



DE
EXPRESSIONE INTEGRALIVM
PER FACTORES.

Auctore

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In integralibus ad series infinitas reuocandis Geometrae adhuc plurimum fuere occupati, cum ad naturam serierum accuratius perspiciendam, tum ob summum usum, quem series praestant ad integralium valores proxime cognoscendos. Iam vero ostendi in Tomo Comment. Petrop. XI. ob easdem rationes reductionem integralium ad producta ex infinitis factoribus constantia non minus esse dignam, quae omni cura excolatur, ibique plurima iam huius reductionis dedi specimina, quae in Analyti haud contemnendum usum afferre videntur; etiamsi ipsa pertractatio nondum satis fuerit polita atque in ordinem digesta. Quam ob rem hoc argumentum denso ita resumere est visum, ut primo fundamenta, quibus innitur, luculentius exponerem, tum vero plures casus, qui imprimis memorabiles videntur, accuratius euoluerem.

Ante omnia autem notari conuenit, hunc modum, integralia per factores exprimendi, non in genere ita tradi posse, ut ad omnes quantitatis variabilis valores aequae pateat, ad quod institutum series infinitae potissimum sunt accommodatae, sed factores tum solum

commode in usum vocari possunt, quando is integralis tantum valor inuestigatur, cum variabili valor quidem determinatus tribuitur. Neque vero hunc valorem pro lubitu assumere licet, sed potius ita comparatum esse oportet, ut iam in formula differentiali singulari gaudeat proprietate, dum eam, vel ad nihilum, vel ad infinitum, redigit.

Huiusmodi autem casus iam prae ceteris notatur imprimis digni, atque in applicatione ad praxin potissimum quaeri solent, ita ut plerumque quaestio versari soleat in valore integralium pro huiusmodi quodam casu inueniundo. Ita si de circuli quadratura agitur, vel huius formulae $\int \frac{dx}{\sqrt{1-xx}}$ valor desideratur, casu, quo $x=1$, vel huius formulae $\int \frac{dx}{1+xx}$, casu, quo $x=\infty$: ibi autem hoc casu differentiale ipsum euadit infinitum, hic vero euanescit.

Quo igitur rem generalius complectar, duplicis generis formulas integrales hic euoluam: quae sint $\int x^{m-1} dx (1-x^n)^k$ et $\int \frac{x^{m-1} dx}{(1+x^n)^k}$, quarum vtramque ita integrari assumo, ut euanescat posito $x=0$. Tum vero prioris integralis $\int x^{m-1} dx (1-x^n)^k$ eum tantum valorem determinare in animo est, quem accipit si ponatur $x=1$: posterioris vero integralis $\int \frac{x^{m-1} dx}{(1+x^n)^k}$ illum valorem, quem casu $x=\infty$ fortitur, tantum inuestigabo. Euidens autem est hos integralium casus prae reliquis tali eminenti praerogatiua gaudere, ut imprimis euolui mereantur.

Quan-

Quoniam hic elegantiae consulens coefficientes omisi, tamen perspicuum est, has formulas aequè late patere, ac si tales coefficientes essent adiecti. Formula namque huiusmodi $\int \gamma y^{m-1} dy (a - \beta y^n)^k$, posito $\frac{\beta y^n}{\alpha} = x^n$ manifesto ad allatam $\int x^{m-1} dx (1 - x^n)^k$, reducitur, neque propterea latius patere est censenda, ac simili reductione haec formula $\int \frac{\gamma y^{m-1} dy}{(\alpha + \beta y^n)^k}$ in altera $\int \frac{x^{m-1} dx}{(1 + x^n)^k}$ continetur, unde omnino superfluum esset, loco formularum nostrarum simpliciori specie expressarum, has magis complicatas adhibere velle.

Verum etiam altera formularum sumtarum in altera continetur, ita ut sufficiat alterutram tantummodo, quam sum traditurus, tractasse.

Si enim ponatur $x = \frac{y}{(1 + y^n)^{\frac{1}{n}}}$, erit $1 - x^n = \frac{1}{1 + y^n}$:
 $x^m = \frac{y^m}{(1 + y^n)^{\frac{m}{n}}}$ et $\frac{dx}{x} = \frac{dy}{y(1 + y^n)}$: quibus valoribus substitutis obtinebitur

$$\int x^{m-1} dx (1 - x^n)^k = \int \frac{y^{m-1} dy}{(1 + y^n)^{k + 1 + \frac{m}{n}}}$$

his integralibus ita sumtis, ut evanescant posito $x = 0$ et $y = 0$; quae conditio hic semper subintelligi debet. Cum igitur posito $y = \infty$, fiat $x = 1$, habebimus sequens Theorema.

Theorema I.

I. Valor formulae integralis $\int x^{m-1} dx (1-x^n)^k$ casu $x=1$, aequalis est valori huius formulae integra-

lis $\int \frac{y^{m-1} dy}{(1+y^n)^{k+1+\frac{m}{n}}}$ casu $y=\infty$.

Cuius aequalitatis ratio est, quod illa forma actu transmutatur in hanc, si ponatur $x = \frac{y}{(1+y^n)^{\frac{1}{n}}}$

Sequens Theorema, quod per similem reductionem oritur, non parum quoque utilitatis habebit, quod ideo cum sua demonstratione apponam.

Theorema 2.

2. Valor huius formulae integralis $\int x^{m-1} dx (1-x^n)^k$ casu $x=1$, aequalis est valori huius formulae integralis $\int y^{nk+n-1} dy (1-y^n)^{\frac{m-n}{n}}$ etiam casu $y=1$.

Demonstratio.

Ponatur $x = (1-y^n)^{\frac{1}{n}}$, ut sit $1-x^n = y^n$; $x^m = (1-y^n)^{\frac{m}{n}}$ et $\frac{dx}{x} = \frac{-y^{n-1} dy}{1-y^n}$, quibus valoribus substitutis habebitur

$$x^{m-1} dx (1-x^n)^k = -y^{nk+n-1} dy (1-y^n)^{\frac{m-n}{n}}$$

Sit $Y = \int y^{nk+n-1} dy (1-y^n)^{\frac{m-n}{n}}$, integrali ita sumto, ut evanescat posito $y=0$; tum posito $y=1$, abeat Y in A . lam cum illas formulas ita integrari oporteat, ut evanescant posito $x=0$, quo casu fit $y=1$, erit:

$$\int x^{m-1} dx (1-x^n)^k = A - Y.$$

Pona-

Ponatur nunc $x = 1$, quo casu fit $y = 0$ ideoque et $Y = 0$, et formula nostra integralis fiet $= A$: seu integrale $\int x^{m-1} dx (1-x^n)^k$ casu $x = 1$ aequale erit integrali $\int y^{nk+n-1} dy (1-y^n)^{\frac{m-n}{n}}$ casu $y = 1$. Q. E. D.

Coroll. 1.

3. Cum igitur hae tres formulae :

I. $\int x^{m-1} dx (1-x^n)^k$; II. $\int \frac{y^{m-1} dy}{(1+y^n)^{k+1+\frac{m}{n}}}$; III. $\int z^{nk+n-1} dz$

$(1-z^n)^{\frac{m-n}{n}}$ ita a se inuicem pendeant, vt prima transeat in secundam posito $x = \frac{y}{(1+y^n)^{\frac{1}{n}}}$, at vero posito $x = (1-z^n)^{\frac{1}{n}}$ ea abeat in tertiam negative sumtam, manifestum est, quoties vna harum fuerit absolute integrabilis, toties et binas reliquas fore absolute integrabiles.

