SVBSIDIVM CALCVLI SINVVM.

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Ex quo calculus ſinuum in analyſin eſt receptus, ita vt ſinus, coſinus ac tangentes angulorum legibus calculi aeque ſint ſubiecti, ac logarithmi atque adeo ipſae quantitates algebraicae, maxima ſine dubio incrementa Analyſis cepiſſe eſt cenſenda. Logarithmi quidem ſtatim a primis fere Analyſeos ſublimioris initiis inter quantitates analyticas referri ſunt coepti, iisque imprimis calculus exponentialium cuius inuentionem Cel. *Ioh. Bernoulli b. m.* iure ſibi vindicauerat, acceptus eſt ferendus: cuius beneficio, vt nunc quidem vel tyronibus conſtat, plurimae praeclarae inuentiones in medium ſunt prolatae. Quae calculi acceſſio in hoc potiſſimum conſtabat, vt logarithmi non ſolum idoneis characteribus in calculum eſſent inducti, ſed etiam certae regulae ſtabilitae, ſecundum quas omnes analyſeos operationes aeque in logarithmis expedire liceat, atque in quantitatibus algebraicis. Simili autem modo mihi equidem angulorum ſinus tangentesque primus in calculum ita transtuliſſe videor, vt inſtar reliquarum quantitatum tractari, cunctaeque operationes ſine vllo impedimento peragi poſſent. Etſi autem haec res haud magni momenti fortaſſe videatur, dum maximam partem in characteribus eſt ſita, quibus in calculo ad eas quantitates deſignandas vti ſoleo; quandoquidem regulae eas tam per differentiationem, quam integrationem, euoluendi iam pridem

pridem ſunt erutae: tamen haec ipſa notandi ratio poſtmodum vniuerſae analyſi tanta attulit adiumenta, vt nouum fere campum pateſeciſſe videatur, in quo Geometrae non ſine notabili elaborauerint fructu. Ac ſi quidem ipſius Analyſis praeſtantiam ſpectamus, eam praecipue ſoli idoneo quantitates ſignis denotandi modo tribuendam eſſe deprehendimus, quo minus erit mirandum, ſi commoda ſinuum in algorithmum introductio tantum lucri attulerit. Neque vero hoc ſubſidio ſolum calculi, qui ſaepenumero fierent maxime prolixi et intricati, mirum in modum contrahuntur, quod quidem iam eſſet eximium commodum: ſed etiam huius calculi ſinuum ope problemata alioquin difficillima ſatis expedite reſolui poſſunt; cuius quidem rei iam complura ſpecimina extant, non ſolum a me, ſed etiam ab aliis, exhibita. Vtilitas autem huius calculi imprimis in problematis mechanicis cernitur, multo maxime autem in Aſtronomia Theoretica, vbi totum negotium ad computum angulorum reducitur, ita vt ſine huius calculi ſubſidio vix quicquam ſit expectandum. Quae nunc certe de Lunae motu anomalo, ac planetarum perturbationibus ab mutua eorum actione oriundis ſunt eruta, huic calculo potiſſimum accepta ſunt ferenda, neque ex hac parte Aſtronomia maiora incrementa ante conſequi poſſe videtur, quam hic ipſe calculus ad maiorem perfectionis gradum fuerit euectus. Minime ergo erunt contemnenda, quae in iſthoc calculo elaborando vlteriusque excolendo verſantur; atque cum reſolutio poteſtatum tam ſinuum, quam coſinuum, in ſinus coſinusue ſimplices maximi ſit momenti, et in Aſtronomicis inveſtivestigationibus absolute necessaria, etiamsi iam hinc inde nonnulla huc spectantia protulerim, tamen haud abs re fore arbitror, si hoc egregium argumentum studiosius pertractauero.

LEMMA.

I. Valor huius formulae imaginariae $(\cos.\ \phi + \sqrt{-1}.\ \sin.\ \phi)^n$ est $\cos.\ n\phi + \sqrt{-1}.\ \sin.\ n\phi$, huius autem formulae imaginariae $(\cos.\phi - \sqrt{-1}.\sin.\phi)^n$ valor est $\cos.\ n\phi - \sqrt{-1}.\ \sin.\ n\phi$.

DEMONSTRATIO.

Si enim habeantur duo anguli ϕ et α, erit harum duarum formularum productum $(\cos.\phi + \sqrt{-1}.\sin.\phi)$ $(\cos.\ \alpha + \sqrt{-1}.\ \sin.\alpha) = \cos.\ \phi \cos.\ \alpha - \sin.\ \phi \sin.\ \alpha + (\sin.\ \phi \cos.\ \alpha + \cos.\ \phi \sin.\ \alpha)\sqrt{-1}$. Constat autem esse $\cos.\ \phi \cos.\ \alpha - \sin.\ \phi \sin.\ \alpha = \cos.(\phi + \alpha)$ et $\sin.\ \phi \cos.\ \alpha + \cos.\ \phi \sin.\ \alpha = \sin.\ (\phi + \alpha)$, vnde illud productum erit $= \cos.\ (\phi + \alpha) + \sqrt{-1}.\ \sin.\ (\phi + \alpha)$. Sit iam $\alpha = \phi$ eritque

$$(\cos.\phi + \sqrt{-1}.\ \sin.\ \phi)^2 = \cos.2\phi + \sqrt{-1}.\sin.\ 2\phi.$$

Haec formula denuo per $\cos.\ \alpha + \sqrt{-1}.\ \sin.\ \alpha$ multiplicata dabit $\cos.\ (2\phi + \alpha) + \sqrt{-1}\sin.(2\phi + \alpha)$, ac posito $\alpha = \phi$

$$(\cos.\ \phi + \sqrt{-1}.\ \sin.\phi)^3 = \cos.\ 3\phi + \sqrt{-1}.\sin.\ 3\phi$$

Quae si denuo per $\cos.\ \alpha + \sqrt{-1}.\ \sin.\ \alpha$ multiplicetur, ac ponatur $\alpha = \phi$, dabit

$$(\cos.\ \phi + \sqrt{-1}.\ \sin.\ \phi)^4 = \cos.4\phi + \sqrt{-1}.\sin.4\phi$$

hocque modo generatim colligitur fore

(cos.

$$(\text{cos}\ \Phi + \sqrt{-1}\,.\,\text{sin}.\,\Phi)^n = \text{cos}.\ n\,\Phi + \sqrt{-1}\,.\,\text{sin}\ n\,\Phi.$$

Cum autem, expressio $\sqrt{-1}$ natura sua signi ambiguitatem inuoluat, erit ob eandem rationem

$$(\text{cos}.\ \Phi - \sqrt{-1}\,.\,\text{sin}.\ \Phi)^n = \text{cos}.\,n\Phi - \sqrt{-1}\,.\ \text{sin}.\,n\Phi$$

COROLL.

2. Si ergo breuitatis gratia ponatur:

$$\text{cos}.\ \Phi + \sqrt{-1}\,.\,\text{sin}.\,\Phi = u \text{ et } \text{cos}.\,\Phi - \sqrt{-1}\,.\,\text{sin}.\ \Phi = v$$

cum sit $u^n = \text{cos}.\,n\Phi + \sqrt{-1}\,.\text{sin}.\,n\Phi$ et $v^n = \text{cos}.\,n\Phi - \sqrt{-1}\,.\text{sin}.\,n\Phi$ erit $u^n + v^n = 2\ \text{cos}.\ n\ \Phi$ et $u^n - v^n = 2\sqrt{-1}\,.\,\text{sin}.\,n\,\Phi$. Constat autem esse $uv = 1$.

PROBLEMA I.

3. Potestatem quamcunque cosinus cuiuspiam anguli in cosinus simplices conuertere, ita vt nusquam duo pluresue occurrant cosinus in se inuicem multiplicati.

SOLVTIO.

Sit $(\text{cos}.\,\Phi)^n$ seu $\text{cos}.\ \Phi^n$ (has enim designationes pro synonimis habeo) potestas proposita ad modum praescriptum conuertenda. Ponatur vt ante

$$\text{cos}.\,\Phi + \sqrt{-1}.\ \text{sin}.\,\Phi = u \text{ et } \text{cos}.\,\Phi - \sqrt{-1}.\ \text{sin}.\,\Phi = v,$$

eritque $\text{cos}.\ \Phi = \frac{1}{2}\,(u+v)$,

ideoque $\text{cos}.\ \Phi^n = \frac{(u+v)^n}{2^n}$ seu $2^n\ \text{cos}.\ \Phi^n = (u+v)^n$

Quae potestas binomialis solito modo euoluatur, vt prodeat:

$$2^n\,\text{cos}.\,\Phi^n = u^n + n\,u^{n-1}v + \tfrac{n(n-1)}{1.\ 2}u^{n-2}v^2 + \tfrac{n(n-1)(n-2)}{1.\ 2.\ 3}u^{n-3}v^3 + \text{etc.}$$

similisque expressio prodit, si litterae u et v permutentur. Additis ergo his duabus expressionibus prodit

$$2^{n+1}\text{cos}.\,\Phi^n = u^n + v^n + \tfrac{n}{1}(u^{n-2} + v^{n-2})uv + \tfrac{n(n-1)}{1.\ 2.}(u^{n-4} + v^{n-4})u^2v^2 + \text{etc.}$$

et

et ob $uv = 1$ habebitur diuidendo per 2

$$2^n \text{cos.}\Phi^n = \tfrac{1}{2}(u^n + v^n) + \tfrac{n}{1}\cdot\tfrac{1}{2}(u^{n-2} + v^{n-2}) + \tfrac{n(n-1)}{1\cdot 2}\tfrac{1}{2}(u^{n-4} + v^{n-4}) + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot\tfrac{1}{2}(u^{n-6} + v^{n-6}) + \text{etc.}$$

Verum cum ſit $u^n + v^n = 2 \text{ coſ. } n\,\Phi$, perſpicuum eſt fore:

$$2^n \text{coſ.}\Phi^n = \text{coſ.}\,n\Phi + \tfrac{n}{1}\text{coſ.}(n-2)\Phi + \tfrac{n(n-1)}{1\cdot 2}\text{coſ.}(n-4)\Phi + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\text{coſ.}(n-6)\Phi + \text{etc.}$$

In qua ſerie cum ad coſinus angulorum negatiuorum peruenitur notandum eſt eos conuenire cum coſinibus eorundem angulorum affirmatiue ſumtorum, ſeu eſſe coſ. $(n-m)\,\Phi = $ coſ. $(m-n)\,\Phi$. Q. E. I.

COROLL. 1.

4. Si ſit $n = 1$. erit $2 \text{ coſ. } \Phi = \text{coſ.}\,\Phi + \text{coſ.}\,\Phi = 2 \text{ coſ. } \Phi$; at ſi $n = 2$ habetur $2^2 \text{ coſ.}\Phi^2 = \text{coſ. } 2\,\Phi + 2 \text{ coſ. } 0\,\Phi + \text{coſ. } 2\,\Phi = 2 \text{ coſ. } 2\,\Phi + 2$. Sit $n = 3$, eritque $2^3 \text{ coſ.}\Phi^3 = \text{coſ. } 3\,\Phi + 3 \text{ coſ.}\,\Phi + 3 \text{ coſ.}\,\Phi + \text{coſ. } 3\,\Phi = 2 \text{ coſ. } 3\,\Phi + 6 \text{ coſ. } \Phi$. Sit $n = 4$ erit $2^4 \text{ coſ. } \Phi^4 = \text{coſ. } 4\,\Phi + 4 \text{ coſ. } 2\,\Phi + 6 \text{ coſ. } 0\,\Phi + 4 \text{ coſ. } 2\,\Phi + \text{coſ. } 4\,\Phi$, ideoque cum ſinguli termini praeter medium bis occurrant, ob coſ $0\,\Phi = 1$ erit:

$$2^4 \text{ coſ. } \Phi^4 = 2 \text{ coſ. } 4\,\Phi + 8 \text{ coſ. } 2\,\Phi + 6.$$

COROLL. 2.

5. Idem hoc ſemper vſu venit, quoties n eſt numerus integer affirmatiuus, vt ſeries a fine ſcripta eadem prodeat, ideoque ſinguli termini praeter medium bis occurrant. Medius autem terminus adeſt quoties n eſt numerus par, coſinusque hoc termino contentus abit in vnitatem.

COROL.

