et

$\cos. A . \frac{i}{n}((n+2k)\pi+\omega) = -\cos. A . \frac{i}{n}(2k\pi+\omega)$.

Exemplum 3.

§. 69. Huius formulae differentialis $\frac{xxdx}{1+2bx^4+x^8}$ integrale inuenire, existente $bb < 1$.

Erit hic $m=2$, et $n=4$, ex quo formula priori erit vtendum. Sit igitur ω arcus; cuius cosinus $=-b$; et quia $n-m-1=i=1$, numero impari, erit

$\sin. A . \frac{i}{n}((n+2k)\pi+\omega) = -\sin. A . \frac{2k\pi+\omega}{4}$ et

$\cos. A . \frac{i}{n}((n+2k)\pi+\omega) = -\cos. A . \frac{2k\pi+\omega}{4}$

Hanc obrem formulae propositae integrale reperietur sequenti modo expressum:

$$+\frac{\sin. A . \frac{\omega}{4}}{8 \sin. A . \omega} l\left(1+2x\cos. A . \tfrac{\omega}{4}+xx\right) + \frac{\cos. A . \frac{\omega}{4}}{4 \sin. A\, \omega} A\,\text{tang.}\frac{x \sin. A . \frac{\omega}{4}}{1+x\cos. A \frac{\omega}{4}}$$

$$+\frac{\sin. A . \frac{2\pi+\omega}{4}}{8 \sin. A . \omega} l\left(1+2x\cos. A \tfrac{2\pi+\omega}{4}+xx\right) + \frac{\cos. A . \frac{2\pi+\omega}{4}}{4 \sin. A\, \omega} A\,\text{tang.}\frac{x \sin. A . \frac{2\pi+\omega}{4}}{1+x\cos. A \frac{2\pi+\omega}{4}}$$

+ sin.

$$+\frac{\sin.A.\frac{4\pi+\omega}{4}}{8\sin.A\omega}l(1+2x\cos.A.\frac{4\pi+\omega}{4}+xx)+\frac{\cos.A.\frac{4\pi+\omega}{4}}{4\sin.A\omega}$$

$$A\,\mathrm{tang}.\frac{x\sin.A.\frac{4\pi+\omega}{4}}{1+x\cos.A\frac{4\pi+\omega}{4}}$$

$$+\frac{\sin.A.\frac{6\pi+\omega}{4}}{8\sin.A\omega}l(1+2x\cos.A\frac{6\pi+\omega}{4}+xx)+\frac{\cos.A.\frac{6\pi+\omega}{4}}{4\sin.A\omega}$$

$$A\,\mathrm{tang}.\frac{x\sin.A.\frac{6\pi+\omega}{4}}{1+x\cos.A\frac{6\pi+\omega}{4}}$$

Scholion 3.

§. 70. Si in formula differentiali propoſita $\frac{x^m dx}{1+2bx^n+x^{2n}}$ foret $bb>1$, tum integratio per problemata praecedentia abſolui poterit. Namque hoc caſu denominator in hos duos factores reales $1+x^n(b+\sqrt{(bb-1)})$ et $1+x^n(b-\sqrt{(bb-1)})$ reſoluitur, ex quo ipſa formula differentialis propoſita diſtribui poterit in binas formulas, quarum denominatores erunt hi duo factores, atque hancobrem earum integralia reperiri poterunt per praecepta ante tradita. Idem praeſtari poterit, ſi formula differentialis propoſita fuerit $\frac{x^m dx}{1+2bx^n-x^{2n}}$, quippe quo caſu denominator pariter reſolui poterit in duos factores reales, cuiusmodi ante tractauimus.