Coroll. 2.

4. Prima autem absolute est integrabilis, vti per se est perspicuum, si sit k numerus integer affirmatiuus: quicunque numerus pro m statuatur. Excipiuntur tamen casus, quibus m aequatur cuiquam numero huius progressionis: 0 ; $-n$; $-2n$; $-3n$; $-kn$; his enim casibus pars integralis pendeat a logarithmis. Casus ergo hi excipiendi buc redeunt, vt integratio absoluta succedat, existente k numero integro affirmatiuo, nisi $-\frac{m}{n}$ sit numerus integer affirmatiuus, vel minor, quam k , vel ipsi k aequalis. Vel nisi $k + \frac{m}{n}$ sit numerus integer affirmatiuus non maior quam k .

Coroll.

Coroll. 3.

5. Simili modo forma secunda erit integrabilis, si $-k-1-\frac{m}{n}$ fuerit numerus integer affirmatiuus, puta i ; casus autem excipiuntur, quibus $-\frac{m}{n}$ pariter est numerus integer affirmatiuus non maior quam i . Vel si denotet ω numerum quemcunque affirmatiuum integrum ex hac serie:

$$0, 1, 2, \dots, i,$$

casus excipiuntur, quibus $-\frac{m}{n} = \omega$.

Coroll. 4.

6. Tertia autem formula absolute erit integrabilis, si $\frac{m-n}{n}$ fuerit numerus integer affirmatiuus, puta i ; excipiuntur autem casus, quibus $-k-1 = \omega$, denotante ω numerum quemcunque integrum affirmatiuum, non maiorem, quam i .

Coroll. 5.

7. His ergo notatis formula $\int x^{m-1} dx (1-x^n)^k$ absolute erit integrabilis, casibus sequentibus, in quibus i numerum affirmatiuum integrum quemcunque denotat, ω autem quemlibet numerum integrum affirmatiuum ipso i non maiorem.

I. Si $k = i$ neque tamen $-\frac{m}{n} = \omega$.

II. Si $k-1-\frac{m}{n} = i$ neque tamen $-\frac{m}{n} = \omega$ (vel $-k-1 = \omega$)

III. Si $\frac{m-n}{n} = i$ neque tamen $-k-1 = \omega$.

Coroll. 6.

8. Manifestum autem est, hos eisdem integrabilitatis casus locum esse habituros in formula hac latius paten.

patente $\int x^{m-1} dx (a + bx^n)^k$, pro quo demonstratio pari modo adornatur. Atque ex his tribus conditionibus casus integrabilitatis omnium huiusmodi formularum diiudicari solent.

Quoniam haec ad meum institutum non pertinent, tamen quia tam facile ex binis Theorematis praemissis fluunt, non incongruum est visum, ea his adicere. Nunc igitur ad verum fundamentum dicendorum progredior, quod reductione integralium ad alias formas nititur. Quam quo distinctius exponam, hanc formam algebraicam contemplor: $x^\alpha (1-x^n)^\gamma = P$, qua differentiatia obtineo:

$$dP = \alpha x^{\alpha-1} dx (1-x^n)^\gamma - \gamma n x^{\alpha+n-1} dx (1-x^n)^{\gamma-1}$$

quae adhuc aliis modis in duo membra dispefci potest, veluti:

$$dP = \alpha x^{\alpha-1} dx (1-x^n)^{\gamma-1} - (\alpha + \gamma n) x^{\alpha+n-1} dx (1-x^n)^{\gamma-1}$$

Tum vero, si in membro posteriori pro x^n scribatur $1 - (1-x^n)$, prior forma dabit

$$dP = (\alpha + \gamma n) x^{\alpha-1} dx (1-x^n)^\gamma - \gamma n x^{\alpha-1} dx (1-x^n)^{\gamma-1}$$

posterior vero eodem redit. Vnde integrando obtineamus

$$P = \alpha \int x^{\alpha-1} dx (1-x^n)^\gamma - \gamma n \int x^{\alpha+n-1} dx (1-x^n)^{\gamma-1}$$

$$P = \alpha \int x^{\alpha-1} dx (1-x^n)^{\gamma-1} - (\alpha + \gamma n) \int x^{\alpha+n-1} dx (1-x^n)^{\gamma-1}$$

$$P = (\alpha + \gamma n) \int x^{\alpha-1} dx (1-x^n)^\gamma - \gamma n \int x^{\alpha-1} dx (1-x^n)^{\gamma-1}$$

Quae integralia cum evanescere debeant posito $x=0$, necesse est, vt eodem casu $P x^\alpha (1-x^n)^\gamma$ evanescat, quod quidem semper fit, si α sit numerus positivus quicunque. Iam si γ quoque fuerit numerus positivus, eui-

dens est posito $x=1$, et hoc casu fieri $P=0$: vnde sequentia elicimus Theoremata:

Theorema. 3.

9. Si α et γ fuerint numeri positivi, ac post integrationem ponatur $x=1$, habebuntur sequentes formularum integralium aequalitates.

$$\text{I. } \alpha \int x^{\alpha-1} dx (1-x^n)^\gamma = \gamma n \int x^{\alpha+n-1} dx (1-x^n)^{\gamma-1}$$

$$\text{II. } \alpha \int x^{\alpha-1} dx (1-x^n)^{\gamma-1} = (\alpha+\gamma n) \int x^{\alpha+n-1} dx (1-x^n)^{\gamma-1}$$

$$\text{III. } (\alpha+\gamma n) \int x^{\alpha-1} dx (1-x^n)^\gamma = \gamma n \int x^{\alpha-1} dx (1-x^n)^{\gamma-1}.$$

Demonstratio.

Cum enim post integrationem ponatur $x=1$, pro hoc casu in superioribus formulis fit $P=0$, indeque aperte sequuntur aequationes hic propositae.
Q. E. D.

Coroll. 1.

10. Harum trium aequationum quaelibet iam in duabus reliquis continetur, vnde eae in hac forma comprehenduntur:

$$\int x^{\alpha+n-1} dx (1-x^n)^{\gamma-1} = \frac{\alpha}{\gamma n} \int x^{\alpha-1} dx (1-x^n)^\gamma = \frac{\alpha}{\alpha+\gamma n} \int x^{\alpha-1} dx (1-x^n)^{\gamma-1}$$

seu sequentes tres formulae integrales inter se aequantur:

$$\frac{\alpha}{\alpha+\gamma n} \int x^{\alpha+n-1} dx (1-x^n)^{\gamma-1} = \frac{\alpha}{\gamma n} \int x^{\alpha-1} dx (1-x^n)^\gamma = \frac{\alpha}{\alpha+\gamma n} \int x^{\alpha-1} dx (1-x^n)^{\gamma-1}$$

si quidem α et γ fuerint numeri positivi.

Coroll. 2.

