

CONSIDERATIO
QVARVMDAM SERIERVM,
QVAE SINGVLARIBVS PROPRIETATIBVS
SVNT PRAEDITAE.
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§. 1.

Saepe numero *consideratio serierum*, quae quasi causa se nobis offerunt, non contemnenda suppeditare solet artificia, quibus deinceps in vniuersa serierum doctrina summo cum fructu vti licet. Cum igitur doctrina *de seriebus* sit maximi momenti in *Analysis*, huiusmodi speculationes omnino dignae sunt habendae, quae omni industria euoluantur. Hunc in finem sequentem seriem offerre constitui, quae, tum ob singulares, quibus praedita deprehenditur proprietate, tum vero propter insignes usus, quos nobis exhibet, omni attentione digna videtur. Series autem ita se habet:

$$\frac{1-x}{1-a} + \frac{(1-x)(a-x)}{a-a^2} + \frac{(1-x)(a-x)(a^2-x)}{a^2-a^3} + \frac{(1-x)(a-x)(a^2-x)(a^3-x)}{a^3-a^4} + \text{etc.}$$

Lex numerorum ex sola inspectione est manifesta, formantur enim ex multiplicatione terminorum huius seriei: $1-x$; $a-x$; a^2-x ; a^3-x ; a^4-x ; a^5-x ; a^6-x ; etc.

Denominatores omnes duobus constant terminis, qui sunt potestates ipsius a , quarum exponentes sunt numeri triangulares. Hinc terminus ordine n seriei propositae erit:

$$\frac{(1-x)(a-x)(a^2-x)(a^3-x) \dots (a^{n-1}-x)}{a^{n(n-1)/2} - a^{n(n+1)/2}}$$

§. 2.

§. 2. Primo quidem patet, si quantitas x potestati cuiquam ipsius a aequalis capiatur, tum seriem alicubi ita obrumpi, ut omnes sequentes termini abeant in nihilum. Ponamus ergo in genere s pro summa seriei propositae, vt sit :

$$s = \frac{1-x}{1-a} + \frac{(1-x)(a-x)}{a-a^2} + \frac{(1-x)(a-x)(a^2-x)}{a^2-a^3} + \frac{(1-x)(a-x)(a^2-x)(a^3-x)}{a^3-a^4} + \text{etc.}$$

ac statnatur primo $x=1$, seu $x=a^0$, eritque ob omnes terminos evanescentes $s=0$. Sit porro $x=a$, vt solus primus terminus superfit, eritque $s=1$. Sit $x=a^2$, fietque $s = \frac{1-a^2}{1-a} + \frac{(1-a^2)(a-a^2)}{a-a^3}$ seu $s=2$. Ponatur $x=a^3$, ac prodibit :

$$s = \frac{1-a^3}{1-a} + \frac{(1-a^3)(a-a^3)}{a-a^4} + \frac{(1-a^3)(a-a^3)(a^2-a^3)}{a^2-a^5}.$$

Horum terminorum primus dat $1+a+aa$; secundus dat $1-a^3$, et tertius dat $x-a-aa+a^3$; quibus collectis fiet $s=3$.

§. 3. Simili modo si ponatur $x=a^4$, operatione instituta reperietur $s=4$; et posito $x=a^5$, prodibit $s=5$. Vnde satis tuto per inductionem concludi posse videtur, quoties x cuicunque potestati ipsius a , cuius exponens sit $= n$, aequatio statuatur, toties hunc ipsum exponentem praebiturum esse valorem ipsius s . At vero haec inducione tantum valet, si n sit numerus integer affirmatiuus. Quod si enim pro quoquis numero fracto valeret, tum foret s logarithmo ipsius x , sumto a pro numero, cuius logarithmus sit $= 1$. Sic si hoc verum esset, posito $a=10$, summa seriei s semper exprimere deberet logarithmum communem ipsius x , essetque :

$$s = -\frac{(1-x)}{9} - \frac{(1-x)(10-x)}{990} - \frac{(1-x)(10-x)(100-x)}{999000} - \frac{(1-x)(10-x)(100-x)(1000-x)}{9999000000} - \text{etc.} = l x.$$

Ex

Ex sequentibus autem perspicuum evadet, hanc aequalitatem non subsistere, nisi sit x potestas ipsius a , exponentem habens integrum affirmatiuum.

§. 4. Quod autem, posito $x = a^n$, non semper sit $s = n$, nisi n sit numerus integer affirmatius, ex casu quo $x = 0$ facile colligitur. Hoc enim casu, si superior inductio se ad omnes omnino numeros extenderet, fieri deberet $s = -\infty$, cum $-\infty$ sit perpetuo logarithmus cyphrae. Verum posito $x = 0$, fiet:

$$s = \frac{1}{1-a} + \frac{1}{1-a^2} + \frac{1}{1-a^3} + \frac{1}{1-a^4} + \frac{1}{1-a^5} + \dots \text{etc.}$$

quae series et si summari non potest, tamen quilibet facile perspiciet, eius summam esse debere finitam, neque propterea logarithmum ipsius $x = 0$ exprimere posse. Simili modo, si posito $a = 10$, atque x non potestati ipsius 10 aequale ponatur, per summationem valor inuenietur, plerumque satis notabiliter a l x discrepans. Sit enim $x = 9$, posito $a = 10$, eritque:

$$s = \frac{8}{9} + \frac{8 \cdot 1}{99} + \frac{8 \cdot 10 \cdot 91}{999} + \frac{8 \cdot 1 \cdot 91 \cdot 991}{9999} + \frac{8 \cdot 1 \cdot 91 \cdot 991 \cdot 9991}{99999} \text{etc.}$$

qui termini si in fractionibus decimalibus exprimantur, prodibit:

$$\begin{aligned} s &= 0,888888888888888888 \\ &\quad 8080808080808080 \\ &\quad \dots 8008008008008008 \\ &\quad \dots 80008000800080 \\ &\quad \dots 800008000080 \\ &\quad \dots 8000008000 \\ &\quad \dots 80000008 \\ &\quad \dots 800000 \\ &\quad \dots 8000 \\ &\quad \dots 80 \\ \hline s &= 0,89705058521067321224 \end{aligned}$$

qui

qui valor vtique minor est, quam logarithmus novarii.

§. 5. Series igitur nostra ita est comparata, ut si pro x substituantur potestates ipsius a rationales, summa seriei aequalis fiat exponenti illius potestatis: scilicet si fit $x = a^0, a^1, a^2, a^3, a^4, a^5, a^6, a^7, a^8$, etc. erit $s = 0, 1, 2, 3, 4, 5, 6, 7, 8$, etc.

quae etsi est proprietas logarithmorum, tamen non nisi exponentes ipsius a sint numeri integri. Quod si ergo concipiatur linea curua, cuius abscissae sint s , et applicatae $= x$, haec curua logarithmicam in punctis innumeris interfecabit, scilicet quoties abscissa s per numerum integrum exprimitur, toties applicata per intersectionem transibit. Vnde patet, curuam logarithmicam ne per infinita quidem puncta determinari; quod etiam in omnibus aliis lineis curuis vsu venit. Hinc itaque intelligitur, quam libet seriem, etsi omnes eius termini indicibus integris respondentes dentur, infinitis modis diuersis interpolari posse, quod argumentum alia occasione uberiori pertractabo.