COROLL. 3.

n. 6. Quodsi ergo termini aequales ab initio et fine coniungantur, et tota series per 2 diuidatur, iidem habebuntur coefficientes qui ante; nisi quod termini constantis, si quis adest, coefficiens in sui semissem sit transmutandus. Vnde hae transformationes, quoties n fuerit numerus integer positiuus, ita se habebunt:

$$1 \text{ cos. } \phi = \text{cos. } \phi$$
$$2 \text{ cos. } \phi^2 = \text{cos. } 2\phi + \tfrac{1}{2}. 2$$
$$4 \text{ cos. } \phi^3 = \text{cos. } 3\phi + 3 \text{ cos. } \phi$$
$$8 \text{ cos. } \phi^4 = \text{cos. } 4\phi + 4 \text{ cos. } 2\phi + \tfrac{1}{2}. 6$$
$$16 \text{ cos. } \phi^5 = \text{cos. } 5\phi + 5 \text{ cos. } 3\phi + 10 \text{ cos. } \phi$$
$$32 \text{ cos. } \phi^6 = \text{cos. } 6\phi + 6 \text{ cos. } 4\phi + 15 \text{ cos. } 2\phi + \tfrac{1}{2}. 20$$
$$64 \text{ cos. } \phi^7 = \text{cos. } 7\phi + 7 \text{ cos. } 5\phi + 21 \text{ cos. } 3\phi + 35 \text{ cos. } \phi$$
$$182 \text{ cos. } \phi^8 = \text{cos. } 8\phi + 8 \text{ cos. } 6\phi + 28 \text{ cos. } 4\phi + 56 \text{ cos. } 2\phi + \tfrac{1}{2} 70$$

etc.

COROLL. 4.

7. Si exponens n sit numerus negatiuus, expressio inuenta in seriem abit infinitam, sicque fiet:

$$\frac{1}{2 \text{ cos. } \phi} = \text{cos. } \phi - \text{cos. } 3\phi + \text{cos. } 5\phi - \text{cos. } 7\phi + \text{cos. } 9\phi - \text{etc.}$$
$$\frac{1}{4 \text{ cos. } \phi^2} = \text{cos. } 2\phi - 2 \text{ cos. } 4\phi + 3 \text{ cos. } 6\phi - 4 \text{ cos. } 8\phi + 5 \text{ cos. } 10\phi - 6 \text{ cos. } 12\phi + \text{etc.}$$
$$\frac{1}{8 \text{ cos. } \phi^3} = \text{cos. } 3\phi - 3 \text{ cos. } 5\phi + 6 \text{ cos. } 7\phi - 10 \text{ cos. } 9\phi + 15 \text{ cos. } 11\phi - 21 \text{ cos. } 13\phi + \text{etc.}$$
$$\frac{1}{16 \text{ cos. } \phi^4} = \text{cos. } 4\phi - 4 \text{ cos. } 6\phi + 10 \text{ cos. } 8\phi - 20 \text{ cos. } 10\phi + 35 \text{ cos. } 12\phi - 56 \text{ cos. } 14\phi + \text{etc.}$$

etc.

COROLL. 5.

8. Quin etiam fi n fuerit numerus fractus, feries notatu dignae prodeunt.

$$\sqrt{2}\,\text{cof.}\,\Phi = \text{cof.}\tfrac{1}{2}\Phi + \tfrac{1}{2}\text{cof.}\tfrac{3}{2}\Phi - \tfrac{1\cdot1}{2\cdot4}\text{cof.}\tfrac{7}{2}\Phi + \tfrac{1\cdot1\cdot3}{2\cdot4\cdot6}\text{cof.}\tfrac{11}{2}\Phi - \tfrac{1\cdot1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}\text{cof.}\tfrac{15}{2}\Phi + \text{etc.}$$

$$\frac{1}{\sqrt{2\,\text{cof.}\,\Phi}} = \text{cof.}\tfrac{1}{2}\Phi - \tfrac{1}{2}\text{cof.}\tfrac{5}{2}\Phi + \tfrac{1\cdot3}{2\cdot4}\text{cof.}\tfrac{9}{2}\Phi - \tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\text{cof.}\tfrac{13}{2} + \tfrac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\text{cof.}\tfrac{17}{2} - \text{etc.}$$

vbi coefficientes plane funt iidem, qui in extractione radicis ex binomio more confueto erui folent.

SCHOLION.

9. Plerumque commodius eft, formulas in coroll. 3 traditas, fi n eft numerus integer pofitiuus, ordine inverfo exhibere. Tum autem conueniet eas in duas claffes diftribui, prout exponens n fuerit numerus par, vel impar. Cafu quidem, quo n eft numerus par, eae ita fe habebunt.

$$2\,\text{cof.}\,\Phi^2 = 1 + \text{cof.}\,2\Phi$$
$$8\,\text{cof.}\,\Phi^4 = 3 + 4\,\text{cof.}\,2\Phi + \text{cof.}\,4\Phi$$
$$32\,\text{cof.}\,\Phi^6 = 10 + 15\,\text{cof.}\,2\Phi + 6\,\text{cof.}\,4\Phi + \text{cof.}\,6\Phi$$
$$128\,\text{cof.}\,\Phi^8 = 35 + 56\,\text{cof.}\,2\Phi + 28\,\text{cof.}\,4\Phi + 8\,\text{cof.}\,6\Phi + \text{cof.}\,8\Phi$$
$$512\,\text{cof.}\,\Phi^{10} = 126 + 210\,\text{cof.}\,2\Phi + 120\,\text{cof.}\,4\Phi + 45\,\text{cof.}\,6\Phi + 10\,\text{cof.}\,8\Phi + \text{cof.}\,10\Phi$$
$$2048\,\text{cof.}\,\Phi^{12} = 462 + 792\,\text{cof.}\,2\Phi + 495\,\text{cof.}\,4\Phi + 220\,\text{cof.}\,6\Phi + 66\,\text{cof.}\,8\Phi + 12\,\text{cof.}\,10\Phi + \text{cof.}\,12\Phi$$

etc.

In genere autem fi fuerit $n = 2\nu$ erit

$$2^{2\nu-1}\,\text{cof.}\,\Phi^{2\nu} = \tfrac{1}{2}\cdot\frac{2\nu(2\nu-1)(2\nu-2)\ldots(\nu+1)}{1\cdot 2\cdot 3\ldots\nu} + \frac{2\nu(2\nu-1)(2\nu-2)\ldots(\nu+2)}{1\cdot 2\cdot 3\ldots(\nu-1)}\text{cof.}\,2\Phi$$
$$+ \frac{2\nu(2\nu-1)(2\nu-2)\ldots(\nu+3)}{1\cdot 2\cdot 3\ldots(\nu-2)}\text{cof.}\,4\Phi + \frac{2\nu(2\nu-1)(2\nu-2)\ldots(\nu+4)}{1\cdot 2\cdot 3\ldots(\nu-3)}\text{cof.}\,6\Phi$$
$$+ \frac{2\nu(2\nu-1)(2\nu-2)\ldots(\nu+5)}{1\cdot 2\cdot 3\ldots(\nu-4)}\text{cof.}\,8\Phi + \frac{2\nu(2\nu-1)(2\nu-2)\ldots(\nu+6)}{1\cdot 2\cdot 3\ldots(\nu-5)}\text{cof.}\,10\Phi$$

etc.

Altero deinde cafu, quo eft n numerus impar, erit

1 cof.

$$1\,\mathrm{cos.}\,\phi = \mathrm{cos.}\,\phi$$
$$4\,\mathrm{cos.}\,\phi^3 = 3\,\mathrm{cos.}\,\phi + \mathrm{cos.}\,3\phi$$
$$16\,\mathrm{cos.}\,\phi^5 = 10\,\mathrm{cos.}\,\phi + 5\,\mathrm{cos.}\,3\phi + \mathrm{cos.}\,5\phi$$
$$64\,\mathrm{cos.}\,\phi^7 = 35\,\mathrm{cos.}\,\phi + 21\,\mathrm{cos.}\,3\phi + 7\,\mathrm{cos.}\,5\phi + \mathrm{cos.}\,7\phi$$
$$256\,\mathrm{cos.}\,\phi^9 = 126\,\mathrm{cos.}\,\phi + 84\,\mathrm{cos.}\,3\phi + 36\,\mathrm{cos.}\,5\phi + 9\,\mathrm{cos.}\,7\phi + \mathrm{cos.}\,9\phi$$
$$1024\,\mathrm{cos.}\,\phi^{11} = 462\,\mathrm{cos.}\,\phi + 330\,\mathrm{cos.}\,3\phi + 165\,\mathrm{cos.}\,5\phi + 55\,\mathrm{cos.}\,7\phi + 11\,\mathrm{cos.}\,9\phi + \mathrm{cos.}\,11\phi$$

etc.

in genere autem si sit $n = 2\nu - 1$ erit

$$2^{2\nu-2}\,\mathrm{cos.}\,\phi^{2\nu-1} = \frac{(2\nu-1)(2\nu-2)\ldots(\nu+1)}{1.\;2.\;\ldots\;(\nu-1)}\,\mathrm{cos.}\,\phi + \frac{(2\nu-1)(2\nu-2)\ldots(\nu+2)}{1.\;2\;\ldots\;(\nu-2)}\,\mathrm{cos.}\,3\phi$$
$$+ \frac{(2\nu-1)(2\nu-2)\ldots(\nu+3)}{1.\;2\;\ldots\;(\nu-3)}\,\mathrm{cos.}\,5\phi + \frac{(2\nu-1)(2\nu-2)\ldots(\nu+4)}{1.\;2\;\ldots\;(\nu-4)}\,\mathrm{cos.}\,7\phi$$
$$+ \frac{(2\nu-1)(2\nu-2)\ldots(\nu+5)}{1.\;2\;\ldots\;(\nu-5)}\,\mathrm{cos.}\,9\phi + \frac{(2\nu-1)(2\nu-2)\ldots(\nu+6)}{1.\;2\;\ldots\;(\nu-6)}\,\mathrm{cos.}\,11\phi$$

etc.

PROBLEMA II.

10. Potestatem quamcunque sinus cuiuspiam anguli in sinus cosinusue simplices conuertere, ita vt nusquam duo sinus vel cosinus occurrant in se inuicem multiplicati.

SOLVTIO.

Hoc problema ex praecedenti facile soluitur. Posito enim $\phi = 90° - \psi$, fit cos. ϕ = sin. ψ, ideoque expressio pro potestate cos. ϕ^n inuenta iam pro potestate sin. ψ^n valebit. Tum autem erit:

$$\mathrm{cos.}\,2\phi = -\mathrm{cos.}\,2\psi \;;\quad \mathrm{cos.}\,3\phi = -\mathrm{sin.}\,3\psi$$
$$\mathrm{cos.}\,4\phi = +\mathrm{cos.}\,4\psi \;;\quad \mathrm{cos.}\,5\phi = +\mathrm{sin.}\,5\psi$$
$$\mathrm{cos.}\,6\phi = -\mathrm{cos.}\,6\psi \;;\quad \mathrm{cos.}\,7\phi = -\mathrm{sin.}\,7\psi \text{ etc.}$$

 Quo-

Quoties ergo n eſt numerus integer, pro fractis enim haec reductio minus commode inſtitui poteſt, ſequentes obtinebuntur reductiones:

$$1 \sin.\psi = \sin.\psi$$
$$2 \sin.\psi^2 = -\cos.2\psi + \tfrac{1}{2}.2$$
$$4 \sin.\psi^3 = -\sin.3\psi + 3\sin.\psi$$
$$8 \sin.\psi^4 = +\cos.4\psi - 4\cos.2\psi + \tfrac{1}{2}.6$$
$$16 \sin.\psi^5 = +\sin.5\psi - 5\sin.3\psi + 10\sin.\psi$$
$$32 \sin.\psi^6 = -\cos.6\psi + 6\cos.4\psi - 15\cos.2\psi + \tfrac{1}{2}.20$$
$$64 \sin.\psi^7 = -\sin.7\psi + 7\sin.5\psi - 21\sin.3\psi + 35\sin.\phi$$
$$128 \sin.\psi^8 = +\cos.8\psi - 8\cos.6\psi + 28\cos.4\psi - 56\cos.2\psi + \tfrac{1}{2}.70$$

etc.