11. Cum sit per Theor. 2. $\int x^{m-1} dx (1-x^n)^k$
 $= \int x^{nk+n-1} dx (1-x^n)^{\frac{m-n}{n}}$ posito itidem $x=1$, aequalitas

litas habebitur inter sex sequentes formulas integrales :

I. $\frac{1}{\alpha} \int x^{\alpha+n-1} dx (1-x^n)^{\gamma-1}$; II. $\frac{1}{\gamma n} \int x^{\alpha-1} dx (1-x^n)^{\gamma}$; III. $\frac{1}{\alpha+\gamma n} \int x^{\alpha-1} dx (1-x^n)^{\gamma-1}$
 IV. $\frac{1}{\alpha} \int x^{n\gamma-1} dx (1-x^n)^{\frac{\alpha}{n}}$; V. $\frac{1}{\gamma n} \int x^{n\gamma-1+n-1} dx (1-x^n)^{\frac{\alpha-n}{n}}$; VI. $\frac{1}{\alpha+\gamma n} \int x^{n\gamma-1} dx (1-x^n)^{\frac{\alpha-n}{n}}$
 dummodo exponentes α et γ fuerint affirmatiui.

Coroll. 3.

12. Si α fuerit numerus infinitus,
 erit $\int x^{\alpha+n-1} dx (1-x^n)^{\gamma-1} = \int x^{\alpha-1} dx (1-x^n)^{\gamma-1}$
 atque ob eandem rationem,
 erit $\int x^{\alpha+2n-1} dx (1-x^n)^{\gamma-1} = \int x^{\alpha+n-1} dx (1-x^n)^{\gamma-1} = \int x^{\alpha-1} dx (1-x^n)^{\gamma-1}$
 vnde generatim colligitur fore $\int x^{\alpha+\mu-1} dx (1-x^n)^{\gamma-1} = \int x^{\alpha-1} dx (1-x^n)^{\gamma-1}$
 dummodo μ fuerit numerus finitus existente α infinito.

Coroll. 4.

13. Pari modo si γ fuerit numerus infinitus,
 erit
 $\int x^{\alpha-1} dx (1-x^n)^{\gamma} = \int x^{\alpha-1} dx (1-x^n)^{\gamma-1}$
 eodemque modo erit $\int x^{\alpha-1} dx (1-x^n)^{\gamma+1} = \int x^{\alpha-1} dx (1-x^n)^{\gamma}$,
 vnde generatim colligitur fore :
 $\int x^{\alpha-1} dx (1-x^n)^{\gamma \pm \mu} = \int x^{\alpha-1} dx (1-x^n)^{\gamma}$
 siquidem μ sit numerus finitus existente γ infinito.

Problema 1.

14. Si m et k sint numeri positivi, atque i denotet numerum integrum affirmatiuum quemcunque, definitur rationem formulae $\int x^{m-1} dx (1-x^n)^{k-1}$ ad formulam $\int x^{m-1} dx (1-x^n)^{k+i-1}$ casu $x=1$.

Q 2

SOLV-

Solutio.

Cum sit $\int x^{\alpha-1} dx (1-x^n)^{\gamma-1} = \frac{\alpha+\gamma n}{\gamma n} \int x^{\alpha-1} dx (1-x^n)^{\gamma}$,
erit ponendo m et k pro α et γ :

$$\int x^{m-1} dx (1-x^n)^{k-1} = \frac{m+kn}{kn} \int x^{m-1} dx (1-x^n)^k$$

si nunc manente $\alpha = m$ ponatur $\gamma = k+1$, erit γ
multo magis numerus affirmatiuus, cum k sit talis; ideo
que pari modo habebitur

$$\int x^{m-1} dx (1-x^n)^k = \frac{m+(k+1)n}{(k+1)n} \int x^{m-1} dx (1-x^n)^{k+1}$$

ac pari modo progrediendo, erit

$$\int x^{m-1} dx (1-x^n)^{k+1} = \frac{m+(k+2)n}{(k+2)n} \int x^{m-1} dx (1-x^n)^{k+2}$$

Hinc ergo ingenere concluditur fore, denotante i nume-
rum, integrum quemcunque ::

$$\frac{\int x^{m-1} dx (1-x^n)^{k-i}}{\int x^{m-1} dx (1-x^n)^{k+i}} = \frac{m+kn}{kn} \cdot \frac{m+kn+n}{kn+n} \cdot \frac{m+kn+2n}{kn+2n} \cdot \frac{m+kn+3n}{kn+3n} \dots \frac{m+kn+in}{kn+in}$$

Q. E. I.

Coroll. 1.

§. Cum sit $\int x^{m-1} dx (1-x^n)^{k-1} = \int x^{kn-1} dx (1-x^n)^{\frac{m-n}{n}}$ ideoque etiam:

$$\int x^{m-1} dx (1-x^n)^{k+i} = \int x^{kn+i-1} dx (1-x^n)^{\frac{m-n}{n}}$$

$$\frac{\int x^{kn-1} dx (1-x^n)^{\frac{m-n}{n}}}{\int x^{kn+i-1} dx (1-x^n)^{\frac{m-n}{n}}} = \frac{m+kn}{m-n} \cdot \frac{m+kn+n}{kn+n} \cdot \frac{m+kn+2n}{kn+2n} \dots \frac{m+kn+in}{kn+in}$$

Coroll. 2.

§. Si hic ponatur $kn = \mu$; et $\frac{m}{n} = x$ seu $m = xn$,
ita ut iam μ et x sint numeri affirmatiui, habebitur
haec reductio:

§.

$$\frac{\int x^{\mu-1} dx (1-x^n)^{\alpha-1}}{\int x^{\mu+m-n-1} dx (1-x^n)^{\alpha-1}} = \frac{\mu+\alpha n}{\mu} \cdot \frac{\mu+\alpha n+n}{\mu+n} \cdot \frac{\mu+\alpha n+2n}{\mu+2n} \dots \frac{\mu+\alpha n+in}{\mu+in}$$

scriptis autem pro μ et x litteris m et k erit:

$$\frac{\int x^{m-1} dx (1-x^n)^{k-1}}{\int x^{m+in+n-1} dx (1-x^n)^{k-1}} = \frac{m+kn}{m} \cdot \frac{m+kn+n}{m+n} \cdot \frac{m+kn+2n}{m+2n} \dots \frac{m+kn+in}{m+in}$$

Coroll. 3.

17. Si haec expressio per expressionem in problema inuentam diuidatur, prodibit:

$$\frac{\int x^{m-1} dx (1-x^n)^{k+i}}{\int x^{m+in+n-1} dx (1-x^n)^{k-1}} = \frac{kn}{m} \cdot \frac{kn+n}{m+n} \cdot \frac{kn+2n}{m+2n} \dots \frac{kn+in}{m+in}$$

in quibus factoribus tam numeratores quam denominatores in arithmetica progressionem progrediuntur, cuius differentia est $=n$.

Problema 2.

18. Valorem formulae $\int x^{m-1} dx (1-x^n)^{k-1}$, quem accipit casu $x=1$, per factores infinitos exprimere, siquidem exponentes m et k sint positui.

Solutio.

