DE REDVCTIONE LINEARVM CVRVARVM AD ARCVS CIRCVLARES.

AVCTORE

L. EVLERO.

Cum dimensio linearum curuarum in geometria sublimiori maximi semper momenti sit habita, Celeb: Ioannes Bernoulli fines huius scientiae mirifice dilatasse censendus est, dum ex consideratione motus reptorii longitudinem cuiusuis lineae curuae per arcum circuli exprimere docuit. Summus quoque Leibnitius hoc inuentum tanti existimauit, vt cum de primo volumine Miscell. Berol. edendo cogitaret, Bernoullium incitauerit, vt specimen huius methodi inserendum secum communicaret. Tanto maius huic inuento pretium est imponendum, cum sola analysis vix vllum aditum ad istam reductionem concedere videatur. Qui enim sola analysi vsi hoc negotium expedire sunt conati, nihil fere praestiterunt, quod non per se esset obuium. Hanc ob rem methodus, quam hic sum expositurus, non parum vtilitatis afferre videtur, cuius beneficio, proposita linea curua quacunque, arcus circuli exhiberi potest ipsi proxime aequalis; et ille quidem ipse quoque, qui per motum reptorium inuenitur.

Definitio.

1. Amplitudinem lineae curuae A M cum Celeb: Bernoullio vocabo angulum A N M, quem normales A N Tab. I. fig. 1.

 et

et MN ad curuae extremitates A et M ductae inter se constituunt.

Coroll. 1.

2. Si ergo curua continua curuatura progrediatur, ita vt nusquam habeat punctum flexus contrarii, crescente curuae longitudine simul eius amplitudo crescet. Scilicet quo maior capiatur arcus AM, eo maior euadet angulus ANM.

Coroll. 2.

3. Si curua AM fuerit circulus, erit punctum N eius centrum, atque $AN = MN$. Cum igitur angulus ANM sit ipsi arcui AM proportionalis, in circulo arcus eorumque amplitudines in eadem ratione crescunt.

Coroll. 3.

4. In omnibus autem aliis curuis amplitudines vel in maiore vel in minore ratione crescunt quam ipsi arcus. Neque in his aequalitas inter normales AN et MN amplius locum inuenit, nisi forte in certis tantum locis.

Scholion.

5. Quod praeter circulum nulla alia detur linea curva, cuius arcus sint ipsorum amplitudinibus proportionales, hoc modo ostendi potest. Consideretur curuae AM euoluta RR, ac ponatur angulus seu amplitudo $ANM = u$, arcus $AM = s$, et radius euolutae puncto M respondens $MR = r$ erit $du = \frac{ds}{r}$, ideoque $r = \frac{ds}{du}$, si ergo arcus AM fuerit proportionalis amplitudini ANM, erit $ds = adu$, hincque $r = a$, vnde cum curuedo vbique sit eadem, curua AM erit circulus.

Pro-

Problema I.

6. Data curuae A M amplitudine A N M vna cum normalibus A N et M N, inuenire limites intra quos longitudo arcus A M contineatur. fig. 2.

Solutio.

Ponatur curuae amplitudo seu angulus $ANM = v$, et normales $AN = p$; $MN = q$; ipseque arcus $AM = s$. Deinde ducatur subtensa A M, quae ex triangulo A N M reperitur fore $= \sqrt{(pp + qq - 2pq \text{ cos. } v)}$. Cum ergo arcus $AM = s$ semper sit maior quam sua subtensa, hinc alterum limitem minorem iam habemus quo erit:

$$s > \sqrt{(pp + qq - 2pq \text{ cos. } v)}$$

Ducantur deinde in punctis A et M tangentes A T et M T, quarum concursus T intra angulum A N M cadet: eritque

$$AT = \frac{q - p \text{ cos. } v}{\text{sin. } v} \text{ et } MT = \frac{p - q \text{ cos. } v}{\text{sin. } v}$$

Manifestum autem est summam tangentium $AT + MT$ fore arcu A M maiorem, vnde prodit alter limes maior.

$$s < \frac{(p + q)(1 - \text{cos. } v)}{\text{sin. } v}$$

seu cum sit $\frac{1 - \text{cos. } v}{\text{sin. } v} = \text{tang. } \frac{1}{2} v$ erit:

$$s < (p + q) \text{ tang. } \tfrac{1}{2} v$$

Erunt ergo limites, intra quos vera arcus A M longitudo continetur, sequentes:

minor . . . . . $\sqrt{(pp + qq - 2pq \text{ cos. } v)}$
maior . . . . . $(p + q) \text{ tang. } \frac{1}{2} v$.

Q. E. I.

Coroll. I.

7. Quia punctum T intra crura anguli A N M producta cadit, si quidem curua continua curuatura progredi-

 tur,

tur, rectarum AT et MT valores semper erunt affirmatiui; eritque ergo $q > p$ cos. v et $p > q$ cos. v. Hinc normalis $MN = q$ intra hos limites continebitur

$$q > p \text{ cos. } v \text{ et } q < \frac{p}{\text{cos. } v}.$$

Coroll. 2.

8. Si angulus ANM bifariam sectus concipiatur recta NO, erit vtique $AO > p$ sin. $\frac{1}{2}v$ et $MO > q$ sin. $\frac{1}{2}v$, vnde addendo alius obtinetur limes minor: scilicet $s > (p + q)$ sin. $\frac{1}{2}v$, qui maiorem habet affinitatem cum altero maiore ante inuento $s < (p + q)$ tang. $\frac{1}{2}v$.

Coroll. 3.

9. Cum sit $AO > p$ sin. $\frac{1}{2}v$ et $MO > q$ sin. $\frac{1}{2}v$ erit quadratis sumendis: $AO^2 > pp$ sin. $\frac{1}{2}v^2$ et $MO^2 > qq$ sin. $\frac{1}{2}v^2$ ideoque

$$2AO^2 + 2MO^2 > 2(pp + qq) \text{ sin. } \tfrac{1}{2}v^2$$

At si partes AO et MO sint aequales erit $2AO^2 + 2MO^2 = S^2$; sin autem sint inaequales, erit semper $2AO^2 + 2MO^2 > ss$, ideoque his casibus expressio $2(pp + qq)$ sin $\frac{1}{2}v^2$ propius accedet ad ss; vnde nouus habetur limes minor:

$$s > \text{sin. } \tfrac{1}{2}v . \sqrt{2(pp + qq)}$$

Coroll. 4.

10. Hic autem limes minor est quam ille, quem ante inuenimus $\sqrt{(pp + qq - 2pq \text{ cos. } v)}$, vnde cum hic sit minor quam s, multo magis ille erit minor. Quod vt appareat ponatur:

sin.

$$\sin.\tfrac{1}{2}v.\sqrt{2(pp+qq)}=P$$
$$\sqrt{(pp+qq-2pq\cos.v.)}=Q$$

ob $\sin.\tfrac{1}{2}v^2=\frac{1-\cos.v}{2}$ erit $PP=(pp+qq)(1-\cos.v)$ et $QQ=pp+qq-2pq\cos.v$, et $QQ-PP=(p-q)^2\cos.v$; vnde est $Q=P+\frac{(p-q)\cos.v}{2P}$, ideoque

$$s>\sin.\tfrac{1}{2}v\sqrt{2(pp+qq)}+\frac{(p-q)^2\cos.v}{2\sin.\tfrac{1}{2}v\sqrt{2(pp+qq)}}$$

Scholion.

11. Facillimum est alios limites inuenire, qui non tam prope ad se inuicem accedunt. Si enim ex M in AN perpendiculum demittatur, erit id $=q\sin.v$, quod cum sit minus arcu AM, erit $s>q\sin.v$; similique modo ex A in MN perpendiculum demittenda erit $s>p\sin v$; vnde conficietur $s>\tfrac{1}{2}(p+q)\sin.v$. Deinde si tangens AT vsque ad occursum cum recta MN producta continuetur, erit ea $=p$ tang. v, quae cum sit maior arcu s, erit $s>p$ tang. v, similiterque $s>q$ tang. v; vnde obtinebitur $s>\tfrac{1}{2}(p+q)$ tang. v. Limites autem ante inuenti multo sunt arctiores quam hi, ideoque ad nostrum institutum magis accommodati.

Theorema.

12. Si angulus $ANM=v$ fuerit valde paruus, positis $AN=p$ et $MN=q$; erit vero proxime arcus AM $=s=\tfrac{1}{2}(p+q)v$.

Demonstratio.

Si angulus v est valde paruus, erit proxime $\sin.\tfrac{1}{2}v=\tfrac{1}{2}v-\tfrac{1}{48}v^3$, et tang. $\tfrac{1}{2}v=\tfrac{1}{2}v+\tfrac{1}{24}v^3$ quibus formulis in limitibus superioribus substitutis erit

$s>$

$s > \frac{1}{2}v(p+q)(1-\frac{1}{24}vv)$ etc.

$s < \frac{1}{2}v(p+q)(1+\frac{1}{12}vv)$

Cum igitur $\frac{1}{2}v(p+q)$ intra hos limites sibi valde propinquos contineatur, erit proxime:

$s = \frac{1}{2}v(p+q)$

Q. E. D.

Coroll. 1.

13. Haec expressio $\frac{1}{2}v(p+q)$ eo exactius praebet longitudinem arcus AM, quo minor fuerit angulus ANM $=v$: si enim hic angulus sit infinite paruus, tum nulla omnino aberratio a veritate locum inuenire potest.

Coroll. 2.

14. Etiamsi autem angulus ANM $=v$ non sit adeo paruus, tamen expressio $\frac{1}{2}v(p+q)$ non multum a longitudine arcus AM differre potest, cum dentur casus, quibus $\frac{1}{2}v(p+q)$ veram arcus AM longitudinem exhibet; quicunque fuerit angulus v. Hoc scilicet euenit, si curua ANM fuerit circulus, tum enim ob $p=q$ erit arcus AM $=pv=qv=\frac{1}{2}v(p+q)$.

Coroll. 3.

fig. 1. 15. Non solum autem circulus hac proprietate gaudet, vt sit arcus AM $=\frac{1}{2}v(p+q)$, sed idem quoque contingit in iis curuis, quarum euolutae sunt circuli. Sit enim euoluta BR circulus, cuius radius $=a$, ac ponatur arcus BR $=z$, quoniam eius amplitudo aequalis est amplitudini curuae AM quam ponimus $=v$, erit $z=av$; sit porro AB $=b$, erit MR $=b+av$, et elementum curuae AM,

AM, $ds=(b+av)dv$, ideoque ipse arcus $AM=bv+\frac{1}{2}avv=\frac{1}{2}v(2b+av)=\frac{1}{2}v(AB+MR)$. Verum ob $BN=RN$ erit $AB+MR=AN+MN=p+q$, ideoque et hoc casu, quo curua AM ex euolutione circuli est nata, erit exacte arcus $AM=\frac{1}{2}v(p+q)$, quantumuis etiam magna fuerit eius amplitudo seu angulus v.

Coroll. 4.