§. 6. Quo autem proprius ad cognitionem nostrae seriei perueniamus, eam in hanc formam transmutare licet:

$$s = \frac{1}{1-a}(1-x) + \frac{x}{1-a^2}(1-x)(1-\frac{x}{a}) + \frac{x}{1-a^3}(1-x)(1-\frac{x}{a})(1-\frac{x}{a^2}) + \frac{x}{1-a^4}(1-x)(1-\frac{x}{a})(1-\frac{x}{a^2})(1-\frac{x}{a^3}) \text{etc.}$$

quae propterea simplicior est praecedente, quod hic numeri trigonales abierint. Ponamus nunc $a.x$ in locum ipsius x ; denotetque t summam seriei hinc resultantis, erit:

$$t = \frac{1}{1-a}(1-ax) + \frac{x}{1-a^2}(1-ax)(1-x) + \frac{x}{1-a^3}(1-ax)(1-x)(1-\frac{x}{a}) + \frac{x}{1-a^4}(1-ax)(1-x)(1-\frac{x}{a})(1-\frac{x}{a^2}) + \text{etc.}$$

Subtrahatur prior series a posteriore, ac reperietur:

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$t - s$

$t-s = x + \frac{x}{a}(1-x) + \frac{x}{a^2}(1-x)(1-\frac{x}{a}) + \frac{x}{a^3}(1-x)(1-\frac{x}{a})(1-\frac{x}{a^2}) + \text{etc.}$
 subtrahatur haec series ab unitate, et cum residuum sit
 per $1-x$ diuisibile erit:

$$1+s-t = (1-x)(1-\frac{x}{a}-\frac{x}{a^2}(1-\frac{x}{a})-\frac{x}{a^3}(1-\frac{x}{a})(1-\frac{x}{a^2})) + \text{etc.}$$

Hic factor posterior autem porro diuisibilis est per $1-\frac{x}{a}$, vnde fit $1+s-t = (1-x)(1-\frac{x}{a})(1-\frac{x}{a^2}-\frac{x}{a^3}(\dots-\frac{x}{a^2})) + \text{etc.}$

Hic denuo factor deprehenditur $1-\frac{x}{a^2}$, hocque seorsim expresso, factor apparebit $1-\frac{x}{a^3}$, et ita porro, vnde tandem reperitur fore:

$$1+s-t = (1-x)(1-\frac{x}{a})(1-\frac{x}{a^2})(1-\frac{x}{a^3})(1-\frac{x}{a^4})(1-\frac{x}{a^5}) + \text{etc.}$$

§. 7. Hinc igitur patet, quoties x aequalis capiatur cuiquam potestati ipsius a , ob unum factorem huius expressionis evanescentem fore $1+s-t=0$, seu $t=s+1$. Quare si posito $x=a^n$, denotante n numerum integrum affirmatiuum, fuerit summa seriei propositae $s=n$, posito $x=a^{n+1}$, erit summa seriei $t=s+1=n+1$. Cum igitur sumato $n=0$, seu $x=1$, sit summa seriei $s=0$, erit, posito $x=a'$, summa seriei $s=1$: hincque porro sequitur, si ponatur $x=a^2$, fore $s=2$, et si $x=a^3$, fore $s=3$. Atque in genere nunc patet, quod ante per solam inductionem eliciimus, si fiat $x=a^n$, denotante n numerum integrum affirmatiuum, fore perpetuo $s=n$. Si autem n non sit numerus integer affirmatius, atque s designet summam seriei initio propositae, facto $x=a^n$, tum posito $x=a^{n+1}$, summa seriei, quae sit $=t$ non erit $=s+1$, sicut enim:

$$t = 1+s-(1-a^n)(1-a^{n+1})(1-a^{n+2})(1-a^{n+3})(1-a^{n+4}) + \text{etc.}$$

Hic

His ergo casibus valor seriei manifeste recedit a natura logarithmorum.

§. 8. Quemadmodum hic valores ipsius x per α multiplicando ex valore ipsius s eliciimus valorem ipsius t , ita vicissim valores ipsius x per α diuidendo ex valore ipsius t obtinebimus valorem ipsius s ; hincque ad valores negatiuos exponentis n descendere poterimus. Scilicet in serie initio proposita, vel ad hanc formam perducta:

$$s = \frac{1}{1-\alpha} (1-x) + \frac{1}{1-\alpha^2} (1-x)(1-\frac{x}{\alpha}) + \frac{1}{1-\alpha^3} (1-x)(1-\frac{x}{\alpha})(1-\frac{x}{\alpha^2}) + \text{etc.}$$

pro sequentibus casibus summam seriei ita indicemus:

$$\text{si } x = 1 \quad \text{fit } s = A = 0$$

$$x = \frac{1}{\alpha} \quad - \quad - \quad - \quad s = B$$

$$x = \frac{1}{\alpha^2} \quad - \quad - \quad - \quad s = C$$

$$\vdots \quad x = \frac{1}{\alpha^3} \quad - \quad - \quad - \quad s = D$$

$$\ast \quad x = \frac{1}{\alpha^4} \quad - \quad - \quad - \quad s = E$$

etc.

Quod si iam ponatur $x = \frac{1}{\alpha}$; fiet $s = B$, et $t = A = 0$, quia t oritur ex s , si loco x scribatur αx : ex praecedentibus oritur:

$$1 + B = (1 - \frac{1}{\alpha})(1 - \frac{1}{\alpha^2})(1 - \frac{1}{\alpha^3})(1 - \frac{1}{\alpha^4})(1 - \frac{1}{\alpha^5}) \text{ etc.}$$

seu $B = -1 + (1 - \frac{1}{\alpha})(1 - \frac{1}{\alpha^2})(1 - \frac{1}{\alpha^3})(1 - \frac{1}{\alpha^4})(1 - \frac{1}{\alpha^5}) \text{ etc.}$

sic si $\alpha = 10$, fiet $B = -1, 109989900000998$.

§. 9. Sit $x = \frac{1}{\alpha^2}$, eritque $s = C$, et $t = B$; vnde habebitur;

$$1 + C - B = (1 - \frac{1}{\alpha^2})(1 - \frac{1}{\alpha^3})(1 - \frac{1}{\alpha^4})(1 - \frac{1}{\alpha^5}) \text{ etc.}$$

ad hanc addatur prior $1 + B$, eritque:

$$2 + C = (2 - \frac{1}{\alpha})(1 - \frac{1}{\alpha^2})(1 - \frac{1}{\alpha^3})(1 - \frac{1}{\alpha^4})(1 - \frac{1}{\alpha^5}) \text{ etc.}$$

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et $C = -2 + (2 - \frac{1}{a})(x - \frac{1}{a^2})(x - \frac{1}{a^3})(x - \frac{1}{a^4})(x - \frac{1}{a^5})$ etc.
Vel ipsa serie eliminata erit :

$$x + B = (x - \frac{1}{a})(x + C - B), \text{ seu } C - 2B = \frac{1}{a}(x + C - B).$$