Statuatur in forma praecedentis problematis numerus i infinitus; et habebitur:

$$\frac{\int x^{m-1} dx (1-x^n)^{k-1}}{\int x^{m+in+n-1} dx (1-x^n)^{k-1}} = \frac{m+kn}{kn} \cdot \frac{m+kn+n}{kn+n} \cdot \frac{m+kn+2n}{kn+2n} \cdot \frac{m+kn+3n}{kn+3n} \dots \text{etc. in infinitum.}$$

Iam manente i eodem numero infinito, loco k alius sumatur numerus finitus, & quicumque; et habebitur simili modo:

$$\frac{\int x^{m-1} dx (1-x^n)^{k-1}}{\int x^{m-1} dx (1-x^n)^{k+i}} = \frac{m+kn}{xn} \cdot \frac{m+kn+n}{xn+n} \cdot \frac{m+kn+2n}{xn+2n} \cdot \frac{m+kn+3n}{xn+3n} \text{ etc.}$$

vbi numerus factorum aequalis est numero factorum praecedentis expressionis, vtrinque scilicet infinitus $= i+1$. At ob i infinitum, est vti §. 13. notauimus $\int x^{m-1} dx (1-x^n)^{k+i} = \int x^{m-1} dx (1-x^n)^{k+i}$ quare priori forma per posteriorem diuisa orietur:

$$\frac{\int x^{m-1} dx (1-x^n)^{k-1}}{\int x^{m-1} dx (1-x^n)^{k+i}} = \frac{x(m+kn)(x+1)(m+kn+n)(x+2)(m+kn+2n)}{k(m+kn)(k+1)(m+kn+n)(k+2)(m+kn+2n)} \text{ etc.}$$

statuatur iam $x=1$, eritque $\int x^{m-1} dx (1-x^n)^{k-1} = \frac{x^m}{m} = \frac{1}{m}$

posito $k=1$, vnde fiet

$$\int x^{m-1} dx (1-x^n)^{k-1} = \frac{1}{m} \cdot \frac{1}{k(m+n)} \cdot \frac{2(m+kn+n)}{(k+1)(m+2n)} \cdot \frac{3(m+kn+2n)}{(k+2)(m+3n)} \cdot \frac{4(m+kn+3n)}{(k+3)(m+4n)} \text{ etc.}$$

Q. E. I.

Aliter.

Tractetur simili modo forma §. 16. inuenta, statuendo i numerum infinitum, eritque:

$$\frac{\int x^{m-1} dx (1-x^n)^{k-1}}{\int x^{m+i-1} dx (1-x^n)^{k-1}} = \frac{m+kn}{m} \cdot \frac{m+kn+n}{m+n} \cdot \frac{m+kn+2n}{m+2n} \cdot \frac{m+kn+3n}{m+3n} \text{ etc.}$$

Iam posito pro m alio numero finito, μ erit pari modo

$$\frac{\int x^{\mu-1} dx (1-x^n)^{k-1}}{\int x^{\mu+i-1} dx (1-x^n)^{k-1}} = \frac{\mu+kn}{\mu} \cdot \frac{\mu+kn+n}{\mu+n} \cdot \frac{\mu+kn+2n}{\mu+2n} \cdot \frac{\mu+kn+3n}{\mu+3n} \text{ etc.}$$

Cum autem sit ob i numerum infinitum:

$$\int x^{m+i-1} dx (1-x^n)^{k-1} = \int x^{\mu+i-1} dx (1-x^n)^{k-1} = \int x^{i-1} dx (1-x^n)^{k-1}$$

euanescentibus quantitibus finitis prae infinitis; et quia vtrinque idem factorum numerus habetur, formam priorem per posteriorem diuidendo orietur:

$\int x^{m-1}$

$$\frac{\int x^{m-1} dx (1-x)^{k-1} \mu(m+kn) (\mu+n)(m+nk+n) (\mu+2n)(\mu+kn+2n)}{\int x^{k-1} dx (1-x)^{k-1} m(\mu+kn) (m+n)(\mu+kn+n) (m+2n)(\mu+kn+2n)} \text{ etc.}$$

statuatur iam $\mu = n$, fiet $\int x^{n-1} dx (1-x^n)^{k-1} = \frac{1 - (1-x^n)^k}{nk}$

integratione ita peracta, vt euanescat posito $x = 0$. Posito nunc $x = 1$, iste valor abit in $\frac{1}{nk}$, vnde obtinebitur:

$$\int x^{m-1} dx (1-x^n)^{k-1} = \frac{1}{nk} \cdot \frac{1(m+kn)}{m(1+k)} \cdot \frac{2(m+kn+n)}{(m+n)(2+k)} \cdot \frac{3(m+kn+2n)}{(m+2n)(3+k)} \text{ etc.}$$

En ergo aliud productum ex infinitis factoribus constans, priori non admodum dissimile, eique adeo aequale, quo valor quaesitus formulae integralis propositae exprimitur. Q. E. I.

Coroll. 1.

19. Has autem duas formas in infinitum excurrentes inter se esse aequales, per se perspicuum est: posteriori enim per priorem diuisa, ob singulorum membrorum numeratores aequales, prodit:

$$1 = \frac{m}{n} \cdot \frac{k(m+n)}{m(k+1)} \cdot \frac{(k+1)(m+2n)}{(m+n)(k+2)} \cdot \frac{(k+2)(m+3n)}{(m+2n)(k+3)} \text{ etc.}$$

At duo factores primi dant $\frac{m+n}{n(k+1)}$; tres $\frac{m+2n}{n(k+2)}$; quatuor $\frac{m+3n}{n(k+3)}$ et infiniti dant $\frac{m+in}{n(k+i)} = \frac{in+m}{in+kn} = 1$.

Coroll. 2.

20. Huiusmodi formae factorum infinitorum innumerabiles formari possunt, quarum valor = 1. Cum enim sit

$$\frac{p+q}{p}$$

$$\frac{p}{p+q} \cdot \frac{p+q}{p+2q} \cdot \frac{p+2q}{p+3q} \cdot \frac{p+3q}{p+4q} \cdot \dots = \frac{p}{p+4q} = \frac{p}{i q}$$

$$\frac{r+s}{r} \cdot \frac{r+2s}{r+s} \cdot \frac{r+3s}{r+2s} \cdot \frac{r+4s}{r+3s} \cdot \dots = \frac{r+4s}{r} = \frac{i s}{r}$$

multiplicando has duas formas habebimus

$$1 = \frac{qr}{ps} \cdot \frac{p(r+s)}{r(p+q)} \cdot \frac{(p+q)(r+2s)}{(r+s)(p+2q)} \cdot \frac{(p+2q)(r+3s)}{(r+2s)(p+3q)} \cdot \text{etc.}$$

Coroll. 3.

21. Si ergo valor formulæ integralis inuentus per hanc expressiönem $= 1$ multiplicetur, prodibit expressiö latius patens eidem aequalis, scilicet:

$$\int x^{m-1} dx (1-x^n)^{k-1} = \frac{qr}{kps} \cdot \frac{1(m+kn)(r+s)}{m(k+1)r(p+q)} \cdot \frac{2(m+kn+n)(p+q)(r+2s)}{(m+n)(k+2)(r+s)(p+2q)} \cdot \frac{3(m+kn+2n)(p+2q)(r+3s)}{(m+2n)(k+3)(r+2s)(p+3q)} \cdot \text{etc.}$$

vbi pro p, q, r, s numeros quoscunque assumere licet. Pluribus modis ergo ita accipi possunt, ut quilibet factor ad formam simpliciozem redigatur.

Coroll. 4.