16. Proposita ergo quacunque curua AM, per terminos A et M describi poterit arcus curuae ex euolutione circuli natae, eiusdem amplitudinis ANM, sicque habebuntur duae lineae curuae AM in A et M ad rectas AN et MN normales et continua curuatura procedentes, vnde in angulis non nimis magnis ne fieri quidem poterit, vt discrimen inter istas binas curuas sit notabile. fig. 2.

Coroll. 5.

17. Cum igitur non solum proxime sed quandoque etiam reuera sit $AM=\frac{1}{2}v(p+q)$, curua AM aequabitur arcui circulari centro N et radio $=\frac{AN+MN}{2}$ intra crura AN et MN descripto.

Scholion. 1.

18. Hac autem ratione dimensio curuae AM per arcum circularem multo accuratius instituitur, quam vllo modo per lineam rectam fieri potest, Vnde ex hoc fonte longe accuratior methodus deduci potest longitudinem curuarum ad arcus circulares reuocandi, quam vulgo hoc fieri solet ad lineas rectas. Linae curuae autem, quae in

eandem plagam vbique fint concauae, quales hic tantum confidero, ratione curuedinis ad fequentia genera referentur. Primum genus arcus tantum circulares complectitur, qui vbique eandem curuaturam tenent, hisque regula data exacte fatisfacit. Ad fecundum genus eas refero curuas, quarum curuedo ab A ad M continuo vel crefcit vel decrefcit, quo cafu dimenfio inuenta vix fenfibiliter a veritate recedere poteft, fi enim curuedo aequabiliter vel crefcit vel decrefcit, quod in curua ex euolutione circuli nata vfu venit, formula $\frac{1}{2}v(p+q)$ exacte fatisfacit: et nifi angulus N fit fatis magnus, curuedinis incrementa vel decrementa ab aequabilitate vix fenfibiliter difcedere poffunt. Tertium genus comprehendit eas curvas A M, quarum curuedo ab A ad punctum aliquod medium O inter A et M crefcit, inde vero ad M vsque iterum decrefcit, quo cafu curua in O gibbum habebit, vnde eam longiorem effe oportet, quam formula noftra indicat. Quarto contra generi adnumeramus eas curuas AM, quarum curuedo ab A ad O decrefcit, ab O vero ad M iterum crefcit, ita vt in O habiturae fint quandam depreffionem, Huiusmodi ergo linearum longitudo minor erit, quam regula declarat, quoniam circa O propius ad lineam rectam accedunt. Quodfi ergo curua ad tertium vel quartum genus pertinens in O fecetur, atque vtra portio AO et MO iam ad genus fecundum referenda ope regulae traditae menfuretur, error neceffario fiet minimus, cum non folum regula ad has partes magis fit accommodata, fed etiam amplitudo iftarum partium minor euadat.

Scho-

Scholion. 2.

19. Cum oſtendiſſem formulam $\frac{1}{2}v(p+q)$ non ſolum longitudinem curuae AM exacte exprimere, ſi ea fuerit circulus, ſed etiam, ſi euolutam habeat circularem, non abs re erit inquirere, vtrum haec proprietas nullis aliis lineis curuis competat. Quae inueſtigatio eo magis erit notatu digna, quod poſt calculum ſatis prolixum tandem ad ſimpliciſſimam ſolutionem perducat, ex quo forte non parum lucis nobis accendetur ad alias quaeſtiones eiusdem generis, quae alias difficillimae videri queant, expedite ſoluendas. Vnde ſequens problema tam ob praeſentem vſum, quam ob propriam elegantiam ſe commendare videtur.

Problema. 2.

20. Inuenire omnes curuas AM huius indolis, vt ductis ad eam normalibus AN, MN, curua AM aequalis ſit arcui circuli centro N radio $= \frac{1}{2}(AN+MN)$ intra rectas AN et MN deſcripto. fig. 3.

Solutio.

Poſitis nimirum $AN=p$, $MN=q$; et angulo $ANM=v$, arcuque $AOM=s$, quaeruntur omnes curvae in quibus ſit $s=\frac{1}{2}v(p+q)$. Ad quas inueniendas ex puncto ipſi M proximo m ducatur normalis mn, in eamque ex N perpendiculum N r demittatur, ob $MN=mr$ erit $nr=dq$ et $Nn=dp$. Quare cum ſit angulus A $nm=v+dv$, erit $nr=dq=dp$ coſ. v et $Nr=dp$ ſin. v. Ducatur deinde $N\mu$ ipſi mn parallela, erit $m\mu=Nr=dp$ ſin v: et ob angulum $MN\mu=dv$ habebitur

 M μ

$M\mu = q\,dv$: vnde cum fit $Mm = ds$ erit $ds = dp$ fin. $v + q\,dv$. Habemus ergo quatuor variabiles p, q, s et v, quarum relationem definire oportebit ope trium sequentium aequationum:

I. $\frac{2s}{v} = p + q$

II. $dq = dp$ cof. v

III. $ds = dp$ fin. $v + q\,dv$

Confultum autem videtur binas variabiles p et q eliminare: quem in finem ponamus $s = tv$, vt fit:

I. $2t = p + q$:

II. $dq = dp$ cof. v

III. $t\,dv + v\,dt = dp$ fin. $v + q\,dv$

quarum prima dat $q = 2t - p$, quae differentiata praebet $dq = 2\,dt - dp = dp$ cof. v, vnde fit $dp = \frac{2\,dt}{1 + cof\,v}$ et tertia aequatio abibit in:

$$t\,dv + v\,dt = \frac{2\,dt\,fin.\,v}{1 + cof.\,v.} + 2t\,dv - p\,dv \text{, feu}$$

$$p\,dv = t\,dv - v\,dt + \frac{2\,dt\,fin.\,v}{1 + cof.\,v.}$$

Differentietur haec denuo pofito dv conftante, et pro dp eius valores $\frac{2\,dt}{1 + cof.\,v}$ fubftituto habebitur:

$$\frac{2\,dt\,dv}{1 + cof.\,v} = -v\,ddt + \frac{2\,ddt\,fin.\,v}{1 + cof.\,v} + \frac{2\,dt\,dv}{1 + cof.\,v}$$

vnde fit $ddt = 0$: $dt = a\,dv$, et $t = b + av$. Confequenter $s = tv = bv + avv$: quae eft aequatio inter arcum s eiusque amplitudinem v, quam fupra (15) naturam curvae ex euolutione circuli natae exprimere vidimus; et quae, fi $a = 0$, ad ipfum eft circulum. Vnde problemati nullae aliae fatisfaciunt lineae praeter circulum et curuas ex euolutione circuli natas. Q. E. I.

Coroll.

Coroll. 1.

21. Quoniam aequatio non mediocriter implicata ad hanc tandem simplicissimam aequationem $ddt = 0$ est reducta, dubium est nullum, quin detur methodus alia hoc problema multo expeditius soluendi.

Scholion.

22. Quod aequatio inuenta $s = bv + avv$ sit ad curvam ex euolutione circuli natam, hoc modo facillime ostenditur. Sit radius euolutae $MR = r$, quoniam est $dv = \frac{ds}{r}$, erit $r = \frac{ds}{dv}$; ideoque hoc casu fit $r = b + 2av$. Sit arcus euolutae $BR = z$, quia est $z = MR - AB$, erit $z = 2av$, huiusque radius osculi $= 2a$: ex quo patet evolutam curuae, quae aequatione $s = bv + avv$ exprimitur, esse circulum. Ceterum data aequatione inter s et v aequatio inter coordinatas orthogonales AP et PM facile reperitur. Sit enim $AP = x$, $PM = y$, ob angulum $AMP = v$ erit $dx = ds$ sin. v et $dy = ds$ cos. v vnde fit $x = \int ds$ sin. v et $y = \int ds$ cos. v. Cum igitur praesenti casu sit $ds = bdv + 2avdv$ erit $x = b\int dv$ sin. v + $2a\int vdv$ sin. v, et $y = b\int dv$ cos. $v + 2a\int vdv$ cos. v. At est $\int dv$ sin. $v = 1 -$ cos. v; $\int dv$ cos. $v =$ sin. v, et $\int vdv$ sin. $v = -v$ cos. $v + \int dv$ cos. $v = -v$ cos. v + sin. v. $\int vdv$ cos. $v = v$ sin. $v - \int dv$ sin. $v = v$ sin. v + cos. v Quocirca habebitur:

fig. 2. fig. 3.

$$x = b - b \text{ cos. } v + 2a \text{ sin. } v - 2av \text{ cos. } v$$

$$y = b \text{ sin. } v + 2a \text{ cos. } v + 2av \text{ sin. } v$$

Hinc fit $yy + (b-x)^2 = bb + 4abv + 4aa + 4aavv$ et $b + 2av = \sqrt{(yy + (b-x)^2 - 4aa)}$

Deinde vero est $ds = \sqrt{(dx^2 + dy^2)} = dv(b + 2av)$ et $dv = \frac{ydy - (b-x)dx}{2a(b+2av)}$ ideoque:

$$2a\sqrt{(dx^2 + dy^2)} = ydy - (b-x)dx$$

Ex hac autem aequatione natura curuae quaesitae non tam facile perspicitur, quam ex praecedente.

Coroll. 2.

fig. 1. 23. Si fuerit B initium euolutae BR, erit $AB = b$, et ducta recta BM erit $BM = \sqrt{(yy + (b-x)^2)}$ sit BM $= u$, erit $2ads = udu$ et $4as = uu - bb$. ideoque $s = \frac{uu - bb}{4a}$. Vnde curua hanc quoque habebit proprietatem, vt sit diameter euolutae BR ad BM-AB ita BM+AB ad arcum AM.

Theorema. 2.

fig. 4. 24. Si arcus AM amplitudinis ANM in duas partes aeque amplas AM′, M′M diuidatur, atque in normalem M′N′ ex N perpendiculum NP′ demittatur; longitudo curuae AM proxime aequabitur arcui circuli centro N inter crura NA, NM descripti radio $= \frac{AN + MN + 2M'P'}{4}$.

Demonstratio.

Ponatur amplitudo totius arcus AM′M seu angulus ANM $= 2v$, erit arcuum AM′, et M′M amplitudo $= v$. Concurrant normales MN′ et MN in V, eruntque anguli AN′M′ $= v$ et M′VM $= v$: ideoque in triangulo VNN′ isoscele NN′ = NV et N′P′ = VP′. Per praecedens autem theorema erit

AM′

$AM' = \frac{1}{2}v(AN' + M'N')$
$M'M = \frac{1}{2}v(M'V + MV)$ proxime
ideoque addendo:
$AM = \frac{1}{2}v(AN' + M'N' + M'V + MV)$ proxime. Est vero $AN' = AN - NN'$ et $MV = MN + NN'$ ergo $AN' + MV = AN + MN$: deinde autem habebimus $M'N' + M'V = 2M'P'$. vnde fiet

$$AM = \frac{1}{2}v(AN + MN + 2M'P') \text{ proxime}$$

Sit nunc r radius circuli, cuius arcus amplitudinis eiusdem $2v$ aequalis sit curuae AM, erit $AM = 2vr$, vnde istius circuli radius fiet:

$$r = \frac{AN + MN + 2M'P'}{4} \text{ proxime}$$

Q. E. D.

Coroll. 1.

