METHODVS FACILIOR ATQVE EXPEDITIOR INTEGRANDI FORMVLAS DIFFERENTIALES RATIONALES.

AVCTORE
L. Eulero.

§. 1.

Cum igitur omnium formularum rationalium integratio (*) per praecepta tradita ſemper abſolui queat, dummodo denominatoris factores ſiue ſimplices ſiue trinomiales habeantur, nihil amplius ad methodum ante expoſitam ſuperaddendum videbatur. Verum tamen hic omnia, quae ante explicuimus, non ſolum modo magis naturali ex propriis fontibus deducemus, verum etiam praecepta ita adornabimus, vt integratio omnium huiusmodi formularum multo facilius atque expeditius perfici queat. Primum enim methodum tam latiſſime patentem, quam facilem aperiemus cuiuscunque denominatoris factores trinomiales inueniendi; dum antea hos factores eruimus ope cuiuspiam theorematis moivraeani, quod tantum valet, quando ſupremae poteſtates variabilis x, iisdem, quibus infimae, coefficientibus ſunt coniunctae: hocque ipſo integrationem ad plurimas alias formulas accommodare poterimus, ad quas prior methodus minus genuina non ſufficit. Deinde inuentis factoribus tam ſimplicibus quam trinomialibus, methodum longe ſimpliciorem ac faciliorem communicabimus ex quolibet factore denominatoris respondentem integralis partem determinandi: in quo negotio ante vſi ſumus methodo cum nimis operoſa, tum ex alienis princi-

 piis

(*) vid. ſuper. meth. integr.

piis deducta. Tertio methodus, quam hic ostendemus, ad omnes formulas differentiales erit aeque accommodata, neque vlla opus erit reductione, quemadmodum ante necesse erat; vbi primum ex denominatore factores, qui erant potestates ipsius x, elicere, atque tum terminum denominatoris absolutum vnitati aequalem reddere oportebat.

§. 2. Sit igitur proposita formula differentialis quaecunque $\frac{M}{N}dx$, cuius integrale requiratur: sintque M et N functiones quaecunque ipsius x, huius formae

$$\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.}$$

tam ratione numeri terminorum, quam potestatum ipsius x vtcunque comparatae. Ad integrationem iam absoluendam oportet fractionem $\frac{M}{N}$ in partes simpliciores reales resoluere, quarum denominatores sint vel binomia $p + qx$, vel trinomia $p + qx + rxx$; quemadmodum ante vidimus. Continebuntur vero etiam in fractione $\frac{M}{N}$ partes integrae, si variabilis x in numeratore M totidem pluresue habeat dimensiones quam in denominatore. Quod si ergo fractio $\frac{M}{N}$ in huiusmodi partes siue integras siue fractas fuerit resoluta, quaelibet pars per dx multiplicata et integrata dabit integralis quaesiti partem; atque omnes istae integralis partes ex singulis partibus, in quas fractio $\frac{M}{N}$ resoluitur, oriundae iunctim sumtae praebebunt integrale formulae $\frac{M}{N}dx$ quaesitum. Totum ergo negotium huc redit, vt fractionis $\frac{M}{N}$ omnes partes simplices eruamus siue integras siue fractas: tum enim singulis per dx multiplicatis integratio facili negotio absoluetur.

§. 3

§. 3. Partes integras autem fractio $\frac{M}{N}$, vti iam monuimus, in se complectitur, si x totidem pluresue habeat dimensiones in numeratore M quam in denominatore N. Contra autem si x pauciores habeat dimensiones in numeratore M quam in denominatore N, tum partes integrae in fractione $\frac{M}{N}$ omnino non continentur, hincque consequenter nullae partes in integrale inducuntur. Ponamus igitur variabilem x in numeratore M non pauciores habere dimensiones quam in denominatore; tum partes integrae in fractione $\frac{M}{N}$ contentae more consueto per diuisionem eliciuntur;

Sit enim $\frac{M}{N} = \frac{Ax^{n+m} + Bx^{n+m-1} + Cx^{n+m-2} + Dx^{n+m-3} + \text{etc}}{\alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \text{etc.}}$

manifestum est partem integram ex diuisione oriundam huiusmodi formam esse habituram

$\mathfrak{A}x^m + \mathfrak{B}x^{m-1} + \mathfrak{C}x^{m-2} + \mathfrak{D}x^{m-3} + \ldots + \mathfrak{M}$

ad cuius coefficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. inueniendos hoc tantum requiritur, vt ista pars integra per denominatorem $\alpha x^n + \beta x^{n-1} + \gamma x^{n-1} +$ etc. multiplicetur, et termini omnes, quibus exponens ipsius x non minor est quam n, terminis respondentibus numeratoris aequentur. Tum igitur orietur

$A = \alpha\mathfrak{A}$

$B = \alpha\mathfrak{B} + \beta\mathfrak{A}$

$C = \alpha\mathfrak{C} + \beta\mathfrak{B} + \gamma\mathfrak{A}$

$D = \alpha\mathfrak{D} + \beta\mathfrak{C} + \gamma\mathfrak{B} + \delta\mathfrak{A}$

etc.

N 3 hinc-

hincque coefficientes quaesiti emergent hoc modo

$\mathfrak{A} = \frac{A}{\alpha}$

$\mathfrak{B} = \frac{B}{\alpha} - \frac{\beta A}{\alpha^2}$

$\mathfrak{C} = \frac{C}{\alpha} - \frac{\beta B}{\alpha^2} + \frac{(\beta^2 - \alpha\gamma) A}{\alpha^3}$

$\mathfrak{D} = \frac{D}{\alpha} - \frac{\beta C}{\alpha^2} + \frac{(\beta^2 - \alpha\gamma) B}{\alpha^3} - \frac{(\beta^3 - 2\alpha\beta\gamma + \alpha^2\delta) A}{\alpha^4}$

etc.

§. 4. Hoc itaque modo, qui diuifioni actuali idem omnino praebiturae anteferendus videtur, facili negotio inuenitur fractionis $\frac{M}{N}$ pars integra

$\mathfrak{A} x^m + \mathfrak{B} x^{m-1} + \mathfrak{C} x^{m-2} + \mathfrak{D} x^{m-3} + \ldots\ldots + \mathfrak{M}$.

definiendis fcilicet coefficientibus $\mathfrak{A}, \mathfrak{B}, \mathfrak{C},$ etc. His autem definitis fimul obtinebitur pars integralis quaefiti, ex ifta parte integra oriunda, quippe quae erit:

$$\frac{\mathfrak{A} x^{m+1}}{m+1} + \frac{\mathfrak{B} x^m}{m} + \frac{\mathfrak{C} x^{m-1}}{m-1} + \ldots\ldots + \mathfrak{M} x + \mathfrak{N}$$

denotante $\mathfrak{N}$ quantitatem quamcunque conftantem. Neque vero opus eft, quemadmodum ante, methodo minus genuina vfi, fecimus, vt fimul partem fractam, quae cum parte integra inuenta coniuncta totam fractionem propofitam $\frac{M}{N}$ conftituat, determinemus; fed fufficiet partem integram tantum inueftigaffe, ex eaque integralis partem conuenientem eruiffe. Reliquas enim integralis partes ex partibus fractis fractionis $\frac{M}{N}$ oriundas immediate ex ipfa fractione $\frac{M}{N}$ elicere docebimus, ita vt non opus habeamus illa faepenumero laborioſa reductione fractiones $\frac{M}{N}$ ad aliam, in qua variabilis x pauciores obtineat dimenfiones in numeratore M quam in denominatore N; quae tamen reductio

ductio necessaria erat visa in methodo praecedente; vbi praeterea factores solitarios denominatoris N formae x^k seorsim elicere, atque reliqui denominatoris terminum absolutum vnitati aequalem efficere coacti fueramus. Methodo autem, quam hic sumus tradituri, nulla istiusmodi praeparatione erit opus.

§. 5. Inuenta parte integra, si quae continetur in fractione $\frac{M}{N}$, ex eaque integralis parte conueniente, progrediamur ad partes fractas singulas simpliciores in fractione $\frac{M}{N}$ contentas eruendas, vt ex his quoque integralis quaesiti partes oriundae obtineantur. Ista autem inuestigatio maximam partem in inuentione factorum simpliciorum denominatoris N absoluitur: qui factores, cum ex instituto nostro, quo totum integrale in forma reali exhibere constituimus, debeant esse reales, erunt illi vel simplices binomiales huius formae $p x + q x$, vel trinomiales $p + q x + r x x$, cuiusmodi factores reales semper exhiberi posse cum docuimus tum in sequentibus fusius docebimus, etiamsi factores simplices sint imaginarii. Primum igitur de factoribus simplicibus $p x + q x$ agemus, qui in denominatore N continentur; inueniuntur hi ex resolutione aequationis $N = 0$; quod si enim huius aequationis radix fuerit inuenta $x = a$, tum simul $x - a$ diuisor erit quantitatis N. Omnia ergo subsidia, quae adhuc sunt inuenta ad radices aequationum algebraicarum eruendas, in praesenti negotio maximam afferent vtilitatem. Probe autem discerni debebunt factores reales ab imaginariis, cum priores solos hoc loco in vsum vocemus, posteriores seorsim tractaturi. Ex resolutione vero aequationum intelligitur, si maximus exponens

ponens ipſius x in N fuerit numerus impar, tum denominatorem N certiſſime vnum eſſe habiturum factorem ſimplicem realem; praeterea vero ſubinde plures habebit, id quod aequationis $N = 0$ reſolutio docebit.

§. 6. Sit igitur $p + qx$ factor denominatoris N isque realis; atque ex eo naſcatur fractionis propoſitae $\frac{M}{N}$ iſta pars $\frac{P}{p+qx}$; cuius numeratorem P, quem quantitatem conſtantem eſſe oportet, ſequenti ratiocinio determinabimus. Cum $p + qx$ ſit factor denominatoris N, ſit $\frac{N}{p+qx} = S$; eritque alterius fractionis, quae inſtar complementi cum $\frac{P}{p+qx}$ coniuncta conſtituit fractionem $\frac{M}{N}$, denominator S. Quare ſi a fractione $\frac{M}{N}$ ſeu $\frac{M}{(p+qx)S}$ ſubtrahamus fractionem ſimplicem $\frac{P}{p+qx}$, reſiduae fractionis $\frac{M-PS}{(p+qx)S}$, numerator $M - PS$ diuiſibilis erit per $p + qx$, quo fractio oriatur denominatorem habens S, vti innuimus. Cum igitur quantitas $M - PS$ ſit diuiſibilis per $p + qx$, fiet ea $= 0$ ſi ponatur $p + qx = 0$ ſiue $x = -\frac{p}{q}$. Subſtituto ergo in M et S vbique $-\frac{p}{q}$ loco x, erit $M - PS = 0$, hincque naſcitur $P = \frac{M}{S}$. Erit itaque numerator ille conſtans aſſumtus $P = \frac{M}{S}$ poſtquam in M et S vbique loco x ſubſtitutum fuerit $-\frac{p}{q}$; quo facto quantitas $\frac{M}{S}$ abibit in quantitatem conſtantem. Ex denominatoris ergo N factore $p + qx$ oritur fractionis $\frac{M}{N}$ pars $\frac{M}{S(p+qx)}$ hincque integralis quaeſiti proueniet pars $\frac{M}{S}\int\frac{dx}{p+qx} = \frac{M}{Sq}\,l(p+qx)$, ſicque ex ſingulis denominatoris N factoribus ſimplicibus conuenientes integralis partes reperientur.

Exemplum 1.

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§. 7. Huius formulae differentialis $\frac{x^3+x^2}{x-1}dx$ integrale inuenire.

Quia hic est $\frac{M}{N}=\frac{x^3+x^2}{x-1}$, in hac fractione partes integrae continentur, quae vel per diuisionem vel modum ante traditum erutae erunt x^2+2x+2 vnde nascitur haec integralis pars $\frac{x^3}{3}+xx+2x+C$. Deinde cum totus denominator N ex vnico factore $x-1$ constet, erit $p=-1$, $q=1$ et $S=\frac{N}{x-1}=1$. Iam ex factore $x-1$ nihilo aequali posito oritur $x=1$, quo valore in $\frac{M}{S}=x^3+x^2$ substituto prodit 2, ideoque ex denominatore N obtinetur integralis pars haec $2\int\frac{dx}{x-1}=2\,l(x-1)$. Quoniam vero totum integrale componitur ex partibus, quae tam ex parte integra quam fracta resultant, erit integrale formulae propositae $\frac{x^3+x^2}{x-1}dx$ haec quantitas finita $\frac{x^3}{3}+x^2+2x+C+2l(x-1)$ cuius veritas per differentiationem faci e comprobatur.

Exemplum. 2.

§. 8. Huius formulae differentialis $\frac{xx+2ax}{aa-xx}dx$ integrale inuenire.

Quia variabilis x in numeratore $xx+2ax$ tot habet dimensiones, quot in denominatore $aa-xx$ pars integra in fractione $\frac{xx+2ax}{aa-xx}$ continetur, quae per diuisionem est -1, vnde integralis pars nascitur $-x+C$. Porro denominator $aa-xx$ in factores $(a-x)(a+x)$ resoluitur: ex priori $a-x$ fit $x=a$, et $\frac{M}{S}=\frac{xx+2ax}{a+x}=\frac{3a}{2}$ posito $x=a$; vnde integralis pars ex factore $a-x$ oriunda est $=\frac{3a}{2}\int\frac{dx}{a-x}=$

$-\frac{3a}{2}\, l(a-x)$. Ex altero factore $a+x$, qui dat $x=-a$, fit $\frac{M}{S}=\frac{xx+2ax}{a-x}=-\frac{a}{2}$; indeque integralis pars oritur haec $-\frac{a}{2}\int\frac{dx}{a+x}=-\frac{a}{2}\, l(a+x)$. Integrale ergo quaesitum repertum est $=C-x-\frac{3a}{2}\, l(a-x)-\frac{a}{2}\, l(a+x)$

Exemplum 3.

§. 9. Huius formulae differentialis $\frac{xx\,dx}{(1-x)(2-x)(3-x)(4-x)}$ integrale inuenire.

Hic variabilis x in numeratore pauciores habet dimensiones, quam in denominatore; ideoque in hac fractione partes integrae non continentur. Ad denominatoris ergo factores aggredimur, qui singuli sunt simplices reales. Primus factor $1-x$ dat $x=1$, et $S=(2-x)(3-x)(4-x)$ hincque $\frac{M}{S}=\frac{xx}{(2-x)(3-x)(4-x)}=\frac{1}{6}$ posito $x=1$: ex primo ergo factore $1-x$ nascitur integralis pars $\frac{1}{6}\int\frac{dx}{1-x}=-\frac{1}{6}\, l(1-x)$. Secundus factor $2-x$ dat $x=2$, et $\frac{M}{S}=\frac{xx}{(1-x)(3-x)(4-x)}=-\frac{4}{2}=-2$ posito $x=2$ hincque nascitur integralis pars $-2\int\frac{dx}{2-x}=2\, l(2-x)$. Tertius factor $3-x$ dat $x=3$ et $\frac{M}{S}=\frac{xx}{(1-x)(2-x)(4-x)}=\frac{9}{2}$ vnde oritur integralis pars $\frac{9}{2}\int\frac{dx}{3-x}=-\frac{9}{2}\, l(3-x)$. Quartus factor $4-x$ dat $x=4$ et $\frac{M}{S}=\frac{xx}{(1-x)(2-x)(3-x)}=\frac{16}{1\cdot2\cdot3}=-\frac{16}{6}=-\frac{8}{3}$; vnde prodit conueniens integralis pars $-\frac{8}{3}\int\frac{dx}{4-x}=\frac{8}{3}\, l(4-x)$. Ex his ergo formulae differentialis propositae $\frac{xx\,dx}{(1-x)(2-x)(3-x)(4-x)}$ integrale colligitur $=C-\frac{1}{6}\, l(1-x)+2\, l(2-x)-\frac{9}{2}\, l(3-x)+\frac{8}{3}\, l(4-x)$.

