DEMONSTRATIO
THEOREMATIS CIRCA ORDINEM IN SVMMIS DIVISORVM OBSERVATVM.

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Iam ab aliquo tempore incidi in theorema, quo natura numerorum non mediocriter illustrari est visa, cum in eo ordo contineatur, quem summae diuisorum, ex numeris serie naturali procedentibus ortae, inter se tenent. Ostendi enim, si singulorum numerorum naturalium 1, 2, 3, 4, 5, 6, 7, 8, etc. omnes diuisores in vnam summam colligantur, haeque diuisorum summae in seriem disponantur, quae erit

1, 3, 4, 7, 6, 12, 8, 15, 13, 18, 12, 28, 14, 24, 24, 31, 18 etc.

hanc seriem esse recurrentem, eiusque singulos terminos ex praecedentibus secundum quandam scalam relationis determinari. Atque hic ordo non solum ideo maxime notatu dignus est visus, quod vix quisquam suspicatus fuerit, hanc seriem certae cuipiam legi esse adstrictam, sed etiam, quod istius ordinis nullam demonstrationem firmam mihi quidem tum temporis reperire licuerit, etiamsi pluribus modis rem tentauerim. Perductus quidem fui ad huius ordinis obseruationem, dum sequentem formulam in infinitum productam sum contemplatus:

$$s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7) \text{ etc.}$$

ex cuius euolutione per inductionem conclusi, fore

$$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \text{ etc.}$$

 vbi

vbi exponentium ipſius x ordo eorum differentiis ſumendis ſit manifeſtus; erit enim ſeries differentiarum

1, 1, 3, 2, 5, 3, 7, 4, 9, 5, 11, 6, 13, 7, 15, 8, etc.

Excerptis enim terminis alternis patet, hanc ſeriem eſſe permixtam ex ſerie numerorum imparium, et ex ſerie numerorum omnium integrorum. Verum quod ſit ſecundum hanc legem: $s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} +$ etc. ſiquidem fuerit $s = (1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)$ etc. per inductionem tantum collegi, neque aequalitatem harum duarum formularum ſolida demonſtratione euincere potui. Quam ob cauſam etiam ordinem illum, quem in ſummis diuiſorum hinc elicui, firmiter demonſtrare non valui; ſed eius demonſtrationem iam tum inniti declaraui demonſtrationi aequalitatis inter binas illas formulas infinitas modo exhibitas. Cum igitur nunc iſtam demonſtrationem ſim adeptus, ordinem quoque illum in ſummis diuiſorum detectum non amplius illis veritatibus, quae agnoſcuntur, neque tamen demonſtrari poſſunt, acenſeri conueniet, quemadmodum tum temporis ſum arbitratus, ſed iam merito ipſi locus inter veritates rigide demonſtratas aſſignari poterit. Cuius rei ne vllum dubium relinquatur, ſingulas propoſitiones, quibus demonſtratio huius veritatis innititur, hic ordine apponam atque demonſtrabo:

PROPOSITIO I.

Si ſit $s = (1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\varepsilon)(1+\zeta)(1+\eta)$ etc. productum hoc, ex infinitis factoribus conſtans, in ſeriem ſequentem conuertitur:

$$s = (1+\alpha) + \beta(1+\alpha) + \gamma(1+\alpha)(1+\beta) + \delta(1+\alpha)(1+\beta)(1+\gamma) + \varepsilon(1+\alpha)(1+\beta)(1+\gamma)(1+\delta) + \zeta(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\varepsilon) + \text{etc.}$$

DEMON-

DEMONSTRATIO.

Cum enim feriei primus terminus fit $(1+\alpha)$ et fecundus $= \beta(1+\alpha)$, erit fumma primi et fecundi $= (1+\alpha)(1+\beta)$: fi iam addatur tertius terminus $\gamma(1+\alpha)(1+\beta)$, prodibit $(1+\alpha)(1+\beta)(1+\gamma)$: addatur infuper terminus quartus, qui eft $\delta(1+\alpha)(1+\beta)(1+\gamma)$, erit fumma $= (1+\alpha)(1+\beta)(1+\gamma)(1+\delta)$. Atque fic in infinitum procedendo, fumma totius feriei, feu omnium eius terminorum, perducetur ad hoc productum: $(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\varepsilon)(1+\zeta)$ etc. Vnde manifeftum eft, fi fuerit

$$s = (1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\varepsilon)(1+\zeta) \text{ etc.}$$

fore viciffim:

$$s = (1+\alpha) + \beta(1+\alpha) + \gamma(1+\alpha)(1+\beta) + \delta(1+\alpha)(1+\beta)(1+\gamma) + \text{ etc.}$$

PROPOSITIO II.