25. Hic ergo valor radii circuli, cuius arcus curvae AM aeque amplus eidem simul sit aequalis, propius ad veritatem accedit, quam ille, qui per theorema primum ex angulo integro ANM definitur, et qui prodierat $r = \frac{AN + MN}{2}$; nisi scilicet vterque sit exactus.

Coroll. 2.

26. Si ergo valor $r = \frac{AN + MN}{2}$ fuerit nimis paruus, necesse est, vt sit iste valor $r = M'P'$ nimis magnus, quia medium arithmeticum ad veritatem proxime accedit, simili modo si valor $r = \frac{AN + MN}{2}$ nimis fuerit magnus, tum iste $r = M'P'$ erit nimis paruus, sicque noui habentur limites $\frac{AN + MN}{2}$ et $M'P'$ inter quos verus ipsius r valor contineatur.

Coroll.

Coroll. 3.

27. Multo pluribus autem casibus formula hic exhibita $r = \frac{AN + MN + 2M'P'}{4}$ veritati prorsus est consentanea, quam praecedens $r = \frac{AN + MN}{2}$. Non solum enim ea pro circulo et curuis ex euolutione circuli natis valet, sed etiam ad innumeras alias insuper lineas curuas extenditur.

Coroll. 4.

28. Quo plures ergo sunt lineae curuae, quarum longitudo per formulam $\frac{1}{2} v (AN + MN + 2M'P')$ sine vllo errore exprimitur, eo minor esse poterit aberratio, etiam si curua AM non ad id genus pertineat.

Theorema. 3.

fig. 5. 29 Si linea curua AM amplitudinis ANM in partes quotcunque aeque amplas AI; I. II; II. III; III. IV; etc. diuidatur, atque in singulas normales ad diuisionum puncta ductas ex puncto N perpendicula demittantur N1, N2, N3, N4, etc. erit radius circuli, cuius arcus aeque amplus ac curua AM simul ipsi longitudini curuae aequalis est $= \frac{AN + MN + 2I1 + 2II2 + 2III3 + etc.}{2n}$ denotante n numerum partium, in quas arcus AM est diuisus.

Demonstratio.

Statuatur amplitudo seu angulus $ANM = nv$, erunt anguli $AN\,I = v$; $AN''II = 2v$; $AN'''III = 3v$; $AN^{IV}IV = 4v$ etc. Si iam singulae arcus propositi AM portiones secundum theorema primum exprimantur, erit:

A I

$$AI = \tfrac{1}{2} v(AN^{I} + IN^{I}) = \tfrac{1}{2} v(AN - NN^{I} + I1 - N^{I}1)$$
$$I.II = \tfrac{1}{2} v(IV^{I} + II\,V^{I}) = \tfrac{1}{2} v(I1 + II2 - V^{I}1 - V^{I}2)$$
$$II.III = \tfrac{1}{2} v(II\,V^{II} + III\,V^{II}) = \tfrac{1}{2} v(II2 + III3 - V^{II}2 + V^{II}3)$$
$$III.IV = \tfrac{1}{2} v(III\,V^{III} + IV\,V^{III} = \tfrac{1}{2} v(III3 + IV4 + V^{III}3 + V^{III}4)$$

et vltima formula, quia in figura assumitur $n = 5$, erit:

$$IV.M = \tfrac{1}{2} v(IV\,V^{IV} + M\,V^{IV}) = \tfrac{1}{2} v(IV4 + MN + V^{IV}4 + V^{IV}N)$$

His ergo in vnam summam collectis prodibit

$$AM = \tfrac{1}{2} v \left\{ \begin{array}{c} AN + MN + 2\,I1 + 2\,II2 + 2III3 + 2IV4 \\ -NN^{I} - N^{I}1 - V^{I}2 + V^{II}3 + V^{III}4 + V^{IV}N \\ -V^{I}1 - V^{II}2 + V^{III}3 + V^{IV}4 \end{array} \right\}$$

Omnes autem hos terminos inferiores se mutuo tollere sequenti modo per analysin breuius ostendetur. Ob angulos singulos cognitos erit:

$$N^{I}V^{I} = \frac{N^{I}N^{II}\,\text{fin.}\,2v}{\text{fin.}\,v}\ N^{II}V^{I} = \frac{N^{I}N^{II}\,\text{fin.}\,v}{\text{fin.}\,v}$$
$$N^{II}V^{II} = \frac{N^{III}N^{II}\,\text{fin.}\,3v}{\text{fin.}\,v};\ N^{III}V^{II} = \frac{N^{II}N^{III}\,\text{fin.}\,2v}{\text{fin.}\,v}$$
$$N^{III}V^{III} = \frac{N^{III}N^{IV}\,\text{fin.}\,4v}{\text{fin.}\,v};\ N^{IV}V^{III} = \frac{N^{III}N^{IV}\,\text{fin.}\,3v}{\text{fin.}\,v}$$

Hinc formulae ex theoremate primo ortae erunt

$$A.I = \tfrac{1}{2} v(AN^{I} + I.N^{I})$$
$$I.II = \tfrac{1}{2} v(IN^{I} + II.N^{II} + \frac{N^{I}N^{II}}{\text{fin.}\,v}(\text{fin.}\,v + \text{fin.}\,2v))$$
$$II.III = \tfrac{1}{2} v(II.N^{II} + III.N^{III} + \frac{N^{II}N^{III}}{\text{fin.}\,v}(\text{fin.}\,2v + \text{fin.}\,3v))$$
$$III.IV = \tfrac{1}{2} v(III.N^{III} + IV.N^{IV} + \frac{N^{III}N^{IV}}{\text{fin.}\,v}(\text{fin.}\,3v + \text{fin.}\,4v))$$
$$IV.M = \tfrac{1}{2} v(IV.N^{IV} + MN + \frac{N^{IV}N}{\text{fin.}\,v}(\text{fin.}\,4v + \text{fin.}\,5v))$$

Addantur hae formulae ac substituatur

 $N^{I}N^{II}$

$$N^{I}N^{II} = NN^{I} - NN^{II}$$
$$N^{II}N^{III} = NN^{II} - NN^{III}$$
$$N^{III}N^{IV} = NN^{III} - NN^{IV}$$
$$\text{et} \quad AN^{I} = AN - NN^{I}$$

prodibitque terminis in ordinem redactis:

$$AM = \tfrac{1}{2}v \left\{ \begin{array}{l} AN + MN + 2I.N^{I} 2II.N^{II} + 2III.N^{III} + 2IV.N^{IV} \\ + \frac{NN^{I}}{\text{fin.}\, v}\,\text{fin.}\, 2v + \frac{NN^{II}}{\text{fin.}\, v}(\text{fin.}\, 3v - \text{fin.}\, v) + \frac{NN^{III}}{\text{fin.}\, v}(\text{fin.}\, 4v - \text{fin.}\, 2v) + \text{etc.} \end{array} \right.$$

Est vero $\frac{\text{fin.}\, 2v}{\text{fin.}\, v} = 2\,\text{cof.}\, v$; $\frac{\text{fin.}\, 3v - \text{fin.}\, v}{\text{fin.}\, v} = 2\,\text{cof.}\, 2v$; $\frac{\text{fin.}\, 4v - \text{fin.}\, 2v}{\text{fin.}\, v} = 2\,\text{cof.}\, 3v$ etc. et $NN^{I}.\ \text{cof.}\, v = N^{I}1$; $NN^{II}\ \text{cof.}\, 2v = N^{II}2$; $NN^{III}\ \text{cof.}\, 3v = N^{III}3$; etc. quibus valoribus introductis erit:

$$AM = \tfrac{1}{2}v(AN + MN + 2I1 + 2II2 + 2III3 + 2IV4 + \text{etc.}$$

vnde, cum amplitudo sit $= nv$, erit radius arcus circuli aeque ampli et aequalis ipsi $AM =$

$$\frac{AN + MN + 2I1 + 2II2 + 2III3 + 2IV4}{2n} \text{ etc.}$$

Q. E. D.

Coroll. 1.

30. Quod si diuisio haec arcus AM in partes aeque amplas in infinitum continuetur, tum formula inuenta, quae in seriem infinitam abibit, longitudinem arcus AM vere exhibebit.

Coroll. 2.

31. Quamuis autem numerus partium sit finitus, tamen plurimi dantur casus, quibus veritas ipsa hoc modo eruitur. Et nisi hoc eueniat, error tamen erit valde par-

paruus eoque minor euadet, quo minoris amplitudinis partes capiantur.

Coroll. 3.

32. Si igitur curua hoc modo per circulum mensuranda proponatur AM, primum ductis ad A et M normalibus AN, MN notetur amplitudo ANM. Tum curua AM diuidatur in quotlibet partes aeque amplas, AB, BC, CD, DE, etc. quarum numerus sit $= n$, et in rectas ad puncta diuisionum normales ex N demittantur perpendicula Nb, Nc, Nd, etc. positoque angulo $ANM = v$, erit longitudo curuae AM

$$= \frac{v(AN + MN + 2Bb + 2Cc + 2Dd + 2Ee + \text{etc.})}{2n} \text{ proxime.}$$

Coroll. 4.

33. Valores isti Bb, Cc, Dd, etc. etiam inueniuntur, si ad puncta B, C, D, E, etc. tangentes ducantur, in easque ex puncto M perpendicula demittantur, tum enim haec perpendicula respectiue aequalia erunt rectis Bb, Cc, Dd, etc.

Problema. 3.

34. Proposita linea curua quacunque AMB, fig. 5. quae vbique ad eandem partem sit concaua, inuenire arcum circuli ab ipsi proxime aequalem.

Solutio.