Simili modo si ponatur $x = \frac{1}{a^6}$. erit $s = D$, et $t = C$,
vnde fiet :

$$x + D - C = (x - \frac{1}{a^4})(x - \frac{1}{a^5})(x - \frac{1}{a^6})(x - \frac{1}{a^7}) \text{ etc.}$$

ad quam prior series addita praebebit :

$$3 + D = (3 - \frac{1}{a} - \frac{1}{a^2} + \frac{1}{a^3})(x - \frac{1}{a^5})(x - \frac{1}{a^6})(x - \frac{1}{a^7}) \text{ etc.}$$

Ac posito $x = \frac{1}{a^6}$ cum fiat :

$$x + E - D = (x - \frac{1}{a^4})(x - \frac{1}{a^5})(x - \frac{1}{a^6}) \text{ etc. erit}$$

$$4 + E = (4 - \frac{1}{a} - \frac{1}{a^2} - \frac{1}{a^3} + \frac{1}{a^4} + \frac{1}{a^5} - \frac{1}{a^6})(x - \frac{1}{a^4})(x - \frac{1}{a^5})(x - \frac{1}{a^6}) \text{ etc.}$$

ficque quoisque libuerit, vltierius progreedi licet.

§. 10. Potest autem inter ternos valores summae
seriei s , pro ternis valoribus ipsius x successivis, relatio per
expressionem finitam exhiberi. Manente enim pro valo-
re x summa $= s$, sit si loco x ponatur αx , summa sé-
riei $= t$, et si loco x ponatur $\alpha \alpha x$, sit summa seriei
 $= u$. Cum igitur inter t et s hanc inuenierimus re-
lationem :

$x + s - t = (x - x)(x - \frac{x}{a})(x - \frac{x}{a^2})(x - \frac{x}{a^3})(x - \frac{x}{a^4})$ etc.
si hic pro x scribamus αx , prodibit relatio similis inter
 u et t :

$$x + t - u = (x - \alpha x)(x - \frac{x}{a})(x - \frac{x}{a^2})(x - \frac{x}{a^3})(x - \frac{x}{a^4}) \text{ etc.}$$

Hinc ergo erit $x + t - u = (x - \alpha x)(x + s - t)$ sive

$$u = 2t - s + \alpha x (x + s - t)$$

$$\text{vel } s = \frac{2t - u + \alpha x (x - t)}{x - \alpha x}$$

Atque hinc pro supra assumtis valoribus A, B, C, D, etc. sequentes prohibunt relationes.

$$\text{Si } x = \frac{1}{a^2}; \text{ erit } A = 2B - C + \frac{1}{a}(x + C - B)$$

$$\text{seu } C = \frac{1+(2a-1)E-A}{a-1} = B + \frac{1+a(B-A)}{a-1}$$

$$\text{si } x = \frac{1}{a^3}; \text{ erit } D = C + \frac{1+a^2(C-B)}{a^2-1}$$

$$\text{si } x = \frac{1}{a^4}; \text{ erit } E = D + \frac{1+a^3(D-C)}{a^3-1}$$

$$\text{si } x = \frac{1}{a^5}; \text{ erit } F = E + \frac{1+a^4(E-D)}{a^4-1}$$

etc.

Hae relationes autem sequenti modo commodius exprimi possunt :

$$C = 2B - A + \frac{1+B-A}{a-1}$$

$$D = 2C - B + \frac{1+C-B}{a^2-1}$$

$$E = 2D - C + \frac{1+D-C}{a^3-1}$$

$$F = 2E - D + \frac{1+E-D}{a^4-1}$$

etc.

Cum ergo sit $A = 0$, si solius litterae B valor fuerit repertus :

$$B = -\frac{1}{a} + \left(\frac{1}{a}\right)\left(\frac{1}{a^2}\right)\left(\frac{1}{a^3}\right)\left(\frac{1}{a^4}\right) \text{ etc.}$$

hinc omnium sequentium litterarum C, D, E, F, etc. valores exacto poterunt assignari.

§. 11. Cum autem denotante n numerum integrum affirmatiuum, si ponatur $x = a^n$, sit $s = n$, ex nostra assumpta serie consequemur hanc summabilem.

$$n = \frac{1-a^n}{1-a} + \frac{(1-a^n)(1-a^{n-1})}{1-a^2} + \frac{(1-a^n)(1-a^{n-1})(1-a^{n-2})}{1-a^3} + \text{etc.}$$

Tum vero hoc casu, quia est $t = n + 1$, erit :

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$x = a^n + a^{n-1}(1-a^n) + a^{n-2}(1-a^n)(1-a^{n-1}) + a^{n-3}(1-a^n)(1-a^{n-1})(1-a^{n-2})$ etc.
cuius veritas omnibus terminis ad eandem partem coniectis
est manifesta, fiet enim :

$$(1-a^n)(1-a^{n-1})(1-a^{n-2})(1-a^{n-3})(1-a^{n-4}) \text{ etc.} = 0.$$

Hinc ansam nanciscimur generalius huiusmodi formas con-
templandi. Sit enim A, B, C, D, E, F, etc. series
quantitatum quarumuis, sitque :

$$(1-A)(1-B)(1-C)(1-D)(1-E) \text{ etc.} = S.$$

Atque hinc obtinebitur :

$1-A-B(1-A)-C(1-A)(1-B)-D(1-A)(1-B)(1-C)$ -etc. $= S$;
haec enim formula facilime reducitur ad illam. Hanc
ob rem habebimus :

$$A+B(1-A)+C(1-A)(1-B)+D(1-A)(1-B)(1-C)+\text{etc.} = S+1.$$

§. 12. Quod si ergo quaepiam harum quantitatum
A, B, C, etc. vnitati fiat aequalis, erit $S = 0$, prod-
ibitque series, cuius summa $= 1$. Sumatur verbi gratia
haec series :

$$A \quad B \quad C \quad D \quad E \quad F$$

$$\frac{1}{2}; \frac{2}{3}; \frac{3}{4}; \frac{4}{5}; \frac{5}{6}; \frac{6}{7}; \text{ etc.}$$

quarum fractionum cum infinitissima sit $= 1$, erit $S = 0$,
et sequens nascetur series :

$$1 = \frac{1}{2} + \frac{2}{2 \cdot 3} + \frac{3}{2 \cdot 3 \cdot 4} + \frac{4}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{ etc.}$$

cuius quidem veritas facile perspicitur, oritur enim ea
hoc modo :

$$\text{sit } z = 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \text{ etc.}$$

$$\text{erit } z - 1 = \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{ etc. hincq. per subtr. prodit}$$

$$1 =$$

$$1 = \frac{1}{2} + \frac{2}{2 \cdot 3} + \frac{3}{2 \cdot 3 \cdot 4} + \frac{4}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.}$$

§. 13. Sit $A = \frac{1}{2}$; $B = \frac{1}{2 \cdot 3}$; $C = \frac{1}{2 \cdot 3 \cdot 4}$; $D = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}$; etc.

$$\text{erit } S = \frac{1}{2} \cdot \frac{2}{2 \cdot 3} \cdot \frac{3}{2 \cdot 3 \cdot 4} \cdot \frac{4}{2 \cdot 3 \cdot 4 \cdot 5} \cdot \text{etc.} = \frac{\pi}{4}$$

denotante π peripheriam circuli, cuius diameter est $= 1$.