Si fuerit $s = (1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc. productum hoc, ex infinitis factoribus conftans, reducetur ad hanc feriem:

$$s = 1-x-xx(1-x)-x^3(1-x)(1-x^2)-x^4(1-x)(1-x^2)(1-x^3) \text{ etc.}$$

DEMONSTRATIO.

Si haec forma $s=(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)$ etc. cum forma praecedente $s = (1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\varepsilon)$ etc. comparetur, manifeftum eft fore: $\alpha = -x$; $\beta = -x^2$; $\gamma = -x^3$; $\delta = -x^4$; $\varepsilon = -x^5$; etc. His igitur valoribus in ferie ibi data, quae producto s aequalis eft inuenta, rite fubftitutis, patebit propofitionis veritas, fcilicet effe:

$$s = 1-x-xx(1-x)-x^3(1-x)(1-x^2)-x^4(1-x)(1-x^2)(1-x^3)- \text{ etc.}$$

PROPOSITIO III.

Si fuerit $s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)$ etc. erit hoc productum infinitum per multiplicationem euoluendo, terminosque secundum potestates ipsius x disponendo:

$$s = 1 - x^1 - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + x^{57} - etc.$$

cuius seriei ratio est ea ipsa, quae supra est exposita.

DEMONSTRATIO.

Cum sit $s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)$ etc. erit $s = 1 - x - xx(1-x) - x^3(1-x)(1-x^2) - x^4(1-x)(1-x^2)(1-x^3) + etc.$

Ponatur $s = 1 - x - Axx$, erit:

$$A = 1 - x + x(1-x)(1-xx) + x^2(1-x)(1-x^2)(1-x^3) + etc.$$

Euoluantur singuli termini tantum secundum factorem $1-x$, ac sequenti modo disponantur:

$$A = \begin{array}{l} \quad\;\; - x \qquad\quad - x^2(1-xx) \qquad\qquad - x^3(1-x^2)(1-x^3) \qquad\qquad - etc. \\ + 1 + x(1-xx) + x^2(1-x^2)(1-x^3) + x^3(1-x^2)(1-x^3)(1+x^4) + etc. \end{array}$$

eritque terminis subscriptis colligendis:

$$A = 1 - x^3 - x^5(1-x^2) - x^7(1-x^2)(1-x^3) - x^9(1-x^2)(1-x^3)(1-x^4) - etc.$$

Ponatur $A = 1 - x^3 - Bx^5$, erit

$$B = 1 - x^2 + x^2(1-x^2)(1-x^3) + x^4(1-x^2)(1-x^3)(1-x^4) + etc.$$

in quibus terminis subscriptis $1 - x^2$ tantum euoluatur, ac fiet

$$B = \begin{array}{l} \quad - x^2 \qquad\qquad - x^4(1-x^3) \qquad\qquad - x^6(1-x^3)(1-x^4) \\ 1 + x^2(1-x^3) + x^4(1-x^3)(1-x^4) + x^6(1-x^3)(1-x^4)(1-x^5) \end{array}$$

denuoque terminis subscriptis colligendis habebitur:

$$B = 1 - x^5 - x^8(1-x^3) - x^{11}(1-x^3)(1-x^4) - x^{14}(1-x^3)(1-x^4)(1-x^5) - etc.$$

Ponatur $B = 1 - x^5 - Cx^8$, erit

$$C = 1 - x^3 + x^3(1-x^3)(1-x^4) + x^6(1-x^3)(1-x^4)(1-x^5) + etc.$$

vbi in singulis terminis factor $1 - x^3$ euoluatur, vt fiat scribendo vt supra:

C =

$$C = \frac{-x^3 - x^6(1-x^4) - x^9(1-x^4)(1-x^5) - \text{etc.}}{1 + x^3(1+x^4) + x^6(1-x^4)(1-x^5) + x^9(1-x^4)(1-x^5)(1-x^6) + \text{etc.}}$$

vnde colligetur:

$$C = 1 - x^7 - x^{11}(1-x^4) - x^{15}(1-x^4)(1-x^5) - x^{19}(1-x^4)(1-x^5)(1-x^6) - \text{etc.}$$