Ad terminos curuae A et B ducantur normales AC, BC quae sibi mutuo occurant in C, erit angulus ACB curuae amplitudo. Vocentur $AC = a$, $BC = b$, et angulus

gulus ACB $=\theta$, tum demisso ex curuae quouis puncto M ad AC perpendiculo MP, positisque coordinatis AP $=x$, PM $=y$ dabitur aequatio inter x et y, ex qua reperietur subnormalis PN $=\frac{ydy}{dx}$, ducatur normalis MN, et vocetur angulus ANM $=v$, qui erit amplitudo arcus AM, erit $\frac{dx}{dy}=$ tang. v, sicque tam abscissa x quam applicata y per angulum v poterit definiri, quibus inuentis erit MN $=\frac{y}{\text{sin.}\,v}$. Deinde in MN, si opus est, productam ex C demittatur perpendiculum CS, ob CN $=a-x-\frac{ydy}{dx}=a-x-\frac{y\,\text{cos.}\,v}{\text{sin.}\,v}$ erit NS $=(a-x)$ cos. $v-\frac{y\,\text{cos.}\,v^2}{\text{sin.}\,v}$, ideoque tota recta MS $=(a-x)$ cos. $v+y$ sin v, ita vt datum quemuis angulum v longitudo rectae MS possit definiri. Indicetur haec recta MS amplitudini v respondens hoc signo $[v]$, ita vt similes rectae angulis $\frac{\theta}{n}$, $\frac{2\theta}{n}$, $\frac{3\theta}{n}$, etc. respondentes exhibeantur his signis $[\frac{\theta}{n}]$, $[\frac{2\theta}{n}]$, $[\frac{3\theta}{n}]$ etc. quibus inuentis sequentes formulae continuo magis ad valorem arcus AM appropinquabunt:

$$AM=\tfrac{1}{2}\theta(a+b)$$

$$AM=\tfrac{1}{4}\theta(a+b+2[\tfrac{\theta}{2}])$$

$$AM=\tfrac{1}{6}\theta(a+b+2[\tfrac{\theta}{3}]+2[\tfrac{2\theta}{3}])$$

$$AM=\tfrac{1}{8}\theta(a+b+2[\tfrac{\theta}{4}]+2[\tfrac{2\theta}{4}]+2[\tfrac{3\theta}{4}])$$

$$AM=\tfrac{1}{10}\theta(a+b+2[\tfrac{\theta}{5}]+2[\tfrac{2\theta}{5}]+2[\tfrac{3\theta}{5}]+2[\tfrac{4\theta}{5}])$$

Generaliter autem si n sumatur pro numero diuisionum erit eo exactius, quo maior fuerit numerus n

$$AM=\tfrac{1}{2n}\theta(a+b+2[\tfrac{\theta}{n}]+2[\tfrac{2\theta}{n}]+2[\tfrac{3\theta}{n}+\ldots+2[\tfrac{(n-1)\theta}{n}])$$

Quodsi ergo radius circuli ab, qui intra crura AC et BC constitutus aequalis sit curuae AB vocetur $=r$, erit arcus

ab

$ab = r\theta$, vnde iſtius circuli radius reperietur:

$$r = \frac{1}{2n}\left(a + b + 2\left[\frac{\theta}{n}\right] + 2\left[\frac{2\theta}{n}\right] + 2\left[\frac{3\theta}{n}\right] + \ldots + 2\left[\frac{(n-1)\theta}{n}\right]\right)$$

Q. E. I.

Coroll. 1.

35. Requirit ergo iſte modus lineas curuas per arcus circulares dimetiendi diuiſionem angulorum in partes quotcunque aequales. Cum igitur non niſi continua biſectio geometrice peragi queat, pro numero n ſucceſſiue aſſumi conueniet terminos progreſſionis geometricae duplae 2, 4. 8, 16, 32, etc.

Coroll. 2.

36. His autem numeris pro n ſucceſſiue aſſumendis id commodi adipiſcimur, vt termini iam ante inuenti omnes in expreſſiones ſequentes ingrediantur, ſicque calculi labor non mediocriter imminuatur.

Coroll. 3.

37. Valores ſcilicit radii circuli quaeſiti r ſequenti modo ex praecedentibus continuo accuratius determinabuntur.

I. $r = \frac{a+b}{2} = P$

II. $r = \frac{1}{2}P + \frac{1}{2}[\theta : 2] = Q$

III. $r = \frac{1}{2}Q + \frac{1}{4}\left[\frac{\theta}{4}\right] + \frac{1}{4}\left[\frac{3\theta}{4}\right] = R$

IV. $r = \frac{1}{2}R + \frac{1}{8}\left[\frac{\theta}{8}\right] + \frac{1}{8}\left[\frac{3\theta}{8}\right] + \frac{1}{8}\left[\frac{5\theta}{8}\right] + \frac{1}{8}\left[\frac{7\theta}{8}\right]$

etc.

Coroll. 4.

fig. 8. 38. Si valor ſecundus Q ſit verus, quantum primus P a vero vel deficit vel excedit, tantumdem quantitas $[\theta : 2]$ excedet vel ſuperabit, ideoque quantitates P et $[\frac{\theta}{2}]$ conſtituent limites, intra quos radius r contineatur. Etiamſi autem Q non ſit verus valor, tamen quia multo minus a veritate differt quam P, eaedem quantitates P et $[\frac{\theta}{2}]$ pro limitibus haberi poſſunt.

Coroll. 5.

39. Simili modo cum valores traditi continuo propius ad veritatem accedant, ſeriemque valde conuergentem conſtituant, limites quoque erunt:

Q et $\frac{1}{2}[\frac{\theta}{4}] + \frac{1}{2}[\frac{3\theta}{4}]$

R et $\frac{1}{4}[\frac{\theta}{8}] + \frac{1}{4}[\frac{3\theta}{8}] + \frac{1}{4}[\frac{5\theta}{8}] + \frac{1}{4}[\frac{7\theta}{8}]$

hique limites continuo ita multo fiunt arctiores, vt mox differentia fiat inſenſibilis.

Scholion.

40. Hos eosdem autem limites praebet motus reptorius Celeb. Io. Bernoullii: eiuſque ergo admirabilem vſum hic ex ſolo calculo ita elicuimus, vt etiam, ſi diviſio arcus non per continuas biſectiones inſtituatur, tamen ſemper applicari poſſit, ideoque multo latius extendatur: neque etiam hoc pacto de ratione, qua motum reptorium adornari conuenit, ſollicitos nos eſſe opus eſt, quod negotium alias non parum ſolertiae requirit.

Problema. 4.

41. Propoſita ellipſi quacunque inuenire radium circuli, cuius peripheria ſit aequalis proxime perimetro ellipſis.

Solutio.

Solutio.

Sit ACB quadrans ellipſis propoſitae, cuius ſemiaxes AC et BC, cum ſint ad curuam normales, erit AC $=a$ et BC$=b$, atque amplitudo θ aequalis erit angulo recto. Denotet ϱ angulum rectum, erit $\theta=\varrho$. Quaeſtio ergo huc redit, vt definiatur quadrans circuli, cuius arcus ſit arcui AMB longitudine aequalis. Ponatur itaque abſciſſa AP$=x$, PM$=y$, erit ex natura ellipſis $yy=\frac{bb}{aa}(2ax-xx)$ ſeu $aa-\frac{aayy}{bb}=(a-x)^2$, vnde fit $a-x=\frac{a}{b}\sqrt{(bb-yy)}$ et $dx=\frac{aydy}{b\sqrt{(bb-yy)}}$. Ducta nunc normali MN vocetur angulus ANM$=v$, erit $\frac{dx}{dy}=\frac{ay}{b\sqrt{(bb-yy)}}=$ tang. v; ideoque $aayy$ coſ. $v^2=b^4$ ſin. v^2-bbyy ſin. v^2, vnde fit $y=\frac{bb\ \text{ſin.}\ v}{\sqrt{(aa\ \text{coſ.}\ v^2+bb\ \text{ſin.}\ v^2)}}$ et conſequenter $\sqrt{(bb-yy)}=\frac{ab\ \text{coſ.}\ v}{\sqrt{(aa\ \text{coſ.}\ v^2+bb\ \text{ſin.}\ v^2)}}$; ita vt ſit $a-x=\frac{aa\ \text{coſ.}\ v}{\sqrt{(aa\ \text{coſ.}\ v^2+bb\ \text{ſin.}\ v^2)}}$. Demiſſo ergo in MN productam ex C perpendiculo CS, erit recta MS$=[v]=(a-x)$ coſ. $v+y$ ſin. $v=\sqrt{(aa\ \text{coſ.}\ v^2+bb\ \text{ſin.}\ v^2)}$. Cum igitur ſit $\theta=\varrho$, ſi pro v ſucceſſiue ſubſtituantur partes aliquotae anguli recti ϱ, erit:

$[\frac{\theta}{2}]=\sqrt{(aa\ \text{coſ.}\ \frac{\varrho}{2}^2+bb\ \text{ſin.}\ \frac{\varrho}{2}^2)}$

$[\frac{\theta}{3}]=\sqrt{(aa\ \text{coſ.}\ \frac{\varrho}{3}^2+bb\ \text{ſin.}\ \frac{\varrho}{3}^2)}$

$[\frac{2\theta}{3}]=\sqrt{(aa\ \text{ſin.}\ \frac{\varrho}{3}^2+bb\ \text{coſ.}\ \frac{\varrho}{3}^2)}$

etc.

His igitur ſingulis valoribus inuentis erit radius circuli propoſitae ellipſi ſecundum perimetrum aequalis

$$r = \frac{a+b}{2n} + \frac{1}{n}\sqrt{(aa\,\text{cos.}\,\frac{\rho}{n}^2 + bb\,\text{sin.}\,\frac{\rho}{n}^2)} + \frac{1}{n}\sqrt{(aa\,\text{cos.}\,\frac{2\rho}{n}^2 + bb\,\text{sin.}\,\frac{2\rho}{n}^2)}$$
$$+ \frac{1}{n}\sqrt{(aa\,\text{cos.}\,\frac{3\rho}{n}^2 + bb\,\text{sin.}\,\frac{3\rho}{n}^2)} + \frac{1}{n}\sqrt{(aa\,\text{cos.}\,\frac{4\rho}{n}^2 + bb\,\text{sin.}\,\frac{4\rho}{n}^2)}$$
$$+ \ldots\ldots + \frac{1}{n}\sqrt{(aa\,\text{cos.}\,\frac{(n-1)\rho}{n}^2 + bb\,\text{sin.}\,\frac{(n-1)\rho}{n}^2)}$$

hicque valor, quo maior accipiatur diuisionum numerus n, eo propius longitudinem radii quaesiti r exhibebit. Q. E. I.

Coroll. 1.

42. Si ponatur semiaxis maior $a = c + d$; semiaxis minor $b = c - d$, fiet $aa\,\text{cos.}\,v^2 + bb\,\text{sin.}\,v^2 = cc + dd + 2cd(\text{cos.}\,v^2 - \text{sin.}\,v^2) = cc + dd + 2cd\,\text{cos.}\,2v$. His ergo valoribus substitutis habebitur ob $2\rho = \pi$, denotante π angulum duobus rectis aequalem

$$r = \frac{c}{n} + \frac{1}{n}\sqrt{(cc+dd+2cd\,\text{cos.}\,\frac{\pi}{n})} + \frac{1}{n}\sqrt{(cc+dd+2cd\,\text{cos.}\,\frac{2\pi}{n})}$$
$$+ \frac{1}{n}\sqrt{(cc+dd+2cd\,\text{cos.}\,\frac{3\pi}{n})} + \frac{1}{n}\sqrt{(cc+dd+2cd\,\text{cos.}\,\frac{4\pi}{n})}$$
$$+ \ldots\ldots + \frac{1}{n}\sqrt{(cc+dd+2cd\,\text{cos.}\,\frac{(n-1)\pi}{n})}$$

Coroll. 2.

43. Hinc ergo patet, si sit $d = 0$, quo casu ellipsis abit in circulum radii $= c$, ob singulos terminos $= c$ fore, quotcunque diuisiones instituantur, semper $r = c$.

Coroll. 3.