Pro valoribus autem negatiuis ipſius n habebitur:

$$\frac{1}{2\sin.\psi} = +\sin.\psi + \sin.3\psi + \sin.5\psi + \sin.7\psi + \sin.9\psi + \text{etc.}$$
$$\frac{1}{4\sin.\psi^2} = -\cos.2\psi - 2\cos.4\psi - 3\cos.6\psi - 4\cos.8\psi - 5\cos.10\psi - \text{etc.}$$
$$\frac{1}{8\sin.\psi^3} = -\sin.3\psi - 3\sin.5\psi - 6\sin.7\psi - 10\sin.9\psi - 15\sin.11\psi - \text{etc.}$$
$$\frac{1}{16\sin.\psi^4} = +\cos.4\psi + 4\cos.6\psi + 10\cos.8\psi + 20\cos.10\psi + 53\cos.12\psi + \text{etc.}$$
$$\frac{1}{32\sin.\psi^5} = +\sin.5\psi + 5\sin.7\psi + 15\sin.9\psi + 35\sin.11\psi + 70\sin.13\psi + \text{etc.}$$

etc.

Hinc ergo qnadruplices formulae generales eliciuntur, prout n fuerit numerus formae vel $4m$, vel $4m-1$, vel $4m-2$, vel $4m-3$, eaeque erunt:

$$2^{4m-1}\sin.\psi^{4m} = \cos.4m\psi - 4m\cos.(4m-2)\psi + \frac{4m(4m-1)}{1.\ 2.}\cos.(4m-4)\psi - \frac{4m(4m-1)(4m-2)}{1.\ 2.\ 3.}\cos.(4m-6)\psi \ldots\ldots + \tfrac{1}{2}.\frac{4m(4m-1)(4m-2)\ldots\ldots(2m+1)}{1.\ 2.\ 3\ \ldots\ldots\ 2m}$$

2^{4m-2}

$$2^{4m-2}\,\text{ſin.}\,\psi^{4m-1} = -\text{ſin.}(4m-1)\psi + (4m-1)\,\text{ſin.}(4m-3)\psi \; \frac{(4m-1)(4m-2)}{1.\;2}\,\text{ſin.}(4m-5)\psi$$
$$+\frac{(4m-1)(4m-2)(4m-3)}{1.\;2.\;3.}\,\text{ſin.}(4m-7)\psi \;\ldots$$
$$+\frac{(4m-1)(4m-2)(4m-3)\ldots\ldots(2m+1)}{1\;.\;2.\;3.\;\ldots\ldots(2m-1)}\,\text{ſin.}\,\psi$$

$$2^{4m-3}\,\text{ſin.}\,\psi^{4m-2} = -\text{coſ.}(4m-2)\psi + (4m-2)\,\text{coſ.}(4m-4)\psi$$
$$-\frac{(4m-2)(4m-3)}{1.\;2.}\,\text{coſ.}(4m-6)\psi + \frac{(4m-2)(4m-3)(4m-4)}{1.\;2.\;3.}\,\text{coſ.}(4m-8)\psi \;\ldots$$
$$+\frac{1}{2}\,\frac{(4m-2)(4\mu-3)(4m-4)\ldots\ldots(2m)}{1.\;2.\;3\;\ldots\ldots(2m-1)}$$

$$2^{4m-4}\,\text{ſin.}\,\psi^{4m-3} = +\text{ſin.}(4m-3)\psi - (4m-3)\,\text{ſin.}(4m-5)\psi$$
$$+\frac{(4m-3)(4m-4)}{1.\;2.}\,\text{ſin.}(4m-7)\psi - \frac{(4m-3)(4m-4)(4m-5)}{1.\;2.\;3.}\,\text{ſin.}(4m-9)\psi \;\ldots$$
$$+\frac{(4m-3)(4m-4)(4m-5)\ldots\ldots(2m)}{1.\;2.\;3.\;\ldots\ldots(2m-2)}\,\text{ſin}\,\psi$$

Simili modo ſi n ſit numerus negatiuus integer quaternas habebimus formulas generales:

$$\frac{1}{2^{4m}\,\text{ſin.}\,\psi^{4m}} = +\text{coſ.}\,4m\psi + 4m\,\text{coſ.}(4m+2)\psi + \frac{4m(4m+1)}{1.\;2.}\,\text{coſ.}(4m+4)\psi$$
$$+\frac{4m(4m+1)(4m+2)}{1.\;2.\;3}\,\text{coſ.}(4m+6)\psi + \text{etc.}$$

$$\frac{1}{2^{4m+1}\,\text{ſin}\,\psi^{4m+1}} = +\text{ſin.}(4m+1)\psi + (4m+1)\,\text{ſin.}(4m+3)\psi$$
$$+\frac{(4m+1)(4m+2)}{1\;.\;2}\,\text{ſin.}(4m+5)\psi + \frac{(4m+1)(4m+2)(4m+3)}{1.\;2.\;3}\,\text{ſin.}(4m+7)\psi + \text{etc.}$$

$$\frac{1}{2^{4m+2}\,\text{ſin}\,\psi^{4m+2}} = -\text{coſ.}(4m+2)\psi - (4m+2)\,\text{coſ.}(4m+4)\psi$$
$$-\frac{(4m+2)(4m+3)}{1.\;2}\,\text{coſ.}(4m+6)\psi - \frac{(4m+2)(4m+3)(4m+4)}{1.\;2.\;3}\,\text{coſ.}(4m+8)\psi - \text{etc.}$$

$$\frac{1}{2^{4m+3}\,\text{ſin}\,\psi^{4m+3}} = -\text{ſin.}(4m+3)\psi - (4m+3)\,\text{ſin.}(4m+5)\psi$$
$$-\frac{(4m+3)(4m+4)}{1.\;2}\,\text{ſin.}(4m+7)\psi - \frac{(4m+3)(4m+4)(4m+6)}{1.\;2.\;3}\,\text{ſin.}(4m+9)\psi - \text{etc.}$$

Sicque quoties n eſt numerus integer, ſiue poſitiuus, ſiue negatiuus, poteſtas ſinus ſin. ψ^n deſiderato modo reſoluitur. Q. E. F.

 COROL.

COROLL. 1.

11. Quoties ergo n est numerus par, siue positiuus, siue negatiuus, potestas sin. ψ^n resoluitur in cosinus simplices angulorum multiplorum ipsius ψ. Sin autem n fuerit numerus impar, potestas sin. ψ^n resoluitur in sinus simplices angulorum multiplorum ipsius ψ.

COROLL. 2.

12. Quodsi n fuerit numerus integer positiuus, atque expressiones inuentae retro disponantur, quaternae formulae supra datae in binas incidunt. Pro paribus enim exponentibus erit:

$$
\begin{aligned}
2\sin.\psi^2 &= 1-\cos.2\psi\\
8\sin.\psi^4 &= 3-4\cos.2\psi+\cos.4\psi\\
32\sin.\psi^6 &= 10-15\cos.2\psi+6\cos.4\psi-\cos.6\psi\\
128\sin.\psi^8 &= 35-56\cos.2\psi+28\cos.4\psi-8\cos.6\psi\\
&\quad+\cos.8\psi\\
512\sin.\psi^{10} &= 126-210\cos.2\psi+120\cos.4\psi-45\cos.6\psi\\
&\quad+10\cos.8\psi-\cos.10\psi\\
2048\sin.\psi^{12} &= 462-792\cos.2\psi+495\cos.4\psi-220\cos.6\psi\\
&\quad+66\cos.8\psi-12\cos.10\psi+\cos.12\psi
\end{aligned}
$$

etc.

atque generatim erit:

$$
\begin{aligned}
2^{2\nu-1}\sin.\psi^{2\nu} = {} & \frac{1}{2}\cdot\frac{2\nu(2\nu-1)(2\nu-2)\ldots(\nu+1)}{1\cdot 2\cdot 3\ldots\nu} - \frac{2\nu(2\nu-1)(2\nu-2)\ldots(\nu+2)}{1\cdot 2\cdot 3\ldots(\nu-1)}\cos.2\psi\\
& + \frac{2\nu(2\nu-1)(2\nu-2)\ldots(\nu+3)}{1\cdot 2\cdot 3\ldots(\nu-2)}\cos.4\psi - \frac{2\nu(2\nu-1)(2\nu-2)\ldots(\nu+4)}{1\cdot 2\cdot 3\ldots(\nu-3)}\cos.6\psi\\
& + \frac{2\nu(2\nu-1)(2\nu2-2)\ldots(\nu+5)}{1\cdot 2\cdot 3\ldots(\nu-4)}\cos.8\psi - \frac{2\nu)2\nu-1)(2\nu-2)\ldots(\nu+6)}{1\cdot 2\cdot 3\ldots(\nu-5)}\cos.10\psi
\end{aligned}
$$

etc.

COROL.

COROLL. 3.

13. Pro imparibus autem exponentibus habebitur:

$$1\sin.\psi = \sin.\psi$$
$$4\sin.\psi^3 = 3\sin.\psi - \sin.3\psi$$
$$16\sin.\psi^5 = 10\sin.\psi - 5\sin.3\psi + \sin.5\psi$$
$$64\sin.\psi^7 = 35\sin.\psi - 21\sin.3\psi + 7\sin.5\psi - \sin.7\psi$$
$$256\sin.\psi^9 = 126\sin.\psi - 84\sin.3\psi + 36\sin.5\psi - 9\sin.7\psi + \sin.9\psi$$
$$1024\sin.\psi^{11} = 462\sin.\psi - 330\sin.3\psi + 165\sin.5\psi - 55\sin.7\psi + 11\sin.9\psi - \sin.11\psi$$

etc.

pro quibus formula generalis eſt

$$2^{2\nu-2}\sin.\psi^{2\nu-1} = \frac{(2\nu-1)(2\nu-2)\ldots(\nu+1)}{1\cdot 2\ldots(\nu-1)}\sin.\psi - \frac{(2\nu-1)(2\nu-2)\ldots(\nu+2)}{1\cdot 2\ldots(\nu-2)}\sin.3\psi$$
$$+ \frac{(2\nu-1)(2\nu-2)\ldots(\nu+3)}{1\cdot 2\ldots(\nu-3)}\sin.5\psi - \frac{(2\nu-1)(2\nu-2)\ldots(\nu+4)}{1\cdot 2\ldots(\nu-4)}\sin.7\psi$$
$$+ \frac{(2\nu-1)(2\nu-2)\ldots(\nu+5)}{1\cdot 2\ldots(\nu-5)}\sin.9\psi - \frac{(2\nu-1)(2\nu-2)\ldots(\nu+6)}{1\cdot 2\ldots(\nu-6)}\sin.11\psi$$

etc.

SCHOLION.

14. Patet ergo, ſi poteſtas ſinus cuiuſpiam anguli velut ſin. Φ^n occurrat, reſolutionem commode inſtitui non poſſe, niſi n ſit numerus integer, ſiue ſit poſitiuus, ſiue negatiuus: hoc autem caſu quadruplices prodire formulas, prout exponens n fuerit numerus formae vel 4α, vel $4\alpha+1$, vel $4\alpha+2$, vel $4\alpha+3$; quae diſtinctio non eſt neceſſaria, ſi quaeſtio eſt de poteſtate coſinus cuiuſpiam. Interim tamen ſi n eſt numerus fractus, formulae pro reſolutione poteſtatum coſinus huc non difficulter traducuntur, cum ſinus in coſinum

num transmutari possit. Posito enim $\phi = 90^\circ - \phi$ erit

$$\sqrt{2}\,\text{sin.}\,\phi = \text{cos.}\tfrac{1}{2}\phi + \tfrac{1}{2}\text{cos.}\tfrac{3}{2}\phi - \tfrac{1\cdot1}{2\cdot4}\text{cos.}\tfrac{7}{2}\phi + \tfrac{1\cdot1\cdot3}{2\cdot4\cdot6}\text{cos.}\tfrac{11}{2}\phi - \text{etc.}$$

$$\frac{1}{\sqrt{2\,\text{sin.}\,\psi}} = \text{cos.}\tfrac{1}{2}\phi - \tfrac{1}{2}\text{cos.}\tfrac{5}{2}\phi + \tfrac{1\cdot3}{2\cdot4}\text{cos.}\tfrac{9}{2}\phi - \tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\text{cos.}\tfrac{13}{2}\phi + \text{etc.}$$

Verum si productum proponatur huiusmodi $\text{sin.}\,\phi^m \cdot \text{cos.}\,\phi^n$, quod in simplices sinus cosinusue sit resoluendum, hoc commode fieri nequit, nisi exponens m sit numerus integer, siue positiuus, siue negatiuus, tumque quatuor constituendi sunt casus, prout m fuerit numerus formae, vel 4α, vel $4\alpha + 1$, vel $4\alpha + 2$, vel $4\alpha + 3$. Secundum hos ergo quaternos casus resolutionem formulae $\text{sin.}\,\phi^m$, $\text{cos.}\,\phi^n$ eruam, vbi quidem notandum est, exponentem n nulli restrictioni esse subiectum, ita vt nonsolum numeros integros, sed etiam fractos, atque adeo irrationales, denotare possit.