22. Sit $p = m$, et $q = n$, eritque:

$$\int x^{m-1} dx (1-x^n)^{k-1} = \frac{r}{mks} \cdot \frac{1(m+kn)(r+s)}{(m+n)(k+1)r} \cdot \frac{2(m+kn+n)(r+2s)}{(m+2n)(k+2)(r+s)} \cdot \frac{3(m+kn+2n)(r+3s)}{(m+3n)(k+3)(r+2s)} \cdot \text{etc.}$$

si porro ponatur $r = k$, et $s = 1$, erit,

$$\int x^{m-1} dx (1-x^n)^{k-1} = \frac{1}{m} \cdot \frac{1(m+kn)}{(m+n)k} \cdot \frac{2(m+kn+n)}{(m+2n)(k+1)} \cdot \frac{3(m+kn+2n)}{(m+3n)(k+2)} \cdot \text{etc.}$$

quæ est expressiö primum inuenta. Sin autem fit

$r = m + kn$, et $s = n$, erit,

$$\int x^{m-1} dx (1-x^n)^{k-1} = \frac{n+kn}{mkn} \cdot \frac{1(m+kn+n)}{(m+n)(k+1)} \cdot \frac{2(m+kn+2n)}{(m+2n)(k+2)} \cdot \frac{3(m+kn+3n)}{(m+3n)(k+3)} \cdot \text{etc.}$$

Coroll. 5.

23. Si ponatur $p = k + 1$, et $q = 1$, erit:

$$\int x^{m-1} dx (1-x^n)^{k-1} = \frac{r}{k(k+1)n} \cdot \frac{1(m+kn)(r+s)}{m r (k+2)} \cdot \frac{2(m+kn+n)(r+2s)}{(m+n)(r+s)(k+3)} \cdot \frac{3(m+kn+2n)(r+3s)}{(m+2n)(r+2s)(k+4)}$$

fit

fit porro $r = 1$, et $s = 1$, erit :

$$\int x^{m-1} dx (1-x^n)^{k-1} = \frac{1}{k(k+1)n} \cdot \frac{2(m+kn)}{m(k+2)} \cdot \frac{3(m+kn+n)}{(m+n)(k+3)} \cdot \frac{4(m+kn+2n)}{(m+2n)(k+4)} \text{ etc.}$$

fit autem ponatur $r = m + kn$, et $s = n$, erit :

$$\int x^{m-1} dx (1-x^n)^{k-1} = \frac{m+kn}{k(k+1)n} \cdot \frac{1(m+kn+n)}{m(k+2)} \cdot \frac{2(m+kn+2n)}{(m+n)(k+3)} \cdot \frac{3(m+kn+3n)}{(m+2n)(k+4)} \text{ etc.}$$

Coroll. 6.

24. Si manente exponente k reliquos exponentes m et n mutemus, habebimus

$$\int x^{\mu-1} dx (1-x^\nu)^{k-1} = \frac{1}{\mu} \cdot \frac{1(\mu+k\nu)}{(\mu+\nu)k} \cdot \frac{2(\mu+k\nu+\nu)}{(\mu+2\nu)(k+1)} \cdot \frac{3(\mu+k\nu+2\nu)}{(\mu+3\nu)(k+2)} \text{ etc.}$$

dummodo μ, ν et k sint numeri affirmatiui. Diuisa ergo illa forma per hanc obtinebimus :

$$\frac{\int x^{\mu-1} dx (1-x^\nu)^{k-1}}{\int x^{m-1} dx (1-x^n)^{k-1}} = \frac{\mu}{m} \cdot \frac{(\mu+\nu)(m+kn)}{(m+n)(\mu+k\nu)} \cdot \frac{(\mu+2\nu)(m+kn+n)}{(m+2n)(\mu+k\nu+\nu)} \cdot \frac{(\mu+3\nu)(m+kn+2n)}{(m+3n)(\mu+k\nu+2\nu)} \text{ etc.}$$

Coroll. 7.

25. Sin autem etiam in altera forma k in x mutetur, habebitur :

$$\frac{\int x^{m-1} dx (1-x^n)^{k-1}}{\int x^{m-1} dx (1-x^n)^{x-1}} = \frac{\mu}{m} \cdot \frac{x(\mu+\nu)(m+kn)}{k(m+n)(\mu+k\nu)} \cdot \frac{(x+1)(\mu+2\nu)(m+kn+n)}{(k+1)(m+2n)(\mu+k\nu+\nu)} \cdot \frac{(x+2)(\mu+3\nu)(m+kn+2n)}{(k+2)(m+3n)(\mu+k\nu+2\nu)} \text{ etc.}$$

posito post integrationem $x = 1$, et existentibus omnibus exponentibus m, n, k ac μ, ν, x affirmatiuis.

Scholion.

26. His conuersionibus formularum integralium in factores infinitos expositis, videamus vicissim quomodo proposita huiusmodi expressio infinita per factores procedens, ad integrationes formularum casu, quo $x = 1$,

Tom. VI. Nou. Com. R reduci

reduci debeat. Hic autem ante omnia spectari debent membra, quae illud productum infinitum constituunt, ex quot illa factoribus sint composita: quae membra primum ita comparata esse debent, ut infinitesima in unitatem abeant. In hunc finem erunt fracta, et ex certo, tam numeratorum, quam denominatorum, numero constabunt, et utriusque per singula membra secundum progressionem arithmeticam procedent, ita ut in illis eadem habeatur differentia; etiamsi enim variae partes diuersas obtineant differentias, eae tamen facile ad eandem reducentur. Cum igitur nihil obstat, quo minus haec differentia unitati aequalis constituatur, pro diuerso factorum cuiusque membri numero, sequentes huiusmodi productorum infinitorum ordines habebimus:

$$\frac{a}{b} \cdot \frac{a+1}{b+1} \cdot \frac{a+2}{b+2} \cdot \frac{a+3}{b+3} \cdot \frac{a+4}{b+4} \cdot \frac{a+5}{b+5} \cdot \text{etc.}$$

$$\frac{ac}{be} \frac{(a+1)(c+1)}{(b+1)(e+1)} \cdot \frac{(a+2)(c+2)}{(b+2)(e+2)} \cdot \frac{(a+3)(c+3)}{(b+3)(e+3)} \cdot \text{etc.}$$

$$\frac{acf}{beg} \frac{(a+1)(c+1)(f+1)}{(b+1)(e+1)(g+1)} \cdot \frac{(a+2)(c+2)(f+2)}{(b+2)(e+2)(g+2)} \cdot \text{etc.}$$

$$\frac{acfb}{begk} \frac{(a+1)(c+1)(f+1)(b+1)}{(b+1)(e+1)(g+1)(k+1)} \cdot \frac{(a+2)(c+2)(f+2)(b+2)}{(b+2)(e+2)(g+2)(k+2)} \cdot \text{etc.}$$

Quomodo ergo cuiusque horum productorum valor per formulas integrales exprimendus sit, videamus.

Problema 3.

27. Per formulas integrales definire valorem huius producti infiniti ex membris simplicibus constantis:

$$P = \frac{a}{b} \cdot \frac{a+1}{b+1} \cdot \frac{a+2}{b+2} \cdot \frac{a+3}{b+3} \cdot \frac{a+4}{b+4} \cdot \frac{a+5}{b+5} \cdot \text{etc.}$$

Solutio.