Tab. II. fig. 1. 44. Ex formulis in coroll. 1. inuentis elegantissima sequitur constructio geometrica similis ei, quam Celeb. Bernoullius dedit. Super diametro $AB = a + b = 2c$ constituatur semicirculus, qui in partes quotcunque aequales diuidatur in punctis a, b, c, d, etc. quarum partium numerus sit $= n$. Tum secta diametro AB in C ita

ita vt sit $AC = a$, $BC = b$; ex C ad singula diuisionum puncta agantur rectae Ca, Cb, Cc, Cd, etc. eritque, si arcus Ac contineat m partes, recta $Cc = \surd(cc + dd + 2\,cd \operatorname{cos.} \frac{m\pi}{n})$. Ducto enim ex centro O radio $Oc = c$, erit angulus $AOc = \frac{m\pi}{n}$, et $CO = AC - AO = d$, ideoque recta $Cc = \surd(cc + dd + 2\,cd \operatorname{cos.} AOc)$. Quamobrem radius circuli ellipsis propositae isoperimetri erit

$$r = \frac{AO + Ca + Cb + Cc + Cd + Ce + Cf + Cg + Ch + Ci + Ck + Cl}{n}$$

Coroll. 4.

45. Cum autem expressiones cosinuum pro diuersis valoribus numeri n sint sequentes:

si $n = 2$: $\operatorname{cos.} \frac{\pi}{2} = 0$;

si $n = 3$; $\operatorname{cos.} \frac{\pi}{3} = \frac{1}{2}$; $\operatorname{cos.} \frac{2\pi}{3} = -\frac{1}{2}$

si $n = 4$ $\operatorname{cos.} \frac{\pi}{4} = \frac{1}{\surd 2}$; $\operatorname{cos.} \frac{2\pi}{4} = 0$; $\operatorname{cos.} \frac{3\pi}{4} = -\frac{1}{\surd 2}$.

si $n = 6$; $\operatorname{cos.} \frac{\pi}{6} = \frac{\surd 3}{2}$; $\operatorname{cos.} \frac{2\pi}{6} = \frac{1}{2}$; $\operatorname{cos.} \frac{3\pi}{6} = 0$; $\operatorname{cos.} \frac{4\pi}{6} = -\frac{1}{2}$; $\operatorname{cos.} \frac{5\pi}{6} = \frac{-\surd 3}{2}$.

si $n = 12$; $\operatorname{cos.} \frac{\pi}{12} = \frac{\surd 3 + 1}{2\surd 2}$; $\operatorname{cos.} \frac{2\pi}{12} = \frac{\surd 3}{2}$; $\operatorname{cos.} \frac{3\pi}{12} = \frac{\surd 2}{2}$; $\operatorname{cos.} \frac{4\pi}{12} = \frac{1}{2}$

$\operatorname{cos.} \frac{5\pi}{12} = \frac{\surd 3 - 1}{2\surd 2}$; $\operatorname{cos.} \frac{6\pi}{12} = 0$; $\operatorname{cos.} \frac{7\pi}{12} = \frac{-\surd 3 + 1}{2\surd 2}$; $\operatorname{cos.} \frac{8\pi}{12} = -\frac{1}{2}$

$\operatorname{cos.} \frac{9\pi}{12} = \frac{-\surd 2}{2}$; $\operatorname{cos.} \frac{10\pi}{12} = \frac{-\surd 3}{2}$; $\operatorname{cos.} \frac{11\pi}{12} = \frac{-\surd 3 - 1}{2\surd 2}$

Sequentes formulae ad valorem radii circuli quaesiti r proxime inueniendum videntur aptissimae:

I. $r = c$

II. $r = \frac{1}{2}(c + \surd(cc + dd))$

III. $r = \frac{1}{3}(c + \surd(cc + dd + cd) + \surd(cc + dd - cd)) = P$

IV. $r = \frac{1}{4}(c + \surd(cc + dd + cd\surd 2) + \surd(cc + dd) + \surd(cc + dd - cd\surd$

V. $r = \frac{1}{6}(3P + \surd(cc + dd + cd\surd 3) + \surd(cc + dd) + \surd(cc + dd - cd\surd 3)) =$

VI. $r = \frac{1}{12}(6Q + \surd(cc + dd + \frac{cd(\surd 3 + 1)}{\surd 2}) + \surd(cc + dd + cd\surd 2)$

$+ \surd(cc + dd + \frac{cd(\surd 3 - 1)}{\surd 2}) + \surd(cc + dd - \frac{cd(\surd 3}{\surd 2}$

$+ \surd(cc + dd - cd\surd 2) + \surd(cc + dd - \frac{cd(\surd 3 -}{\surd 2}$

Exemplum 1.

46. *Inuenire circulum, cuius peripheria proxime aequetur perimetro ellipsis, cuius axes teneant inter se rationem* 5 : 4.

Sit semiaxis maior $a = 10$ et semiaxis minor $b = 8$, quod est exemplum a Celeb. Bernoullio imprimis pertractatum; erit $c = \frac{a+b}{2} = 9$; $d = \frac{a-b}{2} = 1$; hincque $cc + dd = 82$ et $cd = 9$, ergo in fractionibus decimalibus erit:

$$c = 9,000000$$
$$\surd(cc + dd) = 9,055386$$
$$\surd(cc + dd + cd) = 9,539392$$
$$\surd(cc + dd - cd) = 8,544004$$
$$\surd(cc + dd + cd\surd 2) = 9,732827$$
$$\surd(cc + dd - cd\surd 2) = 8,322984$$
$$\surd(cc + dd + cd\surd 3) = 9,878687$$
$$\surd(cc + dd - cd\surd 3) = 8,149328$$
$$\surd(cc + dd + \frac{cd(\surd 3 + 1)}{\surd 2}) = 9,969284$$
$$\surd(cc + dd + \frac{cd(\surd 3 - 1)}{\surd 2}) = 9,309069$$
$$\surd(cc + dd - \frac{cd(\surd 3 - 1)}{\surd 2}) = 8,794386$$
$$\surd(cc + dd - \frac{cd(\surd 3 + 1)}{\surd 2}) = 8,038242$$

vnde

vnde sequentes expressiones radium r continuo accuratius dabunt;

I. $r = 9,000000$
II. $r = 9,027693$
III. $r = 9,027798$
IV. $r = 9,027799$
V. $r = 9,027799$
VI. $r = 9,027799$

vnde patet formulam sextam ad multo plures adhuc figuras valorem ipsius r exhibituram fuisse, si calculum vlterius produxissem: et quidem expressio sexta videtur valorem ipsius r ad 24 notas axactum praebitura fuisse.

Exempl. 2.

47. *Ellipsium non multum a circulo abludentium perimetros per circulum proxime exprimere.*

Quando ellipsis non multum a circulo differt, tum d prae c erit quantitas valde parua: ideoque formulae irrationales commode per approximationem exhiberi poterunt. Sit breuitatis gratia $\sqrt{(cc + dd)} = e$ et angulus $\frac{\pi}{n} = \Phi$ erit $\sqrt{(cc + dd + 2cd \operatorname{cos}. \frac{\pi}{n})} = \sqrt{(ee + 2cd \operatorname{cos}. \Phi)} = e + \frac{cd}{e} \operatorname{cos}. \Phi - \frac{1.1 c^2 d^2}{1.2 e^3} \operatorname{cos}. \Phi^2 + \frac{1.1.3 c^3 d^3}{1.1.3 e^5} \operatorname{cos}. \Phi^3 - \frac{1.1.3.5 c^4 d^4}{1.2.3.4 e^7} \operatorname{cos}. \Phi^4 + \text{etc.}$

His ergo formulis loco irrationalium substitutis erit

$$
\begin{aligned}
r = e - \tfrac{1}{n}(e - c) \\
&+ \frac{cd}{ne}(\operatorname{cos}. \Phi + \operatorname{cos}. 2\Phi + \operatorname{cos}. 3\Phi + \ldots + \operatorname{cos}. (n-1)\Phi) \\
&- \frac{1.1 c^2 d^2}{1.2 n e^3}(\operatorname{cos}. \Phi^2 + \operatorname{cos}. 2\Phi^2 + \operatorname{cos}. 3\Phi^2 + \ldots + \operatorname{cos}. (n-1)\Phi^2) \\
&+ \frac{1.1.3 c^3 d^3}{1.2.3 n e^5}(\operatorname{cos}. \Phi^3 + \operatorname{cos}. 2\Phi^3 + \operatorname{cos}. 3\Phi^3 + \ldots + \operatorname{cos}. (n-1)\Phi^3) \\
&- \frac{1.1.3.5 c^4 d^4}{1.2.3.4 n e^7}(\operatorname{cos}. \Phi^4 + \operatorname{cos}. 2\Phi^4 + \operatorname{cos}. 3\Phi^4 + \ldots + \operatorname{cos}. (n-1)\Phi^4) \\
&\text{etc.}
\end{aligned}
$$

 Est

Eſt vero $\cos.\,\Phi + \cos.\,2\Phi + \ldots + \cos.\,(n-1)\Phi =$
$-\frac{1}{2} + \frac{\cos.\,(n-1)\Phi - \cos.\,n\Phi}{2(1-\cos.\,\Phi)} = \frac{\sin.\,(n-\frac{1}{2})\Phi - \sin.\,\frac{1}{2}\Phi}{2\sin.\,\frac{1}{2}\Phi}$

At cum ſit $n\Phi = \pi$ erit $\cos.\,n\Phi = -1$, et $\cos.\,(n-1)\Phi = \cos.\,(\pi-\Phi) = -\cos.\,\Phi$, ideoque $\cos.\,\Phi + \cos.\,2\Phi + \cos.\,3\Phi \ldots + \cos.\,(n-1)\Phi = 0$. Deinde ob $\cos.\,\Phi^2 = \frac{1}{2} + \frac{1}{2}\cos.\,2\Phi$, fiet $\cos.\,\Phi^2 + \cos.\,2\Phi^2 + \cos.\,3\Phi^2 + \ldots + \cos.\,(n-1)\Phi^2 = \frac{n-1}{2} + \frac{1}{2}(\cos.\,2\Phi + \cos.\,4\Phi + \cos.\,6\Phi + \ldots + \cos.\,(n-1)2\Phi) = \frac{n-1}{2} - \frac{1}{4} + \frac{\cos.\,(n-1)2\Phi - \cos.\,2n\Phi}{4(1-\cos.\,2\Phi)} = \frac{n-2}{2}$. Simili modo reliquas ſeries ad angulos ſimplices reducendo, obtinentur ſequentes ſummationes.

$$\cos.\,\Phi + \cos.\,2\Phi + \cos.\,3\Phi + \ldots + \cos.\,(n-1)\Phi = 0$$
$$\cos.\,\Phi^2 + \cos.\,2\Phi^2 + \cos.\,3\Phi^2 + \ldots + \cos.\,(n-1)\Phi^2 = \tfrac{1}{2}n - 1$$
$$\cos.\,\Phi^3 + \cos.\,2\Phi^3 + \cos.\,3\Phi^3 + \ldots + \cos.\,(n-1)\Phi^3 = 0$$
$$\cos.\,\Phi^4 + \cos.\,2\Phi^4 + \cos.\,3\Phi^4 + \ldots + \cos.\,(n-1)\Phi^4 = \tfrac{3}{8}n - 1$$
$$\cos.\,\Phi^5 + \cos.\,2\Phi^5 + \cos.\,3\Phi^5 + \ldots + \cos.\,(n-1)\Phi^5 = 0$$
$$\cos.\,\Phi^6 + \cos.\,2\Phi^6 + \cos.\,3\Phi^6 + \ldots + \cos.\,(n-1)\Phi^6 = \tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}n - 1$$

etc.