§. 10.

§. 10. Ex his igitur exemplis clare intelligitur, quemadmodum propositae formulae differentialis, cuius denominator in factores simplices reales inter se inaequales est resolubilis, integrale inueniri oporteat, siue variabilis x in numeratore pauciores siue plures habeat dimensiones quam in denominatore. Huius negotii praecipua pars absoluitur in coefficientis inuestigatione per quem formula $\int \frac{dx}{p+qx}$ multiplicari debet, vt integrale ex denominatoris N factore $p+qx$ oriundum obtineatur. Inuenimus autem hunc coefficientem esse $\frac{M}{S}$, postquam vbique loco x eius valor $-\frac{p}{q}$ quem obtinet ex aequatione $p+qx=0$ fuerit substitutus. Hunc igitur valorem $-\frac{p}{q}$ loco x tum in M quam in S substitui oportet. Est autem M numerator formulae differentialis propositae $\frac{M}{N}dx$, qui perpetuo manet idem, at S pro quouis factore denominatoris $p+qx$ variatur, cum sit $S=\frac{N}{p+qx}$: ita vt S habeatur, si totus denominator N per suum factorem $p+qx$ diuidatur. Quodsi ergo denominator N in suos factores iam fuerit vel actu resolutus, vel facile resolubilis, tum omittendo factorem propositum $p+qx$ statim emergit valor litterae S; in quo loco x valorem constantem $-\frac{p}{q}$ substitui oportet: hocque casu expedite reperitur valor coefficientis $\frac{M}{S}$ ponendo vbique $-\frac{p}{q}$ loco x.

§. 11. Sin autem quotus, qui oritur ex diuisione denominatoris N per suum factorem $p+qx$, fiat admodum prolixus, vel etiam indefinitus, vti si fuerit $N=1+x^{50}$ eiusque diuisor $1+x$, vel si sit $N=1-x^n$ eiusque factor $1-x$: priori enim casu quotus S constaret ex 99 terminis, posteriori autem numerus terminorum foret etiam indefinitus n; vnde substitutio loco x facienda fieret admodum operosa, neque valor ipsius S, nisi summatio serierum in subsidium vocetur, commode exhiberi posset. His igitur casibus alium modum tradi conueniet, quo expedite valor ipsius S, quem induit posito $-\frac{p}{q}$ loco x, indicari queat. Cum enim sit $S = \frac{N}{p+qx}$, quaeritur valor fractionis $\frac{N}{p+qx}$ resultans, si loco x ponatur $-\frac{p}{q}$: hoc autem casu non solum denominator fractionis $\frac{N}{p+qx}$ euanescit, sed etiam numerator N, eo quod ipse sit per $p+qx$ diuisibilis. Quocirca valor fractionis $\frac{N}{p+qx}$ posito $-\frac{p}{q}$ loco x idem erit ac fractionis huius $\frac{dN}{q\,dx}$ eadem facta substitutione, quae fractio ex illa oritur differentiando tam numeratorem N quam denominatorem $p+qx$, sumta x pro variabili. Erit itaque $S = \frac{dN}{q\,dx}$ posito vbique $-\frac{p}{q}$ loco x: ac totus coefficiens requisitus $\frac{M}{S}$ erit $\frac{Mq\,dx}{dN}$ posito vbique $-\frac{p}{q}$ loco x, vnde pars integralis ex denominatoris N factore $p+qx$ oriunda erit $\frac{Mq\,dx}{dN}\int\frac{dx}{p+qx} = \frac{M\,dx}{dN}\,l(p+qx)$.

§. 12. Compendium hoc insigni emolumento adhibebitur in inueniendis partibus integralis huiusmodi formularum differentialium $\frac{x^m\,dx}{a^n - b^n x^n}$; ex denominatoris $a^n - b^n x^n$ factoribus. Cum enim denominatoris $a^n - b^n x^n$ factor sit $a - bx$, fiet ex hoc factore $S = \frac{a^n - b^n x^n}{a - bx}$ existente $M = x^m$ et $N = a^n - b^n x^n$. Facto ergo $x = \frac{a}{b}$ fiet

$\frac{a}{b}$ fiet $M = \frac{a^m}{b^m}$ et $S = \frac{-nb^n x^{n-1} dx}{-bdx} = nb^{n-1} x^{n-1} =$ na^{n-1} atque $\frac{M}{S} = \frac{a^{m-n+1}}{nb^m}$, qui valor congruit cum $\frac{Mqdx}{dN}$ seu $-\frac{Mbdx}{dN}$ ob $q = -b$ posito $x = \frac{a}{b}$; est enim $dN = -nb^n x^{n-1} dx$; et $-\frac{Mbdx}{dN} = \frac{bx^m}{nb^n x^{n-1}} =$ $\frac{x^{m-n+1}}{nb^{n-1}} = \frac{a^{m-n+1}}{nb^m}$ posito $x = \frac{a}{b}$. Quocirca ex denominatoris $a^n - b^n x^n$ factore $a - bx$ integralis formulae $\frac{x^m dx}{a^n - b^n x^n}$ nascetur ista pars $\frac{a^{m-n+1}}{nb^m}\int\frac{dx}{a-bx} = \frac{-a^{m-n+1}}{nb^{m+1}}$ $l(a-bx)$, quae priori via sine summatione serierum inveniri non potuisset.

§. 13. Duplicem ergo nacti sumus viam partem integralis, quae ex denominatoris N factore quocunque simplici oritur, assignandi. Sit enim in formula differentiali proposita $\frac{M}{N}dx$ denominatoris N factor simplex $p + qx$, erit integralis pars ex hoc factore oriunda vel $\frac{M}{S}\int\frac{dx}{p+qx}$ existente $S = \frac{N}{p+qx}$, vel $\frac{Mqdx}{dN}\int\frac{dx}{p+qx}$, posito in vtroque coefficiente vbique $-\frac{p}{q}$ loco x, quem valorem x obtinet ex posito factore $p + qx = 0$. Quouis igitur casu oblato ea via vti conueniet, quae fuerit facilior atque ad operationem accommodatior: perpetuo enim vtraque via ad eundem coefficientem deducet. Sic in hac formula differentiali $\frac{dx}{1+x-2x^4}$ est $M = 1$, et $N = 1 + x - 2x^4$, huiusque denominatoris diuisor $1 - x$, ita vt sit $p = 1$, $q =$

$q = -1$. Via ergo priori est $S = 1 + 2x + 2xx + 2x^3$, et $\frac{M}{S} = \frac{1}{7}$ posito $x = 1$; vnde integralis pars ex factore $1 - x$ oriunda erit $\frac{1}{7}\int\frac{dx}{1-x} = -\frac{1}{7}\, l(1-x)$. Via autem posteriori est $dN = dx - 8x^3 dx$, et $\frac{M\, dx}{dN} = \frac{1}{8x^3 - 1} = \frac{1}{7}$ posito $x = 1$, prorsus vt ante.

§. 14. Quamquam haec methodus perpetuo tuta nullisque difficultatibus obnoxia videatur, tamen eius vsus penitus cessat, si denominator N duos pluresue habeat factores inter se aequales. Ponamus enim denominatorem N diuisibilem esse per $(p+qx)^2$; erit coefficiens portionis integralis $\int\frac{dx}{p+qx}$ pro vno factore $p+qx$ vti vidimus $= \frac{M}{S}$ posito $p+qx = 0$ seu $x = -\frac{p}{q}$. Quoniam vero est $S = \frac{N}{p+qx}$; erit S etiamnunc per $p+qx$ diuisibile ideoque facto $x = -\frac{p}{q}$ fiet $S = 0$, hincque coefficiens $\frac{M}{S}$ abibit in infinitum. Vtriusque ergo portionis integralis $\int\frac{dx}{p+qx}$ ex binis factoribus $p+qx$ et $p+qx$ oriundae coefficiens fiet infinitus, alterius quidem affirmatiuus, alterius negatiuus, ita vt integralis portio ex binis coniunctim oriunda sit differentia inter duo infinita, quam finitam esse posse ex natura infiniti satis liquet. Quanta autem sit ea differentia, ex alio fonte decidi oportet, quem mox aperiemus.

§. 15. Ponamus igitur fractionis $\frac{M}{N}$ denominatorem N duos habere factores aequales seu diuisibilem esse per $(p+qx)^2$, ita vt sit $N = (p+qx)^2 S$, atque partem fractionis $\frac{M}{N}$, quae ex hoc factore quadrato $(p+qx)^2$ oritur, seorsim inuestigemus. Sit igitur pars ista $= \frac{A}{p+qx} + \frac{B}{(p+qx)^2}$; ac reliqua pars, quae cum hac fractionem $\frac{M}{N}$ consti-

constituit, sit $\frac{T}{S}$; vbi A et B quantitates constantes, T vero functionem variabilem ipsius x integram denotabit, quam nosse non opus habemus, sufficiet enim coefficientes A et B determinasse. Cum igitur sit $\frac{T}{S} = \frac{M}{N} - \frac{A}{p+qx}$ $- \frac{B}{(p+qx)^2}$, ob $N = (p+qx)^2 S$ erit $\frac{T}{S} = \frac{M - A(p+qx)S - B S}{(p+qx)^2 S}$ ideoque $T = \frac{M - A(p+qx)S - BS}{(p+qx)^2}$, quae cum quantitas integra esse debeat, necesse est vt quantitas $M - A(p+qx)S - BS$ sit diuisibilis per $(p+qx)^2$. Quoniam autem S non amplius per $p+qx$ diuisibilem esse ponimus, eo quod denominatorem N tantum per quadratum $(p+qx)^2$ non vero aliam potestatem superiorem diuisibilem esse assumimus, necesse est vt $\frac{M}{S} - A(p+qx) - B$ sit diuisibile per $(p+qx)^2$. Ex natura igitur aequationum, cum quantitas $\frac{M}{S} - A(p+qx) - B$ duos habeat factores aequales, oportet, vt tam ipsa, quam eius differentiale $d.\frac{M}{S} - Aqdx$ sit diuisibilis per $p+qx$; ergo tam ipsa illa quantitas, quam eius differentiale euanescet posito $p+qx=0$ seu $x = -\frac{p}{q}$. Fiat igitur $x = -\frac{p}{q}$; ac prior aequatio dabit $\frac{M}{S} - B = 0$ seu $B = \frac{M}{S}$; posterior vero $A = \frac{d.\frac{M}{S}}{qdx}$. Determinatis ergo coefficientibus A et B, ex formula differentiali $\frac{M}{N}dx$, cuius denominator N factorem habet $(p+qx)^2$, hic ipse factor praebebit inregralis partem hanc $\frac{d.\frac{M}{S}}{qdx}\int\frac{dx}{p+qx} + \frac{M}{S}\int\frac{dx}{(p+qx)^2}$ posito in coefficientibus vbique $-\frac{p}{q}$ loco x.

§. 16. Si denominator N habeat tres factores aequales seu diuisibilis sit per $(p+qx)^3$; ex eo orietur eiusmodi

modi pars $\frac{A}{(p+qx)^3} + \frac{B}{(p+qx)^2} + \frac{C}{p+qx}$, quae a fractione $\frac{M}{N}$ ablata relinquet fractionem $\frac{T}{S}$ existente $S = \frac{N}{(p+qx)^3}$. Fiet ergo $T = \frac{M - AS - B(p+qx)S - C(p+qx)^2 S}{(p+qx)^3}$, quae, cum quantitas integra esse debeat, oportebit $M - AS - B(p+qx)S - C(p+qx)^2 S$ seu $\frac{M}{S} - A - B(p+qx) - C(p+qx)^2$ diuisibile esse per $(p+qx)^3$: id quod eueniet, si et ipsa illa quantitas, et eius differentiale, et eius differentio-differentiale fuerint per $p+qx$ diuisibilia. Quare sequentes tres quantitates

$$\frac{M}{S} - A - B(p+qx) - C(p+qx)^2$$
$$d.\frac{M}{S} - Bqdx - 2C(p+qx)qdx$$
$$dd.\frac{M}{S} - 2Cqqdx^2$$

diuisibiles esse oportet per $p+qx$; ideoque singulae, si in ipsis ponatur $p+qx=0$ seu $x = -\frac{p}{q}$, euanescent. Ponatur ergo in singulis $x = -\frac{p}{q}$, atque ex prima orietur $A = \frac{M}{S}$; ex secunda $B = \frac{1}{qdx} d.\frac{M}{S}$ et ex tertia $C = \frac{1}{2qqdx^2} dd.\frac{M}{S}$. His coefficientibus inuentis ex denominatoris N factore cubico $(p+qx)^3$ orietur sequens integralis pars

$$\frac{M}{S}.\int\frac{dx}{(p+qx)^3} + \frac{1}{qdx} d.\frac{M}{S}.\int\frac{dx}{(p+qx)^2} + \frac{1}{2q^2dx^2} dd.\frac{M}{S}.\int\frac{dx}{p+qx}$$

posito in coefficientibus vbique $-\frac{p}{q}$ loco x; ac existente $S = \frac{N}{(p+qx)^3}$.

§. 17. Simili modo si ponamus formulae differentialis $\frac{M}{N}dx$ denominatorem N quatuor habere factores aequales seu diuisibilem esse per $(p+qx)^4$, ita vt sit $S = \frac{N}{(p+qx)^4}$ quantitas integra. Quod si iam ex hoc factore $(p+qx)^4$ nasci ponatur ista integralis pars $A\int\frac{dx}{(p+qx)^4} + B\int\frac{dx}{(p+qx)^3}$

+ C

$+ C\int\frac{dx}{(p+qx)^2} + D\int\frac{dx}{p+qx}$; oſtendetur pari, quo ante, modo hanc quantitatem

$\frac{M}{S} - A - B(p+qx) - C(p+qx)^2 - D(p+qx)^3$

diuiſibilem eſſe oportere per $(p+qx)^4$. Hoc autem eveniet ſi praeter hanc ipſam quantitatem eius differentialia primi, ſecundi ac tertii gradus ſingula fuerint diuiſibilia per $p+qx$. Hinc itaque per $p+qx$ diuiſibiles erunt quatuor ſequentes quantitates

$\frac{M}{S} - A - B(p+qx) - C(p+qx)^2 - D(p+qx)^3$

$d.\frac{M}{S} - Bqdx - 2C(p+qx)qdx - 3D(p+qx)^2qdx$

$dd.\frac{M}{S} - 2Cqqdx^2 - 6D(p+qx)q^2dx^2$

$d^3.\frac{M}{S} - 6Dq^3dx^3$.

ſingulae ergo euaneſcent, poſito $x = \frac{-p}{q}$. Facto autem vbique $x = \frac{-p}{q}$, prima aequatio dabit $A = \frac{M}{S}$; ſecunda dabit $B = \frac{1}{qdx}\, d.\frac{M}{S}$; tertia dabit $C = \frac{1}{2q^2dx^2}\, dd.\frac{M}{S}$, et quarta dabit $D = \frac{1}{6q^3dx^3}\, d^3.\frac{M}{S}$. Ex his colligitur integralis pars ex denominatoris factore $(p+qx)^4$ oriunda $=$ $\frac{M}{S}\int\frac{dx}{(p+qx)^4} + \frac{1}{qdx}\, d.\frac{M}{S}\int\frac{dx}{(p+qx)^3} + \frac{1}{2q^2dx^2}\, dd.\frac{M}{S}\int\frac{dx}{(p+qx)^2}$ $+ \frac{1}{6q^3dx^3}\, d^3.\frac{M}{S}\int\frac{dx}{p+qx}$ poſito in omnibus coefficientibus $x = \frac{-p}{q}$.