PROBLEMA 3.

15. Huiusmodi productum $\text{sin.}\,\phi^m \cdot \text{cos.}\,\phi^n$, in quo exponens m est numerus integer formae 4α, in sinus cosinusue simplices resoluere.

SOLVTIO.

Ponatur $\text{cos.}\,\phi + \sqrt{-1}\cdot\text{sin.}\,\phi = u$ et $\text{cos.}\,\phi - \sqrt{-1}\cdot\text{sin.}\,\phi = v$, erit:

$$\text{cos.}\,\phi = \frac{u + v}{2} \text{ et sin.}\,\phi = \frac{u - v}{2\sqrt{-1}}$$

$$\text{et cos.}\,\nu\phi = \frac{u^\nu + v^\nu}{2} \text{ et sin.}\,\nu\phi = \frac{u^\nu - v^\nu}{2\sqrt{-1}}$$

propter-

propterea quod per lemma habemus:

$\cos.\, \nu\Phi + \sqrt{-1}.\, \sin.\, \nu\Phi = u^\nu$ et $\cos.\, \nu\Phi - \sqrt{-1}.\, \sin.\, \nu\Phi = v^\nu$

Formula ergo proposita $\sin.\, \Phi^m \cos.\, \Phi^n$ abit in $\frac{(u-v)^m}{2^m(\sqrt{-1})^m} \cdot \frac{(u+v)^n}{2^n}$

et quia m est numerus integer formae 4α erit $(\sqrt{-1})^m = +1$; ideoque habebitur:

$$\sin.\, \Phi^m \cos.\, \Phi^n = \frac{(u-v)^m (u+v)^n}{2^{m+n}} \text{ siue}$$

$$2^{m+n} \sin.\, \Phi^m \cos.\, \Phi^n = (u-v)^m (u+v)^n = u^{m+n} \left(1-\frac{v}{u}\right)^m \left(+\frac{v}{u}\right)^n$$

Sit breuitatis gratia $\frac{v}{u} = z$, atque in seriem conuerti oportet hanc expressionem $(1-z)^m (1+z)^n$, quae vocetur $= S$, eritque

$$lS = m\,l(1-z) + n\,l(1+z) \text{ et differeutiando}$$

$$\frac{dS}{S} = -\frac{m\,dz}{1-z} + \frac{n\,dz}{1+z} = \frac{(n-m)\,dz - (m+n)\,z\,dz}{1-zz}$$

Ponatur $n - m = f$ et $m + n = g$, vt sit

$$(1-zz)\frac{dS}{dz} - fS + gSz = 0$$

Iam statuatur

$$S = 1 + Az + Bzz + Cz^3 + Dz^4 + Ez^5 + Fz^6 + Gz^7 + \text{etc.}$$

ac facta substitutione prodibit:

$$\left.\begin{array}{l} A + 2Bz + 3Cz^2 + 4Dz^3 + 5Ez^4 + 6Fz^5 + 7Gz^6 + \text{etc.} \\ \quad\quad - A \quad - 2B \quad - 3C \quad - 4D \quad - 5E \quad - \text{etc.} \\ -f - fA - fB - fC - fD - fE - fF - \text{etc.} \\ \quad + g + gA + gB + gC + gD + gE + \text{etc.} \end{array}\right\} = 0$$

Coefficientes ergo assumti A, B, C, etc. ita determinabuntur, vt sit

$$A = f$$
$$2B = fA - g$$
$$3C = fB - (g-1)A$$
$$4D = fC - (g-2)B$$
$$5E = fD - (g-3)C$$
$$6F = fE - (g-4)D$$
etc.

hisque valoribus inuentis erit: 2^{m+n} fin. Φ^m cof. $\Phi^n =$ $u^g + Au^{g-1}v + Bu^{g-2}v^2 + Cu^{g-3}v^3 + Du^{g-4}v^4 +$ etc. Cum autem ob m numerum parem fit 2^{m+n} fin. Φ^m cof. $\Phi =$ $(v-u)^m (v+u)^n$ erit fimili modo 2^{m+n} fin. Φ^n cof. $\Phi =$ $v^g + Av^{g-1}u + Bv^{g-2}u^2 + Cv^{g-3}v^3 + Dv^{g-4}u^4 +$ etc.

His igitur formulis addendis erit ob $vu = 1$;

$$2 \cdot 2^{m+n} \text{fin.} \Phi^m \text{cof.} \Phi^n = u^g + v^g + A(u^{g-2} + v^{g-2}) + B(u^{g-4} + v^{g-4}) + C(u^{g-6} + v^{g-6}) + D(u^{g-8} + v^{g-8}) \text{ etc.}$$

et cum in genere fit $u^\nu + v^\nu = 2 \,\text{cof.}\, \nu\Phi$ erit:

$$2^{m+n} \text{fin.} \Phi^m \text{cof.} \Phi^n = \text{cof.}\, g\Phi + A\,\text{cof.}(g-2)\Phi + B\,\text{cof.}(g-4)\Phi + C\,\text{cof.}(g-6)\Phi + \text{etc.}$$

pofito breuitatis gratia $m+n=g$ et $n-m=f$, fubftitutisque in locum coefficientium A, B, C, D, etc. valoribus ante indicatis.

Q. E. F.

PROBLEMA. IV.

16. *Si* exponens m fuerit numerus huius formae $4\alpha+2$ feu impariter par, productum fin. Φ^m cof. Φ^n in finus cofinusue fimplices refoluere.

SOLV-

SOLVTIO.

Posito vt ante $\cos.\Phi + \sqrt{-1}.\sin.\Phi = u$ et $\cos.\Phi - \sqrt{-1}.\sin.\Phi = v$, prodibit

$$\sin.\Phi^m \cos.\Phi^n = \frac{(u-v)^m}{2^m(\sqrt{-1})^m} \cdot \frac{(u+v)^n}{2^n}$$

Quia autem m est numerus impariter par, erit $(\sqrt{-1})^m = -1$ erit $-2^{m+n}\sin.\Phi^m\cos.\Phi^n = (u-v)^m(u+v)^n$

et ob m numerum parem erit quoque

$$-2^{m+n}\sin.\Phi^m\cos.\Phi^n = (v-u)^m(v+u)^n$$

quarum formularum vtraque vt ante resoluatur; scilicet posito $n-m=f$ et $m+n=g$, et coefficientibus A, B, C etc. ita assumtis, vt sit

$$\begin{array}{ll} A = f & 2B = fA - g \\ 3C = fB-(g-1)A & 4D = fC-(g-2)B \\ 5E = fD-(g-3)C & 6F = fE-(g-4)D \end{array}$$

etc.

summa illarum formularum praebebit:

$$-2.2^{m+n}\sin.\Phi^m\cos.\Phi^n = u^g + v^g + A(u^{g-2}+v^{g-2}) + B(u^{g-4}+v^{g-4}) + \text{etc.}$$

quae progressio vt ante ad cosinus simplices angulorum multiplorum ipsius Φ reducitur, ita vt sit:

$$2^{m+n}\sin.\Phi^m\cos.\Phi^n = -\cos.g\Phi - A\cos.(g-2)\Phi - B\cos.(g-4)\Phi - C\cos.(g-6)\Phi - \text{etc.}$$

Q. E. F.

PROBLEMA V.

17. Si exponens m fuerit numerus impar formae $4\alpha+1$, productum fin. Φ^m cof. Φ^n in finus cofinusue fimplices refoluere.

SOLVTIO.

Pofito iterum cof. $\Phi+\sqrt{-1}$. fin. $\Phi=u$ et cof. $\Phi-\sqrt{-1}$. fin. $\Phi=v$, vt fiat

$$\text{fin.}\,\Phi^m\,\text{cof.}\,\Phi^n=\frac{(u-v)^m}{2^m(\sqrt{-1})^m}\cdot\frac{(u+v)^n}{2^n}$$

ob m numerum formae $4\alpha+1$ erit $(\sqrt{-1})^m=\sqrt{-1}$, ideoque habebitur:

$$2^{m+n}\sqrt{-1}.\,\text{fin.}\,\Phi^m\,\text{cof.}\,\Phi^n=(u-v)^m(u+v)^n.$$

At ob m numerum imparem erit $(u-v)^m=-(v-u)^m$ hincque

$$2^{m+n}\sqrt{-1}.\,\text{fin.}\,\Phi^m\,\text{cof.}\,\Phi^n=-(v-u)^m(v+u)^n$$

Hanc ob rem his formulis addendis et per $2\sqrt{-1}$ diuidendis, fiet

$$2^{m+n}\,\text{fin.}\,\Phi^m\,\text{cof.}\,\Phi^n=\frac{(u-v)^m(u+v)^n-(v-u)^m(v+u)^n}{2\sqrt{-1}}$$

pofitoque $m+n=g$ et $n-m=f$, fumtisque

$$\begin{array}{ll} A=f & 2B=fA-g \\ 3C=fB-(g-1)A & 4D=fC-(g-2)B \\ 5E=fD-(g-3)C & 6F=fE-(g-4)D \end{array}$$

etc.

obtinebitur:

$$2^{m+n}\,\text{fin.}\,\Phi^m\,\text{cof.}\,\Phi^n=\frac{u^g-v^g}{2\sqrt{-1}}+\frac{A(u^{g-2}-v^{g-2})}{2\sqrt{-1}}+\frac{B(u^{g-4}-v^{g-4})}{2\sqrt{-1}}+\text{etc.}$$

Verum

Verum ex lemmate eſt $\frac{u^\nu - v^\nu}{2\sqrt{-1}} = \text{ſin.}\, \nu\,\Phi$, vnde oritur

$$2^{m+n}\text{ſin.}\,\Phi^m \text{coſ.}\,\Phi^n = \text{ſin.}\, g\,\Phi + A\,\text{ſin.}\,(g-2)\Phi + B\,\text{ſin.}\,(g-4)\Phi + C\,\text{ſin.}\,(g-6)\,\Phi + \text{etc.}$$

Q. E. F.

PROBLEMA VI.

18. Si exponens m ſit numerus impar formae $4\alpha-1$, reſoluere hoc productum ſin. Φ^m coſ. Φ^n in ſinus coſinusue ſimplices.

SOLVTIO.

Poſito denuo coſ. $\Phi + \sqrt{-1}$. ſin. $\Phi = u$ et coſ. $\Phi - \sqrt{-1}$. ſin. $\Phi = v$, vt ſit

$$\text{ſin.}\,\Phi^m \text{coſ.}\,\Phi^n = \frac{(u-v)^m}{2^m(\sqrt{-1})^m} \cdot \frac{(u+v)^n}{2^n}$$

ob m numerum formae $4\alpha-1$ erit $(\sqrt{-1})^m = -\sqrt{-1}$, ideoque

$$2^{m+n}\,\text{ſin.}\,\Phi^m\, \text{coſ.}\,\Phi^n = -\frac{(u-v)^m\,(u+v)^n}{\sqrt{-1}}$$

At ob m numerum imparem erit $(u-v)^m = -(v-u)^m$; ergo

$$2^{m+n}\,\text{ſin.}\,\Phi^m\, \text{coſ.}\,\Phi^n = +\frac{(v-u)^m(v+u)^n}{\sqrt{-1}}$$

quarum expreſſionum ſemiſumma eſt:

$$2^{m+n}\,\text{ſin.}\,\Phi^m\, \text{coſ.}^n = -\frac{(u-v)^m\,(u+v)^n + (v-u)^m\,(v+u)^n}{2\sqrt{-1}}$$

Si iam ponatur $n-m=f$, $m+n=g$, coefficientesque A, B, C etc. per ſequentes formulas determinentur

 A = f

$$A = f \qquad 2B = fA - g$$
$$3C = fB - (g-1)A \qquad 4D = fC - (g-2)B$$
$$5E = fD - (g-3)C \qquad 6F = fE - (g-4)D$$
etc.

reperietur:

$$2^{m+n} \text{fin.}\,\Phi^m \text{cof.}\,\Phi^n = \frac{-(u^g - v^g)}{2\sqrt{-1}} - \frac{A(u^{g-2} - v^{g-2})}{2\sqrt{-1}} - \frac{B(u^{g-4} - v^{g-4})}{2\sqrt{-1}} - \text{etc.}$$

atque ob $\frac{u^\nu - v^\nu}{2\sqrt{-1}} = \text{fin.}\,\nu\,\Phi$ obtinebitur tandem

$$2^{m+n}\text{fin.}\,\Phi^m \text{cof.}\,\Phi^n = -\text{fin.}\,g\,\Phi - A\,\text{fin.}(g-2)\Phi - B\,\text{fin.}(g-4)\Phi$$
$$- C\,\text{fin.}(g-6)\,\Phi - \text{etc.}$$

Q. E. F.

COROLL. 1.