Quibus valoribus ſubſtitutis habebitur:

$$r = e - \tfrac{1}{n}(e-c) - \frac{1\cdot1\,c^2d^2}{2\cdot2\,e^3} + \frac{1\cdot1\,c^2d^2}{1\cdot2\,ne^3}$$
$$- \frac{1\cdot3\cdot5\cdot1\cdot3\cdot c^4d^4}{2\cdot3\cdot4\cdot2\cdot4\,e^7} + \frac{1\cdot1\cdot3\cdot5\,c^4d^4}{1\cdot2\cdot3\cdot4\,ne^7}$$
$$- \frac{1\cdot3\cdot5\cdot7\cdot9\cdot1\cdot3\cdot5\cdot c^6d^6}{2\cdot3\cdot4\cdot5\cdot6\cdot2\cdot4\cdot6\,e^{11}} + \frac{1\cdot1\cdot3\cdot5\cdot7\cdot9\,c^6d^6}{1\cdot2\cdot3\cdot4\cdot5\cdot6\,ne^{11}}.$$

quae aequatio hanc formam induit ſimpliciorem:

$r =$

$$r = e - \tfrac{1}{n}(e-c) - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{c^2d^2}{e^3} + \frac{1\cdot 1}{1\cdot 2}\cdot\frac{c^2d^2}{ne^3}$$
$$- \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4}\,\frac{c^4d_4}{e^7} + \frac{1\cdot 1\cdot 3\cdot 5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{c^4d^7}{ne^7}$$
$$- \frac{1\cdot 1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}\,\frac{c^6d^6}{e^{11}} + \frac{1\cdot 1\cdot 3\cdot 5\cdot 7\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\frac{c^6d^6}{ne^{11}}$$
$$- \frac{1\cdot 1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8}\,\frac{c^8d^8}{e^{15}} + \frac{1\cdot 1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}\,\frac{c^8d^8}{ne^{15}}$$
$$\text{etc.}$$

At vltima ſeries $-\frac{1\cdot 1}{1\cdot 2}\frac{c^2d^2}{ne^3} + \frac{1\cdot 1\cdot 3\cdot 5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{c^4d^4}{ne^7} +$ etc. ſummam habet $\frac{1}{n}(e - \frac{1}{2}\sqrt{(ee+2cd)} - \frac{1}{2}\sqrt{(ee-2cd)}) = \frac{1}{n}(e-c)$ ita vt numerus n prorſus ex calculo euaneſcat fiatque accurate

$$r = e - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{c^2d^2}{e^3} - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4}\cdot\frac{c^4d^4}{e^7} - \frac{1\cdot 1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}\cdot\frac{c^6d^6}{e^{11}} - \text{etc.}$$

quae ſeries eo celerius conuergit, quo minor fuerit valor d ratione ipſius $e = \sqrt{(cc+dd)}$. Eſt autem, dum ellipſis ſemiaxes ponuntur a et b, $c = \frac{a+b}{2}$ et $d = \frac{a-b}{2}$.

Exempl. 3.

48. *Si axis minor ellipſis prorſus euaneſcat, tum eius quadrans ſemiaxi maiori aequabitur*, cui ergo quadrantem *circuli aequalem inueniri oporteat.*

Sit ergo ſemiaxis maior $= a$, et radius circuli, cuius quarta pars ipſi a aequetur ſit $= r$, eritque $r\varrho = a$: ideoque $\varrho = \frac{a}{r}$: ita vt radio r proxime inuento peripheria circuli per lineam rectam exprimi poſſit proxime. Ex formula ergo praecedente erit $c = \frac{a}{2}$; $d = \frac{a}{2}$ et $e = \frac{a}{\sqrt{2}}$: quibus valoribus ſubſtitutis in vltima ſerie prodit:

$$r = \frac{a}{\sqrt{2}}\left(1 - \frac{1\cdot 1}{4\cdot 4} - \frac{1\cdot 1\cdot 3\cdot 5}{4\cdot 4\cdot 8\cdot 8} - \frac{1\cdot 1\cdot 3\cdot 5\cdot 7\cdot 9}{4\cdot 4\cdot 8\cdot 8\cdot 12\cdot 12} - \text{etc.}\right)$$

Cum autem ſit $\varrho = \frac{\pi}{2}$, denotante $1 : \pi$ rationem diametri ad peripheriam, erit $r = \frac{a}{\varrho} = \frac{2a}{\pi}$ vnde ſequentis ſeriei ſumma habebitur.

$1 - \frac{1.1}{4.4} - \frac{1.1.3.5}{4.4.8.8} - \frac{1.1.3.5.7.9}{4.4.8.8.12.12} - \text{etc.} = \frac{2\sqrt{2}}{\pi}$.

Aliter autem valor ipſius r ex formulis in ipſa ſolutione problematis inuentis expreſſus reperietur, namque ob $b = 0$ erit;

$$r = \frac{a}{2n} + \frac{a}{n}\left(\text{coſ.}\,\frac{\varrho}{n} + \text{coſ.}\,\frac{2\varrho}{n} + \text{coſ.}\,\frac{3\varrho}{n} + \text{coſ.}\,\frac{4\varrho}{n} + \ldots + \text{coſ.}\,\frac{(n-1)\varrho}{n}\right)$$

vnde ſequentes formulae ad valorem r appropinquabunt:

ſi $n = 1$, $r = \frac{1}{2}a$

ſi $n = 2$; $r = \frac{1}{4}a(1 + 2\,\text{coſ.}\,\frac{\varrho}{2})$

ſi $n = 3$; $r = \frac{1}{6}a(1 + 2\,\text{coſ.}\,\frac{\varrho}{3} + 2\,\text{coſ.}\,\frac{2\varrho}{3})$

etc.

At vero ſeries illa coſinuum expreſſione finita exhiberi poteſt; erit enim:

$$\text{coſ.}\,\frac{\varrho}{n} + \text{coſ.}\,\frac{2\varrho}{n} + \text{coſ.}\,\frac{3\varrho}{n} + \ldots + \text{coſ.}\,\frac{(n-1)\varrho}{n} = -\frac{1}{2} + \frac{\text{coſ.}\,\frac{(n-1)\varrho}{n} - \text{coſ.}\,\varrho}{2(1 - \text{coſ.}\,\frac{\varrho}{n})}$$

Iam ob ϱ angulum rectum, erit $\text{coſ.}\,\varrho = 0$, et $\text{coſ.}\,\frac{(n-1)\varrho}{n} = \text{coſ.}\,(\varrho - \frac{\varrho}{n}) = \text{ſin.}\,\frac{\varrho}{n}$: vnde praecedens ſumma abit in $-\frac{1}{2} + \frac{\text{ſin.}\,\frac{\varrho}{n}}{2(1-\text{coſ.}\,\frac{\varrho}{n})}$. Eſt vero $\frac{1-\text{coſ.}\,\frac{\varrho}{n}}{\text{ſin.}\,\frac{\varrho}{n}} = \text{tang}\,\frac{\varrho}{2n}$; ideoque ſumma ſeriei

$$\text{coſ.}\,\frac{\varrho}{n} + \text{coſ.}\,\frac{2\varrho}{n} + \text{coſ.}\,\frac{3\varrho}{n} + \ldots + \text{coſ.}\,\frac{(n-1)\varrho}{n} = -\frac{1}{2} + \frac{1}{2\,\text{tang.}\,\frac{\varrho}{2n}}$$

quo valore ſubſtituto fiet

$r =$

$r = \frac{m}{2n \tan. \frac{\varrho}{2n}}$, hincque $\varrho = 2n \tan. \frac{\varrho}{2n}$

Hic autem manifestum est, posito $n = \infty$, aequationem perfecte satisfacere, ceterum vero eo magis ad veritatem accedere, quo maior fuerit n. Est enim $\tan. \frac{\varrho}{2n} = \frac{\varrho}{2n} + \frac{\varrho^3}{24n^3} + \text{etc.}$ ideoque erit quidem $\varrho < 2n \tan. \frac{\varrho}{2n}$, at defectus circiter erit $= \frac{\varrho^3}{12nn}$. Ex hoc autem casu nihil deducitur, quod non aliunde esset notissimum.

Problema. 5.

49. *Longitudinem arcus parabolici per arcum circuli proxime exhibere.*

Solutio.

Sit AMB parabola ad axem AC relata, cuius natura inter coordinatas $AP = x$ et $PM = y$ hac aequatione contineatur $yy = 2cx$. Ducatur ad M normalis MN, et vocetur angulus $ANM = v$, qui simul amplitudinem arcus AM metietur. Iam ob subnormalem $PN = c$, erit $y = c \tan. v$ et $x = \frac{yy}{2c} = \frac{c \tan. v^2}{2}$ atque $MN = \frac{c}{\cos. v}$ et $AN = c(1 + \frac{1}{2} \tan. v^2)$ Sit iam arcus AMB, quem metiri oporteat, amplitudo $= \theta$, erit ducta normali BC angulus $ACB = \theta$, et $AC = c(1 + \frac{1}{2} \tan. \theta^2)$, atque $BC = \frac{c}{\cos. \theta}$ Vocetur iam vt in probl. 3. $AC = a = c(1 + \frac{1}{2} \tan. \theta^2)$ et $BC = b = \frac{c}{\cos. \theta}$: et demisso ex C in normalem MN productam perpendiculo CS, erit $MS = (a - x) \cos. v + y \sin. v = c(1 + \frac{1}{2} \tan. \theta^2 - \frac{1}{2} \tan. v^2) \cos. v + c \tan. v \sin. v$ vel succinctius $MS = c \cos. v + \frac{1}{2} c (\tan. \theta^2 + \tan. v^2) \cos. v$. Sit nunc ab arcus circuli centro Tab. I. fig. 7.

centro C inter normales AC et BC descriptus, ipsique arcui parabolico AMB aequalis, voceturque eius radius $aC = r$: si pro v successiue substituantur partes anguli θ indeterminatae $\frac{\theta}{n}, \frac{2\theta}{n}, \frac{3\theta}{n} \ldots \frac{(n-1)\theta}{n}$ ob $a + b = c$ $\left(1 + \frac{1}{\cos.\theta} + \frac{\sin.\theta^2}{2\cos.\theta^2}\right) = \frac{c(1+\cos.\theta)^2}{2\cos.\theta^2}$ prodibit

$$r = \frac{c}{2n}\left\{\begin{array}{l} \frac{(1+\cos.\theta)^2}{2\cos.\theta^2} + 2\left(\cos.\frac{\theta}{n} + \cos.\frac{2\theta}{n} + \cos.\frac{3\theta}{n} + \ldots + \cos.\frac{(n-1)\theta}{n}\right. \\ \quad + \tan g.\,\theta^2\left(\cos.\frac{\theta}{n} + \cos.\frac{2\theta}{n} + \cos.\frac{3\theta}{n} + \ldots + \cos.\frac{(n-1)\theta}{n}\right. \\ \quad + \tan g.\frac{\theta}{n}\sin.\frac{\theta}{n} + \tan g.\frac{2\theta}{n}\sin.\frac{2\theta}{n} + \ldots + \tan g.\frac{(n-1)\theta}{n}\sin.\frac{(n-1)\theta}{n} \end{array}\right\}$$