Denotante i numerum infinitum vidimus esse

$$\frac{\int x^{m-1} dx (1-x^n)^{k-1}}{\int x^m dx (1-x^n)^{k-1}} = \frac{m+kn}{m} \cdot \frac{m+kn+n}{m+n} \cdot \frac{m+kn+2n}{m+2n} \cdot \text{etc.}$$

quae forma ad propositam reducetur ponendo $n=1$; $m+k=a$, et $m=b$, vnde fit $k=a-b$. Cum ergo k debeat esse numerus affirmatiuus, si fuerit $a > b$, erit

$$P = \frac{\int x^{b-1} dx (1-x)^{a-b-1}}{\int x^b dx (1-x)^{a-b-1}} = \frac{\int x^{a-b-1} dx (1-x)^{b-1}}{\int x^{a-b-1} dx (1-x)^i}$$

sin autem sit $b > a$, erit inuerse:

$$P = \frac{\int x^i dx (1-x)^{b-a-1}}{\int x^{a-1} dx (1-x)^{b-a-1}} = \frac{\int x^{b-a-1} dx (1-x)^i}{\int x^{b-a-1} dx (1-x)^{a-1}}. \quad \text{Q. E. I.}$$

Corollarium.

28. Manifestum autem est, si sit $a > b$, valorem P fore infinitum, sin autem sit $b > a$, fore $P=0$. Casu autem $a=b$ fit $P=1$: qui casus cum ad vtrumque expositorum seque pertineat, euidentis est, esse

$$\int \frac{x^{a-1} dx}{1-x} = \int \frac{x^i dx}{1-x} \text{ quae integralia casu } x=1 \text{ vtiq;}$$

fiunt ita infinita, vt rationem aequalitatis obtineant,

$$\text{Est autem in genere } \int \frac{x^{a-1} dx}{1-x} = \int \frac{x^{b-1} dx}{1-x}.$$

R 2

Proble-

Problema 4.

29. Per formulas integrales definire valorem huius producti infiniti ex membris duplicatis constantis:

$$P = \frac{ac}{be} \cdot \frac{(a+1)(c+1)}{(b+1)(e+1)} \cdot \frac{(a+2)(c+2)}{(b+2)(e+2)} \cdot \frac{(a+3)(c+3)}{(b+3)(e+3)} \cdot \text{etc.}$$

Solutio.

Cum sit per §. 24. denotantibus m, n, k, μ, ν numeros positivos

$$\frac{(\mu+\nu)(m+kn)}{(\mu+n)(\mu+kv)} \cdot \frac{(\mu+2\nu)(m+kn+n)}{(\mu+2n)(\mu+kv+\nu)} \cdot \text{etc.} = \frac{m}{\mu} \cdot \frac{\int x^{m-1} dx (1-x^n)^{k-1}}{\int x^{\mu-1} dx (1-x^\nu)^{k-1}}$$

ponatur $n=1$; $\nu=1$; $\mu+1=a$; $m+k=c$, $m+i=b$; $\mu+k=e$; erit $\mu=a-1$; $m=b-1$; et $k=c-b+1=e-a+1$. Quare, quo haec forma ad propositam possit reuocari, necesse est, ut sit $c-b=e-a$; nisi enim haec conditio locum habeat, valor producti propositi P esset vel infinitus, vel euanescebat. Quod incommodum ne locum habeat, sit $c-b=e-a$, seu $a+c=b+e$; atque dum sint $a-1$; $b-1$; et $c-b$, vel $e-a$, numeri affirmatiui, erit:

$$P = \frac{b-1}{a-1} \frac{\int x^{b-2} dx (1-x)^{c-b}}{\int x^{a-2} dx (1-x)^{e-a}}$$

Vel consideretur haec forma:

$$\frac{\mu(m+kn-n)}{m(\mu+kv-\nu)} \cdot \frac{(\mu+\nu)(m+kn)}{(\mu+n)(\mu+kv)} \cdot \text{etc.} = \frac{m+kn-n}{\mu+kv-\nu} \frac{\int x^{m-1} dx (1-x^n)^{k-1}}{\int x^{\mu-1} dx (1-x^\nu)^{k-1}}$$

quae ex illa luculenter nascitur, ac ponatur $n=1$; $\nu=1$, $\mu=a$; $m=b$; $c=m+k-1$; et $e=\mu+k-1$; erit-

eritque $k-1=c-b=e-a$; iterum ergo esse debet $a+c=b+e$. Nunc ergo, dummodo sint a, b , et $c-b+1$, vel $e-a+1$, numeri positivi, erit

$$P = \frac{c \int x^{b-1} dx (1-x)^{c-b}}{e \int x^{a-1} dx (1-x)^{e-a}}$$

Quoties ergo fuerit $a+c=b+e$, valor quaesitus P est finitus: ac per has formulas integrales casu $x=1$ innotescit. Q. E. I.

Coroll. 1.

30. Cum sit $a+c=b+e$, si sit $c > b$, erit quoque $e > a$, et a et b in primo membro $\frac{a^b}{b^e}$ denotant factores minores numeratoris et denominatoris. Requiritur autem tantum, ut $c-b+1$ sit numerus positivus. Quare si etiam $c-e+1$ sit numerus positivus, alio insuper modo valor quaesitus P exprimi poterit: scilicet permutandis b et e hoc modo:

$$P = \frac{c \int x^{e-1} dx (1-x)^{c-e}}{b \int x^{a-1} dx (1-x)^{b-a}}$$

Coroll. 2.

31. Atque quaelibet harum formularum locum habebit:

$$P = \frac{c \int x^{e-1} dx (1-x)^{c-e}}{b \int x^{a-1} dx (1-x)^{b-a}} = \frac{c \int x^{b-1} dx (1-x)^{c-b}}{e \int x^{a-1} dx (1-x)^{e-a}} = \frac{a \int x^{e-1} dx (1-x)^{a-c}}{b \int x^{c-1} dx (1-x)^{b-c}} = \frac{a \int x^{b-1} dx (1-x)^{a-b}}{e \int x^{c-1} dx (1-x)^{e-c}}$$

Quarum prima locum habet, si $c-e+1=b-a+1$ sit > 0 , secunda si $c-b+1=e-a+1 > 1$, tertia si $a-e+1=b-c+1 > 0$, et quarta, si $a-b+1=e-c+1 > 0$.

R 3

Coroll.

Coroll. 3.

32. Prima forma et quarta simul valebunt, si differentia inter a et b sit unitate minor, ideoque et inter c et e . Atque omnes quatuor simul locum habebunt, si insuper differentia inter a et e fuerit unitate minor.

Coroll. 4.

33. Si ergo ponatur $a = p + m$; $b = p + n$; $c = p - m$ et $e = p - n$, ut sit $a + c = b + e = 2p$, fueritque $m + n < 1$, erit

$$P = \frac{p-m \int x^{p-n-1} dx (1-x)^{n-m}}{p+n \int x^{p+n-1} dx (1-x)^{n-m}} - \frac{p+m \int x^{p-n-1} dx (1-x)^{n+m}}{p+n \int x^{p-m-1} dx (1-x)^{n+m}}$$

$$P = \frac{p-m \int x^{p+n-1} dx (1-x)^{-n-m}}{p-n \int x^{p+m-1} dx (1-x)^{-n-m}} - \frac{p+m \int x^{p+n-1} dx (1-x)^{-n-m}}{p-n \int x^{p-m-1} dx (1-x)^{-n-m}}$$

Atque hae quatuor formulae inter se erunt aequales.

Problema 5.