Hinc ergo oriatur haec series pro quadratura circuli.

$$\frac{\pi}{4} + 1 = \frac{1}{2} + \frac{8}{9 \cdot 25} + \frac{8 \cdot 24}{9 \cdot 25 \cdot 49} + \frac{8 \cdot 24 \cdot 48}{9 \cdot 25 \cdot 49 \cdot 81} + \text{etc.}$$

$$\text{seu } \frac{\pi}{4} + 8 = \frac{2 \cdot 4}{5 \cdot 5} + \frac{2 \cdot 4 \cdot 1 \cdot 6}{5 \cdot 5 \cdot 7 \cdot 7} + \frac{2 \cdot 4 \cdot 1 \cdot 6 \cdot 6 \cdot 8}{5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9} + \text{etc.}$$

Cum ergo huiusmodi producta, quorum valor S exhiberi potest, innumerabilia habeantur: ex quolibet hoc modo series infinita, cuius summa assignari queat, deriuabitur. Amplissimus ergo hinc aperitur campus, series summabiles, quotquot libuerit, inueniendi.

§. 14. Reuertor autem ad seriem initio assumtam

$$s = \frac{1}{1-x} (1-x) + \frac{1}{1-x^2} (1-x)(1-\frac{x}{a}) + \frac{1}{1-a^2} (1-x)(1-\frac{x}{a})(1-\frac{x}{a^2}) + \text{etc.}$$

quam in aliam formam, in qua termini secundum potestates ipsius x procedant, transfundere animus est. Hoc primo quidem per euolutionem singulorum terminorum fieri posset, at quia hoc pacto prodituri essent singuli coefficientes in seriebus infinitis, commodissime in hunc finem adhibebitur formula supra inuenta $u = 2t - s + ax(1-t+s)$, seu $u - 2t + s = ax + ax(s-t)$, vbi ex s nascitur t , si loco x ponatur $a x$, parique modo ex t fit u , si loco x denuo ponatur $a x$. Quare si pro serie quaesta assumamus

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc. erit:}$$

$$s = A + Bax + Ca^2x^2 + Da^3x^3 + Ea^4x^4 + Fa^5x^5 + \text{etc. et}$$

$$u = A + Ba^2x + Ca^3x^2 + Da^4x^3 + Ea^5x^4 + Fa^6x^5 + \text{etc.}$$

Ex

Ex his ergo conficietur:

$$u - 2t + s = B(1-a)^2 x + C(1-aa)^2 x^2 + D(1-a^3)^2 x^3 + E(1-a^4)^2 x^4 + \text{etc.}$$

$$ax(1+s-t) = ax + Ba(1-a)x^2 + Ca(1-aa)x^3 + Da(1-a^3)x^4 + \text{etc.}$$

Ex quarum serierum aequalitate concluditur fore:

$$B = \frac{a}{(1-a)^2}; C = \frac{Ba(1-a)}{(1-aa)^2}; D = \frac{Ca(1-aa)}{(1-a^3)^2}; E = \frac{Da(1-a^3)}{(1-a^4)^2}; \text{ etc.}$$

§. 15. Hinc ergo sequentes coefficientium assumptum valores obtinebuntur:

$$B = \frac{a}{(1-a)^2}$$

$$C = \frac{a^2}{(1-a)(1-aa)}$$

$$D = \frac{a^3}{(1-a)(1-aa)(1-a^3)^2}$$

$$E = \frac{a^4}{(1-a)(1-aa)(1-a^3)(1-a^4)^2}$$

$$F = \frac{a^5}{(1-a)(1-aa)(1-a^3)(1-a^4)(1-a^5)^2}$$

etc.

Primus autem terminus A hinc non definitur. At quia A praebet valorum ipsius s, si ponatur $x = 0$, perspicuum est fore:

$$A = \frac{1}{1-a} + \frac{1}{1-a^2} + \frac{1}{1-a^3} + \frac{1}{1-a^4} + \frac{1}{1-a^5} + \text{etc.}$$

His ergo valoribus definitis, series initio proposita:

$$s = \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)(1-\frac{x}{a}) + \frac{1}{1-a^3}(1-x)(1-\frac{x}{a})(1-\frac{x}{aa}) + \text{etc.}$$

transmutabitur in hanc formam:

$$s = \left\{ \frac{1}{1-a} + \frac{1}{1-a^2} + \frac{1}{1-a^3} + \frac{1}{1-a^4} + \frac{1}{1-a^5} + \text{etc.} \right. \\ \left. + \frac{ax}{(1-a)^2} + \frac{a^2x^2}{(1-a)(1-aa)^2} + \frac{a^3x^3}{(1-a)(1-aa)(1-a^3)^2} + \frac{a^4x^4}{(1-a)(1-aa)(1-a^3)(1-a^4)^2} + \text{etc.} \right\}$$

§. 16. Cum igitur posito $x = a^n$, denotante n numerum integrum affirmatum, fiat $s = n$, habebitur haec summatio:

$n =$

$$n + \frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \frac{1}{a^5-1} + \text{etc.} = \\ \frac{a^{n+1}}{(a-1)^2} - \frac{a^{2n+2}}{(a-1)(a^2-1)^2} + \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)^2} - \frac{a^{4n+4}}{(a-1)(a^2-1)(a^3-1)(a^4-1)^2} + \text{etc.}$$

Quod si ergo fuerit $n = 0$, erit :

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \text{etc.} = \frac{a^2}{(a-1)^2} - \frac{a^3}{(a-1)(a^2-1)^2} + \frac{a^4}{(a-1)(a^2-1)(a^3-1)^2} + \text{etc.}$$

ac, si ponatur $n = 1$, erit :

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \text{etc.} = \frac{a^2}{(a-1)^2} - \frac{a^4}{(a-1)(a^2-1)^2} + \frac{a^6}{(a-1)(a^2-1)(a^3-1)^2} + \text{etc.} - \ddots$$

Generaliter ergo erit :

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \text{etc.} = \frac{a^{n+1}}{(a-1)^2} - \frac{a^{2n+2}}{(a-1)(a^2-1)^2} + \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)^2} - \text{etc.} - \ddots$$

denotante n numerum integrum quemcumque affirmati-
vum.

§. 17. Si loco n ponatur $n = 1$, habebitur :

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \text{etc.} = \frac{a^n}{(a-1)^2} - \frac{a^{2n}}{(a-1)(a^2-1)^2} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)^2} - \ddots$$

a qua, si series superior auferatur, proueniet :

$$x = \frac{a^n}{a-1} - \frac{a^{2n}}{(a-1)(a^2-1)} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)} - \frac{a^{4n}}{(a-1)(a^2-1)(a^3-1)(a^4-1)} + \text{etc.}$$

Huius ergo seriei summa semper aequalis est vnitati, qui-
cumque valor ipfi a tribuatur, et quicunque numerus integer
affirmatiuus pro n substituatur. Casu autem quo $n = 1$
haec summatio facile perspicitur. Quod enim sit :

$$x = \frac{a}{a-1} - \frac{a^2}{(a-1)(a^2-1)} + \frac{a^3}{(a-1)(a^2-1)(a^3-1)} - \text{etc.}$$

sequitur luculenter ex consideratione huius seriei :

$$z = 1 - \frac{1}{a-1} + \frac{1}{(a-1)(a^2-1)} - \frac{1}{(a-1)(a^2-1)(a^3-1)} + \text{etc. Vnde fit :}$$

$$x-z = \frac{1}{a-1} - \frac{1}{(a-1)(a^2-1)} + \frac{1}{(a-1)(a^2-1)(a^3-1)} - \frac{1}{(a-1)(a^2-1)(a^3-1)(a^4-1)} + \text{etc.}$$

quae inuicem additae dabunt :

Tom. III. Nov. Comment.