Ponatur $C = 1 - x^7 - D x^{11}$, erit

$$D = 1 - x^4 + x^4(1-x^4)(1-x^5) + x^8(1-x^4)(1-x^5)(1-x^6) + \text{etc.}$$

quae abit in hanc formam:

$$D = \frac{-x^4 - x^8(1-x^5) - x^{12}(1-x^5)(1-x^6) - \text{etc.}}{1 + x^4(1+x^5) + x^8(1-x^5)(1-x^6) + x^{12}(1-x^5)(1-x^6)(1-x^7) + \text{etc.}}$$

ficque erit

$$D = 1 - x^9 - x^{14}(1-x^5) - x^{19}(1-x^5)(1-x^6) - x^{24}(1-x^5)(1-x^6)(1-x^7) - \text{etc.}$$

Quodfi porro ponatur $D = 1 - x^9 - Ex^{14}$, reperietur fimili modo:

$E = 1 - x^{11} - Fx^{17}$; hincque vltra:

$F = 1 - x^{17} - Gx^{20}$; $G = 1 - x^{15} - Hx^{23}$; $H = 1 - x^{17} - Ix^{26}$;

etc.

Reftituamus iam fucceffiue hos valores, eritque:

$$s = 1 - x - Axx$$
$$Ax^2 = x^2(1 - x^3) - Bx^7$$
$$Bx^7 = x^7(1 - x^5) - Cx^{15}$$
$$Cx^{15} = x^{15}(1 - x^7) - Dx^{26}$$
$$Dx^{26} = x^{26}(1 - x^9) - Ex^{40}$$

etc.

Quamobrem habebimus:

$$s = 1 - x - x^2(1-x^3) + x^7(1-x^5) - x^{15}(1-x^7) + x^{26}(1-x^9) - x^{40}(1-x^{11}) + \text{etc.}$$

fiue id ipfum, quod demonftrari oportet:

$$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \text{etc.}$$

vnde fimul lex exponentium fupra indicata per differentias luculente perfpicitur.

PRO-

PROPOSITIO IV.

SEV

THEOREMA PRINCIPALE DEMONSTRANDVM.

Si haec ſcribendi formula $\int n$ denotet ſummam omnium diuiſorum numeri n, ſimilique modo numerorum minorum, veluti $n-\alpha$, deſignentur per $\int(n-\alpha)$, ſumma diuiſorum numeri n, ſeu $\int n$, ita pendebit a ſummis diuiſorum numerorum minorum, vt ſit

$$\int n = \int(n-1)+\int(n-2)-\int(n-5)-\int(n-7)+\int(n-12)+\int(n-15)$$
$$-\int(n-22)-\int(n-26)+\int(n-35)+\int(n-40)-\int(n-51)-\int(n-57)\text{ etc.}$$

Vbi ſequentia ſunt notanda:

1°. Signa + et − geminata terminos huius progreſſionis alternatim afficere.

2°. Legem numerorum 1, 2, 5, 7, 12, 15, 22, 26, etc. ex eorum differentiis, quae ſunt 1, 3, 2, 5, 3, 7, 4, etc. fieri manifeſtam; vnde colligitur hos numeros omnes in formula hac generali $\frac{3zz \pm z}{2}$ contineri.

3°. Quouis caſu iſtius progreſſionis eos tantum terminos ab initio eſſe accipiendos, qui poſt ſignum $\int$ numeros affirmatiuos retineant; reliquos vero omnes, quibus ſignum $\int$ numeris negatiuis praefigitur, eſſe omittendos; ita ſi ſit $n = 10$, erit $\int 10 = \int 9 + \int 8 - \int 5 - \int 3 = 13 + 15 - 6 - 4 = 18$.

4°. Quibus caſibus occurrit terminus $\int(n-n)$, quod euenit, ſi n fuerit numerus huius ſeriei 1, 2, 5, 7, 12, 15 etc. iis caſibus pro valore huius termini $\int(n-n)$, ſeu $\int 0$ aſſumi oportere ipſum numerum propoſitum n; ſic ſi

ſit

fit $n = 7$, erit $\int 7 = \int 6 + \int 5 - \int 2 - \int 0 = 12 + 6 - 3 - 7 = 8$, et fi fit $n = 12$, erit $\int 12 = \int 11 + \int 10 - \int 7 - \int 5 + \int 0 = 12 + 18 - 8 - 6 + 12 = 28$.

DEMONSTRATIO.