34. Per formulas integrales exprimere valorem huius producti infiniti ex memoris triplicatis constantes.

$$P = \frac{acf}{beg} \cdot \frac{(a+1)(c+1)(f+1)}{(b+1)(e+1)(g+1)} \cdot \frac{(a+2)(c+2)(f+2)}{(b+2)(e+2)(g+2)} \cdot \text{etc.}$$

Solutio.

Cum inuenerimus §. 25.

$$\frac{\mu(\mu+\nu)(\mu+\nu+k)}{\mu(\mu+\nu)(\mu+\nu)} \cdot \frac{(x+1)(\mu+\nu)(\mu+\nu+k)}{(k+1)(\mu+2\nu)(\mu+\nu+k)} \cdot \text{etc.} = \frac{\mu \int x^{\mu-1} dx (1-x^{\nu})^{k-1}}{\mu \int x^{\mu-1} dx (1-x^{\nu})^{k-1}}$$

erit

erit membrum antierius adhuc adiciendo :

$$\frac{(k-1)\mu(m+kn-n)}{(k-1)m(\mu+kn-v)} \cdot \frac{\mu(\mu+v)(m+kn)}{k(m+n)(\mu+kv)} \cdot \text{etc.} = \frac{(k-1)(m+kn-n)}{(k-1)(\mu+kv+v)} \cdot \frac{\int x^{m-1} dx (1-x^k)^{k-1}}{\int x^{\mu-1} dx (1-x^k)^{k-1}}$$

quae forma quo ad propositam reducat, fiatnatur :

$$k-1 = a; k-1 = b; \mu = c; m = e; n = 1; v = 1;$$

$$\text{ac } m+k-1 = e+b+f; \mu+k-1 = c+a=g.$$

Cum ergo haec reductio non succedat, nisi ista conditione sit $f=b+e$ et $g=a+c$; vt habeatur hoc productum infinitum :

$$P = \frac{ac(b+e)}{be(a+c)} \cdot \frac{(a+1)(c+1)(b+e+1)}{(b+1)(e+1)(a+c+1)} \cdot \frac{(a+2)(c+2)(b+e+2)}{(b+2)(e+2)(a+c+2)} \cdot \text{etc.}$$

Quare cum hoc casu sit $m=e$; $k=b+1$; $\mu=c$ et $k=a+1$; existentibus $n=v=1$, erit

$$P = \frac{a(b+e)}{b(a+c)} \cdot \frac{\int x^{e-1} dx (1-x)^b}{\int x^{c-1} dx (1-x)^a}$$

dummodo sint $c, e, b+1$ et $a+1$ numeri positivi.

Q. E. I.

Coroll. I.

35. Cum per §. 9. sit $\int x^{\alpha-1} dx (1-x)^{\gamma-1} = \frac{\alpha+\gamma}{\alpha}$ $\int x^{\alpha} dx (1-x)^{\gamma-1}$, erit

$$\int x^{e-1} dx (1-x)^b = \frac{b+e+1}{b+e} \int x^e dx (1-x)^b, \text{ ideoque}$$

$$P = \frac{ac(b+e)(b+e+1)}{be(a+c)(a+c+1)} \cdot \frac{\int x^e dx (1-x)^b}{\int x^c dx (1-x)^a}$$

Et cum sit $\int x^{\alpha-1} dx (1-x)^{\gamma} = \frac{\gamma}{\alpha+\gamma} \int x^{\alpha-1} dx (1-x)^{\gamma-1}$, erit

$$\int x^{e-1} dx (1-x)^b = \frac{b}{b+e} \int x^{e-1} dx (1-x)^{b-1}, \text{ habebitur quoque}$$

$$P = \frac{\int x^{e-1} dx (1-x)^{b-1}}{\int x^{c-1} dx (1-x)^{a-1}}$$

Coroll.

Coroll. 2.

36. Formula haec autem locum habet, dummodo a, b, c et e sint numeri affirmatiui: et quia iam a et c item b et e inter se permutari possunt, erit quoque $P = \frac{\int x^{b-1} dx (1-x)^{e-1}}{\int x^{a-1} dx (1-x)^{c-1}}$: quae conuersio autem ex Th. 2. per se est manifesta.

Scholion 1.

37. Problema ergo propositum non in genere est solutum, sed tantum casu, quo $f = b + e$; et $g = a + c$; ficque solutio nostra duplici limitatione restringitur. Vnica vero tantum restrictione est opus, ne valor ipsius P vel fiat infinitus, vel euanesceat, qua requiritur, ut sit $a + c + f = b + e + g$. Quo autem problema pro hac vnica limitatione soluatur, necesse est plures formulas integrales in computum ducere, quod hoc modo praestari poterit. Posito igitur $a + c + f = b + e + g$, cum sit

$$P = \frac{acf}{beg} \cdot \frac{(a+1)(c+1)(f+1)}{(b+1)(e+1)(g+1)} \cdot \frac{(a+2)(c+2)(f+2)}{(b+2)(e+2)(g+2)} \text{ etc.}$$

statuatur $P = QR$, sitque

$$Q = \frac{(p+q)(p-q)}{(p+r)(p-r)} \cdot \frac{(p+q+1)(p-q+1)}{(p+r+1)(p-r+1)} \text{ etc.} \cdot \frac{p+q}{p+r} \cdot \frac{\int x^{p-r-1} dx (1-x)^{q+r}}{\int x^{p-q-1} dx (1-x)^{q+r}}$$

$$\text{et } R = \frac{\alpha\gamma(\beta+\epsilon)}{\beta\epsilon(\alpha+\gamma)} \cdot \frac{(\alpha+1)(\gamma+1)(\beta+\epsilon+1)}{(\beta+2)(\epsilon+1)(\alpha+\gamma+1)} \text{ etc.} = \frac{\int x^{\beta-1} dx (1-x)^{\epsilon-1}}{\int x^{\alpha-1} dx (1-x)^{\gamma-1}}$$

Fiat iam primum membrum producti QR aequale primo membro formae propositae P , scilicet

$$\frac{\alpha\gamma}{\beta\epsilon}$$

$\frac{\alpha\gamma(\beta+\varepsilon)(p+q)(p-q)}{\beta\varepsilon(\alpha+\gamma)(p+r)(p-r)} = \frac{acf}{beg}$, quod pluribus modis fieri potest. Primum enim illud pluribus modis ad tres factores potest reduci, ponatur scilicet $\beta+\varepsilon=p+r$ et $\alpha+\gamma=p+q$; ut habeatur $q=\alpha+\gamma-p$; et $r=\beta+\varepsilon-p$; eritque:

$$\frac{\alpha\gamma(2p-\alpha-\gamma)}{\beta\varepsilon(2p-\beta-\varepsilon)} = \frac{acf}{beg}$$

Quod si ergo statuatur $\alpha=a$; $\beta=b$; $\gamma=c$; $\varepsilon=e$; et $2p=a+c+f=b+e+g$; erit $q=a+c-p$; et $r=b+e-p$. Sicque nulla alia restrictio hic involuitur, nisi ut sit $a+c+f=b+e+g=2p$. Pro hoc ergo casu erit producti infiniti propositi valor

$$P = \frac{a+c}{b+e} \cdot \frac{\int x^{2p-b-e-1} dx (1-x)^{a+b+c+e-2p} \int x^{b-1} dx (1-x)^{e-a}}{\int x^{2p-a-c-1} dx (1-x)^{a+b+c+e-2p} \int x^{a-1} dx (1-x)^{c-e}}$$

vbi iam tam litteras a et e , quam b et e pro lubitu inter se permutari licet.