N

17

$$x = \frac{a}{a-1} - \frac{aa}{(a-1)(a^2-1)} + \frac{a^3}{(a-1)(a^2-1)(a^3-1)} - \frac{a^4}{(a-1)(a^2-1)(a^3-1)(a^4-1)} + \text{etc.}$$

§ 18. Deinde autem veritas istius seriei pro reliquis ipsis n valoribus sequentem in modum ostendi potest. Si fuerit :

$$x = \frac{a^n}{a-1} - \frac{a^{2n}}{(a-1)(a^2-1)} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.}$$

dico fore quoque :

$$x = \frac{a^{n+1}}{a-1} - \frac{a^{2n+2}}{(a-1)(a^2-1)} + \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.}$$

Nam cum sit per hypothesin :

$$x = \frac{a^n}{a-1} - \frac{a^{2n}}{(a-1)(a^2-1)} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)} - \text{etc. erit quoque}$$

$$0 = a^n - \frac{a^{2n}}{a-1} + \frac{a^{3n}}{(a-1)(a^2-1)} - \text{etc.}$$

quae series inuicem additae dabunt :

$$x = \frac{a^{n+1}}{a-1} - \frac{a^{2n+2}}{(a-1)(a^2-1)} + \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.}$$

Quare cum haec series :

$$x = \frac{a^n}{a-1} - \frac{a^{2n}}{(a-1)(a^2-1)} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.}$$

vera sit ostensio casu $n=1$, erit quoque vera casu $n=2$, hincque porro casibus $n=3$, $n=4$, etc. ita ut quicunque numerus integer affirmatiuus pro n substituatur, summa seriei perpetuo futura sit $= 1$.

§ 19. Quoniam seriem initio propositam $s = \frac{x}{1-x}$ ($1-x$) etc. secundum dimensiones ipsius x hic disposui, ope proprietatis supra demonstratae $u-2t+s=ax+ax(s-t)$; non incongruum erit eandem transmutationem immediate

ex

ex ipsa serie s derivare; sic enim ad summationem innumerabilium nouarum serierum pertingemus. Oportebit ergo singulos seriei s terminos per multiplicationem evolvvi, quod ut expeditius fieri possit, considerabo terminum

quemcumque: $\frac{1}{(1-a^m)} (1-x)(1-\frac{x}{a})(1-\frac{x}{a^2})(1-\frac{x}{a^3}) \dots (1-\frac{x}{a^{m-1}})$

Ponam ergo $P = (1-x)(1-\frac{x}{a})(1-\frac{x}{a^2})(1-\frac{x}{a^3}) \dots (1-\frac{x}{a^{n-1}})$

$$\text{eritque } P = l(1-x) + l\left(1-\frac{x}{a}\right) + l\left(1-\frac{x}{a^2}\right) + \dots + l\left(1-\frac{x}{a^{m-1}}\right)$$

et differentiando fiet :

$$\frac{dP}{P} = \frac{-dx}{x-x} - \frac{dx}{a-x} - \frac{dx}{aa-x} - \dots - \frac{dx}{a^{m-1}-x}. \quad \text{ferr}$$

$$1 + x + x^2 + x^3 + x^4 + x^5 + \text{etc. infin.}$$

$$\left| \frac{1}{a} + \frac{x}{a^2} + \frac{x^2}{a^3} + \frac{x^3}{a^4} + \frac{x^4}{a^5} + \frac{x^5}{a^6} + \dots \right. \quad \text{etc.}$$

$$\frac{1}{a^2} + \frac{x}{a^4} + \frac{x^2}{a^6} + \frac{x^3}{a^8} + \frac{x^4}{a^{10}} + \frac{x^5}{a^{12}} + \text{etc.}$$

singulas nunc series verticales summando orietur:

$$dP = -P dx \left(\frac{a^{m-1}}{a^m - a^{m-1}} + \frac{a^{2m-1}}{a^{2m} - a^{2m-1}} x + \frac{a^{3m-1}}{a^{3m} - a^{3m-1}} x^2 + \frac{a^{4m-1}}{a^{4m} - a^{4m-1}} x^3 + \text{etc.} \right)$$

§ 20. Fingatur nunc pro P haec series:

$P = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc. eritque:}$

$$\frac{dp}{dx} = \gamma + 2\alpha x + 3\beta x^2 + 4\delta x^3 + 5\epsilon x^4 + \text{etc.}$$

N^2

Fact

Facta iam substitutione fiet:

$$\beta + \frac{a^{m-1}}{a^m - a^{m-1}}, \alpha = 0$$

$$2\gamma + \frac{a^{m-1}}{a^m - a^{m-1}} \beta + \frac{a^{2m-1}}{a^{2m} - a^{2m-2}} \alpha = 0$$

$$3\delta + \frac{a^{m-1}}{a^m - a^{m-1}} \gamma + \frac{a^{2m-1}}{a^{2m} - a^{2m-2}} \beta + \frac{a^{3m-1}}{a^{3m} - a^{3m-2}} \alpha = 0$$

etc.

atque cum posito $x = 0$, fiat $P = 1$, patet esse $\alpha = 1$.

$$\text{Erit ergo } \beta = \frac{-a^{m-1}}{a^m - a^{m-1}} \text{ et } 2\gamma = \frac{(a^{m-1})^2}{(a^m - a^{m-1})^2} + \frac{a^{2m-1}}{a^{2m} - a^{2m-2}} = 0$$

$$\text{Ieu. } 2\gamma = \frac{a^{m-1}}{a^m - a^{m-1}} \left(\frac{a^{m-1}}{a^m - a^{m-1}} - \frac{a^{m-1}}{a^{m+1} - a^{m-1}} \right) = \frac{2a^m(a^{m-1}-1)(a^{m-1})}{(a^m - a^{m-1})(a^{2m} - a^{2m-2})}$$

ideoque $\gamma = \frac{(a^{m-1})(a^{m-1}-1)}{(a^m - a^{m-1})(a^{m-1})}$. Simili modo reliqui coefficientes, verum tamen non sine ingenti labore eruuntur, atque tandem satis concinne exprimi deprehendentur.

§. 2. I. Quo igitur hanc coefficientium determinatio- nem commodius expediam, methodum hic iam aliquoties usurpatam adhibeo. Scilicet in serie $P = \alpha + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4 + \text{etc.}$ loco x pono $\frac{x}{a}$, serieque resultantis summa sit $= Q$, nempe;

$$Q = \alpha + \frac{\beta x}{a} + \frac{\gamma x^2}{a^2} + \frac{\delta x^3}{a^3} + \frac{\epsilon x^4}{a^4} + \text{etc.}$$

$$\text{Cum autem sit } P = (1-x)(1-\frac{x}{a})(1-\frac{x}{a^2}) \dots (1-\frac{x}{a^{m-1}})$$

$$\text{erit } Q = (1-\frac{x}{a})(1-\frac{x}{a^2})(1-\frac{x}{a^3}) \dots (1-\frac{x}{a^m}), \text{ ideoque}$$

$$P(x -$$

QVARVMDAM SERIERN.