At est $\cos.\frac{\theta}{n} + \cos.\frac{2\theta}{n} + \ldots + \cos.\frac{(n-1)\theta}{n} = -\frac{1}{2} + \frac{\cos.\left(1-\frac{1}{n}\right)\theta - \cos.\theta}{2\left(1-\cos.\frac{\theta}{n}\right)}$

$$= -\frac{1}{2} - \frac{\cos.\theta}{2} + \frac{\sin.\theta\sin.\frac{\theta}{n}}{2\left(1-\cos.\frac{\theta}{n}\right)} = -\frac{1}{2} - \frac{1}{2}\cos.\theta + \frac{1}{2}\sin.\theta\ \cot.\frac{\theta}{2n}$$

quo valore substituto habebitur:

$$r = \frac{c}{2n}\left\{\begin{array}{l} \frac{1}{2}\tan g.\,\theta\sin.\theta + \sin.\theta\cot.\frac{\theta}{2n} + \frac{1}{2}\tan g.\,\theta^2\sin.\theta\cot.\frac{\theta}{2n} \\ + \tan g\frac{\theta}{n}\sin.\frac{\theta}{n} + \tan g.\frac{2\theta}{n}\sin.\frac{2\theta}{n} + \tan g.\frac{3\theta}{n}\sin.\frac{3\theta}{n} + \ldots + \tan g.\frac{(n-1)\theta}{n}\sin.\frac{(n-1)\theta}{n} \end{array}\right\}$$

Ponamus nunc amplitudinem arcus parabolici AMB esse 60°, vt sit $\sin.\theta = \frac{\sqrt{3}}{2}$; $\cos.\theta = \frac{1}{2}$; $\tan g.\,\theta = \sqrt{3}$ atque $\sin.\frac{1}{2}\theta = \frac{1}{2}$; $\cos.\frac{1}{2}\theta = \frac{\sqrt{3}}{2}$; $\tan g.\frac{1}{2}\theta = \frac{\sqrt{3}}{3}$ $\sin.\frac{1}{4}\theta = \frac{\sqrt{3}-1}{2\sqrt{2}}$; $\cos.\frac{1}{4}\theta = \frac{\sqrt{3}+1}{2\sqrt{2}}$; $\tan g.\frac{1}{4}\theta = 2-\sqrt{3}$; $\sin.\frac{1}{8}\theta = \sqrt{\frac{2\sqrt{2}-\sqrt{3}-1}{4\sqrt{2}}}$; $\cos.\frac{1}{8}\theta = \sqrt{\frac{2\sqrt{2}+\sqrt{3}+1}{4\sqrt{2}}}$; $\tan g.\frac{1}{8}\theta = (\sqrt{2}-1)(\sqrt{3}-\sqrt{2})$ $\sin.\frac{3}{4}\theta = \frac{\sqrt{2}}{2}$; $\cos.\frac{3}{4}\theta = \frac{\sqrt{2}}{2}$; $\tan g.\frac{3}{4}\theta = 1$ ex quibus valoribus oritur:

$r =$

$$r = \frac{c}{2n}\left(\frac{3}{4} + \frac{5\sqrt{3}}{4}\,\text{cot.}\,\frac{\theta}{2n} + \text{tang.}\,\frac{\theta}{n}\,\text{fin.}\,\frac{\theta}{n} + \text{tang.}\,\frac{2\theta}{n}\,\text{fin.}\,\frac{2\theta}{n} + \text{tang.}\,\frac{3\theta}{n}\,\text{fin.}\,\frac{3\theta}{n} + \ldots + \text{tang.}\,\frac{(n-1)\theta}{n}\,\text{fin.}\,\frac{(n-1)\theta}{n}\right)$$

Subftituantur nunc pro n valores 1, 2, 4, et prodibunt fequentes expreffiones ad r appropinquantes:

I. $r = \frac{9}{4}c$

II. $r = \frac{9}{8}c + \frac{2}{\sqrt{3}}c$

III. $r = \frac{9}{16}c + \frac{1}{\sqrt{3}}c + \frac{3\sqrt{2}}{8}c + \frac{\sqrt{6}}{4}c$

Pofita ergo ratione diametri ad peripheriam $= 1 : \pi$ ob angulum $60° = \frac{1}{3}\pi$, fequentes expreffiones arcum parabolicum AMB, cuius amplitudo ACB eft $60°$, proxime exhibebunt:

I. $AMB = \frac{\pi c}{3}\,\frac{9}{4} = \frac{3}{4}\pi c$

II. $AMB = \frac{\pi c}{3}\left(\frac{9}{8} + \frac{2}{\sqrt{3}}\right)$

III. $AMB = \frac{\pi c}{3}\left(\frac{9}{16} + \frac{1}{\sqrt{3}} + \frac{3\sqrt{2}}{8} + \frac{\sqrt{6}}{4}\right)$

Conftat autem rectificationem parabolae a logarithmis pendere, vnde arcus AMB veram longitudinem inueftigemus. Cum igitur fit $AC = c\left(1 + \frac{3}{2}\right)$ erit abfciffa arcui AMB refpondens $= \frac{3}{2}c$, qui valor ipfi x in aequatione $yy = 2cx$ tribuatur, eritque $yy = 3cc$ Iam ob $x = \frac{yy}{2c}$ erit $dx = \frac{ydy}{c}$, et elementum arcus $AMB = \frac{dy}{c}\sqrt{(cc + yy)}$, cuius integrale eft $= \frac{y}{2c}\sqrt{(cc+yy)} + \frac{c}{2}\,l\,\frac{y+\sqrt{(cc+yy)}}{c}$. Ponatur nunc $y = c\sqrt{3}$, fietque arcus $AMB = c\sqrt{3} + \frac{c}{2}\,l(2+\sqrt{3})$; vnde fequentes aequationes proximae inter quadraturam circuli et logarithmos obtinebuntur;

I. $3\sqrt{3} + \frac{3}{2}\,l(2+\sqrt{3}) = \frac{9}{4}\pi$

II. $3\sqrt{3} + \frac{3}{2}\,l(2+\sqrt{3}) = \left(\frac{9}{8} + \frac{2}{\sqrt{3}}\right)\pi$

III. $3\sqrt{3} + \frac{3}{2}\,l(2+\sqrt{3}) = \left(\frac{9}{16} + \frac{1}{\sqrt{3}} + \frac{3\sqrt{2}}{8} + \frac{\sqrt{6}}{4}\right)\pi$

etc. Q. E. I.

Scholion.

Tab. I. fig. 8.

50. Quemadmodum ex parabola logarithmi ad quadraturam circuli reuocantur, ita, ſi pro curua AMB lineа rectificabilis accipitur, linea recta ad arcum circularem reducetur, ſicque viciſſim linea recta exhiberi poterit, quae arcui circuli proxime ſit aequalis. Ita ſi curua AMB hac aequatione exprimatur $y = \sqrt{x} - \frac{1}{3}x\sqrt{x}$: erit ipſe curuae arcus $AM = \sqrt{x} + \frac{1}{3}x\sqrt{x}$; et ſumta $x = 1 = AC$, erit applicata $BC = \frac{2}{3}$, normalis ad curuam, ideoque arcus AMB amplitudo angulus rectus ACB; et curva $AMB = \frac{4}{3}$. Quod ſi iam capiatur arcus AM amplitudinis $ANM = v$, atque in MN demittatur perpendiculum CS reperietur $MS = \frac{2}{3}\left(\text{cof.}\, v + \frac{1}{1 + \text{cof.}\, v}\right)$ vnde poſito radio circuli, $= r$ cuius quadrans $= AMB = \frac{4}{3}$ ob $a = 1$ et $b = \frac{2}{3}$ erit:

$$r = \frac{1}{2n}\Big(\frac{5}{3} + \frac{4}{3}\left(\text{cof.}\,\frac{\rho}{n} + \text{cof.}\,\frac{2\rho}{n} + \text{cof.}\,\frac{3\rho}{n} + \ldots + \text{cof.}\,\frac{(n-1)\rho}{n}\right)$$
$$+ \frac{4}{3}\left(\frac{1}{1+\text{cof.}\,\frac{\rho}{n}} + \frac{1}{1+\text{cof.}\,\frac{2\rho}{n}} + \frac{1}{1+\text{cof.}\,\frac{3\rho}{n}} + \ldots + \frac{1}{1+\text{cof.}\,\frac{(n-1)\rho}{n}}\right)\Big)$$

ſeu priorem ſeriem ſummando:

$$r = \frac{1}{2n} + \frac{1}{3n\,\text{tang.}\,\frac{\rho}{2n}} + \frac{2}{2n}\left(\frac{1}{1+\text{cof.}\,\frac{\rho}{n}} + \frac{1}{1+\text{cof.}\,\frac{2\rho}{n}} + \ldots + \frac{1}{1+\text{cof.}\,\frac{(n-1)\rho}{n}}\right)$$

At poſito $\pi = 2\rho$, ſeu $1 : \pi$, ratione diametri ad peripheriam, erit $\frac{4}{3} = \frac{\pi r}{2}$, ideoque $r = \frac{8}{3\pi}$, vnde peripheria π per ſeriem algebraicam exprimitur.

Prob-

Problema. 6.

51. Si curua propoſita fuerit cyclois AMB, inuenire radium circuli, cuius quarta peripheriae pars proxime ſit aequalis arcui cycloidico AMB. Tab. II. fig. 2.

Solutio.

Sit BQC ſemiſſis circuli generatoris eiusque centrum in O; vocetur radius $OB = OC = c$, et poſita ratione diametri ad peripheriam $= 1 : \pi$ erit ſemicircumferentia $BQC = \pi c$, cui aequalis eſt baſis AC, ita vt ſit $AC = a = \pi c$ et $BC = b = 2c$. Ducatur iam recta quaecunque PM baſi AC parallela, iunctaeque cordae CQ per M parallela ducatur MN, erit haec ad cycloidem normalis, ac propterea angulus ANM menſurabit amplitudinem arcus AM. Sit angulus $ANM = v$, erit ducta chorda BQ angulus $CBQ = v$, ideoque chorda $CQ = 2c \text{ ſin. } v = MN$. Deinde ob angulum ad centrum $COQ = 2v$ erit arcus $CQ = 2cv$ et arcus $BQ = \pi c - 2cv$, cui aequalis eſt recta QM huicque CN, ita vt ſit $CN = \pi c - 2cv$. Quare ſi in MN productam ex C demittatur perpendiculum CS erit $NS = \pi c \text{ coſ. } v - 2cv \text{ coſ. } v$; ideoque $MS = [v] = 2c \text{ ſin. } v + \pi c \text{ coſ. } v - 2cv \text{ coſ. } v$; ſubſtituamus iam pro v partes anguli $ACB = \varrho$, denotante ϱ angulum rectum, ita vt ſit $\varrho = \frac{1}{2}\pi$, erit

$$\left[\frac{\varrho}{n}\right] = 2c \text{ ſin. } \frac{\varrho}{n} + \pi c \text{ coſ. } \frac{\varrho}{n} - \frac{2c\varrho}{n} \text{ coſ. } \frac{\varrho}{n}$$

$$\left[\frac{2\varrho}{n}\right] = 2c \text{ ſin. } \frac{2\varrho}{n} + \pi c \text{ coſ. } \frac{2\varrho}{n} - \frac{4c\varrho}{n} \text{ coſ. } \frac{2\varrho}{n}$$

etc.