19. Productum ergo $\text{fin.}\,\Phi^m \text{cof.}\,\Phi^n$ in cofinus fimplices refoluitur, fi exponens m fuerit numerus par: in finus autem fimplices, fi exponens m fuerit numerus impar. Atque fi exponens m fit vel 4α, vel $4\alpha + 1$, finguli termini erunt pofitiui, fin autem m fit vel $4\alpha + 2$ vel $4\alpha - 1$, feu $4\alpha + 3$, termini figno negatiuo funt affecti.

COROLL. 2.

20. His regulis tam ratione fignorum, quam vtrum finus, an cofinus, accipi debeant, obferuatis, refolutio horum quaternorum cafuum requirit eandem coefficientium A, B, C etc. determinationem, quae ita fe habet, vt pofito $n - m = f$ et $m + n = g$ effe debeat:

$A = f$

$A = f$
$2B = fA - g$
$3C = fB - (g-1)A$
$4D = fC - (g-2)B$
$5E = fD - (g-3)C$
$6F = fE - (g-4)D$
etc.

COROLL. 3.

21. Vel hos coefficientes ita definiri oportet, vt fit $(1-z)^m(1+z)^n = 1 + Az + Bzz + Cz^3 + Dz^4 + Ez^5 + Fz^6 + \text{etc.}$ ex qua reſolutione hi coefficientes ſaepe facilius eruentur.

COROLL. 4

22. Quoniam hi coefficientes in genere in fractiones abeunt, ſi hoc incommodum vitare velimus, ſtatim ponatur

$$(1-z)^m(1+z)^n = 1 + \frac{\alpha}{1}z + \frac{\beta}{1\cdot 2}z^2 + \frac{\gamma}{1\cdot 2\cdot 3}z^3 + \frac{\delta}{1\cdot 2\cdot 3\cdot 4}z^4 + \text{etc.}$$

vt ſit

$$A = \frac{\alpha}{1}; \quad B = \frac{\beta}{1\cdot 2}; \quad C = \frac{\gamma}{1\cdot 2\cdot 3}$$
$$D = \frac{\delta}{1\cdot 2\cdot 3\cdot 4}; \quad E = \frac{\varepsilon}{1\cdot 2\cdot 3\cdot 4\cdot 5}; \quad E = \frac{\zeta}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$$

tum autem erit:

$\alpha = f$
$\beta = f\alpha - g$
$\gamma = f\beta - 2(g-1)\alpha$
$\delta = f\gamma - 3(g-2)\beta$
$\varepsilon = f\delta - 4(g-3)\gamma$
$\zeta = f\varepsilon - 5(g-4)\delta$
etc.

COROL.

COROLL. 5.

23. Si hi valores euoluuntur, reperietur

$$\alpha = f$$
$$\beta = ff - g$$
$$\gamma = f^3 - (3g-2)f$$
$$\delta = f^4 - (6g-8)ff + 3g(g-2)$$
$$\varepsilon = f^5 - (10g-20)f^3 + (15gg-50g+24)f$$
$$\zeta = f^6 - (15g-40)f^4 + (45gg-210g+184)ff - 15g(g-2)(g-4)$$

etc.

Verum difficile eſt harum formularum progreſſionem perſpicere eamque continuare, niſi determinationes ante indicatae in ſubſidium vocentur.

COROLL. 6.

24. Valores coefficientum A, B, C, D, etc. etiam ita ex omnibus praecedentibus definiri poſſunt, vt ſit.

$$A = f$$
$$2B = fA - g$$
$$3C = fB - gA + f$$
$$4D = fC - gB + fA - g$$
$$5E = fD - gC + fB - gA + f$$
$$6F = fE - gD + fC - gB + fA - g$$

etc.

COROL.

COROLL. 7.

25. Hinc statim patet, si sit $g = f$ seu $m = 0$, et $g = n$, fore $A = f$; $2B = (f-1)A$; $3C = (f-2)B$; $4D = (f-3)C$; etc.

ideoque:

$$A = n; B = \frac{n(n-1)}{1 \cdot 2}; \; C = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}; \; D = \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}.$$

At si sit $g = -f$, seu $n = 0$, et $g = m$, erit

$A = f$; $2B = (f+1)A$; $3C = (f+2)B$; $4D = (f+3)C$, etc.

ideoque

$$A = -m; \; B = \frac{m(m-1)}{1 \cdot 2}; \; C = -\frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}; \; D = \frac{m(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3 \cdot 4};$$

Hi autem sunt casus iam ante euoluti.

SCHOLION I.

26. Quodsi pro exponentibus m et n successiue numeri integri affirmatiui capiantur, sequentes prodibunt resolutiones:

$1 \cos. \Phi = \cos. \Phi$

$1 \sin. \Phi = \sin. \Phi$

$2 \cos. \Phi^2 = + \cos. 2\Phi + 1$

$2 \sin. \Phi \cos. \Phi = + \sin. 2\Phi$

$2 \sin. \Phi^2 = - \cos. 2\Phi + 1$

$4 \cos. \Phi^3 = + \cos. 3\Phi + 3 \cos. \Phi$

$4 \sin. \Phi \cos. \Phi^2 = + \sin. 3\Phi + \sin. \Phi$

$4 \sin. \Phi^2 \cos. \Phi = - \cos. 3\Phi + \cos. \Phi$

$4 \sin. \Phi^3 = - \sin. 3\Phi + 3 \sin. \Phi$

$8\cos.\Phi^4 = +\cos.4\Phi + 4\cos.2\Phi + 3$
$8\sin.\Phi\cos.\Phi^3 = +\sin.4\Phi + 2\sin.2\Phi$
$8\sin.\Phi^2\cos.\Phi^2 = -\cos.4\Phi \quad \ast \quad + 1$
$8\sin.\Phi^3\cos.\Phi = -\sin.4\Phi + 2\sin.2\Phi$
$8\sin.\Phi^4 = +\cos.4\Phi - 4\cos.2\Phi + 3$

$16\cos.\Phi^5 = +\cos.5\Phi + 5\cos.3\Phi + 10\cos.\Phi$
$16\sin.\Phi\cos.\Phi^4 = +\sin.5\Phi + 3\sin.3\Phi + 2\sin.\Phi$
$16\sin.\Phi^2\cos.\Phi^3 = -\cos.5\Phi - \cos.3\Phi + 2\cos.\Phi$
$16\sin.\Phi^3\cos.\Phi^2 = -\sin.5\Phi + \sin.3\Phi + 2\sin.\Phi$
$16\sin.\Phi^4\cos.\Phi = +\cos.5\Phi - 3\cos.3\Phi + 2\cos.\Phi$
$16\sin.\Phi^5 = +\sin.5\Phi - 5\sin.3\Phi + 10\sin.\Phi$

$32\cos.\Phi^6 = +\cos.6\Phi + 6\cos.4\Phi + 15\cos.2\Phi + 10$
$32\sin.\Phi\cos.\Phi^5 = +\sin.6\Phi + 4\sin.5\Phi + 5\sin.2\Phi$
$32\sin.\Phi^2\cos.\Phi^4 = -\cos.6\Phi - 2\cos.4\Phi + \cos.2\Phi + 2$
$32\sin.\Phi^3\cos.\Phi^3 = -\sin.6\Phi \quad \ast \quad + 3\sin.2\Phi$
$32\sin.\Phi^4\cos.\Phi^2 = +\cos.6\Phi - 2\cos.4\Phi - \cos.2\Phi + 2$
$32\sin.\Phi^5\cos.\Phi = +\sin.6\Phi - 4\sin.4\Phi + 5\sin.2\Phi$
$32\sin.\Phi^6 = -\cos.6\Phi + 6\cos.4\Phi - 15\cos.2\Phi + 10$

$64\cos.\Phi^7 = +\cos.7\Phi + 7\cos.5\Phi + 21\cos.3\Phi + 35\cos.\Phi$
$64\sin.\Phi\cos.\Phi^6 = +\sin.7\Phi + 5\sin.5\Phi + 9\sin.3\Phi + 5\sin.\Phi$
$64\sin.\Phi^2\cos.\Phi^5 = -\cos.7\Phi - 3\cos.5\Phi - \cos.3\Phi + 5\cos.\Phi$
$64\sin.\Phi^3\cos.\Phi^4 = -\sin.7\Phi - \sin.5\Phi + 3\sin.3\Phi + 3\sin.\Phi$
$64\sin.\Phi^4\cos.\Phi^3 = +\cos.7\Phi - \cos.5\Phi - 3\cos.3\Phi + 3\cos.\Phi$
$64\sin.\Phi^5\cos.\Phi^2 = +\sin.7\Phi - 3\sin.5\Phi + \sin.3\Phi + 5\sin.\Phi$
$64\sin.\Phi^6\cos.\Phi = -\cos.7\Phi + 5\cos.5\Phi - 9\cos.3\Phi + 5\cos.\Phi$
$64\sin.\Phi^7 = -\sin.7\Phi + 7\sin.5\Phi - 21\sin.3\Phi + 35\sin.\Phi$

Que-

Quemadmodum autem formulas has commodius eruere liceat, deinceps docebo, inde quod cuiusque ordinis prima series ex praecedentibus est cognita.

SCHOLION 2.

27. Sin autem alter exponentium m et n sit numerus negatiuus, expressio inuenta seriem exhibebit infinitam, cuius formam in aliquot casibus inuestigare operae erit pretium. In hunc finem sequentia exempla adiungere visum est.

EXEMPLVM 1.

28. *Tangentem cuiusque anguli Φ, seu hanc expressionem $\frac{\sin.\Phi}{\cos.\Phi}$, in seriem conuertere, quae secundum sinus simplices procedat*

Forma hac comparata cum generali $\sin.\Phi^m \cos.\Phi^n$ erit $m=1$ et $n=-1$, vnde fit $f=-2$, et $g=o$, hincque elicientur valores sequentes:

$A=-2$	$A=-2$
$2B=+4$	$B=+2$
$3C=-4-2=-6$	$C=-2$
$4D=+4+4=+8$	$D=+2$
$5E=-4-6=-10$	$E=-2$
$6F=+4+8=+12$	$F=+2$

etc.

Cum nunc sit $m=1$, casus ad probl. V pertinet, eritque

$$\frac{2^o \sin.\Phi}{\cos.\Phi}=\sin.o\Phi-2\sin.(-2\Phi)+2\sin.(-4\Phi)-2\sin.(-6\Phi)+\text{etc.}$$

vnde concluditur fore:

$$\frac{\sin.\Phi}{\cos.\Phi} = 2\sin.2\Phi - 2\sin.4\Phi + 2\sin.6\Phi - 2\sin.8\Phi + 2\sin.10\Phi - \text{etc.}$$

EXEMPLVM 2.

29. *Cotangentem cuiusuis anguli* Φ, *seu hanc expressionem* $\frac{\cos.\Phi}{\sin.\Phi}$, *in seriem conuertere, quae secundum sinus simplices procedat.*

Pro hoc casu erit $m = -1$ et $n = 1$, vnde $f = 2$, et $g = 0$; ideoque obtinebitur

$A = 2$	$A = 2$
$2B = 4 - 0;$	$B = 2$
$3C = 4 + 2 = 6$	$C = 2$
$4D = 4 + 4 = 8$	$D = 2$
$5E = 4 + 6 = 10$	$E = 2$

etc.

At ob $m = -1$ hic casus ad Probl. VI. pertinet; eritque

$$\frac{2^0 \cos.\Phi}{\sin.\Phi} = -\sin.0\Phi - 2\sin.(-2\Phi) - 2\sin.(-4\Phi) - 2\sin.(-6\Phi) - \text{etc.}$$

quae reducitur ad

$$\frac{\cos.\Phi}{\sin.\Phi} = 2\sin.2\Phi + 2\sin.4\Phi + 2\sin.6\Phi + 2\sin.8\Phi + 2\sin.6\Phi + \text{etc.}$$

EXEMPLVM 3.

30. *Hanc expressionem* $\frac{\sin.\Phi}{\cos.\Phi^2}$ *in seriem conuertere.*

Ob $m = 1$ et $n = -2$ erit $f = -3$ et $g = -1$, vnde

$A = -3$

$A = -3$	$A = -3$
$2B = +9 + 1 = 10$	$B = +5$
$3C = -15 - 6 = -21$	$C = -7$
$4D = +21 + 15 = 36$	$D = +9$
$5E = -27 - 28 = -55$	$E = -11$

etc.