702

$$P(1 - \frac{x}{a^m}) = Q(1 - x) \text{, seu } a^m P - Px - a^m Q + a^m Qx = 0.$$

substituantur hic series pro P et Q assumtae, fietque

$$\begin{aligned} & \alpha a^m + \beta a^m x + \gamma a^m x^2 + \delta a^m x^3 + \text{etc.} \\ & -ax - \beta x^2 - \gamma x^3 - \text{etc.} \\ & -\alpha a^m - \beta a^{m-1} x - \gamma a^{m-2} x^2 - \delta a^{m-3} x^3 - \text{etc.} \\ & + \alpha a^m x + \beta a^{m-1} x^2 + \gamma a^{m-2} x^3 + \text{etc.} \end{aligned} \quad \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} = 0.$$

Ex comparatione terminorum homogeneorum hinc invenitur:

$$\begin{aligned} \beta &= \frac{-\alpha(a^m - 1)}{a^{m-1}(a - 1)}; \quad \delta = \frac{-\gamma(a^{m-2} - 1)}{a^{m-3}(a^2 - 1)} \\ \gamma &= \frac{-\beta(a^{m-1} - 1)}{a^{m-2}(aa - 1)}; \quad \varepsilon = \frac{-\delta(a^{m-3} - 1)}{a^{m-4}(a^4 - 1)} \\ &\text{etc.} \end{aligned}$$

§. 22. Cum igitur sit $a = x$, coefficientes ita libebunt;

$$a = x$$

$$\beta = \frac{-(a^m - 1)}{a^{m-1}(a - 1)}$$

$$\gamma = \frac{+(a^m - 1)(a^{m-1} - 1)}{a^{2m-2}(a - 1)(aa - 1)}$$

$$\delta = \frac{-(a^m - 1)(a^{m-1} - 1)(a^{m-2} - 1)}{a^{3m-6}(a - 1)(aa - 1)(a^2 - 1)}$$

$$\varepsilon = \frac{+(a^m - 1)(a^{m-1} - 1)(a^{m-2} - 1)(a^{m-3} - 1)}{a^{4m-10}(a - 1)(a^2 - 1)(a^3 - 1)(a^4 - 1)}$$

etc.

N. 31

Termi-

Terminus ergo seriei s , quicunque $\frac{1}{1-a^m}(1-x)(1-\frac{x}{a})(1-\frac{x}{a^2}) \dots (1-\frac{x}{a^{m-1}})$ evolutus, dabit hanc progressionem:

$$\frac{1}{1-a^m} - \frac{1}{a^{m-1}(1-a)}x + \frac{(1-a^{m-1})x^2}{a^{2m-2}(1-a)(1-a^2)} - \frac{(1-a^{m-1})(1-a^{m-2})x^3}{a^{3m-6}(1-a)(1-a^2)(1-a^3)}.$$

Si igitur successivè pro m numeri 1, 2, 3, 4, etc. substituantur, prodibunt sequentes formulae, seu termini seriei s .

$$\text{Primus} = \frac{1}{1-a} - \frac{x}{1-a}$$

$$\text{Secund:} = \frac{1}{1-a^2} - \frac{x}{a(1-a)} + \frac{(1-x)x^2}{a(1-a)(1-a^2)}$$

$$\text{Tert:} = \frac{1}{1-a^3} - \frac{x}{a^2(1-a)} + \frac{(1-a^2)x^2}{a^3(1-a)(1-a^2)} - \frac{(1-a)(1-a^2)x^3}{a^3(1-a)(1-a^2)(1-a^3)}$$

$$\text{Quart:} = \frac{1}{1-a^4} - \frac{x}{a^3(1-a)} + \frac{(1-a^3)x^2}{a^5(1-a)(1-a^2)} - \frac{(1-a^2)(1-a^3)x^3}{a^6(1-a)(1-a^2)(1-a^3)} + \frac{(1-a)(1-a^2)(1-a^4)x^4}{a^6(1-a)(1-a^2)(1-a^3)(1-a^4)}$$

etc.

§. 23. Si ergo omnes isti termini in unam summam colligantur, prodibit congeries infinitarum serierum, quae simul sumtae, seriei initio propositae, erunt aequales. Scilicet cum sit:

$$s = \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)(1-\frac{x}{a}) + \frac{1}{1-a^3}(1-x)(1-\frac{x}{a})(1-\frac{x}{a^2}) + \text{etc. erit:}$$

$$s = \frac{1}{1-a} + \frac{1}{1-a^2} + \frac{1}{1-a^3} + \frac{1}{1-a^4} + \frac{1}{1-a^5} + \text{etc.}$$

$$\frac{-x}{1-a} \left(1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \text{etc.} \right)$$

$$\frac{-x^2}{a(1-a)(1-a^2)} \left(\frac{1-x}{1} + \frac{1-x^2}{a^2} + \frac{1-x^3}{a^4} + \frac{1-x^4}{a^5} + \text{etc.} \right)$$

$$\frac{-x^3}{a^3(1-a)(1-a^2)(1-a^3)} \left(\frac{(1-a)(1-a^2)}{1} + \frac{(1-a^2)(1-a^3)}{a^3} + \frac{(1-a^3)(1-a^4)}{a^6} + \text{etc.} \right)$$

$$\frac{-x^4}{a^6(1-a)(1-a^2)(1-a^3)(1-a^4)} \left(\frac{(1-a)(1-a^2)(1-a^3)}{1} + \frac{(1-a^2)(1-a^3)(1-a^4)}{a^4} + \text{etc.} \right)$$

etc.

Cum

Cum igitur haec series congruere debeat cum ante inuenita, ex consensu singularum harum serierum summa reperientur.

$$\begin{aligned}
 & \frac{1}{1-a} + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \dots \text{etc.} & = & \frac{1}{1-a} \\
 & \frac{1-a^2}{1-a} + \frac{1-a^2}{a^2} + \frac{1-a^2}{a^4} + \frac{1-a^2}{a^6} + \dots \text{etc.} & = & \frac{1-a^2}{1-a} \\
 & \frac{(1-a)(1-a^2)}{1-a} + \frac{(1-a^2)(1-a^2)}{a^2} + \frac{(1-a^2)(1-a^4)}{a^4} + \dots \text{etc.} & = & \frac{1-a^2}{1-a^2} \\
 & \frac{(1-a^2)(1-a^2)(1-a^2)}{1-a^2} + \frac{(1-a^2)(1-a^2)(1-a^4)}{a^4} + \dots \text{etc.} & = & \frac{1-a^2}{1-a^4} \\
 & \frac{(1-a)(1-a^2)(1-a^3)(1-a^4)}{1-a^5} + \frac{(1-a^2)(1-a^3)(1-a^4)(1-a^5)}{a^5} + \dots \text{etc.} & = & \frac{1-a^5}{1-a^5}
 \end{aligned}$$