§. 18. Facili igitur negotio hos coefficientes determinamus, quos in ſuperiori tractatione per prolixiſſimos calculos eruimus, ac pro altioribus poteſtatibus tantum per inductionem concluſimus; haecque determinatio pro factoribus ſimplicibus cuiuscunque formae valet, cum ſuperior ad hanc tantum formam $1+qx$ eſſet accommodata. Hic autem vlterius progreſſuri inductione non indigemus; ſi enim denominatoris N factor ſit $(p+qx)^n$; hincque integralis pars oriunda ponatur $= A\int\frac{dx}{(p+qx)^n} + B\int\frac{dx}{(p+qx)^{n-1}} + C\int\frac{dx}{(p+qx)^{n-2}} + D\int\frac{dx}{(p+qx)^{n-3}} + \text{etc.}$ ratiocinio ſupra adhibito patebit, poſito $S = \frac{N}{(p+qx)^n}$ per $(p+qx)^n$ diuiſibilem eſſe debere hanc expreſſionem $\frac{M}{S} - A - B(p+qx) - C(p+qx)^2 - D(p+qx)^3 - E(p+qx)^4 - \text{etc.}$ Tam igitur haec ipſa expreſſio, quam eius differentialia ordinis primi, ſecundi, tertii, etc. vsque ad ordinem $n-1$ incluſiue ſingula per $p+qx$ diuiſibilia eſſe oportet,

$$\frac{M}{S} - A - B(p+qx) - C(p+qx)^2 - D(p+qx)^3 - E(p+qx)^4 - \text{etc.}$$
$$d.\frac{M}{S} - Bq\,dx - 2C(p+qx)q\,dx - 3D(p+qx)^2 q\,dx - \text{etc.}$$
$$dd.\frac{M}{S} - 2Cq^2dx^2 - 6D(p+qx)q^2dx^2 - 12E(p+qx)^2q^2dx^2 - \text{etc.}$$
$$d^3.\frac{M}{S} - 6Dq^3dx^3 - 24E(p+qx)q^3dx^3 - 60F(p+qx)q^3dx^3 - \text{etc.}$$
$$d^4.\frac{M}{S} - 24Eq^4dx^4 - 120F(p+qx)q^4dx^4 - \text{etc.}$$
$$d^5.\frac{M}{S} - 120Fq^5dx^5 - \text{etc.}$$

Quod ſi iam ponatur $p+qx=0$ ſeu $x=\frac{-p}{q}$, ſingulae iſtae expreſſiones euaneſcunt; indeque reperitur

$$A = \frac{M}{S} \qquad\qquad D = \frac{1}{6q^3dx^3}\,d^3.\frac{M}{S}$$
$$B = \frac{1}{q\,dx}\,d.\frac{M}{S} \qquad\qquad E = \frac{1}{24q^4dx^4}\,d^4.\frac{M}{S}$$
$$C = \frac{1}{2q^2dx^2}\,dd.\frac{M}{S} \qquad\qquad F = \frac{1}{120q^5dx^5}\,d^5.\frac{M}{S}$$

etc.

Ex

Ex his igitur colligitur integralis quaesiti pars ex denominatoris N factore $(p+qx)^n$ oriunda fore

$$\frac{M}{S}\int\frac{dx}{(p+qx)^n}+\frac{1}{qdx}d.\frac{M}{S}\int\frac{dx}{(p+qx)^{n-1}}+\frac{1}{2q^2dx^2}dd.\frac{M}{S}\int\frac{dx}{(p+qx)^{n-2}}$$

$$+\frac{1}{6q^3dx^3}d^3.\frac{M}{S}\int\frac{dx}{(p+qx)^{n-3}}+\frac{1}{24q^4dx^4}d^4.\frac{M}{S}\int\frac{dx}{(p+qx)^{n-4}}$$

$$+\ldots\ldots\ldots+\frac{1}{1.2.3\ldots(n-1)q^{n-1}dx^{n-1}}d^{n-1}.\frac{M}{S}\int\frac{dx}{p+qx}$$

existente $S=\frac{N}{(p+qx)^n}$ atque in coefficientibus vbique posito $x=\frac{-p}{q}$.

Exemplum 4.

§. 19. Huius formulae differentialis $\frac{(1-x)dx}{x^4(2x-1)^3(3x-2)^2(4x-3)}$ integrale inuenire.

Hic est $M=1-x$ et $N=x^4(2x-1)^3(3x-2)^2(4x-3)$ et cum variabilis x in numeratore M pauciores habeat dimensiones quam in denominatore N, nulla pars integra in fractione $\frac{M}{N}$ continetur, nullaque inde nascitur integralis pars. Consideremus ergo factores denominatoris, ac primo quidem x^4, erit $S=(2x-1)^3(3x-2)^2(4x-3)$. et $p=0$, atque $q=1$; vnde ponendum erit $x=\frac{-p}{q}=0$. Iam ad coefficientes requisitos inueniendos erit

$$\frac{M}{S}=\frac{1-x}{(2x-1)^3(3x-2)^2(4x-3)}=\frac{1}{12}$$

$$d.\frac{M}{S}=\frac{120x^3-288xx+223x-56}{(2x-1)^4(3x-2)^3(4x-3)^2}dx=\frac{7}{9}.dx$$

$$dd.\frac{M}{S}=\frac{-17280x^5+65664x^4-98016x^3+72068x^2-26162x+1758}{(2x-1)^5(3x-2)^4(4x-3)^3}dx^2=\frac{879}{216}dx^2.$$

$$d^3.\frac{M}{S}=\left(\frac{-26162}{432}+\frac{37580}{432}+\frac{45096}{864}+\frac{45396}{1296}\right)dx^3=\frac{24499}{216}dx^3$$

 Hinc

Hinc ex denominatoris factore x^4 nafcitur integralis pars haec $\frac{1}{12}\int\frac{dx}{x^4} + \frac{11}{9}\int\frac{dx}{x^3} + \frac{1879}{432}\int\frac{dx}{xx} + \frac{24499}{1296}\int\frac{dx}{x}$ feu $C - \frac{1}{36x^3} - \frac{7}{18xx} - \frac{1879}{432x} + \frac{24499}{1296}\, l\, x$.

Sumamus alterum factorem $(2x-1)^3$, quo eft $q = 2$; $p = -1$; eft valor pro x fubftituendus $= \frac{1}{2}$: deinde eft $S = x^4(3x-2)^2(4x-3)$ atque coefficientes quaefiti

$\frac{M}{S} = \frac{1-x}{x^4(3x-2)^2(4x-3)} = -32.$

$\frac{1}{dx}\, d.\frac{M}{S} = \frac{72x^3 - 161x^2 + 112x - 24}{x^5(3x-2)^3(4x-3)^2} = -192.$

$\frac{1}{dx^2}\, dd.\frac{M}{S} = -3200.$

Integralis ergo pars ex factore $(2x-1)^3$ oriunda eft $-32\int\frac{dx}{(2x-1)^3} - 96\int\frac{dx}{(2x-1)^2} - 400\int\frac{dx}{2x-1}$ fiue $+\frac{8}{(2x-1)^2} + \frac{48}{2x-1} - 200\, l(2x-1)$

Tertius denominatoris factor $(3x-2)^2$ dat $p = -2$ et $q = 3$, atque $S = x^4(2x-1)^3(4x-3)$ vnde ponendo $x = \frac{2}{3}$ oriuntur coefficientes

$\frac{M}{S} = \frac{1-x}{x^4(2x-1)^3(4x-3)} = -\frac{2187}{16}$

$\frac{1}{dx}\, d.\frac{M}{S} = \frac{+32805}{16}$

integralis ergo pars ex factore $(3x-2)^2$ oriunda erit $-\frac{2187}{16}\int\frac{dx}{(3x-2)^2} + \frac{10935}{16}\int\frac{dx}{3x-2}$ fiue $+\frac{729}{16(3x-2)} + \frac{3645}{16}\, l(3x-2)$

Tandem vltimus factor $4x-3$, dat $x = \frac{3}{4}$ atque $S = x^4(2x-1)^3(3x-2)^2$ vnde erit

$\frac{M}{S} = \frac{1-x}{x^4(2x-1)^3(3x-2)^2} = \frac{8192}{81}$

vnde integrale ex hoc factore oriundum erit

$\frac{8192}{81}\int\frac{dx}{4x-3} = \frac{2048}{81}\, l(4x-3)$

Formulae

Formulae itaque differentialis propositae huius

$$\frac{(1-x)dx}{x^4(2x-1)^3(3x-2)^2(4x-3)}$$

integrale completum erit:

$$C-\frac{1}{36x^3}-\frac{7}{18xx}-\frac{1879}{432x}+\frac{24499}{1296}lx-200l(2x-1)+\frac{3645}{16}l(3x-2)$$
$$+\frac{8}{(2x-1)^2}+\frac{48}{2x-1}+\frac{729}{16(3x-2)}+\frac{2048}{81}l(4x-3)$$

§. 20. Conuenit quandoque loco differentiationum ipsius $\frac{M}{S}$ ipso principio vti, vnde eas deduximus; hocque modo facilius peruenietur ad numeratorem quaesitum. Scilicet si denominatoris N factor fuerit R, ita vt sit $N=RS$, in fractione $\frac{M}{N}$ seu $\frac{M}{RS}$ continebitur fractio simplicior $\frac{V}{R}$, si ea fuerit summa harum $\frac{V}{R}+\frac{T}{S}$. Fiet ergo $M=VS+TR$, vnde oritur $T=\frac{M-VS}{R}$. Quare cum T sit quantitas integra, pro V eiusmodi quantitatem integram quaeri oportet, vt $M-VS$ diuisibile fiat per R, quod autem ita effici debet, vt variabilis x, pauciores obtineat dimensiones in V quam in R. Haec vero quantitatis V inventio interdum sine differentiationibus facilius absoluitur, solo ratiocinio. Sit enim pro $\frac{M}{N}$ proposita ista fractio

$\frac{1}{x^m(1+x^n)}$, vbi est $M=1$, $R=x^m$, et $S1+x^n$;

atque ad quaerendam fractionem $\frac{V}{x^m}$ in illa fractione contentam, in cuius numeratore V variabilis x pauciores habeat dimensiones quam m; oportet pro V eius modi functionem inuestigare, vt $1-V(1+x^n)$ fiat diuisibile per x^m.

 Patet

Patet autem, vt 1 tollatur, esse debere $V = 1 + X$, quo substituto haec quantitas $X + x^n + Xx^n$ diuisibilis est reddenda per x^m. Perspicuum autem est si fuerit $m < n$, tum diuisionem succedere si $X = 0$, ideoque casu $m < n$ in fractione $\frac{1}{x^m(1+x^n)}$ continetur haec simplicior $\frac{1}{x^m}$. Quod si autem sit $m > n$ tum ponatur $X = Y - x^n$, habebiturque $Y - x^{2n} + Yx^n$ diuisibile per x^m: euenit hoc si $m < 2n$ posito $Y = 0$; quare si $m > n$ et $m < 2n$, tum erit $X = -x^n$ et $V = 1 - x^n$. Hinc facile consequimur generatim fractionem $\frac{V}{x^m}$ fore $\frac{1 - x^n + x^{2n} - x^{3n} + x^{4n} - \text{etc.}}{x^m}$ in cuius numeratore tot capiendi sunt termini, quoad ad exponentem ipsius x maiorem quam m perueniatur. In formula ergo differentiali $\frac{dx}{x^m(1+x^n)}$ ex denominatoris factore x^m haec elicitur integralis pars

$$\int \frac{dx}{xm} - \int \frac{dx}{xm^{-n}} + \int \frac{dx}{x^{m-2n}} - \int \frac{dx}{x^{m-3n}} + \text{etc.}$$

eousque continuanda, donec exponentes ipsius x fiant negatiui: ista autem integralis pars hoc pacto multo facilius reperitur, quam per differentiationes ante indicatas.

§. 21. Exposuimus igitur modum facilem atque expeditum ex factoribus simplicibus denominatoris N in formula differentiali $\frac{M}{N}dx$, ac potestatibus eorum, quae quidem in N continentur, partes integralis quaesiti respondentes inueniendi. Totum enim integrale formulae $\frac{Mdx}{N}$ componitur ex partibus, quae cum ex quantitatibus integris

integris in fractione $\frac{M}{N}$ contentis oriuntur, tum ex singulis factoribus denominatoris N. Eae quidem integralis partes, quae ex quantitate integra in fractione $\frac{M}{N}$ contenta nascuntur, sunt perpetuo quantitates algebraicae; illae autem quae ex factoribus simplicibus denominatoris N proficiscuntur, sunt quantitates logarithmicae, cum quibus etiam algebraicae coniunguntur, si potestas cuiuspiam factoris simplicis in denominatore N contineatur; hocque casu subinde euenire potest, vt in integrali pars logarithmica penitus euanescat, solaeque quantitates algebraicae superstites maneant. Quod si igitur denominator N omnes factores simplices habeat reales, tum integrale formulae differentialis $\frac{M\,dx}{N}$ nisi est quantitas algebraica, per logarithmos exhiberi potest. Sin autem in denominatore N contineantur factores simplices imaginarii, tum quidem per methodum integrandi hic expositam perueniretur ad logarithmos imaginarios, quos autem, siquidem quantitatem realem prae se ferant, ad arcus circulares reduci posse constat. Supra autem iam obseruauimus, si denominator N habeat factores simplices imaginarios, tum eorum numerum semper esse parem, atque ex iis binos semper ita esse comparatos, vt eorum productum fiat expressio realis. Hanc obrem loco factorum simplicium imaginariorum formari poterunt factores trinomiales reales, quorum numerus erit duplo minor, ex hisque factoribus peruenietur ad integralis partes a quadratura circuli pendentes.

§. 22. Praecipuum igitur negotium, si denominator N habeat factores simplices imaginarios, in hoc versabitur, vt ipsius denominatoris N factores trinomiales reales exhibeantur

hibeantur, in quibus factores imaginarii contineantur. Sit itaque $p + rx + qxx$ huiusmodi factor trinomialis ipsius N, cuius factores simplices sint imaginarii, erit $4pq > rr$ seu $\frac{r}{2\sqrt{pq}} < 1$. Denotabit igitur $\frac{r}{2\sqrt{pq}}$ cosinum cuiuspiam anguli qui sit Φ, ita vt sit $\frac{r}{2\sqrt{pq}} = \text{cos. A} . \Phi$ et $r = 2\sqrt{pq} . \text{cos. A}\,\Phi$. Quamobrem generalis forma huiusmodi factoris trinomialis erit

$$p - 2x\sqrt{pq} . \text{cos. A}\,\Phi + qxx$$

atque ideo in hoc nobis erit elaborandum, vt inueniamus an huiusmodi factores trinomiales in denominatore N contineantur, et quot sint futuri, et quales. Patet autem in hac forma trinomiali etiam factores simplices reales comprehendi, si fiat $\Phi = 0$; tum enim ob $\text{cos. A} . \Phi = 1$, erit factor ille trinomialis $= (\sqrt{p} - x\sqrt{q})^2$, indicabitque denominatorem N diuisibilem esse per $\sqrt{p} - x\sqrt{q}$; etsi concludi non potest, etiam ipsius quadratum $(\sqrt{p} - x\sqrt{q})^2$ esse diuisorem ipsius N: inuestigatio enim diuisorum aequalium ex alio fonte est petenda. Quamobrem si determinauerimus, quot variis modis expressio $p - 2x\sqrt{pq} . \text{cos. A}\,\Phi + qxx$ tanquam factor in denominatore N contineatur, tum simul tum omnes factores trinomiales in imaginarios resolubiles, quam etiam ipsos factores simplices reales assequemur. Atque hinc etiam, si ista inuestigatio perpetuo poterit absolui, intelligetur, quod supra iam probauimus, omnes factores simplices imaginarios ad factores trinomiales reales reduci posse.