Vnde ſi r ſit radius circuli quaeſiti, erit:

$$r = \frac{c}{2n}\Big(\pi + 2 + 4\left(\text{sin.}\tfrac{\varrho}{n} + \text{sin.}\tfrac{2\varrho}{n} + \ldots + \text{sin.}\tfrac{(n-1)\varrho}{n}\right) + 2\pi\left(\text{cos.}\tfrac{\varrho}{n} + \text{cos.}\tfrac{2\varrho}{n} + \ldots + \text{cos}\tfrac{(n-1)\varrho}{n}\right) - \tfrac{2\pi}{n}\left(\text{cos.}\tfrac{\varrho}{n} + 2\,\text{cos.}\tfrac{2\varrho}{n} + 3\,\text{cos.}\tfrac{3\varrho}{n} + \ldots + (n-1)\,\text{cos.}\tfrac{(n-1)\varrho}{n}\right)\Big)$$

At iam supra ostendimus esse:

$$\text{cos.}\tfrac{\varrho}{n} + \text{cos.}\tfrac{2\varrho}{n} + \text{cos.}\tfrac{3\varrho}{n} + \ldots + \text{cos.}\tfrac{(n-1)\varrho}{n} = -\tfrac{1}{2} + \frac{\text{cos.}\left(\varrho - \frac{\varrho}{n}\right) - \text{cos.}\,\varrho}{2\left(1 - \text{cos.}\frac{\varrho}{n}\right)} = -\tfrac{1}{2} + \frac{\text{sin.}\frac{\varrho}{n}}{2\left(1 - \text{cos.}\frac{\varrho}{n}\right)}$$

ob ϱ angulum rectum. Simili modo reperietur fore:

$$\text{sin.}\tfrac{\varrho}{n} + \text{sin.}\tfrac{2\varrho}{n} + \text{sin.}\tfrac{3\varrho}{n} + \ldots + \text{sin.}\tfrac{(n-1)\varrho}{n} = \frac{\text{sin.}\frac{\varrho}{n} + \text{sin.}\frac{(n-1)}{n}\varrho - \text{sin.}\,\varrho}{2\left(1 - \text{cos.}\frac{\varrho}{n}\right)}$$

: differentietur vtrinque, considerata ϱ tamquam variabili: erit

$$\tfrac{1}{n}\left(\text{cos.}\tfrac{\varrho}{n} + 2\,\text{cos.}\tfrac{2\varrho}{n} + 3\,\text{cos.}\tfrac{3\varrho}{n} + \ldots + (n-1)\,\text{cos.}\tfrac{(n-1)\varrho}{n}\right) = \frac{\text{cos.}\frac{\varrho}{n} + (n-1)\,\text{cos.}\frac{(n-1)\varrho}{n} - n\,\text{cos.}\,\varrho}{2n\left(1 - \text{cos.}\frac{\varrho}{n}\right)} - \frac{\text{sin.}\frac{\varrho}{n}}{2n\left(1 - \text{cos.}\frac{\varrho}{n}\right)^2}\left(\text{sin.}\tfrac{\varrho}{n} + \text{sin.}\tfrac{(n-1)\varrho}{n} - \text{sin.}\,\varrho\right)$$

Cum iam sit ϱ angulus rectus erit:

$$\text{sin.}\tfrac{\varrho}{n} + \text{sin.}\tfrac{2\varrho}{n} + \ldots\,\text{sin.}\tfrac{(n-1)\varrho}{n} = -\tfrac{1}{2} + \frac{\text{sin}\,\frac{\varrho}{n}}{2\left(1 - \text{cos.}\frac{\varrho}{n}\right)}\ \text{etc.}$$

$$\text{cos.}\tfrac{\varrho}{n} + 2\,\text{cos.}\tfrac{2\varrho}{n} + 3\,\text{cos.}\tfrac{3\varrho}{n} + \ldots + (n-1)\,\text{cos.}\tfrac{(n-1)\varrho}{n} = \frac{\text{cos.}\frac{\varrho}{n} + n\,\text{sin.}\frac{\varrho}{n} - \text{sin.}\frac{\varrho}{n}}{2\left(1 - \text{cos.}\frac{\varrho}{n}\right)} - \frac{\text{sin.}\frac{\varrho}{n}}{2\left(1 - \text{cos.}\frac{\varrho}{n}\right)^2}\left(\text{sin.}\tfrac{\varrho}{n} + \text{cos.}\right.$$

$\frac{\varrho}{n}$ —

$$\frac{\varrho}{n} - 1) = \frac{\operatorname{cof.}\frac{\varrho}{n} + n \operatorname{fin.}\frac{\varrho}{n} - \operatorname{fin.}\frac{\varrho}{n} - 1 - \operatorname{cof.}\frac{\varrho}{n} + \operatorname{fin.}\frac{\varrho}{n}}{2(1 - \operatorname{cof.}\frac{\varrho}{n})}$$

$$= \frac{-1 + n \operatorname{fin.}\frac{\varrho}{n}}{2(1 - \operatorname{cof.}\frac{\varrho}{n})}$$

His valoribus fubftitutis prouenict:

$$r = \frac{c}{2n}\left(\pi + 2 - 2 + \frac{2 \operatorname{fin.}\frac{\varrho}{n}}{1 - \operatorname{cof.}\frac{\varrho}{n}} - \pi + \frac{\pi \operatorname{fin.}\frac{\varrho}{n}}{1 - \operatorname{cof.}\frac{\varrho}{n}} + \frac{\pi}{n(1 - \operatorname{cof.}\frac{\varrho}{n})} - \frac{\pi \operatorname{fin.}\frac{\varrho}{n}}{1 - \operatorname{cof.}\frac{\varrho}{n}}\right)$$ feu $$r = \frac{c}{2n}\left(\frac{2 \operatorname{fin.}\frac{\varrho}{n}}{1 - \operatorname{cof.}\frac{\varrho}{n}} + \frac{\pi}{n(1 - \operatorname{cof.}\frac{\varrho}{n})}\right).$$ In qua formula, quo maior accipitur numerus n, eo propius valor ipfius r inuenietur. Q. E. I.

Coroll. 1.

52. Inuento radio r erit arcus cycloidalis AMB $=$ $r\varrho = \frac{\pi r}{2}$: ideoque habebitur arcus

$$AMB = \frac{\pi c}{4n} \cdot \frac{2n \operatorname{fin.}\frac{\varrho}{n} + \pi}{n(1 - \operatorname{cof.}\frac{\varrho}{n})} = \frac{\pi c(2n \operatorname{fin.}\frac{\varrho}{n} + \pi)}{4nn(1 - \operatorname{cof.}\frac{\varrho}{n})}$$

qui valor erit exactus, fi n ftatuatur numerus infinitus. Hoc autem cafu erit $2n \operatorname{fin.}\frac{\varrho}{n} = 2\varrho = \pi$ et $1 - \operatorname{cof.}\frac{\varrho}{1}$ $= \frac{\varrho\varrho}{2nn} = \frac{\pi\pi}{8nn}$: vnde fit $AMB = 4c = 2BC$, vti ex natura cycloidis conftat.

Coroll. 2.

53. Cum igitur fit arcus cycloidis AMB reuera $=$ $4c$, habebitur fequens aequatio eo propius vera, quo maior fuerit numerus n

$$4 = \frac{\pi(\pi + 2n\,\text{fin.}\,\frac{\varrho}{n})}{4nn(1 - \text{cof.}\,\frac{\varrho}{n})}$$

seu $\pi\pi + 2n\pi\,\text{fin.}\,\frac{\varrho}{n} = 16nn(1 - \text{cof.}\,\frac{\varrho}{n})$ vnde fit
$\pi = -n\,\text{fin.}\,\frac{\varrho}{n} + n\sqrt{(16 - 16\,\text{cof.}\,\frac{\varrho}{n} + \text{fin.}\,\frac{\varrho}{n}^2)}$ seu
$\pi = -n\,\text{fin.}\,\frac{\varrho}{n} + n\sqrt{(1 - \text{cof.}\,\frac{\varrho}{n})(17 + \text{cof.}\,\frac{\varrho}{n})}$

Coroll. 3.

54. Cum sit $1 - \text{cof.}\,\frac{\varrho}{n} = 2\,\text{fin.}\,\frac{\varrho}{2n}^2$ et $\text{fin.}\,\frac{\varrho}{n} = 2\,\text{fin.}\,\frac{\varrho}{2n}\,\text{cof.}\,\frac{\varrho}{2n}$
erit $\pi = -2n\,\text{fin.}\,\frac{\varrho}{2n}\,\text{cof.}\,\frac{\varrho}{2n} + 2n\,\text{fin.}\,\frac{\varrho}{2n}\sqrt{(9 - \text{fin.}\,\frac{\varrho}{2n}^2)}$
fit nunc $2n = m$ et $\frac{\varrho}{2n} = \frac{\varrho}{m} = \Phi$, seu $\Phi = \frac{90^\circ}{m}$
erit $\pi = -m\,\text{fin.}\,\Phi\,\text{cof.}\,\Phi + m\,\text{fin.}\,\Phi\sqrt{(9 - \text{fin.}\,\Phi^2)}$

Coroll. 4.

55. Si ergo circulus describatur radio $= 1$, erit in eo quadrans $= \varrho = \frac{1}{2}\pi = m\Phi$; denotante Φ partem quamcunque, vnde ipse arcus Φ sequenti modo proxime definietur

$$\Phi = -\frac{\text{fin.}\,\Phi\,\text{cof.}\,\Phi + \text{fin.}\,\Phi\sqrt{(9 - \text{fin}\,\Phi^2)}}{2}$$

Tab. II. fig. 3. seu $\Phi = \text{fin.}\,\Phi(\sqrt{(2 + \frac{1}{4}\,\text{cof.}^2)} - \frac{1}{2}\,\text{cof.}\,\Phi)$

Proposito ergo in quadrante ACB arcu AM, cuius sinus PM, capiatur CD = chordae quadrantis AB, et bisecta CP in O iungatur DO, vnde resecetur OI = CO et in radio CM producto capiatur CL = DI, demissumque in AC perpendiculum LQ eo propius aequabitur arcui AM, quo minor fuerit iste arcus.