§. 24. Hae series in sequentes formas transfundi possunt, ex quibus lex progressionis clarius perspicietur:

$$\begin{aligned}
 \frac{a}{a-1} &= 1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \dots \text{etc.} \\
 \frac{a^2}{a^2-1} &= (1 - \frac{1}{a}) + \frac{1}{a}(1 - \frac{1}{a^2}) + \frac{1}{a^2}(1 - \frac{1}{a^3}) + \frac{1}{a^3}(1 - \frac{1}{a^4}) + \dots \text{etc.} \\
 \frac{a^3}{a^3-1} &= (1 - \frac{1}{a})(1 - \frac{1}{a^2}) + \frac{1}{a}(1 - \frac{1}{a^2})(1 - \frac{1}{a^3}) + \frac{1}{a^2}(1 - \frac{1}{a^3})(1 - \frac{1}{a^4}) + \dots \text{etc.} \\
 \frac{a^4}{a^4-1} &= (1 - \frac{1}{a})(1 - \frac{1}{a^2})(1 - \frac{1}{a^3}) + \frac{1}{a}(1 - \frac{1}{a^2})(1 - \frac{1}{a^3})(1 - \frac{1}{a^4}) + \dots \text{etc.} \\
 \frac{a^5}{a^5-1} &= (1 - \frac{1}{a})(1 - \frac{1}{a^2})(1 - \frac{1}{a^3})(1 - \frac{1}{a^4}) + \frac{1}{a}(1 - \frac{1}{a^2})(1 - \frac{1}{a^3})(1 - \frac{1}{a^4})(1 - \frac{1}{a^5}) + \dots \text{etc.}
 \end{aligned}$$

Vnde colligitur fore generaliter

$$\begin{aligned}
 & \frac{a^{m+1}}{a^{m+1}-1} = \frac{\frac{1}{a}}{1 - \frac{1}{a^{m+1}}} \\
 & = (1 - \frac{1}{a})(\frac{1}{a^2}) \dots (1 - \frac{1}{a^m}) + \frac{1}{a}(1 - \frac{1}{a^2})(1 - \frac{1}{a^3}) \dots (1 - \frac{1}{a^{m+1}}) + \\
 & \quad \frac{1}{a^2}(1 - \frac{1}{a^3})(1 - \frac{1}{a^4}) \dots (1 - \frac{1}{a^{m+2}}) + \frac{1}{a^3}(1 - \frac{1}{a^4})(1 - \frac{1}{a^5}) \dots (1 - \frac{1}{a^{m+3}}) + \dots \text{etc.}
 \end{aligned}$$

§. 25. Summa huius seriei etiam hoc modo inuestigari potest. Sit breuitatis gratia $\frac{1}{a} = b$, atque ponatur summa quaesita:

$$z =$$

$$z = (1-b)(1-b^2) \dots (1-b^m) + b(1-b^2)(1-b^3) \dots (1-b^{m+1}) + \\ b^2(1-b^3)(1-b^4) \dots (1-b^{m+2}) + b^3(1-b^4)(1-b^5) \dots (1-b^{m+3}) + \text{etc.}$$

Multiplicetur utrinque per $1-b^{m+1}$, atque prodibit:

$$(1-b^{m+1})z = (1-b)(1-b^2) \dots (1-b^m)(1-b^{m+1}) + (1-b^2)(1-b^3) \dots (1-b^{m+1})(b-b^{m+2}) \\ + (1-b^3)(1-b^4) \dots (1-b^{m+2})(b^2-b^{m+3}) + \text{etc.}$$

At est $b-b^{m+2} = 1-b^{m+2}-(1-b)$; $b^2-b^{m+3} = 1-b^{m+3}-(1-bb)$
 $b^3-b^{m+4} = 1-b^{m+4}-(1-b^3)$, etc. qui valores loco ultimorum factorum substituti dabuntur:

$$(1-b^{m+1})z = (1-b)(1-b^2) \dots (1-b^{m+1}) + (1-b^2)(1-b^3) \dots (1-b^{m+2}) \\ - (1-b)(1-b^2) \dots (1-b^{m+1}) - (1-b^2)(1-b^3) \dots (1-b^{m+2}) \\ + (1-b^3)(1-b^4) \dots (1-b^{m+3}) + (1-b^4)(1-b^5) \dots (1-b^{m+4}) + \text{etc.} \\ - (1-b^5)(1-b^6) \dots (1-b^{m+5}) - \text{etc.}$$

Cum ergo omnes termini destruantur, solus remanebit ultimus, $(1-b^{m+1})z = (1-b^m)(1-b^{m+1}) \dots (1-b^{m+1})$, unde patet, si fuerit $b < 1$, hoc est $a > 1$, ut assumimus,

fore $(1-b^{m+1})z = 1$, ideoque $z = \frac{1}{1-b^{m+1}} = \frac{a^{m+1}}{a^{m+1}-1}$, ut inuenieramus.

§ 26. Ex iis, quae §. XXI. sunt tradita, facile reperitur series secundum dimensiones ipsius x procedens, quae aequalis sit huic producto infinitorum Factorum.

$$P = 1-x\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\left(1-\frac{x}{a^4}\right) \text{ etc.}$$

Posito enim $P = 1-ax+\xi x^2-\gamma x^3+\delta x^4-\varepsilon x^5+\text{etc.}$
scribatur $a x$ loco x , et valor resultans sit $= Q$, erit:

$$Q = (1-ax)\left(1-a\left(1-\frac{x}{a}\right)\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\left(1-\frac{x}{a^4}\right) \text{ etc.} = P-axP$$

et $Q = 1 - \alpha ax + \alpha^2 x^2 - \gamma \alpha^3 x^3 + \delta \alpha^4 x^4 - \epsilon \alpha^5 x^5 + \text{etc.}$

sed $\alpha x P = \alpha x - \alpha ax^2 + \alpha x^3 - \gamma \alpha x^4 + \delta \alpha x^5 - \text{etc.}$

$-P = 1 + \alpha x - \alpha x^2 + \gamma x^3 - \delta x^4 + \epsilon x^5 - \text{etc.}$

Vnde fit $\alpha = \frac{a}{a-1}$; $\beta = \frac{\alpha a}{a^2-1}$; $\gamma = \frac{\beta a}{a^3-1}$; $\delta = \frac{\gamma a}{a^4-1}$ etc.

Quam ob rem productum infinitum $P = (1-x)(1-\frac{x}{a})(1-\frac{x}{a^2})\dots$ etc.

resoluetur in hanc seriem infinitam:

$$P = 1 - \frac{ax}{a-1} + \frac{a^2 x^2}{(a-1)(a^2-1)} - \frac{a^3 x^3}{(a-1)(a^2-1)(a^3-1)} + \frac{a^4 x^4}{(a-1)(a^2-1)(a^3-1)(a^4-1)} \text{ etc.}$$

§. 27. Si igitur istud productum P nihilo aequale ponatur haec aequatio infinita:

$$0 = 1 - \frac{ax}{a-1} + \frac{a^2 x^2}{(a-1)(a^2-1)} - \frac{a^3 x^3}{(a-1)(a^2-1)(a^3-1)} + \text{etc.}$$

omnes suas radices x habebit reales, eruntque valores ipsius x terminis istius progressionis Geometricae:

$$1, a, a^2, a^3, a^4, a^5, a^6, a^7, \text{etc.}$$

vnde si ponatur $x = a^n$, denotante n numerum integrum affirmatiuum quemicunque, erit:

$$0 = 1 - \frac{a^{n+1}}{a-1} + \frac{a^{2n+2}}{(a-1)(a^2-1)} - \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)} + \text{etc.}$$

cuius veritas iam supra §. XVIII. est demonstrata.