Ergo ob $m = 1$ ex Probl. V. habebitur:

$$\frac{\sin.\Phi}{2\cos.\Phi^2} = \sin.(-\Phi) - 3\sin.(-3\Phi) + 5\sin.(-5\Phi) - 7\sin.(-7\Phi) + \text{etc.}$$

quae reducitur ad hanc formam:

$$\frac{\sin.\Phi}{\cos.\Phi^2} = -2\sin\Phi + 6\sin.3\Phi - 10\sin 5.\Phi + 14\sin 7\Phi - 18\sin.9\Phi + \text{etc.}$$

cuius progressionis lex sponte patet.

SCHOLION 3.

31. Quoniam in his seriebus coefficientes A, B, C, D, etc. progressionem, vel terminorum aequalium, vel arithmeticam, constituere sunt inuenti, in genere obseruo; primo hos coefficientes secundum terminos aequales progredi, quoties fuerit $2 - f + g = 0$, seu $g - f = -2$, hoc est: si sit $m = -1$, hocque casu omnes terminos eodem signo fore affectos. Sin autem sit $n = -1$, terminos quidem fore aequales, sed signis alternis praeditos. Deinde noto, si sit vel $m = -2$, vel $n = -2$, seriem coefficientium A, B, C, etc. fore arithmeticam, priorique casu omnes terminos paribus, posteriori vero alternantibus signis affectos. Sin autem sit vel $m = -3$, vel $n = -3$, seriem prodire secundi ordinis, vel eodem, vel alternantibus, signis progredientem, et ita porro. Verum hic est animaduertendum, vt huiusmodi series, quales

 dixi,

dixi, proueniant, si pro m, vel n, numerus negatiuus integer accipiatur, alterum numerum oportere esse affirmatiuum integrum illo non maiorem.

PROBLEMA VII.

32. Si fuerit $S = A + B\cos.2\phi + C\cos.4\phi + D\cos.6\phi + \text{etc.}$ inuenire seriem ipsi $\frac{S\,\sin.\phi}{\cos.\phi}$ aequalem.

SOLVTIO.

Ponatur $\frac{S\,\sin.\phi}{\cos.\phi} = \beta\sin.2\phi + \gamma\sin.4\phi + \delta\sin.6\phi + \varepsilon\sin.8\phi + \text{etc.}$ eritque per $2\cos.\phi$ multiplicando:

$$2S\sin.\phi = \beta\sin.\phi + \beta\sin.3\phi + \gamma\sin.5\phi + \delta\sin.7\phi \;\text{etc.}$$
$$\qquad\qquad\qquad\quad +\gamma \qquad\quad +\delta \qquad\quad +\varepsilon$$

Quodsi autem ipsa series proposita per $2\sin.\phi$ multiplicetur prodibit:

$$2S\sin.\phi = 2A\sin.\phi + B\sin.3\phi + C\sin.5\phi + D\sin.7\phi$$
$$\qquad\qquad -B \qquad\quad -C \qquad\quad -D \qquad\quad -E \quad \text{etc.}$$

Similibus ergo terminis inter se aequandis obtinebitur

$$\beta = 2A - B$$
$$\gamma = B - C - \beta = -C + 2B - 2A$$
$$\delta = C - D - \gamma = -D + 2C - 2B + 2A$$
$$\varepsilon = D - E - \delta = -E + 2D - 2C + 2B - 2A$$
$$\text{etc.}$$

vnde valores coefficientium β, γ, δ, etc. facile definiuntur. Q. E. F.

COROLL. 1.

33. Si series S finito terminorum numero constet, altera series S tang. ϕ vel in infinitum excurret, vel

vel alicubi terminabitur, quod posterius eueniet, si fuerit $A - B + C - D +$ etc. $= 0$.

COROLL. 2.

34. At $A - B + C - D +$ etc. est valor seriei propositae S casu quo angulus Φ fit rectus; series ergo S tang. Φ non abrumpitur, nisi series S ita sit comparata, vt casu $\Phi =$ angulo recto in nihilum abeat.

PROBLEMA VIII.

35. Si proposita fuerit haec series:

$$S = B \sin. 2\Phi + C \sin. 4\Phi + D \sin. 6\Phi + E \sin. 8\Phi + \text{etc.}$$

inuenire seriem, quae exprimat valorem formulae $\frac{S \sin. \Phi}{\cos. \Phi}$.

SOLVTIO.

Ponatur series quaesita:

$$\frac{S \sin. \Phi}{\cos. \Phi} = \alpha + \beta \cos. 2\Phi + \gamma \cos. 4\Phi + \delta \cos. 6\Phi + \varepsilon \cos. 8\Phi + \text{etc.}$$

quae per $2 \cos. \Phi$ multiplicata dat:

$$\begin{aligned} 2S \sin. \Phi = 2\alpha \cos. \Phi &+ \beta \cos. 3\Phi + \gamma \cos. 5\Phi + \delta \cos. 7\Phi + \varepsilon \cos. 9\Phi + \text{etc.} \\ &+ \beta \qquad\quad + \gamma \qquad\quad + \delta \qquad\quad + \varepsilon \qquad\quad + \zeta \end{aligned}$$

vnde terminis similibus aequandis elicitur:

$$\begin{aligned} \beta &= B - 2\alpha \\ \gamma &= C - B - \beta = C - 2B + 2\alpha \\ \delta &= D - C - \gamma = D - 2C + 2B - 2\alpha \\ \varepsilon &= E - D - \delta = E - 2D + 2C - 2B + 2\alpha \\ &\text{etc.} \end{aligned}$$

Coefficiens ergo α manet indeterminatus, pro eoque pro lubitu valor assumi potest. Q. E. F.

COROL.

COROLL. 1.

36. Si ergo in ferie propofita ponatur $B=o$; $C=o$; $D=o$; etc. ita vt quoque fit $S=o$, fiet $\beta=-2\alpha$; $\gamma=+2\alpha$; $\delta=-2\alpha$; $\varepsilon=+2\alpha$; etc. in infinitum: vnde prodibit:

$$o=\alpha-2\alpha\operatorname{cof}.2\Phi+2\alpha\operatorname{cof}.4\Phi-2\alpha\operatorname{cof}.6\Phi+2\alpha\operatorname{cof}.8\Phi-\text{etc.}$$

Huiusmodi ergo feries, feriei cuicunque addita, eius fummam non mutat, vnde ratio patet, cur valor ipfius α non determinetur.

COROLL. 2.

37. Si feries S non in infinitum excurrat, tum femper pro α eiusmodi valor accipi poterit, vt etiam feries pro $\frac{S\,\text{fin}.\Phi}{\text{cof}.\Phi}$ non in infinitum excurrat. Scilicet fi feriei S omnes termini euanefcant, vt fit $S=o$, tum capiatur $\alpha=o$, fietque etiam $\frac{S\,\text{fin}.\Phi}{\text{cof}.\Phi}=o$

COROLL. 3.

38. Si feries S vnico termino conftet, feu fit $S=B\,\text{fin}.2\Phi$, fiat $\alpha=B$, vt fit $\beta=-B$, reperienturque $\gamma=o$, $\delta=o$, $\varepsilon=o$, etc. ficque prodibit $S\,\text{tang}.\Phi=B-B\,\text{cof}.2\Phi$.

COROLL. 4.

39. Si feries S duos tantum habeat terminos, vt fit

$$S=B\,\text{fin}.2\Phi+C\,\text{fin}.4\Phi$$

capiatur $\alpha=B-C$, fientque coefficientes δ, ε, ζ, etc. nihilo aequales, ita vt fit

S tang.

$$S \text{ tang.} \Phi = \alpha + \beta \cos . 2\Phi + \gamma \cos . 4\Phi$$

COROLL. 5.

40. Hinc igitur patet, si series S finito terminorum numero constet, vt etiam series S tang. $\Phi = \frac{S \sin . \Phi}{\cos . \Phi}$ fiat finita, tum valorem ipsius α ita capi oportere, vt sit

$$\alpha = B - C + D - E + F - G \text{ etc.}$$

quo assumto reliqui coefficientes facile reperientur.

PROBLEMA. IX.

41. Si proposita sit haec series

$$S = A \cos . \Phi + B \cos . 3\Phi + C \cos . 5\Phi + D \cos . 7\Phi + E \cos . 9\Phi + \text{etc.}$$

inuenire seriem, quae exprimat valorem formulae $\frac{S \sin \Phi}{\cos . \Phi}$.

SOLVTIO.

Ponatur series quaesita

$$\frac{S \sin . \Phi}{\cos . \Phi} = \alpha \sin . \Phi + \beta \sin . 3\Phi + \gamma \sin . 5\Phi + \delta \sin . 7\Phi + \varepsilon \sin . 9\Phi + \text{etc.}$$

quae per $2 \cos . \Phi$ multiplicata dat:

$$\begin{aligned} 2S \sin . \Phi = \alpha \sin . 2\Phi &+ \beta \sin . 4\Phi + \gamma \sin . 6\Phi + \delta \sin . 8\Phi + \varepsilon \sin . 10\Phi \text{ etc.} \\ +\beta \quad &+\gamma \qquad\quad +\delta \qquad\quad +\varepsilon \qquad\quad +\zeta \end{aligned}$$

At ipsa series proposita per $2 \sin . \Phi$ multiplicata dat:

$$\begin{aligned} 2S \sin . \Phi = A \sin . 2\Phi &+ B \sin . 4\Phi + C \sin 6\Phi + D \sin . 8\Phi + E \sin . 10\Phi \text{ etc.} \\ -B \quad &-C \qquad\quad -D \qquad\quad -E \qquad\quad -F \end{aligned}$$

vnde sequentes prodeunt determinationes

$$\beta = A - B - \alpha$$
$$\gamma = B - C - \beta = -C + 2B - A + \alpha$$
$$\delta = C - D - \gamma = -D + 2C - 2B + A - \alpha$$
$$\varepsilon = D - E - \delta = -E + 2D - 2C + 2B - A + \alpha$$
$$\zeta = E - F - \varepsilon = -F + 2E - 2D + 2C - 2B + A - \alpha$$
etc.

vbi iterum coefficiens α non determinatur, sed arbitrio nostro relinquitur. Q. E. I.

COROLL. 1.

42. Si omnes coefficientes A, B, C, etc. euanescant, vt sit $S=0$, fiet $\beta=-\alpha$; $\gamma=+\alpha$; $\delta=-\alpha$; $\varepsilon=+\alpha$; etc. ideoque erit

$$0=\alpha \sin.\Phi-\alpha \sin.3\Phi+\alpha \sin.5\Phi-\alpha \sin.7\Phi+\alpha \sin.9\Phi-\text{etc.}$$

seu $\sin.\Phi-\sin.3\Phi+\sin.5\Phi-\sin.7\Phi+\sin.9\Phi-\text{etc.}=0$

Supra autem inuenimus (34) esse

$$\cos.2\Phi-\cos.4\Phi+\cos.6\Phi-\cos.8\Phi+\cos.10\Phi-\text{etc.}=\tfrac{1}{2}$$

COROLL. 2.

43. Si ergo series proposita S finito terminorum numero constet, pro α eiusmodi valor accipi potest, vt etiam serie S tang. Φ finito terminorum numero constet. Capi scilicet debet:

$$\alpha=A-2B+2C-2D+2E-\text{etc.}$$

PROBLEMA X.

44. Si proposita sit haec series

$$S=A\sin.\Phi+B\sin.3\Phi+C\sin.5\Phi+D\sin.7\Phi+E\sin.9\Phi+\text{etc.}$$

inuenire seriem, quae exprimat valorem formulae $\frac{S\sin.\Phi}{\cos.\Phi}$

SOLVTIO.