§. 23. Ponamus ergo denominatoris N factorem esse $p - 2x\sqrt{pq} . \text{cos. A}\,\Phi + qxx$

is

is itaque in se complectitur hos binos factores simplices imaginarios:

$$x\sqrt{q}-\sqrt{p}\,.\,\text{cos. A}\,\Phi+\sqrt{-p}\,.\,\text{sin. A}\,\Phi$$
$$x\sqrt{q}-\sqrt{p}\,.\,\text{cos. A}\,\Phi-\sqrt{-p}\,.\,\text{sin. A}\,\Phi$$

si igitur hi factores simplices nihilo aequales ponantur, et valores ipsius x inde oriundi in N substituantur, vtroque casu valor ipsius N euanescet. Fiet autem, valores ipsius x coniunctim exprimendo

$$x=\sqrt{\tfrac{p}{q}}\,.\,\text{cos. A}\,\Phi\pm\sqrt{-\tfrac{p}{q}}\,.\,\text{sin. A}\,\Phi.$$

vel si ponamus commoditatis gratia $f=\sqrt{\frac{p}{q}}$, erit

$$x=f\,\text{cos. A}\,\Phi\pm f\sqrt{-1}\,.\,\text{sin. A}\,\Phi$$

vterque igitur valor ipsius x in N substitutus ad nihilum perducere debet. Colligitur autem cum ex ipsa operatione instituenda, tum ex proprietatibus de multiplicatione arcuum cognitis, singulas ipsius x potestates sequenti modo expressum iri.

$$x^2=ff\,\text{cos. A}\,.\,2\,\Phi\pm ff\sqrt{-1}\,.\,\text{sin. A}\,.\,2\,\Phi$$
$$x^3=f^3\,\text{cos. A}\,.\,3\,\Phi\pm f^3\sqrt{-1}\,.\,\text{sin. A}\,.\,3\,\Phi$$
$$x^4=f^4\,\text{cos. A}\,.\,4\,\Phi\pm f^4\sqrt{-1}\,.\,\text{sin. A}\,.\,4\,\Phi$$

et generaliter

$$x^k=f^k\,\text{cos. A}\,.\,k\,\Phi\pm f^k\sqrt{-1}\,.\,\text{sin. A}\,.\,k\,\Phi$$

Cum igitur loco cuiusuis potestatis ipsius x duo tantum termini substitui debeant, substitutio vtraque pro vtroque signorum ambiguo facile absoluitur. Quod, quo facilius perspiciatur, scribatur in N primo f^k cos. A . $k\Phi$ loco cuiusuis potestatis x^k, sitque quod prodit $=$ P; deinde loco x^k

 scribatur

ſcribatur f^k ſin. A. $k\Phi$; et quod prodit, ſit $= Q$: atque manifeſtum eſt per ſubſtitutionem

$$x^k = f^k \text{ coſ. A.} k\Phi \pm f^k \sqrt{-1}.\text{ ſin. A.} k\Phi$$

denominatorem abiturum eſſe in

$$P \pm Q\sqrt{-1}.$$

quae duplex expreſſio cum debeat eſſe $= 0$, erit tam $P = 0$, quam $Q = 0$.

§. 24. Ad valores igitur tam pro p et q quam pro arcu Φ inueniendos, qui reddat

$$p - 2x\sqrt{pq}.\text{ coſ. A}\Phi + qxx$$

factorem denominatoris N, poſito $f = \sqrt{\frac{p}{q}}$, duplicem nanciſcimur aequationem, primo ſcilicet loco x^k ponendo f^k coſ. A$k\Phi$, oritur aequatio $P = 0$; ac deinde loco x^k ponendo f^k ſin. A$k\Phi$, orietur altera aequatio $Q = 0$, ex quibus duabus aequationibus tum quantitatem f, quum arcum Φ determinari oportebit. Hoc autem pluribus modis ſemper praeſtari poterit, tot ſcilicet, quot varios factores tam ſimplices quam trinomiales reales denominator N in ſe complectitur. Simplices quidem prodeunt, ſi $\Phi = 0$, quo caſu alter valor Q ſponte fit 0, ob ſin. A. $k\Phi = 0$; tum autem erit coſ. A. $k\Phi = 1$, ac valor P ex N naſcetur, ponendo ſimpliciter f loco x. Quare, quot iſta aequatio $P = 0$ habebit radices reales, tot prodibunt factores ſimplices reales denominatoris N; ac ſi omnes radices aequationis $P = 0$ fuerint reales, tum vlteriori inveſtigatione non erit opus. Sin autem radices imaginariae contineantur, tum alios quaeri oportet valores pro arcu Φ, qui aequationibus $P = 0$ et $Q = 0$ ſatisfaciant, hincque con-

conuenienter valores pro f elicientur; atque ſic factores trinomiales obtinentur, factores ſimplices imaginarios complectentes. Vſus autem huius regulae clarius apparebit, ſi eius ope factores trinomiales inueſtigemus denominatorum, quos deinceps in exemplis ſumus tractaturi. Sit igitur primum ſequens propoſita forma, cuius factores reales ſiue ſimplices ſiue trinomiales inueſtigari oporteat

$$\alpha + \beta x^n$$

§. 25. Quia ſubſtitutiones praeſcriptae loco poteſtatum ipſius x ſunt faciendae, terminus abſolutus α ita eſt ſpectandus, quaſi eſſet αx^0. Poſito ergo loco poteſtatis ipſius x generalis x^k tam f^k coſ. A $k\Phi$, quam f^k ſin. A. $k\Phi$, et vtraque expreſſione reſultante facta $=0$, ſequentes duae aequationes habebuntur

$$\alpha + \beta f^n \text{ coſ. A} . n\Phi = 0$$

$$\beta f^n \text{ ſin. A} . n\Phi = 0.$$

Primum igitur poni poteſt $\Phi = 0$, quo poſteriori aequationi ſatisfiet, prior vero dabit

$$\alpha + \beta f^n = 0 \text{ ſeu } f = \sqrt[n]{-\tfrac{\alpha}{\beta}} = \sqrt{\tfrac{p}{q}}$$

vnde oritur diuiſor ſimplex $\sqrt{p} - x\sqrt{q}$ ſeu $\sqrt[n]{\alpha} - x\sqrt[n]{-\beta}$ ſiue $\sqrt[n]{-\frac{\alpha}{\beta}} - x$. Quod ſi ergo ſit n numerus impar, ſemper vnus habetur factor ſimplex realis $\sqrt[n]{\alpha} - x\sqrt[n]{-\beta}$. At ſi n ſit numerus par, factor ſimplex realis non dabitur, niſi $\frac{-\alpha}{\beta}$ fuerit quantitas affirmatiua, hoc vero caſu duplex habebitur factor ſimplex realis, nempe $\pm \sqrt[n]{-\frac{\alpha}{\beta}} - x$; ſiue huius expreſſionis $\alpha - \beta x^n$ hi duo erunt factores ſimplices reales $\sqrt[n]{\alpha} + x\sqrt[n]{\beta}$ et $\sqrt[n]{\alpha} - x\sqrt[n]{\beta}$, ſi quidem n

 eſt

eſt numerus par; hi autem caſus per ſe ſunt noti, atque in ſequenti inueſtigatione denuo occurrent.

§. 26. Non igitur ſit $\Phi = 0$, atque aequatio poſterior dabit ſin. A. $n\Phi = 0$: ex quo poſita ſemiperipheria circuli $= \pi$, exiſtente radio $= 1$; erit $n\Phi =$ multiplo cuicunque ſemiperipheriae π, quod ſit $k\pi$, hincque $\Phi = \frac{k\pi}{n}$. Hinc autem fiet coſ. A $n\Phi =$ coſ A $k\pi = \pm 1$: erit nempe coſ. A. $n\Phi = +1$, ſi k fuerit numerus par, et coſ. A. $n\Phi = -1$ ſi k fuerit numerus impar. Subſtituto hoc valore in priori aequatione habebimus $\alpha \pm \beta f^n = 0$. Hinc duos caſus euolui conueniet, prout α et β ſint quantitates vel iisdem ſignis vel diuerſis affectae. Sint primo iisdem ſignis affectae

$$\alpha + \beta x^n$$

atque ſumatur k numerus impar, $2k - 1$, vt ſit $\Phi = \frac{(2k-1)\pi}{n}$ et coſ. A. $n\Phi = -1$, erit $\alpha - \beta f^n = 0$ et $f = \sqrt[n]{\frac{\alpha}{\beta}} = \sqrt{\frac{p}{q}}$; vnde $p = \sqrt[n]{\alpha^2}$ et $q = \sqrt[n]{\beta^2}$

Formae igitur propoſitae $\alpha + \beta x^n$ habebimus hunc factorem trinomialem generalem:

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta}\,.\,\text{coſ. A}.\,\frac{(2k-1)\pi}{n} + xx\sqrt[n]{\beta^2}$$

Atque hinc tot factores diuerſi reſultabunt, quot loco k numeris integris ſubſtituendis diuerſi valores pro coſ. A. $\frac{(2k-1)\pi}{n}$ oriuntur. Quod ſi autem loco k ſucceſſiue omnes numeros integros $1, 2, 3, \ldots\ldots n$ ſubſtituamus, tum quilibet factor trinomialis bis occurret, ſi n fuerit numerus par, ſin autem n fuerit numerus impar, tum in medio ſolitarius factor relinquetur, poſito $2k - 1 = n$; hocque caſu fit coſ. A. $\pi = -1$; et ex hoc factor ſimplex

plex realis nascitur $\sqrt[n]{\alpha} + x\sqrt[n]{\beta}$. Factores autem trinomiales obtinentur, ponendo loco $2k-1$ omnes numeros impares minores quam n.

§. 27. Ex his igitur omnes factores tam simplices quam trinomiales reales exhiberi possunt formae

$$\alpha + \beta x^n$$

si enim n sit numerus par, omnes erunt trinomiales, eorumque numerus $= \frac{n}{2}$: qui erunt

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta}\,.\,\mathrm{cos.\,A.}\,\frac{\pi}{n} + xx\sqrt[n]{\beta^2}$$
$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta}\,.\,\mathrm{cos.\,A.}\,\frac{3\pi}{n} + xx\sqrt[n]{\beta^2}$$
$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta}\,.\,\mathrm{cos.\,A.}\,\frac{5\pi}{n} + xx\sqrt[n]{\beta^2}$$
$$\vdots$$
$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta}\,.\,\mathrm{cos.\,A.}\,\frac{(n-1)\pi}{n} + xx\sqrt[n]{\beta^2}$$

Quod si autem n fuerit numerus impar, tum vnus factor erit simplex, reliqui trinomiales, horumque numerus $= \frac{n-1}{2}$: omnes autem erunt

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta}\,.\,\mathrm{cos.\,A.}\,\frac{\pi}{n} + xx\sqrt[n]{\beta^2}$$
$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta}\,.\,\mathrm{cos.\,A.}\,\frac{3\pi}{n} + xx\sqrt[n]{\beta^2}$$
$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta}\,.\,\mathrm{cos.\,A.}\,\frac{5\pi}{n} + xx\sqrt[n]{\beta^2}$$
$$\vdots$$
$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta}\,.\,\mathrm{cos.\,A}\,\frac{(n-2)\pi}{n} + xx\sqrt[n]{\beta^2}$$
$$\sqrt[n]{\alpha} + x\sqrt[n]{\beta}$$

vtroque autem casu factores exhibiti actu in se ducti formam propositam $\alpha + \beta x^n$ producent.

§. 28. Sint iam quantitates α et β diuersis signis affectae, seu quaerantur factores huius expressionis

$$\alpha - \beta x^n$$

atque pro k accipi oportebit numerum parem $2k$, ita vt sit $\Phi = \frac{2k\pi}{n}$ et $\alpha - \beta f^n = 0$ seu $f = \sqrt[n]{\frac{\alpha}{\beta}} = \sqrt{\frac{p}{q}}$, vnde

$p = \sqrt[n]{\alpha^2}$ et $q = \sqrt[n]{\beta^2}$

Factor igitur trinomialis realis in genere erit

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta}\,.\,\text{cos. A}\,.\,\tfrac{2k\pi}{n} + xx\sqrt[n]{\beta^2}$$

atque tot erunt huiusmodi factores, quot varii prodibunt valores pro cos. A. $\frac{2k\pi}{n}$. Omnes autem diuersi prodibunt valores, si pro $2k$ substituantur omnes numeri pares vsque ad n. Et quidem $a\,o$ loco $2k$ substituendo oritur factor simplex

$$\sqrt[n]{\alpha} + x\sqrt[n]{\beta}$$

Praeterea vero, si n numerus par et fiat $2k = n$, denuo factor simplex realis oritur

$$\sqrt[n]{\alpha} + x\sqrt[n]{\beta}.$$

Quare si n fuerit numerus par huius formulae

$$\alpha - \beta x^n$$

sequentes erunt factores reales siue simplices siue trinomiales;

$$\sqrt[n]{\alpha} - x\sqrt[n]{\beta}$$

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta}\,.\,\text{cos. A}\,.\,\tfrac{2\pi}{n} + xx\sqrt[n]{\beta^2}$$

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta}\,.\,\text{cos. A}\,.\,\tfrac{4\pi}{n} + xx\sqrt[n]{\beta^2}$$

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta}\,.\,\text{cos. A}\,.\,\tfrac{6\pi}{n} + xx\sqrt[n]{\beta^2}$$

$\sqrt[n]{}$

$$\vdots$$

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta} \,.\, \text{cof. A} \frac{(n-2)\pi}{n} + xx\sqrt[n]{\beta^2}$$

$$\sqrt[n]{\alpha} + x\sqrt[n]{\beta}$$

Quod si autem n fuerit numerus impar, tum formulae
$\alpha - \beta x^n$
factores reales erunt sequentes:

$$\sqrt[n]{\alpha} - x\sqrt[n]{\beta}$$

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta} \,.\, \text{cof. A} \,.\, \frac{2\pi}{n} + xx\sqrt[n]{\beta^2}$$

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta} \,.\, \text{cof. A} \,.\, \frac{4\pi}{n} + xx\sqrt[n]{\beta^2}$$

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta} \,.\, \text{cof. A} \,.\, \frac{6\pi}{n} + xx\sqrt[n]{\beta^2}$$

$$\vdots$$

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta} \,.\, \text{cof. A} \,.\, \frac{(n-1)\pi}{n} + xx\sqrt[n]{\beta^2}$$

Atque vtroque casu productum ex his omnibus factoribus ortum producet formulam $\alpha - \beta x^n$.

§. 29. Ex his perspicitur, si loco k omnes numeri integri ab 1, 2, 3 .... vsque ad n inclusiue substituantur; tum omnes factores trinomiales ex ista forma generali resultantes

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta} \,.\, \text{cof. A} \frac{(2k-1)\pi}{n} + xx\sqrt[n]{\beta^2}$$

si in se inuicem ducantur, producturos expressionem hanc:
$(\alpha + \beta x^n)^2$
ideoque, si ex singulis illis factoribus radices quadratae extrahantur, productum ex his omnibus radicibus dabit formulam

mulam $\alpha + \beta x^n$. Simili modo ſi, vt ante, loco k omnes numeri integri 1, 2, 3, ... n ſubſtituantur, tum omnes factores trinomiales, quorum numerus erit $= n$, qui reſultant ex forma generali

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta}. \text{ coſ A. } \frac{2k\pi}{n} + xx\sqrt[n]{\beta^2}$$

ſi in ſe mutuo ducantur, dabunt productum

$$(\alpha - \beta x^n)^2$$

Atque idcirco, ſi ex ſingulis his factoribus radices quadratae extrahantur, earum productum dabit ipſam expreſſionem $\alpha - \beta x^n$.

§. 30. Hoc igitur pacto reſolui poteſt formula $\alpha + \beta x^n$ in n factores, quorum quilibet eſt radix quadrata ex expreſſione trinomiali huiusmodi.

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta}, \text{ coſ. A. } \Phi + xx\sqrt[n]{\beta^2}$$

Poteſt autem radix quadrata ex huiusmodi expreſſione admodum ſuccincte geometrice conſtrui;

Erit enim $\sqrt{(\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta}. \text{ coſ. A.}\Phi + xx\sqrt[n]{\beta^2})} = \sqrt{((\sqrt[n]{\alpha}. \text{ coſ. A}.\Phi - x\sqrt[n]{\beta})^2 + (\sqrt[n]{\alpha}. \text{ ſin. A}.\Phi)^2)}$.