§. 28. Praecipue autem est notatu digna series, cui supra innumerabiles aliae aequales sunt inuentae (§. XVI.), quae est

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \frac{1}{a^5-1} + \text{etc.}$$

cuius summa; si $a > 1$, et si est finita et per approximations facile assignatur, tamen neque numeris rationalibus, neque irrationalibus exprimi potest. Quo circa ea imprimita digna videtur, vt Geometriæ naturam illius qua-

Tom. III. Nov. Comment. O titia-

CONSIDERATIO

tatis transcendentis, inuestigent, qua: eius summa ex-
primatur.

§. 29. Monstrabo autem, quem ad modum summa
huiusmodi serierum vero proxime expedite inueniri posset,
et quidem hanc seriem in aliquanto latiori sensu conde-
rabo. Sit:

$$s = \frac{1}{a-z} + \frac{1}{a^2-z} + \frac{1}{a^3-z} + \frac{1}{a^4-z} + \frac{1}{a^5-z} + \text{etc.}$$

Conuertantur singuli termini in series Geometricas, eritque:

$$s = \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \frac{1}{a^5} + \text{etc.}$$

$$+ z \left(\frac{1}{a^2} + \frac{1}{a^4} + \frac{1}{a^6} + \frac{1}{a^8} + \frac{1}{a^{10}} + \text{etc.} \right)$$

$$+ z^2 \left(\frac{1}{a^3} + \frac{1}{a^6} + \frac{1}{a^9} + \frac{1}{a^{12}} + \frac{1}{a^{15}} + \text{etc.} \right),$$

etc.

quae series denuo summatae dabunt:

$$s = \frac{z}{a-1} + \frac{z^2}{a^2-1} + \frac{z^3}{a^3-1} + \frac{z^4}{a^4-1} + \frac{z^5}{a^5-1} + \text{etc.}$$

Quod si ergo fuerit $z = 1$, hae ambae series in eandem
recidunt, neque haec transmutatio nullum affert discriumen.

§. 30. Ad seriem hanc summandam ponamus pri-
oris formae iam n terminos acti: esse summatos, quorum
summa sit $= A$, ita vt sit:

$$A = \frac{1}{a-z} + \frac{1}{a^2-z} + \frac{1}{a^3-z} + \frac{1}{a^4-z} + \dots + \frac{1}{a^n-z}$$

Erit ergo tota summa quaesita:

$$s = A + \frac{I}{a^{n+1}-z} + \frac{I}{a^{n+2}-z} + \frac{I}{a^{n+3}-z} + \frac{I}{a^{n+4}-z} + \text{etc.}$$

Iam istae fractiones in series Geometricas evoluantur,
eritque:

$$s = A$$

$$\begin{aligned}
 s = A + & \frac{1}{a^{n+1}} + \frac{1}{a^{n+2}} + \frac{1}{a^{n+3}} + \frac{1}{a^{n+4}} + \text{etc.} \\
 & + z \left(\frac{1}{a^{2n+2}} + \frac{1}{a^{2n+4}} + \frac{1}{a^{2n+6}} + \frac{1}{a^{2n+8}} + \text{etc.} \right) \\
 & + z^2 \left(\frac{1}{a^{3n+3}} + \frac{1}{a^{3n+6}} + \frac{1}{a^{3n+9}} + \frac{1}{a^{3n+12}} + \text{etc.} \right) \\
 & \quad \text{etc.}
 \end{aligned}$$

quae series denio summatae dabunt :

$$s = A + \frac{1}{a^n(a-1)} + \frac{z}{a^{2n}(aa-1)} + \frac{z^2}{a^{3n}(a^2-1)} + \frac{z^3}{a^{4n}(a^4-1)} + \text{etc.}$$

quae eo citius conuergit, quam prima, quo maior fuerit numerus n .

$$\text{§. 31. Sit } a=2, \text{ vt sit } s = \frac{1}{z-z} + \frac{1}{4-z} + \frac{1}{8-z} + \frac{1}{16-z} + \text{etc.}$$

Si igitur fuerit $A = \frac{1}{z-z} + \frac{1}{4-z} + \frac{1}{8-z} + \dots + \frac{1}{z^n-z}$

$$\text{erit: } s = A + \frac{1}{1 \cdot 2^n} + \frac{z}{3 \cdot 2^{2n}} + \frac{z^2}{7 \cdot 2^{3n}} + \frac{z^3}{15 \cdot 2^{4n}} + \frac{z^4}{31 \cdot 2^{5n}} + \text{etc.}$$

Ponamus autem $z=1$, ita vt quaeratur summa huius seriei :

$$s = 1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \frac{1}{63} + \text{etc.}$$

Addantur exempli causa quatuor termini initiales actu, vt sit $n=4$; erit :

$$1 = 1, 00000000000000$$

$$\frac{1}{3} = 0, 33333333333333$$

$$\frac{1}{7} = 0, 142857142857142$$

$$\frac{1}{15} = 0, 06666666666666$$

$$A = 1, 542857142857141$$

$$\text{Hinc erit } s = A + \frac{1}{16 \cdot 1} + \frac{1}{16 \cdot 2 \cdot 3} + \frac{1}{16 \cdot 3 \cdot 7} + \frac{1}{16 \cdot 4 \cdot 15} + \text{etc.}$$

O 2 atque

168 CONSIDERATIO QVARVM DAM SERIERV M.

atque isti termini in fractionibus decimalibus dabunt :

$$0,063838009558149$$

$$A = 1,542857142857142$$

$$\text{Ergo } s = 1,006095152415291$$

§. 32. Ceterum si seriei $s = \frac{1}{a^1} + \frac{2}{a^2} + \frac{3}{a^3} + \frac{4}{a^4} + \dots$ etc.
 singuli termini in series Geometricas resoluantur, atque potestates similes ipsius a colligantur, reperietur haec forma :
 $s = \frac{1}{a} + \frac{2}{a^2} + \frac{3}{a^3} + \frac{4}{a^4} + \frac{2}{a^5} + \frac{4}{a^6} + \frac{2}{a^7} + \frac{4}{a^8} + \frac{3}{a^9} + \dots$ etc.
 quia series hanc habet proprietatem, ut cuiusvis fractionis numerator indicet, quot divisores habeat exponens ipsius a in denominatore. Sic fractionis $\frac{4}{a^6}$ numerator est = 4, quia exponens 6 quatuor habet divisores 1, 2, 3, 6. Vnde si exponens ipsius a in denominatore sit numerus primus, numerator perpetuo erit = 2 : pro numeris autem non primis erit is binario maior. Hinc facile pater, si $a = 10$ fore :

$$s = 0,122324243426244526264428344628.$$

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