Ponatur series quaesita:

$$\frac{S\sin.\Phi}{\cos.\Phi}=\alpha\cos.\Phi+\beta\cos.3\Phi+\gamma\cos.5\Phi+\delta\cos.7\Phi+\varepsilon\cos.9\Phi+\text{etc.}$$

quae per $2\cos.\Phi$ multiplicata dat:

2 S

$$2S\sin.\Phi = \alpha + \alpha\cos.2\Phi + \beta\cos.4\Phi + \gamma\cos.6\Phi + \delta\cos.8\Phi + \varepsilon\cos.10\Phi \text{ etc.}$$
$$\quad\quad + \beta \quad\quad + \gamma \quad\quad + \delta \quad\quad + \varepsilon \quad\quad + \zeta$$

Si autem ipsa series proposita per $2\sin.\Phi$ multiplicetur habebitur.

$$2S\sin\Phi = A - A\cos.2\Phi - B\cos.4\Phi - C\cos.6\Phi - D\cos.8\Phi - E\cos.10\Phi \text{ etc.}$$
$$\quad\quad + B \quad\quad + C \quad\quad + D \quad\quad + E \quad\quad + F$$

vnde sequentes eliciuntur coefficientium quaesitorum determinationes.

$$\alpha = A$$
$$\beta = B - A - \alpha = B - 2A$$
$$\gamma = C - B - \beta = C - 2B + 2A$$
$$\delta = D - C - \gamma = D - 2C + 2B - 2A$$
$$\varepsilon = E - D - \delta = E - 2D + 2C - 2B + 2A$$
$$\zeta = F - E - \varepsilon = F - 2E + 2D - 2C + 2B - 2A$$
etc.

Hoc igitur casu omnes coefficientes quaesiti determinantur, nullusque eorum arbitrio nostro relinquitur. Q. E. I.

COROLL. 1.

45. Si series proposita S finito terminorum numerorum constet, fieri potest, vt series $\frac{S\sin.\Phi}{\cos.\Phi}$ vel quoque sit finita, vel in infinitum excurrat. Prius eueniet, si coefficientes A, B, C, etc. ita fuerint comparati, vt sit

$$A - B + C - D + E - \text{etc.} = 0$$

COROLL. 2

46. Series autem proposita S abit in A-B+C-D+ etc. si angulus Φ statuatur rectus; quare si valor seriei S

 euanescat,

euanefcat pofito $\Phi = 90^\circ$, tum feries $\frac{S\,fin.\Phi}{cof.\Phi}$ finito conftabit terminorum numero, fiquidem feries S fuerit talis:

SCHOLION. 1.

47. Quatuor haec problemata methodum fuppeditant, formulas fupra (26) exhibitas facilius inueniendi: atque haec problemata ita adornaui, vt has formulas ordine retrogrado fcriptas praeberent. Cum enim valor expreffionis 2^{n-1} cof. Φ^n iam fupra per progreffionem cofinuum fimplicium fit erutus, inde horum problematum ope iftae formulae:

2^{n-1}fin.Φcof.Φ^{n-1}; 2^{n-1}fin.Φ^2cof.Φ^{n-2}; 2^{n-1}fin.Φ^3cof.Φ^{n-3}; etc.

in fimiles progreffiones conuerti poterunt: ac fi quidem exponens n fuerit numerus par, negotium per bina problemata priora conficietur, fi autem n fit numerus impar, per bina pofteriora. Quoniam has formulas iam ad poteftatem feptimam exhibuimus; fumamus poteftatem octauam ex §. 9.

$$128\,\text{cof.}\Phi^8 = 35 + 56\,\text{cof.}2\Phi + 28\,\text{cof.}4\Phi + 8\,\text{cof.}6\Phi + \text{cof.}8\Phi$$

et in problemate VII. fit $S = 128\,\text{cof.}\Phi^8$, erit

$A = 35$; $B = 56$, $C = 28$, $D = 8$, $E = 1$

vnde eruitur:

$$\beta = 70 - 56 = 14$$
$$\gamma = 56 - 28 - 14 = 14$$
$$\delta = 28 - 8 - 14 = 6$$
$$\varepsilon = 8 - 1 - 6 = 1$$

ficque erit:

$$128\,\text{fin.}\Phi\,\text{cof.}\Phi^7 = 14\,\text{fin.}2\Phi + 14\,\text{fin.}4\Phi + 6\,\text{fin.}6\Phi + \text{fin.}8\Phi$$

Sit

Sit nunc in problemate VIII. $S = 128 \sin.\Phi \cos.\Phi^7$ ideoque

$$B = 14;\ C = 14;\ D = 6;\ E = 1;$$

et capiatur $\alpha = B - C + D - E = 5$ (§. 40.) erit

$$\beta = 14 - 10 \quad = \ 4$$
$$\gamma = 14 - 14 - 4 \quad = -4$$
$$\delta = 6 - 14 + 4 \quad = -4$$
$$\varepsilon = 1 + 6 + 4 \quad = -1$$

et hanc ob rem:

$$128 \sin.\Phi^2 \cos.\Phi^6 = 5 + 4\cos.2\Phi - 4\cos.4\Phi - 4\cos.6\Phi - \cos.8\Phi$$

fit denuo in problemate VII. $S = 128 \sin.\Phi^2 \cos.\Phi^6$ et

$$A = 5;\ B = 4;\ C = -4;\ D = -4;\ E = -1$$

reperieturque:

$$\beta = 10 - 4 \quad = \ 6$$
$$\gamma = 4 + 4 - 6 \quad = \ 2$$
$$\delta = -4 + 4 - 2 \quad = -2$$
$$\varepsilon = -4 + 1 + 2 \quad = -1$$

Ergo

$$128 \sin.\Phi^3 \cos.\Phi^5 = 6\sin.2\Phi + 2\sin.4\Phi - 2\sin.6\Phi - \sin.8\Phi$$

Nunc in problemate VIII. fit $S = 128 \sin.\Phi^3 \cos.\Phi^5$ atque

$$B = 6;\ C = 2;\ D = -2;\ E = -1$$

et capiatur $\alpha = B - C + D - E = 3$

$$\beta = 6 - 6 \quad = \ 0$$
$$\gamma = 2 - 6 - 0 \quad = -4$$
$$\delta = -2 - 2 + 4 \quad = \ 0$$
$$\varepsilon = -1 + 2 - 0 \quad = \ 1 \text{ : ergo}$$

 128 sin.

$$128\,\text{fin.}\Phi^4\,\text{cof.}\Phi^4 = 3 * -4\,\text{cof.}4\Phi * +1\,\text{cof.}8\Phi.$$

Sit nunc in problemate VII. $S = 128\,\text{fin.}\Phi^4\,\text{cof.}\Phi^4$ et

$$A = 3;\ B = 0;\ C = -4;\ D = 0;\ E = 1.$$

sumaturque

$$\begin{aligned} \beta &= 6-0 &&= 6 \\ \gamma &= 0+4-6 &&= -2 \\ \delta &= -4-0+2 &&= -2 \\ \varepsilon &= 0-1+2 &&= +1 \text{ ergo} \end{aligned}$$

$$128\,\text{fin.}\Phi^5\,\text{cof.}\Phi^3 = 6\,\text{fin.}2\Phi\ \ 2\,\text{fin.}4\Phi - 2\,\text{fin.}6\Phi + \text{fin.}8\Phi$$

Sicque vlterius progrediendo obtinebimus in supplementum §. 26. has formulas inuertendo.

$$\begin{aligned}
128\,\text{cof.}\Phi^8 &= +\text{cof.}8\Phi + 8\,\text{cof.}6\Phi + 28\,\text{cof.}4\Phi + 56\,\text{cof.}2\Phi + 35 \\
128\,\text{fin.}\Phi\,\text{cof.}\Phi^7 &= +\text{fin.}8\Phi + 6\,\text{fin.}6\Phi + 14\,\text{fin.}4\Phi + 14\,\text{fin.}2\Phi \\
128\,\text{fin.}\Phi^2\,\text{cof.}\Phi^6 &= -\text{cof.}8\Phi - 4\,\text{cof.}6\Phi - 4\,\text{cof.}4\Phi + 4\,\text{cof.}2\Phi + 5 \\
128\,\text{fin.}\Phi^3\,\text{cof.}\Phi^5 &= -\text{fin.}8\Phi - 2\,\text{fin.}6\Phi + 2\,\text{fin.}4\Phi + 6\,\text{fin.}2\Phi \\
128\,\text{fin.}\Phi^4\,\text{cof.}\Phi^4 &= +\text{cof.}8\Phi \quad * \quad - 4\,\text{cof.}4\Phi \quad * \quad + 3 \\
128\,\text{fin.}\Phi^5\,\text{cof.}\Phi^3 &= +\text{fin.}8\Phi - 2\,\text{fin.}6\Phi - 2\,\text{fin.}4\Phi + 6\,\text{fin.}2\Phi \\
128\,\text{fin.}\Phi^6\,\text{cof.}\Phi^2 &= -\text{cof.}8\Phi + 4\,\text{cof.}6\Phi - 4\,\text{cof.}4\Phi - 4\,\text{cof.}2\Phi + 5 \\
128\,\text{fin.}\Phi^7\,\text{cof.}\Phi &= -\text{fin.}8\Phi + 6\,\text{fin.}6\Phi - 14\,\text{fin.}4\Phi + 14\,\text{fin.}2\Phi \\
128\,\text{fin.}\Phi^8 &= +\text{cof.}8\Phi - 8\,\text{cof.}6\Phi + 28\,\text{cof.}4\Phi - 36\,\text{cof.}2\Phi + 35
\end{aligned}$$

SCHOLION. 2.

48. Simili modo, pro vsu probl. IX. et X. ostendendo, sit

$$256\,\text{cof.}\Phi^9 = 126\,\text{cof.}\Phi + 84\,\text{cof.}3\Phi + 36\,\text{cof.}5\Phi + 9\,\text{cof.}7\Phi + \text{cof.}9\Phi$$

Sitque in probl. IX. $S = 256\,\text{cof.}\Phi^9$, atque

A =

$A = 126$; $B = 84$, $C = 36$, $D = 9$; $E = 1$

capiatur $\alpha = A - 2B + 2C - 2D + 2E = 14$

$\beta = 126 - 84 - 14 = 28$

$\gamma = 84 - 36 - 28 = 20$

$\delta = 36 - 9 - 20 = 7$

$\varepsilon = 9 - 1 - 7 = 1$ ergo

$$256 \sin.\Phi \cos.\Phi^8 = 14 \sin.\Phi + 28 \sin.3\Phi + 20 \sin.5\Phi + 7 \sin.7\Phi + \sin.9\Phi$$

In probl. X. fit $S = 256 \sin.\Phi \cos.\Phi^8$ et

$A = 14$; $B = 28$, $C = 20$, $D = 7$; $E = 1$

$\alpha = 14$

$\beta = 28 - 14 - 14 = 0$

$\gamma = 20 - 28 - 0 = -8$

$\delta = 7 - 20 + 8 = -5$

$\varepsilon = 1 - 7 + 5 = -1$ ergo

$$256 \sin.\Phi^2 \cos.\Phi^7 = 14 \cos.\Phi * - 8 \cos.5\Phi - 5 \cos.7\Phi - \cos.9\Phi$$

In probl. IX. fit. $S = 256 \sin,\Phi^2 \cos,\Phi^7$ et

$A = 14$; $B = 0$; $C = -8$, $D = -5$, $E = -1$

capiatur $\alpha = 14 - 0 - 16 + 10 - 2 = +6$

$\beta = 14 - 0 - 6 = +8$

$\gamma = 0 + 8 - 8 = 0$

$\delta = -8 + 5 - 0 = -3$

$\varepsilon = -5 + 1 + 3 = -1$ ergo

$$256 \sin.\Phi^3 \cos.\Phi^6 = 6 \sin.\Phi + 8 \sin.3\Phi * - 3 \sin.7\Phi - \sin.9\Phi$$

In

In probl. X. fit $S = 256 \sin.\Phi^3 \cos.\Phi^6$, et

$$A=6;\; B=8;\; C=0;\; D=-3;\; E=-1$$

fiet

$$\alpha = 6$$
$$\beta = 8-6-6 = -4$$
$$\gamma = 0-8+4 = -4$$
$$\delta = -3-0+4 = +1$$
$$\varepsilon = -1+3-1 = -1 \quad \text{ergo}$$

$$256 \sin.\Phi^4 \cos.\Phi^5 = 6\cos.\Phi\ 4\cos.3\Phi - 4\cos.5\Phi + \cos.7\Phi + \cos.9\Phi$$

vnde in supplementum §. 26. habebitur

$$256 \cos.\Phi^9 = +\cos.9\Phi + 9\cos.7\Phi + 36\cos.5\Phi + 84\cos.3\Phi + 126\cos.\Phi$$
$$256 \sin.\Phi \cos.\Phi^8 = +\sin.9\Phi + 7\sin.7\Phi + 20\sin.5\Phi + 28\sin.3\Phi + 14\sin.\Phi$$
$$256 \sin.\Phi^2 \cos.\Phi^7 = -\cos.9\Phi - 5\cos.7\Phi - 8\cos.5\Phi \quad * \quad + 14\cos.\Phi$$
$$256 \sin.\Phi^3 \cos.\Phi^6 = -\sin.9\Phi + 3\sin.7\Phi \quad * \quad + 8\sin.3\Phi + 6\sin.\Phi$$
$$256 \sin.\Phi^4 \cos.\Phi^5 = +\cos.9\Phi - \cos.7\Phi - 4\cos.5\Phi - 4\cos.3\Phi + 6\cos.\Phi$$
$$256 \sin.\Phi^5 \cos.\Phi^4 = +\sin.9\Phi - \sin.7\Phi - 4\sin.5\Phi + 4\sin.3\Phi + 6\sin.\Phi$$
$$256 \sin.\Phi^6 \cos.\Phi^3 = -\cos.9\Phi + 3\cos.7\Phi \quad * \quad - 8\cos.3\Phi + 6\cos.\Phi$$
$$256 \sin.\Phi^7 \cos.\Phi^2 = -\sin.9\Phi + 5\sin.7\Phi - 8\sin.5\Phi \quad * \quad + 14\sin.\Phi$$
$$256 \sin.\Phi^8 \cos.\Phi = +\cos.9\Phi - 7\cos.7\Phi + 20\cos.5\Phi - 28\cos.3\Phi + 14\cos.\Phi$$
$$256 \sin.\Phi^9 = +\sin.9\Phi - 9\sin.7\Phi + 36\sin.5\Phi - 84\sin.3\Phi + 126\sin.\Phi$$

Hoc igitur modo istas formulas, quo vsque libuerit, continuare licet.