Erit ergo quilibet eorum factorum hypothenuſa trianguli rectanguli, cuius alter cathetus $= \sqrt[n]{\alpha}. \text{ coſ. A}. \Phi - x\sqrt[n]{\beta}$ et alter $\sqrt[n]{\alpha}. \text{ ſin. A}. \Phi$, quae expreſſiones in circulo, cuius radius $= \sqrt[n]{\alpha}$, commodiſſime exhiberi poſſunt.

Tab. I. fig. 5. Fiat nempe circulus P Q R S T V centro C et radio CP $= \sqrt[n]{\alpha}$; diuidatur eius peripheria in $2n$, ſeu ſemiperipheria

pheria in n partes, erit $Pp = \frac{\pi}{n}$; $PQ = \frac{2\pi}{n}$; $Pq = \frac{3\pi}{n}$; $PR = \frac{4\pi}{n}$; $Pr = \frac{5\pi}{n}$; etc.

Capiatur porro in radio CP diſtantia $CO = x\sqrt[n]{\beta}$; atque ex puncto O ad ſingula diuiſionis puncta ducantur rectae OP, Op, OQ, Oq, etc. quae, quantae futurae ſint, ex recta indefinita OM colligi poterit. Sit arcus $PM = \Phi$ et ducta perpendiculari MN erit $MN = \sqrt[n]{\alpha} \,.\, \text{ſin.} \, A \,.\, \Phi$; $CN = \sqrt[n]{\alpha} \,.\, \text{coſ.} \, A\Phi$; ideoque $ON = \sqrt[n]{\alpha} \,.\, \text{coſ.} \, A\,\Phi - x\sqrt[n]{\beta}$; vnde fiet $OM = \sqrt{(\sqrt[n]{\alpha^2} - 2\sqrt[n]{\alpha\beta} \, \text{coſ.} \, A\Phi + x^2\sqrt[n]{\beta^2})}$. Ex his ergo ſumendis diuiſionum punctis paribus erit

$$OP \,.\, OQ \,.\, OR \,.\, OS \,.\, OT \,.\, OV = \alpha - \beta x^n$$

ſumendis autem diuiſionibus imparibus erit

$$Op \,.\, Oq \,.\, Or \,.\, Os \,.\, Ot \,.\, Ov = \alpha + \beta x^n$$

hocque eſt theorema elegantiſſimum a Coteſio inuentum, cuius adeo demonſtratio per methodum noſtram inueſtigandi factores trinomiales a priori eſt data.

§. 31. Progrediamur ad exemplum magis intricatum atque quaeramus factores reales tum ſimplices quam trinomiales huius expreſſionis

$$\alpha + \beta x^n + \gamma x^{2n}$$

cuius factor trinomialis quicunque ſi fuerit

$$p - 2x\sqrt{pq} \,.\, \text{coſ.} \, A\,\Phi + qxx$$

poſito $f = \sqrt{\frac{p}{q}}$, incognitae f et Φ ex his duabus aequationibus erui debebunt.

$$\alpha + \beta f^n \, \text{coſ.} \, A \,.\, n\Phi + \gamma f^{2n} \, \text{coſ.} \, A \,.\, 2n\Phi = 0$$

$$\beta f^n \, \text{ſin.} \, A \,.\, n\Phi + \gamma f^{2n} \, \text{ſin.} \, A \,.\, 2n\Phi = 0$$

Cum iam ſit ſin. A. $2n\Phi = 2$ ſin. A. $n\Phi$. coſ. A. $n\Phi$, erit ex aequatione poſteriori vel ſin. A. $n\Phi = 0$ vel $\beta + 2\gamma f^n$ coſ. A. $n\Phi = 0$. Sit primo ſin. A. $n\Phi = 0$ erit vel $n\Phi = 2k\pi$ vel $n\Phi = (2k-1)\pi$; ponamus ergo $n\Phi = 2k\pi$ erit coſ. A. $n\Phi = 1$, et coſ. A. $2n\Phi = 1$, vnde $\alpha + \beta f^n + \gamma f^{2n} = 0$; hincque $f^n = \frac{-\beta \pm \sqrt{(\beta^2 - 4\alpha\gamma)}}{2\gamma}$ ſin autem $n\Phi = (2k-1)\pi$ erit coſ. A. $n\Phi = -1$ et coſ. A. $2n\Phi = +1$ vnde $\alpha - \beta f^n + \gamma f^{2n} = 0$ hincque $f^n = \frac{\beta \pm \sqrt{(\beta^2 - 4\alpha\gamma)}}{2\gamma}$. Iſtae ergo ſolutiones locum habere non poſſunt, niſi ſit $\beta^2 > 4\alpha\gamma$. Sit ergo $\beta^2 > 4\alpha\gamma$ atque ſequentes caſus erunt notandi.

I. $\alpha + \beta x^n + \gamma x^{2n}$

Vt f^n affirmatiuum obtineat valorem, ſumi debet $\Phi = \frac{(2k-1)\pi}{n}$ eritque $f = \sqrt[n]{\frac{\beta \pm \sqrt{(\beta^2 - 4\alpha\gamma)}}{2\gamma}} = \sqrt{\frac{p}{q}}$ ſit $\frac{\beta + \sqrt{(\beta^2 - 4\alpha\gamma)}}{2} = \zeta$ et $\frac{\beta - \sqrt{(\beta^2 - 4\alpha\gamma)}}{2} = \eta$ eruntque factores trinomiales huius formae hi bini

$$\sqrt[n]{\zeta^2} - 2x \sqrt[n]{\gamma\zeta}.\ \text{coſ. A.}\ \tfrac{(2k-1)\pi}{n} + xx \sqrt[n]{\gamma\gamma}$$

$$\sqrt[n]{\eta^2} - 2x \sqrt[n]{\gamma\eta}.\ \text{coſ. A.}\ \tfrac{(2k-1)\pi}{n} + xx \sqrt[n]{\gamma\gamma}$$

vtraque expreſſio vt praecedenti caſu tractata dabit factores reales vel ſimplices (nempe ſi n numerus impar) vel trinomiales, qui omnes in ſe inuicem ducti expreſſionem propoſitam producunt.

§. 32. Maneat $\beta^2 > 4\alpha\gamma$, ſitque haec forma propoſita.

II. $\alpha - \beta x^n + \gamma x^{2n}$

Vt f^n affirmatiuum valorem obtineat, ſumi debet $\Phi = \frac{2k\pi}{n}$ eritque $f = \sqrt[n]{\frac{\beta \pm \sqrt{(\beta^2 - 4\alpha\gamma)}}{2\gamma}} = \sqrt{\frac{p}{q}}$, ſit vt ante

$\beta +$

$\frac{\beta+\sqrt{(\beta^2-4\alpha\gamma)}}{2}=\zeta$ et $\frac{\beta-\sqrt{(\beta^2-4\alpha\gamma)}}{2}=\eta$, hincque orientur ſequentes duae formae pro factoribus trinomialibus quaeſitis

$$\sqrt[n]{\zeta^2}-2x\sqrt[n]{\gamma\zeta}.\,\text{coſ.}\,A.\tfrac{2k\pi}{n}+xx\sqrt[n]{\gamma\gamma}$$
$$\sqrt[n]{\eta^2}-2x\sqrt[n]{\gamma\eta}.\,\text{coſ.}\,A.\tfrac{2k\pi}{n}+xx\sqrt[n]{\gamma\gamma}$$

qui, quoties fiunt quadrata, radices praebent ſimplices reales, ceteris caſibus factores trinomiales reſultant. Sit iam propoſita iſta expreſſio

III. $\alpha+\beta x^n-\gamma x^{2n}$

in qua ſemper eſt $\beta^2+4\alpha\gamma$ quantitas poſitiua. Praebet autem caſus $\Phi=\frac{2k\pi}{n}$ vnum valorem poſitiuum pro $f^n=\frac{\beta+\sqrt{(\beta^2+4\alpha\gamma)}}{2\gamma}$; alterque caſus $\Phi=\frac{(2k-1)\pi}{n}$ pariter vnum $f^n=\frac{-\beta+\sqrt{(\beta^2+4\alpha\gamma)}}{2\gamma}$. Ponatur $\frac{\beta+\sqrt{(\beta^2+4\alpha\gamma)}}{2}=\zeta$ et $\frac{-\beta+\sqrt{(\beta^2+4\alpha\gamma)}}{2}=\eta$, atque ſequentes duae formulae dabunt omnes factores reales tam ſimplices (quando ſcilicet fiunt quadrata) quam trinomiales:

$$\sqrt[n]{\zeta^2}-2x\sqrt[n]{\gamma\zeta}.\,\text{coſ.}\,A.\tfrac{2k\pi}{n}+xx\sqrt[n]{\gamma\gamma}$$
$$\sqrt[n]{\eta^2}-2x\sqrt[n]{\gamma\eta}.\,\text{coſ.}\,A.\tfrac{(2k-1)\pi}{n}+xx\sqrt[n]{\gamma\gamma}$$

§. 33. Quartus caſus, quo ſponte fit $\beta^2>4\alpha\gamma$, eſt haec forma

IV. $\alpha-\beta x^n-\gamma x^{2n}$

Hic iterum caſus $\Phi=\frac{2k\pi}{n}$ vnum praebet caſum poſitivum pro $f^n=\frac{-\beta+\sqrt{(\beta^2+4\alpha\gamma)}}{2\gamma}$ alterque caſus $\Phi=\frac{(2k-1)\pi}{n}$ pariter vnum $f^n=\frac{\beta+\sqrt{(\beta^2+4\alpha\gamma)}}{2\gamma}$. Ponatur ergo $\frac{-\beta+\sqrt{(\beta^2+4\alpha\gamma)}}{2}=\zeta$ et $\frac{\beta+\sqrt{(\beta^2+4\alpha\gamma)}}{2}=\eta$, atque omnes factores formulae pro-

propositae tam simplices, quam trinomiales continebuntur in his binis sequentibus expressionibus:

$$\sqrt[n]{}\,\zeta^2 - 2x\sqrt[n]{}\,\gamma\zeta.\,\text{cos. A.}\,\tfrac{2k\pi}{n} + xx\sqrt[n]{}\,\gamma\gamma$$

$$\sqrt[n]{}\,\eta^2 - 2x\sqrt[n]{}\,\gamma\eta.\,\text{cos. A.}\,\tfrac{(2k-1)\pi}{n} + xx\sqrt[n]{}\,\gamma\gamma$$

Ceterum de his casibus, quibus $\beta^2 > 4\alpha\gamma$, notandum est, iis formulam propositam

$$\alpha + \beta x^n + \gamma x^{2n}$$

actu posse resolui in binas formulas reales duobus terminis constantes

$$\sqrt{\alpha} + x^n\left(\frac{\beta + \sqrt{(\beta^2 - 4\alpha\gamma)}}{2\sqrt{\alpha}}\right)$$

$$\sqrt{\alpha} + x^n\left(\frac{\beta - \sqrt{(\beta^2 - 4\alpha\gamma)}}{2\sqrt{\alpha}}\right)$$

quae cum similes sint iis, quas primo loco tractauimus, vtraque seorsim modo iam exposito in suos factores resolui poterit. Prouenient autem hoc pacto illi ipsi factores, quos hic exhibuimus.

§. 34. Pro casibus iam, quibus non est $\beta^2 > 4\alpha\gamma$, alteram solutionem aequationis

$$\beta f^n\,\text{sin. A.}\,n\Phi + \gamma f^{2n}\,\text{sin. A.}\,2n\Phi = 0$$

accipi conueniet, quae dat $\beta + 2\gamma f^n\,\text{cos. A.}\,n\Phi = 0$ Sit cos. A. $n\Phi = z$, erit $f^n = \frac{-\beta}{2\gamma z}$; qui valor ob cos. A. $2n\Phi = 2zz - 1$ in priori aequatione substitutus dat $\alpha - \frac{\beta\beta}{2\gamma} + \frac{\beta\beta(2zz-1)}{4\gamma zz} = 0$ seu $4\alpha\gamma zz = \beta\beta$; hinc erit $z = \frac{-\beta}{2\sqrt{\alpha\gamma}}$; et $f^n = \sqrt{\frac{\alpha}{\gamma}}$ ideoque $p = \sqrt[n]{}\,\alpha$ et $q = \sqrt[n]{}\,\gamma$. Ponamus eum arcum minimum $= \omega$, cuius cosinus est $\frac{\beta}{2\sqrt{\alpha\gamma}}$

eritque

eritque cos. $A((2k-1)\pi \mp \omega) = \frac{-\beta}{2\sqrt{\alpha\gamma}} = z$; ideoque obtinetur $\phi = \frac{(2k-1)\pi \pm \omega}{n}$. Quocirca casu $\beta^2 < 4\alpha\gamma$, si arcus, cuius cosinus est $= \frac{\beta}{2\sqrt{\alpha\gamma}}$ ponatur $= \omega$ erit formulae propositae

$$\alpha + \beta x^n + \gamma x^{2n}$$

quilibet factor trinomialis in hac forma contentus

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\gamma}\,.\,\text{cos. A}\,.\,\frac{(2k-1)\pm\omega}{n} + xx\sqrt[n]{\gamma\gamma}$$

Huiusmodi autem factores habebuntur numero n, qui prodibunt si loco k successiue omnes numeri integri $1, 2, 3 \ldots$ vsque ad n substituantur, tribuendo ipsi ω siue signum $+$ siue $-$; vtroque enim casu arcus prodibunt, quorum cosinus congruent. Signum scilicet $-$ arcu ω praefixum eosdem dabit cosinus, quos signum $+$, ordine tantum retrogrado, siquidem loco k numeri $1, 2, 3 \ldots n$ substituantur.

vnde factores ipsi erunt sequentes

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\gamma}\,.\,\text{cos. A}\,.\,\frac{\pi-\omega}{n} + xx\sqrt[n]{\gamma\gamma}$$

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\gamma}\,.\,\text{cos. A}\,.\,\frac{3\pi-\omega}{n} + xx\sqrt[n]{\gamma\gamma}$$

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\gamma}\,.\,\text{cos. A}\,.\,\frac{5\pi-\omega}{n} + xx\sqrt[n]{\gamma\gamma}$$

$$\vdots$$

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\gamma}\,.\,\text{cos. A}\,.\,\frac{(2n-1)\pi-\omega}{n} + xx\sqrt[n]{\gamma\gamma}$$

§. 35. Si coefficiens β fuerit negatiuus, seu, si huius formae, existente $\beta\beta < 4\alpha\gamma$,

$$\alpha - \beta x^n + \gamma x^{2n}$$

 facto-

factores debeant inuestigari; tum calculo vt ante subducto, erit $f^n = \frac{\beta}{2\gamma z}$ et $z = \frac{\beta}{2\sqrt{\alpha\gamma}} =$ cos. A. $n\,\Phi$. Quodsi ergo arcus, cuius cosinus $= \frac{\beta}{2\sqrt{\alpha\gamma}}$ ponatur ω, fiet etiam cos. A $(2k\pi \pm \omega) = \frac{\beta}{2\sqrt{\alpha\gamma}}$ ex quo $\Phi = \frac{2k\pi \pm \omega}{n}$. Factor igitur quicunque formulae propositae continebitur in hac forma

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\gamma}\,.\,\text{cos. A}.\,\frac{2k\pi \mp \omega}{n} + xx\sqrt[n]{\gamma\gamma}$$

Ipsi ergo factores trinomiales, quorum numerus est n, erunt sequentes:

$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\gamma}\,.\,\text{cos. A}.\,\frac{2\pi - \omega}{n} + xx\sqrt[n]{\gamma\gamma}$$
$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\gamma}\,.\,\text{cos. A}.\,\frac{4\pi - \omega}{n} + xx\sqrt[n]{\gamma\gamma}$$
$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\gamma}\,.\,\text{cos. A}.\,\frac{6\pi - \omega}{n} + xx\sqrt[n]{\gamma\gamma}$$
$$\vdots$$
$$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\gamma}\,.\,\text{cos. A}.\,\frac{2n\pi - \omega}{n} + xx\sqrt[n]{\gamma\gamma}$$

§. 36. Hi etiam factores simili modo, quo casu praecedenti, commode per circulum construi possunt. Construatur enim circulus PQRSTV radio CA $= \sqrt[n]{\alpha}$ eiusque peripheria diuidatur in $2n$ partes, in punctis P, p, Q, q, etc.. Tum a puncto primo diuisionis P capiatur arcus PA $= \frac{\omega}{n}$, ita vt arcus n. PA cosinus sit $= \frac{\beta}{2\sqrt{\alpha\gamma}}$. Tum per A ducatur diameter AB, et in eo ex centro C capiatur CO $= x\sqrt[n]{\gamma}$; atque ex puncto hoc O ad singula diuisionis puncta ducantur rectae. Erit autem Ap $= \frac{\pi - \omega}{n}$; AQ $= \frac{2\pi - \omega}{n}$; A$q$ $= \frac{3\pi - \omega}{n}$, etc. Quoniam vero

Tab. I. fig. 6.

sumto

quocunque arcu $AM = \Phi$, et demisso sinu MN est $MN = \sqrt[n]{\alpha} \,.\, \text{sin.} A \,.\, \Phi$ et $ON = \sqrt[n]{\alpha} \,.\, \text{cos.} A \Phi - x \sqrt[n]{\gamma}$; erit $OM = \sqrt{(\sqrt[n]{\alpha^2} - 2x \sqrt[n]{\alpha\gamma} \,.\, \text{cos.} A \Phi + xx \sqrt[n]{\gamma\gamma})}$ erit loco Φ arcubus Ap, AQ, Aq, etc. substituendo $\alpha + \beta x^n + \gamma x^{2n} = Op^2 \, Oq^2 \, Or^2 \, Os^2 \, Ot^2 \, Ov^2$; atque $\alpha - \beta x^n + \gamma x^{2n} = OQ^2 \, OR^2 \, OS^2 \, OT^2 \, OV^2 \, OP^2$. Quae sunt theoremata a Celeb. Moivraeo demonstrata.