Formetur feries $z = x \int 1 + x^2 \int 2 + x^3 \int 3 + x^4 \int 4 + x^5 \int 5 +$ etc. vbi quaelibet poteftas ipfius x multiplicata fit per fummam diuiforum exponentis eius poteftatis. Quodfi iam fingulae diuiforum fummae refoluantur, manifeftum eft, hanc feriem transformari in hanc formam

$$z = 1(x + x^2 + x^3 + x^4 + x^5 + \text{etc.}) + 2(x^2 + x^4 + x^6 + x^8 + x^{10} + \text{etc.})$$
$$+ 3(x^3 + x^6 + x^9 + x^{12} + x^{15} + \text{etc.}) + 4(x^4 + x^8 + x^{12} + x^{16} + x^{20} + \text{etc}).$$
$$+ 5(x^5 + x^{10} + x^{15} + x^{20} + x^{25} + \text{etc.} + 6(x^6 + x^{12} + x^{18} + x^{24} + x^{30} + \text{etc.})$$

etc.

quibus feriebus geometricis fummatis fiet:

$$z = \frac{1x}{1-x} + \frac{2xx}{1-xx} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \frac{6x^6}{1-x^6} + \text{etc.}$$

Multiplicetur haec forma per $-\frac{dx}{x}$, ac producti integrale erit

$$-\int \frac{zdx}{x} = l(1-x) + l(1-xx) + l(1-x^3) + l(1-x^4) + l(1-x^5) + \text{etc.}$$

feu $-\int \frac{zdx}{x} = l(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc.

quae expreffio poft fignum logarithmicum, cum fit eadem, quae in propofitione praecedente vocata eft $= s$, erit $-\int \frac{zdx}{x} = l\,s$, ideoque alterum valorem pro s fumendo, erit quoque:

$$-\int \frac{zdx}{x} = l(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{etc.})$$

 cuius

cuius differentiale per $\frac{-dx}{x}$ diuifum, dabit alium valorem pro z, nempe

$$z = \frac{1x + 2x^2 - 5x^5 - 7x^7 + 12x^{12} + 15x^{15} - 22x^{22} - etc.}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + etc.}$$

qui valor fi aequalis ponatur affumto, et vtrinque per denominatorem $1 - x - x^2 + x^5 + x^7 - x^{12}$ etc. multiplicetur, reperietur terminis fecundum poteftates ipfius x difponendis, omnibusque ad eandem partem collocandis:

$$\begin{array}{lcccccccccc}
0 = & x\int 1 & + x^2\int 2 & + x^3\int 3 & + x^4\int 4 & + x^5\int 5 & + x^6\int 6 & + x^7\int 7 & + x^8\int 8 & + x^9\int 9 & + x^{10}\int 10 \; etc. \\
 & \cdot & - \int 1 & - \int 2 & - \int 3 & - \int 4 & - \int 5 & - \int 6 & - \int 7 & - \int 8 & - \int 9 \\
 & \cdot & \cdot & - \int 1 & - \int 2 & - \int 3 & - \int 4 & - \int 5 & - \int 6 & - \int 7 & - \int 8 \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & + \int 1 & + \int 2 & + \int 3 & + \int 4 & + \int 5 \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + \int 1 & + \int 2 & + \int 3 \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & -1 & -2 & * & * & +5 & * & +7 & * & * & *
\end{array}$$

vnde fingularum poteftatum ipfius x coefficientibus nihilo aequatis, fequitur fore

$$\begin{array}{ll}
\int 1 = 1 & \int 6 = \int 5 + \int 4 - \int 1 \\
\int 2 = \int 1 + 2 & \int 7 = \int 6 + \int 5 - \int 2 - 7 \\
\int 3 = \int 2 + \int 1 & \int 8 = \int 7 + \int 6 - \int 3 - \int 1 \\
\int 4 = \int 3 + \int 2 & \int 9 = \int 8 + \int 7 - \int 4 - \int 2 \\
\int 5 = \int 4 + \int 3 - 5 & \int 10 = \int 9 + \int 8 - \int 5 - \int 3
\end{array}$$

atque indolem illius aequationis vel leuiter attendenti patebit, effe generatim:

$$\int n = \int(n-1) + \int(n-2) - \int(n-5) - \int(n-7) + \int(n-12) + \int(n-15) \text{ - etc.}$$

hac progreffione quouis cafu eousque continuata, donec perueniatur ad fummas numerorum negatiuorum. Deinde per fe eft perfpicuum, numeros abfolutos 1, 2, 5, 7, etc.

etc. qui in illis formulis conspiciuntur, vicem tenere termini $\int o$; vnde concluditur, in casibus quibus pro $\int n$ in progressione illa reperta occurrit terminus $\int(n-n)$, seu $\int o$, valorem eius semper ipsi numero proposito n aequalem esse capiendum: sicque habetur plena ac perfecta demonstratio theorematis propositi, quae, cum praeter tractationem serierum infinitarum, per logarithmos et differentialia procedat, minus quidem naturalis, sed ob hoc ipsum multo magis notabilis est aestimanda.