THEOREMA.

49. Si assignari queat summa huius seriei

$$Az^m + Bz^{m+n} + Cz^{m+2n} + Dz^{m+3n} Ez^{m+4n} + \text{etc.} = Z$$

semper quoque exhiberi poterunt summae harum serierum:

$$A\cos.m\Phi + B\cos.(m+n)\Phi + C\cos.(m+2n)\Phi + D\cos.(m+3n)\Phi + \text{etc.}$$

$$A\sin.m\Phi + B\sin.(m+n)\Phi + C\sin.(m+2n)\Phi + D\sin.(m+3n)\Phi + \text{etc.}$$

DEMON-

DEMONSTRATIO.

Ponantur ſummae harum ſerierum:

$$A\operatorname{cof}. m\Phi + B\operatorname{cof}.(m+n)\Phi + C\operatorname{cof}.(m+2n)\Phi + D\operatorname{cof}.(m+3n)\Phi + \text{etc.} = S$$

$$A\operatorname{fin}. m\Phi + B\operatorname{fin}.(m+n)\Phi + C\operatorname{fin}(m+2n)\Phi + D\operatorname{fin}.(m+3n)\Phi + \text{etc.} = T$$

ſitque vt ſupra

$$\operatorname{cof}.\Phi + \sqrt{-1}.\operatorname{fin}.\Phi = u \text{ et } \operatorname{cof}.\Phi - \sqrt{-1}.\operatorname{fin}.\Phi = v$$

erit $\operatorname{cof}.\nu\Phi + \sqrt{-1}.\operatorname{fin}.\nu\Phi = u^\nu$ et $\operatorname{cof}.\nu\Phi - \sqrt{-1}.\operatorname{fin}.\nu\Phi = v^\nu$

Hinc ergo erit

$$S + T\sqrt{-1} = Au^m + Bu^{m+n} + Cu^{m+2n} + Du^{m+3n} + \text{etc.} = U$$

$$S - T\sqrt{-1} = Av^m + Bv^{m+n} + Cv^{m+2n} + Dv^{m+3n} + \text{etc.} = V$$

Summae ſcilicet harum ſerierum U et V per hypotheſin dantur, cum U et V tales ſint functiones ipſarum u et v, qualis functio Z eſt ipſius z. Hinc itaque elicitur $S = \frac{U+V}{2}$ et $T = \frac{U-V}{2\sqrt{-1}}$; ideoque ſummae propoſitarum ſerierum S et T innoteſcunt. Q. E. D.

COROLL. 1.

50. Cum ſit $z^m + az^{m+n} + a^2z^{m+2n} + a^3z^{m+3n} + \text{etc.}$

$$= \frac{z^m}{1 - a z^n}$$

erit $U = \frac{u^m}{1 - au^n}$ et $V = \frac{v^m}{1 - av^n}$;

Hinc $U + V = \frac{u^m + v^m - a(u^{m-n} + v^{m-n})u^n v^n}{1 - a(u^n + v^n) + aau^n v^n}$

$$U - V = \frac{u^m - v^m - a(u^{m-n} - v^{m-n})u^n v^n}{1 - a(v^n + v^n) + aau^n v^n}$$

At est $uv = 1$, $u^\nu + v^\nu = 2\,\text{cos}.\,\nu\Phi$; $u^\nu - v^\nu = 2\sqrt{-1}.\,\text{sin}.\,\nu\Phi$, vnde fit

$$\frac{U+V}{2} = \frac{\text{cos}.\,m\Phi - a\,\text{cos}.\,(m-n)\Phi}{1 + aa - 2a\,\text{cos}.\,n\Phi} = S \quad \text{et}$$

$$\frac{U-V}{2\sqrt{-1}} = \frac{\text{sin}.\,m\Phi - a\,\text{sin}.\,(m-n)\Phi}{1 + aa - 2a\,\text{cos}.\,n\Phi} = T$$

Ex quo sequentes habentur summationes:

$$\text{cos}.\,m\Phi + a\,\text{cos}.(m+n)\Phi + a^2\text{cos}.(m+2n)\Phi + a^3\text{cos}.(m+3n)\Phi + \text{etc.} = \frac{\text{cos}\,m\Phi - a\,\text{cos}.(m-n)\Phi}{1 + aa - 2a\,\text{cos}.\,n\Phi}$$

$$\text{sin}.\,m\Phi + a\,\text{sin}.(m+n)\Phi + a^2\,\text{sin}.(m+2n)\Phi + a^3\,\text{sin}\,(m+3n)\Phi + \text{etc.} = \frac{\text{sin}.\,m\Phi - a\,\text{sin}.(m-n)\Phi}{1 + aa - 2a\,\text{cos}.\,n\Phi}$$

COROLL. 2.

51. Sit $m = 1$ et $n = 1$ erit

$$\text{cos}.\,\Phi + a\,\text{cos}.\,2\Phi + a^2\text{cos}.\,3\Phi + a^3\text{cos}.\,4\Phi + \text{etc.} = \frac{\text{cos}.\,\Phi - a}{1 + aa - 2a\,\text{cos}.\,\Phi}$$

$$\text{sin}\,\Phi + a\,\text{sin}.\,2\Phi + a^2\text{sin}.\,3\Phi + a^3\text{sin}.\,4\Phi + \text{etc.} = \frac{\text{sin}.\,\Phi}{1 + aa - 2a\,\text{cos}.\,\Phi}$$

Si insuper sit $a = 1$ erit

$$\text{cos}.\,\Phi + \text{cos}.\,2\Phi + \text{cos}.\,3\Phi + \text{cos}.\,4\Phi + \text{etc.} = \frac{\text{cos}.\,\Phi - 1}{2 - 2\,\text{cos}.\,\Phi} = -\tfrac{1}{2}$$

$$\text{sin}.\,\Phi + \text{sin}.\,2\Phi + \text{sin}.\,3\Phi + \text{sin}.\,4\Phi + \text{etc.} = \frac{\text{sin}.\,\Phi}{2 - 2\,\text{cos}.\,\Phi} = \frac{1}{2\,\text{tang}.\,\frac{1}{2}\Phi}$$

Sin autem sit $a = -1$ erit

$$\text{cos}.\,\Phi - \text{cos}.\,2\Phi + \text{cos}.\,3\Phi - \text{cos}.\,4\Phi + \text{etc.}\ \frac{\text{cos}.\,\Phi + 1}{2 + 2\,\text{cos}.\,\Phi} = \tfrac{1}{2}$$

$$\text{sin}.\,\Phi - \text{sin}.\,2\Phi + \text{sin}.\,3\Phi - \text{sin}.\,4\Phi + \text{etc.} = \frac{\text{sin}.\,\Phi}{2 + 2\,\text{cos}.\,\Phi} = \tfrac{1}{2}\,\text{tang}.\,\tfrac{1}{2}\Phi$$

COROL.

COROLL. 3.

52. Sit $m=1$, et $n=2$, erit

$$\text{cof.}\Phi+a\,\text{cof.}3\Phi+a^2\text{cof.}5\Phi+a^3\text{cof.}7\Phi+\text{etc.}=\frac{\text{cof.}\Phi-a\,\text{cof.}\Phi}{1+aa-2a\,\text{cof.}2\Phi}$$

$$\text{fin.}\Phi+a\,\text{fin.}3\Phi+a^2\text{fin.}5\Phi+a^3\text{fin.}7\Phi+\text{etc.}=\frac{\text{fin.}\Phi+a\,\text{fin.}\Phi}{1+aa-2a\,\text{cof.}2\Phi}$$

Quodfi ergo fit $a=1$, erit

$$\text{cof.}\Phi+\text{cof.}3\Phi+\text{cof.}5\Phi+\text{cof.}7\Phi+\text{etc.}=0$$

$$\text{fin.}\Phi+\text{fin.}3\Phi+\text{fin.}5\Phi+\text{fin.}7\Phi+\text{etc.}=\frac{\text{fin.}\Phi}{1-\text{cof.}2\Phi}=\frac{1}{2\,\text{fin.}\Phi}$$

Sin autem fit $a=-1$, erit

$$\text{cof.}\Phi-\text{cof.}3\Phi+\text{cof.}5\Phi-\text{cof.}7\Phi+\text{etc.}=\frac{2\,\text{cof.}\Phi}{2+2\,\text{cof.}2\Phi}=\frac{1}{2\,\text{cof.}\Phi}$$

$$\text{fin.}\Phi-\text{fin.}3\Phi+\text{fin.}5\Phi-\text{fin.}7\Phi+\text{etc.}=0.$$

SCHOLION.

53. Ope huius theorematis ergo, cuius vfus latiffime patet, innumerabiles exhiberi poffunt feries fecundum, vel finus, vel cofinus, multiplorum cuiuspiam anguli progredientes, quarum fumma conftat. Cafum hic quidem tantum euolui, quo coefficientes A, B, C, D etc. in geometrica progreffione progrediuntur, verum pari modo calculus ad alias feries accommodatur. Praeterea autem hic notaffe fufficiat, ex feriebus iam inuentis, innumerabiles alias, tam per differentiationem, quam integrationem, elici poffe. Veluti cum fit

$$\text{cof.}\Phi-\text{cof.}2\Phi+\text{cof.}3\Phi-\text{cof.}4\Phi+\text{cof.}5\Phi-\text{etc.}=\tfrac{1}{2}$$

erit differentiando:

$$\text{fin.}\Phi-2\,\text{fin.}2\Phi+3\,\text{fin.}3\Phi-4\,\text{fin.}4\Phi+5\,\text{fin.}5\Phi-\text{etc.}=0$$

denuoque differentiando

$\cos.\Phi - 4\cos.2\Phi + 9\cos.3\Phi - 16\cos.4\Phi + 25\cos.5\Phi - \text{etc.} = 0$

et ita porro.

Illa autem series per $d\Phi$ multiplicata et integrata dat:

$\sin.\Phi - \frac{1}{2}\sin.2\Phi + \frac{1}{3}\sin.3\Phi - \frac{1}{4}\sin.4\Phi + \frac{1}{5}\sin.5\Phi - \text{etc.} = \frac{\Phi}{2}$

vbi additione constantis non est opus, cum posito $\Phi = 0$ summa sponte euanescat. Si haec per $-d\Phi$ multiplicata denuo integretur, prodibit

$$\cos.\Phi - \tfrac{1}{4}\cos.2\Phi + \tfrac{1}{9}\cos.3\Phi - \tfrac{1}{16}\cos.4\Phi + \text{etc.} = \alpha - \frac{\Phi\Phi}{4}$$

ideoque posito $\Phi = 0$

$$1 - \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{16} + \text{etc.} = \alpha = \frac{\pi\pi}{12} \text{ vt aliunde constat.}$$

Quare si sit $\Phi = \frac{\pi}{\sqrt{3}} = 103^{0}, 55^{I}, 22^{II}, 58^{III}, 27^{IV}$, summa istius seriei euanescit. Plurimas alias autem insignes huiusmodi serierum affectiones, ne nimis sim longus, hic praetermitto.

DE