§. 37. Antequam istam formulae $\alpha + \beta x^n + \gamma x^{2n}$ resolutionem in factores trinomiales dimittamus, non abs re erit annotare, quod cum terminus x^1 desit in producto, summa omnium coefficientium ipsius x in factoribus aequalis nihilo esse debeat. Erit ergo

$$\text{cos.} A. \tfrac{\pi-\omega}{n} + \text{cos.} A. \tfrac{3\pi-\omega}{n} + \text{cos.} A. \tfrac{5\pi-\omega}{n} + \ldots + \text{cos.} A. \tfrac{(2n-1)\pi-\omega}{n} = 0$$

et

$$\text{cos.} A. \tfrac{2\pi-\omega}{n} + \text{cos.} A. \tfrac{4\pi-\omega}{n} + \text{cos.} A. \tfrac{6\pi-\omega}{n} + \ldots + \text{cos.} A. \tfrac{2n\pi-\omega}{n} = 0$$

Quod quidem, si n est numerus par, sponte patet, tum enim alii cosinus fiunt negatiui atque aequales ratione reliquorum. Quando autem n est numerus impar, puta $n = 2m - 1$; tum cosinus negatiui affirmatiuos singuli singulos non destruunt, interim tamen omnes negatiui simul sumti affirmatiuis simul sumtis aequales erunt. Est vero $\text{cos.} A. \Phi = \frac{1}{2}$ chord. $A(\pi + 2\Phi)$ et quia est $\Phi = \frac{(2k-1)\pi - \omega}{2m-1}$, erit $\text{cos.} A \frac{(2k-1)\pi-\omega}{2m-1} = \frac{1}{2}$ chord. $A. (\pi - \frac{2\omega}{2m-1} + \frac{2(2k-1)\pi}{2m-1})$; omnesque hae chordae, quarum numerus est $= 2m - 1$ simul sumtae erunt $= 0$. Sit $\xi = \pi - \frac{2\omega}{2m-1}$, et ponatur $\frac{2\pi}{2m-1} = \epsilon$, ita vt ϵ sit vna pars totius peripheriae, si

ea

ea fuerit in numerum quemcunque imparem partium aequalium diuisa. Hanc obrem erit chord. $A(\xi+\varrho)+$ chord. $A(\xi+3\varrho)+$ chord. $A(\xi+5\varrho)+\ldots+$ chord. $A(\xi+(4m-3)\varrho)=0$. Si ergo peripheria circuli in numerum quemcunque imparem partium aequalium verbi gratia in nouem partes aequales Pp, pQ, Qq, qR, Rr, rS, Ss, sT, TP diuidatur atque in peripheria punctum capiatur quodcunque A; ex quo ad singula peripheriae puncta chordae ducantur, erit posita vna peripheriae parte nona $=\varrho$, et arcu AP pro arbitrio assumto $=\xi$ erit

Tab. II. fig. I.

chord. $A(\xi+\varrho)=Ap$	chord. $A(\xi+11\varrho)=-AQ$
chord. $A(\xi+3\varrho)=Aq$	chord. $A(\xi+13\varrho)=-AR$
chord. $A(\xi+5\varrho)=Ar$	chord. $A(\xi+15\varrho)=-AS$
chord. $A(\xi+7\varrho)=As$	chord. $A(\xi+17\varrho)=-AT$

$$\text{chord. } A(\xi+9\varrho)=-AP;$$

chordae enim arcuum tota peripheria maiorum, pariter ac sinus arcuum semiperipheria maiorum, fiunt negatiuae. Erit ergo ductis vti praecepimus chordis,

$$Ap+Aq+Ar+As=AP+AQ+AR+AS+AT$$

siue

$$AP-Ap+AQ-Aq+AR-Ar+AS-As+AT=0$$

Quod est theorema circa chordas non inelegans, notum quidem, at ex traditis praeceptis sponte quasi deriuatum.

§. 38. Resolutionem formularum $\alpha+\beta x^n$ et $\alpha+\beta x^n+\gamma x^{2n}$ in factores fusius exposuimus, eo quod formulae quantumuis compositae in eiusmodi formulas resolui possunt; ex quo secundum haec praecepta formularum magis compositarum factores

res ſimplices vel trinomiales inueniri poterunt, conceſſa reſolutione aequationum altiorum dimenſionum. *S*it nempe propoſita iſta formula

$$\alpha+\beta x^n+\gamma x^{2n}+\delta x^{3n}$$

poſito $x^n=z$, ea abibit in $\alpha+\beta z+\gamma z^2+\delta z^3$ quae perpetuo vnum diuiſorem ſimplicem habebit realem, quoniam maximus ipſius z exponens eſt impar. Quamobrem formula propoſita vel in tres binomiales huiusmodi $a+bx^n$ vel in vnam huiusmodi $a+bx^n$ et in vnam trinomialem reſoluitur $a+bx^n+ex^{2n}$; hincque eius factores vel trinomiales vel ſimplices facile per praecepta praecedentia aſſignantur. Loco formulae igitur $\alpha+\beta x^n+\gamma x^{2n}+\delta x^{3n}$ ſemper ſubſtitui poteſt huiusmodi expreſſio

$$(\mathfrak{a}+\mathfrak{b}x^n)(\mathfrak{A}+\mathfrak{B}x^n+\mathfrak{C}x^{2n})$$

cuius omnes factores reales tam ſimplices quam trinomiales ope regulae traditae exhibebuntur.

§. 39. Propoſita iam ſit haec expreſſio

$$\alpha+\beta x^n+\gamma x^{2n}+\delta x^{3n}+\varepsilon x^{4n}$$

cuius factores reales tam ſimplices quam trinomiales inueſtigari oporteat. Ponamus $x^n=z$, et habebimus hanc expreſſionem quatuor dimenſionum

$$\alpha+\beta z+\gamma z^2+\delta z^3+\varepsilon z^4$$

quae nihilo aequalis poſita vel omnes radices habebit imaginarias vel duas tantum vel nullam. Poſterioribus binis caſibus reſolutio propoſita in factores nulla laborat difficultate; eo quod iis expreſſio propoſita in binas reales huius formae

$$\mathfrak{A}+\mathfrak{B}x^n+\mathfrak{C}x^{2n}$$

 diſtribui

distribui potest. At si omnes quatuor radices aequationis $0 = \alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4$ sint imaginariae, tum resolutio in factores per regulas traditas absolui non potest, nisi constiterit expressionem propositam

$$\alpha + \beta x^n + \gamma x^{2n} + \delta x^{3n} + \varepsilon x^{4n}$$

resolui posse in huiusmodi binos factores reales

$$(\mathfrak{A} + \mathfrak{B} x^n + \mathfrak{C} x^{2n})(\mathfrak{a} + \mathfrak{b} x^n + \mathfrak{c} x^{2n})$$

quod, etsi fieri posse supra iam docuimus, tamen idem pro hoc casu data opera seorsim demonstrabimus, quo simul magis fiet perspicuum, omnem expressionem algebraicam in factores reales vel simplices vel trinomiales resolui posse.

§. 40. Ad hanc demonstrationem concinnandam praemitto istud lemma:

> *Aequatio quaecunque algebraica parium demensionum, in qua maxima incognitae potestas et terminus absolutus seu cognitus disparia habent signa, duas ad minimum habet radices, quarum altera erit affirmatiua, altera negatiua.*

Sit enim huiusmodi aequatio

$$z^{2m} + a z^{2m-1} + b z^{2m-2} + c z^{2m-3} + \ldots - p = 0$$

in qua maxima ipsius z potestas habet signum $+$ terminus absolutus autem p signum $-$. Transferatur terminus absolutus p ad alteram signi $=$ partem, ita vt ex altera parte omnes termini incogita z affecti maneant, habebiturque haec aequatio

$$z^{2m} + a z^{2m-1} + b z^{2m-2} + c z^{2m-3} + \ldots + n z = p$$

vocemus totum membrum incognitum, seu omnes termini-
nos

nos z inuoluentes iunctim sumtos breuitatis gratia $= Z$, vt sit $Z = p$; atque manifestum est, si ponatur $z = +\infty$, fieri $Z = +\infty$; sin autem ponatur $z = 0$, fiet $Z = 0$. Loco z igitur omnes valores intra o et $+\infty$, hoc est, omnes numeros affirmatiuos substituendo, pro Z omnes possibiles numeri affirmatiui resultabunt. Quare cum p sit numerus affirmatiuus, dabitur numerus affirmatiuus, qui loco z substitutus efficiat $Z = p$, ideoque aequatio proposita vnam certe habebit radicem affirmatiuam. Si iam loco z ponamus $-\infty$, fiet iterum $Z = +\infty$, ex quo si loco z omnes numeri negatiui seu intra o et $-\infty$ contenti substituantur, tum denuo pro Z omnes numeri possibiles affirmatiui prodibunt; quare dabitur quoque numerus negatiuus, qui loco z substitutus faciet $Z = p$, hincque aequatio proposita $Z - p = 0$ habebit quoque radicem negatiuam. Aequatio igitur

$$z^{2m} + az^{2m-1} + bz^{2m-2} + cz^{2m-3} + \ldots\ldots - p = 0$$

si p fuerit quantitas positiua, certo duas habet radices reales, quarum vna est affirmatiua.

§. 41. Vt iam ostendamus expressionum

$$\alpha + \beta x^{n} + \gamma x^{2n} + \delta x^{3n} + \varepsilon x^{4n}$$

perpetuo in duos factores reales resolubilem esse; sufficiet breuitatis gratia ostendisse, hanc expressionem $z^4 + pz^2 + qz + r$ resolui posse in duos factores $zz + uz + A$ et $zz + uz + B$, qui sint reales. Ponendo enim $x^n = z$, et termino secundo tollendo, illa expressio in hanc transmutatur. Cum igitur pro coefficientibus r, A; et B valores reales inueniri debeant, vt sit

$$z^4 + pz^2 + qz + r = (zz + uz + A)(zz - uz + B)$$

erit $p = A + B - uu$; $q = Bu - Au$; et $r = AB$, binae priores aequationes vero dant

$$B = p + uu - A = \frac{q + Au}{u} \text{ vnde fit}$$

$$A = \frac{u^3 + pu - q}{2u} \text{ et } B = \frac{u^3 + pu + q}{2u}$$

qui valores in tertia aequatione ſubſtituti dant

$$4ruu = u^6 + 2pu^4 + ppuu - qq \text{ ſeu}$$

$$u^6 + 2pu^4 + (pp - 4r)uu - qq = 0,$$

quae cum ſit aequatio parium dimenſionum, atque terminus abſolutus qq ſemper poſitiuus ſignum habeat oppoſitum ſummae poteſtati incognitae u, haec aequatio certo pro u vnam radicem realem dabit. Inuento autem pro u valore reali, valores quoque pro A et B fient reales, hincque factores $zz + uz + A$ et $zz - uz + B$ ipſi prodibunt reales. Simili autem modo demonſtrabitur omnem expreſſionem parium dimenſionum ſemper reſolubilem eſſe in factores trinomiales reales. Hoc certe euictum eſt hanc expreſſionem multo latius patentem

$$\alpha + \beta x^n + \gamma x^{2n} + \delta x^{3n} + \varepsilon x^{4n}$$

reſolubilem eſſe in factores reales vel ſimplices vel trinomiales; quotcunque ea contineat dimenſiones.

§. 42. Tradito igitur non ſolum modo factores trinomiales inueniendi, ſed etiam actu euolutis factoribus formularum principalium $\alpha + \beta x^n$ et $\alpha + \beta x^n + \gamma x^{2n}$ exponendum eſt, quemadmodum, ſi cognitus fuerit factor trinomialis denominatoris N, in formula differentiali $\frac{Mdx}{N}$ integralis pars ex hoc factore oriunda debeat determinari.

minari. Sit igitur denominatoris N factor trinomialis quicunque

$$p - 2x\sqrt{pq} \cdot \text{cos.} A\Phi + qxx$$

qui resoluitur in hos simplices imaginarios

$$x\sqrt{q} - \sqrt{p} \cdot \text{cos.} A\Phi - \sqrt{-p} \cdot \text{sin.} A\Phi$$

$$x\sqrt{q} - \sqrt{p} \cdot \text{cos.} A\Phi + \sqrt{-p} \cdot \text{sin.} A\Phi$$

Ex his per methodum ante expositam resultant integralis quaesiti binae sequentes partes

$$+ \int \frac{A\,dx}{x\sqrt{q} - \sqrt{p} \cdot \text{cos.} A\Phi - \sqrt{-p} \cdot \text{sin.} A\Phi}$$

$$+ \int \frac{B\,dx}{x\sqrt{q} - \sqrt{p} \cdot \text{cos.} A\Phi + \sqrt{-p} \cdot \text{sin.} A\Phi}$$

vbi coeffitientes A et B determinantur ex $\frac{M\,dx}{dN}$ substituendo loco x valorem, quem ex vtroque denominatore nihilo aequali posito obtinet. Sit igitur $\frac{dN}{dx} = L$, eritque $A = \frac{M}{L}$ posito vbique loco x valore $\frac{\sqrt{p}}{\sqrt{q}}$ cos. AΦ + $\frac{\sqrt{-p}}{\sqrt{q}}$ sin. A. Φ. Simili vero modo est $B = \frac{M}{L}$ posito loco x hoc valore $\frac{\sqrt{p}}{\sqrt{q}}$. cos. AΦ − $\frac{\sqrt{-p}}{\sqrt{q}}$. sin. AΦ.

§ 43. Ponatur $f = \frac{\sqrt{p}}{\sqrt{q}}$; et cum sit

$$x = f \text{ cos.} A\Phi \pm f\sqrt{-1} \cdot \text{sin.} A \cdot \Phi$$

erit, vt supra vidimus, potestas quaecunque

$$x^k = f^k \text{ cos.} A k\Phi \pm f^k\sqrt{-1} \cdot \text{sin.} A \cdot k\Phi$$

Ad substitutiones ergo faciendas, ponatur primo f^k cos. A. $k\Phi$ vbique loco x^k, hocque facto abeat M in $\mathfrak{M}$ et L in $\mathfrak{L}$. Deinde ponatur f^k sin. A. $k\Phi$ loco x^k, abeatque M in $\mathfrak{m}$ et L in $\mathfrak{l}$. Hoc facto fiet $A = \frac{\mathfrak{M} + \mathfrak{m}\sqrt{-1}}{\mathfrak{L} + \mathfrak{l}\sqrt{-1}}$, et

$B = \frac{\mathfrak{M} - \mathfrak{m}\sqrt{-1}}{\mathfrak{L} - \mathfrak{l}\sqrt{-1}}$. Ex his colligitur $A + B =$
$= \frac{2\mathfrak{L}\mathfrak{M} + 2\mathfrak{l}\mathfrak{m}}{\mathfrak{L}\mathfrak{L} + \mathfrak{l}\mathfrak{l}}$, et $A - B = \frac{2\mathfrak{L}\mathfrak{m} - 2\mathfrak{l}\mathfrak{M}}{\mathfrak{L}\mathfrak{L} + \mathfrak{l}\mathfrak{l}} \sqrt{-1}$

His inuentis ſumma illarum duarum fractionum integralium erit

$$\int \frac{2(\mathfrak{L}\mathfrak{M} + \mathfrak{l}\mathfrak{m})x\sqrt{q} - 2(\mathfrak{L}\mathfrak{M} + \mathfrak{l}\mathfrak{m})\sqrt{p},\ \text{coſ.}\,A\Phi + 2(\mathfrak{l}\mathfrak{M} - \mathfrak{L}\mathfrak{m})\sqrt{p},\ \text{ſin.}\,A\Phi}{(p - 2x\sqrt{pq},\ \text{coſ.}\,A\Phi + qxx)(\mathfrak{L}\mathfrak{L} + \mathfrak{l}\mathfrak{l})}\, dx$$

vnde ipſum integrale ex factore trinomiali

$$p - 2x\sqrt{pq},\ \text{coſ.}\,A\Phi + qxx$$

oriundum per logarithmos et arcus circulares erit

$$\frac{\mathfrak{L}\mathfrak{M} + \mathfrak{l}\mathfrak{m}}{(\mathfrak{L}^2 + \mathfrak{l}^2)\sqrt{q}}\, l(p - 2x\sqrt{pq},\ \text{coſ.}\,A\Phi + qxx) + \frac{2(\mathfrak{l}\mathfrak{M} - \mathfrak{L}\mathfrak{m})}{(\mathfrak{L}^2 + \mathfrak{l}^2)\sqrt{q}}\, A\ \text{tang}\, \frac{x\sqrt{q},\ \text{ſin.}\,A.\Phi}{\sqrt{p} - x\sqrt{q},\ \text{coſ.}\,A.\Phi}$$

vbi $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{l}$ et $\mathfrak{M}$ ex datis L et M ita determinantur, vt fiat

$\left.\begin{matrix} M = \mathfrak{M} \\ L = \mathfrak{L} \end{matrix}\right\}$ poſito $x^k = f^k\ \text{coſ.}\,A.\,k\Phi$

$\left.\begin{matrix} M = \mathfrak{m} \\ L = \mathfrak{l} \end{matrix}\right\}$ poſito $x^k = f^k\ \text{ſin.}\,A.\,k\Phi$

eſtque $L = \frac{dN}{dx}$ et $f = \sqrt{\frac{p}{q}}$, vti aſſumſimus. Ponatur porro:

$$\frac{\mathfrak{L}}{\sqrt{(\mathfrak{L}\mathfrak{L} + \mathfrak{l}\mathfrak{l})}} = \text{coſ.}\,A.\lambda \ \Big|\ \frac{\mathfrak{M}}{\sqrt{(\mathfrak{M}^2 + \mathfrak{m}^2)}} = \text{coſ.}\,A.\mu$$
$$\frac{\mathfrak{L}}{\sqrt{(\mathfrak{L}\mathfrak{L} + \mathfrak{l}\mathfrak{l})}} = \text{ſin.}\,A.\lambda \ \Big|\ \frac{\mathfrak{m}}{\sqrt{(\mathfrak{M}^2 + \mathfrak{m}^2)}} = \text{ſin.}\,A.\mu$$

ita vt ſit $\lambda = A\ \text{tang.}\frac{\mathfrak{l}}{\mathfrak{L}}$ et $\mu = A\ \text{tang.}\frac{\mathfrak{m}}{\mathfrak{M}}$ reperienturque hinc arcus λ et μ facili negotio.

Ex his vero colligetur fore:

$$\text{ſin.}\,A(\lambda - \mu) = \frac{\mathfrak{l}\mathfrak{M} - \mathfrak{L}\mathfrak{m}}{\sqrt{(\mathfrak{L}^2 + \mathfrak{l}^2)(\mathfrak{M}^2 + \mathfrak{m}^2)}}$$
$$\text{coſ.}\,A(\lambda - \mu) = \frac{\mathfrak{L}\mathfrak{M} + \mathfrak{l}\mathfrak{m}}{\sqrt{(\mathfrak{L}^2 + \mathfrak{l}^2)(\mathfrak{M}^2 + \mathfrak{m}^2)}}$$

Iam ponatur $\frac{\sqrt{(\mathfrak{M}^2 + \mathfrak{m}^2)}}{\sqrt{(\mathfrak{L}^2 + \mathfrak{l}^2)}} = \mathfrak{K}$, eritque integralis pars ex factore hoc

$$p = 2x\sqrt{pq}\,.\,\text{coſ.}\,A\Phi + qxx$$

oriunda,

oriunda, simplicius hoc modo expressa

$$\frac{\mathfrak{R}\,\text{cos}.\,A(\lambda-\mu)}{\sqrt{q}}\,l(p-2x\sqrt{pq}\,.\,\text{cos}.\,A\Phi+qxx)$$
$$+\frac{2\mathfrak{R}\,\text{sin}.\,A(\lambda-\mu)}{\sqrt{q}}\,A\,\text{tang}.\,\frac{x\sqrt{q}.\,\text{sin}.\,A.\Phi}{\sqrt{p}-x\sqrt{q}.\,\text{cos}.\,A\Phi}$$

Pro quouis igitur casu ea forma, quae commodior visa fuerit, vti licebit.

Problema I.

§. 44. Inuenire integrale huius formulae differentialis $\frac{x^m dx}{\alpha+\beta x^n}$, existentibus α et β quantitatibus positiuis.

Solutio.

Hic est $M=x^m$ et $N=\alpha+\beta x^n$ et $\frac{dN}{dx}=L=n\beta x^{n-1}$ Factor autem generalis denominatoris N est per §. 26.

$$\sqrt[n]{\alpha^2}-2x\sqrt[n]{\alpha\beta}.\,\text{cos}.\,A\,\frac{(2k-1)\pi}{n}+xx\sqrt[n]{\beta^2}$$

vnde est $p=\sqrt[n]{\alpha^2}$; $q=\sqrt[n]{\beta^2}$, et $\Phi=\frac{(2k-1)\pi}{n}$ atque $f=\sqrt[n]{\frac{\alpha}{\beta}}$. Ex his fit:

$\mathfrak{M}=f^m\ \text{cos}.\,A.\,m\Phi$	$\mathfrak{L}=n\beta f^{n-1}\ \text{cos}.\,A\,(n-1)\Phi$
$\mathfrak{m}=f^m\ \text{sin}.\,A.\,m\Phi$	$\mathfrak{l}=n\beta f^{n-1}\ \text{sin}.\,A\,(n-1)\Phi$

atque porro $\frac{\mathfrak{l}}{\mathfrak{L}}=\text{tang}.\,A\,(n-1)\Phi=\text{tang}.\,A\lambda$, vnde $\lambda=(n-1)\Phi$. Simili modo est $\frac{\mathfrak{m}}{\mathfrak{M}}=\text{tang}.\,A.\,m\Phi=\text{tang}.\,A.\mu$, ergo $\mu=m\Phi$. et $\lambda-\mu=(n-m-1)\Phi$. Deinde est $\sqrt{(\mathfrak{M}^2+\mathfrak{m}^2)}=f^m$ et $\sqrt{(\mathfrak{L}^2+\mathfrak{l}^2)}=n\beta f^{n-1}$ ergo $\mathfrak{R}=\frac{1}{n\beta f^{n-m-1}}=\frac{f^{m+1}}{n\alpha}$. Ex quolibet ergo factore trinomiali denominatoris oritur haec integralis pars.

$$\frac{\text{cos}.\,A\,\frac{(n-m-1)(2k-1)\pi}{n}}{n\alpha^{\frac{n-m-1}{n}}\beta^{\frac{m+1}{n}}}\,l(\sqrt[n]{\alpha^2}-2x\sqrt[n]{\alpha\beta}.\text{cos}.A\tfrac{(2k-1)\pi}{n}+xx\sqrt[n]{\beta^2})$$

+ 2

$$+\frac{2\,\text{fin.}\,A\frac{(n-m-1)(2k-1)\pi}{n}}{n\alpha^{\frac{n-m-1}{n}}\beta^{\frac{m+2}{n}}}\,A\,\text{tang.}\,\frac{x\sqrt[n]{\beta}.\,\text{fin.}\,A\frac{(2k-1)\pi}{n}}{\sqrt[n]{\alpha}-x\sqrt[n]{\beta}.\,\text{cof.}\,A\frac{(2k-1)\pi}{n}}$$

quae etiam hoc modo exhiberi poteſt:

$$-\frac{\text{cof.}\,A\frac{(m+1)(2k-1)\pi}{n}}{n\alpha^{\frac{n-m-1}{n}}\beta^{\frac{m+2}{n}}}\,l\left(\sqrt[n]{\alpha^2}-2x\sqrt[n]{\alpha\beta}.\,\text{cof.}\,A.\frac{(2k-1)\pi}{n}+xx\sqrt[n]{\beta^2}\right)$$

$$+\frac{2\,\text{fin.}\,A.\frac{(m+1)(2k-1)\pi}{n}}{n\alpha^{\frac{n-m-1}{n}}\beta^{\frac{m+2}{n}}}\,A\,\text{tang.}\,\frac{x\sqrt[n]{\beta}.\,\text{fin.}\,A.\frac{(2k-1)\pi}{n}}{\sqrt[n]{\alpha}-x\sqrt[n]{\beta}.\,\text{cof.}\,A.\frac{(2k-1)\pi}{n}}$$

*S*i iam loco $2k-1$ ſubſtituantur omnes numeri impares 1, 3, 5, 7, etc. vsque ad n; atque adeo etiam ipſe numerus n, ſi ſit impar, prodibunt omnes integralis partes, quae quidem ex denominatore $N=\alpha+\beta x^n$ proficiſcuntur. Si igitur m fuerit numerus affirmatiuus minor quam n, his partibus in vnam ſummam colligendis completum oritur integrale quaeſitum: hicque etiam comprehendetur ea integralis pars, quae oritur ex factore ſimplici reali denominatoris, ſi n fuerit numerus impar: dummodo eius integralis, quod hoc caſu oritur, tantum ſemiſſis capiatur; vel loco quadrati, quod hoc caſu ſignum log. l habebit prefixum, eius tantum radix quadrata ſubſtituatur. Alterum enim membrum arcum circuli inuoluens hoc caſu euaneſcit.

Locum autem habent haec integralia ex denominatoris $\alpha+\beta x^n$ factoribus orta, ſiue m ſit maior quam n ſiue minor, ſiue etiam numerus negatiuus. Niſi autem m ſit numerus affirmatiuus minor quam n, ad integrale ex factoribus denominatoris $\alpha+\beta x^n$ inuentum quidpiam inſuper eſt adiiciendum.

Sit

Sit m numerus maior, quam n, ac ponatur $m = \sigma n + \tau$, exiſtente τ numero minore, quam n. Per regulam igitur primum datam ex fractione $\frac{x^{\sigma n+\tau}}{\alpha+\beta x^n}$ pars integra eſt extrahenda, quae erit huiusmodi $\frac{1}{\beta} x^{(\sigma-1)n+\tau} - \frac{\alpha}{\beta^2} x^{(\sigma-2)n+\tau} + \frac{\alpha^2}{\beta^3} x^{(\sigma-3)n+\tau} - \ldots \ldots \mp \frac{\alpha^{\sigma-1}}{\beta^\sigma} x^\tau$, in vltimo termino ſignum $+$ valet, ſi σ fuerit numerus impar, ſignum $-$ autem, ſi σ ſit numerus par. Hoc ergo caſu ad integrale ante ex factoribus denominatoris $\alpha+\beta x^n$ inuentum adiici debet hoc integrale:

$$\frac{x^{\sigma n-n+\tau+1}}{\beta(\sigma n-n+\tau+1)} - \frac{\alpha x^{\sigma n-2n+\tau+1}}{\beta^2(\sigma n-2n+\tau+1)} + \frac{\alpha^2 x^{\sigma n-3n+\tau+1}}{\beta^3(\sigma n-3n+\tau+1)}$$
$$- \frac{\alpha^3 x^{\sigma n-4n+\tau+1}}{\beta^4(\sigma n-4n+\tau+1)} \ldots\ldots \mp \frac{\alpha^{\sigma-1} x^{\tau+1}}{\beta^\sigma(\tau+1)}.$$

Sit iam m numerus negatiuus, ac ponatur $m = -\sigma n - \tau$, exiſtente τ numero minore quam n; atque ad integrale ex factoribus ipſius $\alpha+\beta x^n$ inuentum inſuper addi debet integrale, quod oritur ex $x^{-\sigma n-\tau}$, quod membrum in denominatorem ingredietur. Habebimus ſcilicet $\frac{M}{N} = \frac{1}{x^{\sigma n+\tau}(\alpha+\beta x^n)}$ ponamus ex $x^{\sigma n+\tau}$ reſultare hanc fractionis partem $\frac{V}{x^{\sigma n+\tau}}$, atque manifeſtum eſt $1 - V(\alpha+\beta x^n)$ diuiſibile eſſe oportere per $x^{\sigma n+\tau}$. Erit ergo $V = \frac{1}{\alpha} - \frac{\beta}{\alpha^2} x^n + \frac{\beta^2}{\alpha^3} x^{2n} - \ldots \pm \frac{\beta^\sigma}{\alpha^{\sigma+1}} x^{\sigma n}$; fiet enim $1 - V(\alpha+\beta x^n) = \mp \beta \frac{\sigma+1}{\alpha^{\sigma+1}} x^{\sigma n+n}$, vtique diuiſibile per $x^{\sigma n+\tau}$; vbi signorum ambiguorum superius valet, si σ sit numerus par, inferius si impar. Quare ad integrale ex factoribus ipsius $\alpha+\beta x^n$ inuentum adiici debet insuper

$$\int \frac{V\,dx}{x^{\sigma n+\tau}} = \frac{-1}{\alpha(\sigma n+\tau-1)x^{\sigma n+\tau-1}} + \frac{\beta}{\alpha^2(\sigma n-n+\tau-1)x^{\sigma n-n+\tau-1}}$$

$$-\frac{\beta^2}{\alpha^3(\sigma n-2n+\tau-1)x^{\sigma n-2n+\tau-1}} + \ldots\ldots \mp \frac{\beta^\sigma}{\alpha^{\sigma+1}(\tau-1)x^{\tau-1}}$$

De cetero casus, quo m est numerus negatiuus, ad praecedentem reduci potest, ita vt peculiari solutione non sit opus. Si enim proposita sit haec formula differentialis $\frac{dx}{x^m(\alpha+\beta x^n)}$, ponatur $x=\frac{1}{y}$, atque habebitur $-\frac{y^{m+n-2}dy}{\alpha y^n+\beta}$, cuius integratio per extractionem partis integrae ex fractione $\frac{y^{m+n-2}}{\alpha y^n+\beta}$ absoluitur, vti docuimus. Dedimus ergo integrale formulae $\frac{x^m dx}{\alpha+\beta x^n}$ pro valore quocunque exponentis m siue affirmatiuo siue negatiuo. Q. E. I.

Problema 2.

§. 45. Inuenire integrale huius formulae differentialis $\frac{x^m dx}{\alpha-\beta x^n}$, existente m quocunque numero integro siue affirmatiuo siue negatiuo.

Solutio.

Hic est $M=x^m$; $N=\alpha-\beta x^n$ et $L=\frac{dN}{dx}=-n x \beta^{n-1}$.

Factor

Factor autem generalis trinomialis denominatoris $N = \alpha - \beta x^n$ eſt per §. 28.

$\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta} \,.\, \text{coſ.A.}\frac{2k\pi}{n} + xx\sqrt[n]{\beta^2}$

vnde eſt $p = \sqrt[n]{\alpha^2}$; $q = \sqrt[n]{\beta^2}$; $\Phi = \frac{2k\pi}{n}$; et $f = \sqrt[n]{\frac{\alpha}{\beta}}$

Ex his autem fiet porro

$\mathfrak{M} = f^m \text{ coſ. A} m\Phi \mid \mathfrak{L} = -n\beta f^{n-1} \text{ coſ. A}(n-1)\Phi$
$\mathfrak{m} = f^m \text{ ſin. A} m\Phi \mid \mathfrak{l} = -n\beta f^{n-1} \text{ ſin. A}(n-1)\Phi$

hincque $\frac{\mathfrak{l}}{\mathfrak{L}} = \text{tang. A}(n-1)\Phi$, et $\lambda = (n-1)\Phi = \frac{2k(n-1)\pi}{n}$ atque $\frac{\mathfrak{m}}{\mathfrak{M}} = \text{tang. A}.m\Phi$; et $\mu = m\Phi = \frac{2km\pi}{n}$; ideoque $\lambda - \mu = \frac{(n-m-1)2k\pi}{n}$; quare coſ. $A(\lambda - \mu) = \text{coſ.A}\frac{(n-m-1)2k\pi}{n} = \text{coſ.A}\frac{(m+1)2k\pi}{n}$, et ſin. $A(\lambda-\mu) = \text{ſin.A}\frac{(n-m-1)2k\pi}{n} = \text{ſin. A}\frac{(m+1)2k\pi}{n}$. Deinde eſt $\sqrt{(\mathfrak{M}^2 + \mathfrak{m}^2)} = f^m$ et $\sqrt{(\mathfrak{L}^2 + \mathfrak{l}^2)} = -n\beta f^{n-1}$

ergo $\mathfrak{K} = \frac{-1}{n\beta f^{n-m-1}} = \frac{-\beta^{\frac{n-m-1}{n}}}{n\beta\alpha^{\frac{n-m-1}{n}}} = \frac{-1}{n\alpha^{\frac{n-m-1}{n}}\beta^{\frac{m+1}{n}}}$

et $\frac{\mathfrak{K}}{\sqrt{q}} = \frac{-1}{n\alpha^{\frac{n-m-1}{n}}\beta^{\frac{m+2}{n}}}$. Ex factore ergo trinomiali generali naſcitur ſequens integralis pars:

$$\frac{-\text{coſ. A}\frac{(m+1)2k\pi}{n}}{n\alpha^{\frac{n-m-1}{n}}\beta^{\frac{m+2}{n}}}\, l\left(\sqrt[n]{\alpha^2} - 2x\sqrt[n]{\alpha\beta}\,.\,\text{coſ.A.}\frac{2k\pi}{n} + xx\sqrt[n]{\beta^2}\right)$$

$$+\frac{2\,\text{ſin A}\frac{(m+1)2k\pi}{n}}{n\alpha^{\frac{n-m-1}{n}}\beta^{\frac{m+2}{n}}}\ \text{A tang.}\frac{x\sqrt[n]{\beta}\,.\,\text{ſin. A.}\frac{2k\pi}{n}}{\sqrt[n]{\alpha} - x\sqrt[n]{\beta}\,\text{coſ. A}\frac{2k\pi}{n}}$$

Si iam loco $2k$ ſucceſſiue omnes numeri pares 0, 2, 4, 6, etc. vsque ad n, atque adeo ipſe numerus n, ſi ſit par, ſubſtituantur, prodibunt omnes integralis partes ex deno-

 minatore

minatore $\alpha-\beta x^n$ oriundae. Quoties autem fit cos. A. $\frac{2k\pi}{n}$ vel $+1$ vel -1, quorum illud euenit, si $2k=0$, hoc vero casu $2k=n$, si quidem n est numerus par, tum factor prodit simplex, et membrum a quadratura circuli pendens euanescit: membri autem logarithmici his casibus semissis debet capi, seu quod eodem redit, loco quadrati, quod his casibus signum log. l habebit praefixum, eius tantum radix quadrata scribi debet. Hocque pacto colligendis omnibus istis integralibus resultabit completum integrale quaesitum, si quidem m fuerit numerus affirmatiuus minor quam n. Quodsi autem m fuerit numerus maior quam m puta $m=\sigma n+\tau$, tum ad integrale illud insuper addi debet $\frac{-x^{\sigma n-n+\tau+1}}{\beta(\sigma n-n+\tau+1)} - \frac{\alpha x^{\sigma n-2n+\tau+1}}{\beta^2(\sigma n-2n+\tau+1)} - \frac{\alpha^2 x^{\sigma n-3n+\tau+1}}{\beta^3(\sigma n-3n+\tau+1)} \ldots\ldots \frac{-\alpha^{\sigma-1}x^{\tau+1}}{\beta^{\sigma}(\tau+1)}$. Sin autem m fuerit numerus negatiuus, puta $m=-\sigma n-\tau$, tum ad integrale ex factoribus denominatoris $\alpha-\beta x^n$ inuentum adiici oportebit:

$$\frac{-1}{\alpha(\sigma n+\tau-1)x^{\sigma n+\tau-1}} - \frac{\beta}{\alpha^2(\sigma n-n+\tau-1)x^{\sigma n-n+\tau-1}} - \frac{\beta^2}{\alpha^3(\sigma n-2n+\tau-1)x^{\sigma n-2n+\tau-1}} - \ldots - \frac{\beta^{\sigma}}{\alpha^{\sigma+1}(\tau-1)x^{\tau-1}}$$

Q. E. I.

Problema. 3.

§. 46. Inuenire integrale huius formulae differentialis $\frac{x^m dx}{\alpha+\beta x^n+\gamma x^{2n}}$, denotante m numerum quemcunque integrum siue affirmatiuum siue negatiuum.

Solutio.

Solutio.

Hic eſt $M = x^m$; $N = \alpha + \beta x^n + \gamma x^{2n}$ et $L = \frac{dN}{dx} = n\beta x^{n-1} + 2n\gamma x^{2n-1}$. Factor autem ipſius N quicunque trinomialis ſit

$$p - 2x\sqrt{pq}.\ \text{coſ. A.}\ \Phi + qxx$$

ſingulos enim factores huius formae ſupra inuenire docuimus; ſit porro $f = \sqrt{\frac{p}{q}}$; eritque

$$\mathfrak{M} = f^m \text{ coſ. A.} m\Phi \quad \Big| \quad \mathfrak{L} = n\beta f^{n-1} \text{ coſ. A.}(n-1)\Phi + 2n\gamma f^{2n-1} \text{ coſ. A}(2n-1)\Phi$$
$$\mathfrak{m} = f^m \text{ ſin. A.} m\Phi \quad \Big| \quad \mathfrak{l} = n\beta f^{n-1} \text{ ſin. A.}(n-1)\Phi + 2n\gamma f^{2n-1} \text{ ſin. A}(2n-1)\Phi$$

hincque $\frac{\mathfrak{m}}{\mathfrak{M}} = \text{tang. A.}\, m\Phi$ et $\mu = m\Phi$; ſimilique modo $\frac{\mathfrak{l}}{\mathfrak{L}} = \frac{\beta \text{ ſin. A}(n-1)\Phi + 2\gamma f^n \text{ ſin. A}(2n-1)\Phi}{\beta \text{ coſ. A}(n-1)\Phi + 2\gamma f^n \text{ coſ. A}(2n-1)\Phi} = \text{tang. A.}\,\lambda$ vnde arcus λ innoteſcit. Porro eſt $\sqrt{(\mathfrak{M}^2 + \mathfrak{m}^2)} = f^m$ et $\sqrt{(\mathfrak{L}^2 + \mathfrak{l}^2)} = nf^{n-1}$ $\sqrt{(\beta^2 + 4\beta\gamma f^n \text{ coſ. A.}\, n\Phi + 4\gamma\gamma f^{2n})}$ ideoque $\mathfrak{R} =$

$$\frac{1}{nf^{n-m-1}\sqrt{(\beta^2 + 4\beta\gamma f^n \text{ coſ. A.}\, n\Phi + 4\gamma\gamma f^{2n})}}$$

Ex iſto ergo denominatoris factore generali orietur ſequens integralis pars

$$\frac{\mathfrak{R} \text{ coſ. A}(\lambda - m\Phi)}{\sqrt{q}}\, l(p - 2x\sqrt{pq} \text{ coſ. A}\,\Phi + qxx) + \frac{2\mathfrak{R} \text{ ſin. A}(\lambda - m\Phi)}{\sqrt{q}} \text{ A tang.} \frac{x\sqrt{q}.\text{ ſin. A}\,\Phi}{\sqrt{p} - x\sqrt{q}.\text{ coſ. A}\,\Phi}$$

Hoc igitur modo ex ſingulis denominatoris factoribus trinomialibus reſpondentes integralis partes reperiuntur: quoniam vero etiam factores ſimplices in factoribus trinomialibus continentur, quando hi in quadrata abeunt: etiam integralis partes ex his reſultantes obtinebuntur, ſi prioris membri logarithmici ſemiſſis ſumatur: alterum enim membrum a quadratura circuli pendens ſponte euaneſcit. Quodſi ergo exponens m fuerit numerus affirmatiuus minor quam $2n$, tum hoc modo completum integrale reperietur. At si m sit numerus maior, quam $2n$, tum in fractione $\frac{M}{N}$ pars integra continebitur, ex qua peculiaris integralis pars nascitur. Inuenitur autem haec pars integra per diuisionem, vti supra ostendimus. Quodsi autem m fuerit numerus negatiuus, tum ponatur $x = \frac{1}{y}$, atque formula proposita $\frac{dx}{x^m(\alpha + \beta x^n + \gamma x^{2n})}$ abibit in hanc $\frac{-y^{2n+m-2}dy}{\alpha y^{2n} + \beta y^n + \gamma}$, ex qua si partes integrae eliciantur, atque integrentur, tum ea ipsa integralis pars reperietur, quae ex x^m, quatenus in denominatore versatur, resultat. Ope adeo regulae traditae omnino formularum differentialium integralia concessa aequationum quotcunque dimensionum resolutione actu exhiberi poterunt; ita vt non solum sint quantitates reales, sed etiam alias quadraturas praeter hyperbolae ac circuli non requirant.

Q. E. I.

Theo-

Fig. 1.

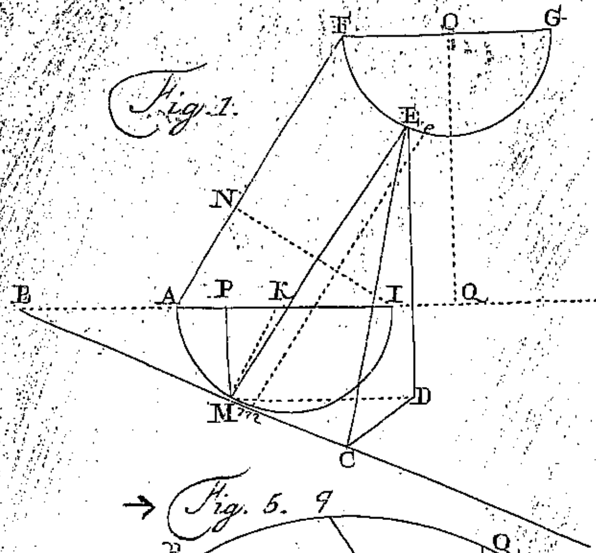

Fig. 2.

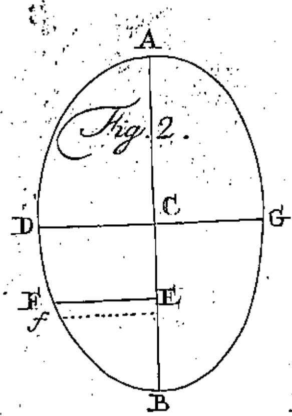

Fig. 3.

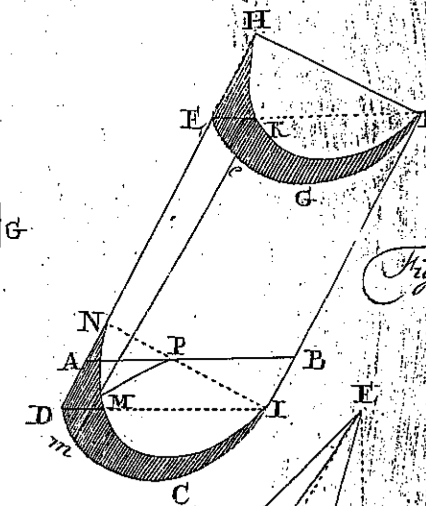

Fig. 4.

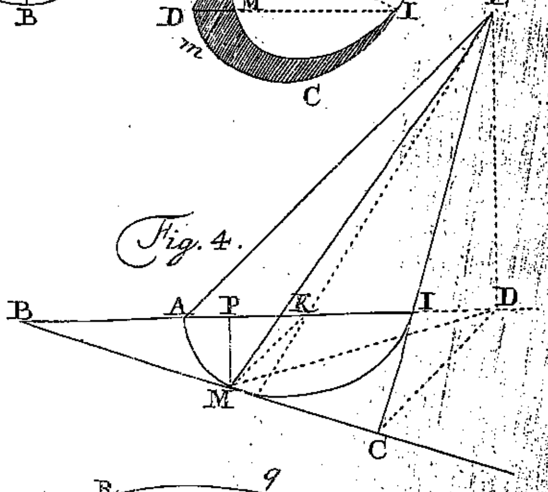

Fig. 5.

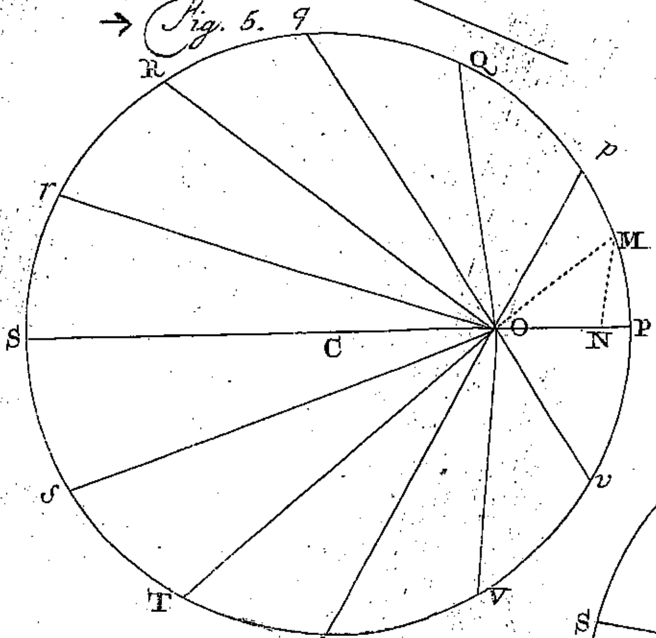

Fig. 6.