Alio modo statuatur $\gamma=p+r$ et $\varepsilon=p-q$; ut sit

$$\frac{\alpha(\beta+\varepsilon)(p+q)}{\beta(\alpha+\gamma)(p-r)} = \frac{acf}{beg}$$

Iam sit $\alpha=a$; $\beta=b$; $\varepsilon=c-b$; $\gamma=e-a$; erit $q=p-c+b$ et $r=e-a-p$; hincque $f=2p-c+b$ et $g=2p-e+a$. Sin autem ponatur summa $a+c+f=b+e+g=s$; erit $a+b+2p=s$, et $2p=s-a-b$; sicque $p+q=s-a-c=f$; $p-q=c-b$; $p+r=e-a$; $p-r=s-b-e=g$; et $q+r=b+e-a-c$. Atque hinc oritur:

$$P = \frac{s-a-c}{e-a} \cdot \frac{\int x^{s-b-e-1} dx (1-x)^{b+e-a-c} \int x^{b-1} dx (1-x)^{c-b-a}}{\int x^{c-b-1} dx (1-x)^{b+e-a-c} \int x^{a-1} dx (1-x)^{e-a-a}}$$

Vbi iterum tam litteras a et c , quam b et e inter se permutare licet. Vel erit etiam ob plures valores ipsius Q

$$P = \frac{c-b}{e-a} \frac{\int x^{s-1} dx (1-x)^{c+e-s}}{\int x^{f-1} dx (1-x)^{c+e-s}} \frac{\int x^{b-1} dx (1-x)^{c-b-1}}{\int x^{a-1} dx (1-x)^{e-a-1}}$$

At formula prius inuenta, ponendo s pro $2p$, abit in hanc,

$$P = \frac{a+c}{b+e} \frac{\int x^{g-1} dx (1-x)^{b+e-f}}{\int x^{f-1} dx (1-x)^{b+e-f}} \frac{\int x^{b-1} dx (1-x)^{e-1}}{\int x^{a-1} dx (1-x)^{e-a}}$$

Scholion 2.

38. Quodsi iam omnes istae permutationes adhibeantur, quae pro formula Q obtinent, atque formulae proposita fuerit:

$$P = \frac{aef}{beg} \cdot \frac{(a+1)(c+1)(f+1)}{(b+1)(e+1)(g+1)} \cdot \frac{(a+2)(c+2)(f+2)}{(b+2)(e+2)(g+2)} \cdot \text{etc.}$$

fueritque $a+c+f=b+e+g$, reperientur sequentes valores pro valore P , scilicet:

$$P = \frac{f \int x^{e-a-1} dx (1-x)^{a+f-e}}{g \int x^{c-b-1} dx (1-x)^{a+f-e}} \frac{\int x^{b-1} dx (1-x)^{c-b-1}}{\int x^{a-1} dx (1-x)^{e-a-1}}$$

$$P = \frac{f \int x^{g-1} dx (1-x)^{f-g}}{e-a \int x^{c-b-1} dx (1-x)^{f-g}} \frac{\int x^{b-1} dx (1-x)^{c-b-1}}{\int x^{a-1} dx (1-x)^{e-a-1}}$$

$$P = \frac{c-b}{g} \frac{\int x^{e-a-1} dx (1-x)^{g-f}}{\int x^{f-1} dx (1-x)^{g-f}} \frac{\int x^{b-1} dx (1-x)^{c-b-1}}{\int x^{a-1} dx (1-x)^{e-a-1}}$$

$$P = \frac{c-b}{e-a} \frac{\int x^{g-1} dx (1-x)^{e-a-f}}{\int x^{f-1} dx (1-x)^{e-a-f}} \frac{\int x^{b-1} dx (1-x)^{c-b-1}}{\int x^{a-1} dx (1-x)^{e-a-1}}$$

$$P = \frac{f}{g} \frac{\int x^{b+e-1} dx (1-x)^{f-b-e}}{\int x^{a+c-1} dx (1-x)^{f-b-e}} \frac{\int x^{b-1} dx (1-x)^{e-1}}{\int x^{a-1} dx (1-x)^{e-a-1}}$$

$P =$

$$P = \frac{f \int x^{g-1} dx (1-x)^{f-g} \int x^{b-1} dx (1-x)^{e-1}}{b+e \int x^{a+c-1} dx (1-x)^{f-g} \int x^{a-1} dx (1-x)^{c-1}}$$

$$P = \frac{a+c \int x^{b+e-1} dx (1-x)^{g-f} \int x^{b-1} dx (1-x)^{e-1}}{g \int x^{f-1} dx (1-x)^{g-f} \int x^{a-1} dx (1-x)^{c-1}}$$

$$P = \frac{a+c \int x^{g-1} dx (1-x)^{b+e-f} \int x^{b-1} dx (1-x)^{e-1}}{b+e \int x^{f-1} dx (1-x)^{b+e-f} \int x^{a-1} dx (1-x)^{c-1}}$$

Porro autem hic tam ternas litteras a, c, f , quam has b, e, g , pro lubitu inter se permutare licet, ex quo maxima copia formularum, quae omnes eidem valori P sunt aequales, enascetur.

Scholion 3.

39. Hinc etiam pro producto simpliciori

$$P = \frac{ac}{be} \cdot \frac{(a+1)(c+1)}{(b+1)(e+1)} \cdot \frac{(a+2)(c+2)}{(b+2)(e+2)} \text{ etc.}$$

si fuerit $a+c=b+e$, praeter valores supra inuentos plures alii exhiberi poterunt. Primum enim quia $a+c=b+e$ valor in problemate § inuentus huc pertinet:

$$P = \frac{\int x^{e-1} dx (1-x)^{b-1}}{\int x^{c-1} dx (1-x)^{a-1}} \text{ Deinde si in serie §}$$

praec. vna litterarum a, c, f vni ex b, e, g aequalis statuatur, vel haec eadem expressio, vel aliae obtinebuntur, quae cum praecedentibus erunt:

$$P = \frac{\int x^{e-1} dx (1-x)^{a-e-1}}{\int x^{c-1} dx (1-x)^{b-c-1}}; \quad P = \frac{\int x^{b-1} dx (1-x)^{a-b-1}}{\int x^{c-1} dx (1-x)^{e-c-1}}$$

$$P = \frac{\int x^{e-1} dx (1-x)^{c-e-1}}{\int x^{a-1} dx (1-x)^{b-a-1}}; \quad P = \frac{\int x^{b-1} dx (1-x)^{c-b-1}}{\int x^{a-1} dx (1-x)^{e-a-1}}$$

S 2

P =

$$P = \frac{\int x^{b-1} dx (1-x)^{e-1}}{\int x^{a-1} dx (1-x)^{c-1}}; \quad P = \frac{\int x^{b-1} dx (1-x)^{e-1}}{\int x^{a-1} dx (1-x)^{c-1}}$$

vbi est $e-a=c-b$ et $c-e=b-a$.

In sequentibus est n numerus arbitrarius :

$$P = \frac{\int x^{e-n-1} dx (1-x)^{n+a-e-1}}{\int x^{c-n-1} dx (1-x)^{n+b-c-1}} \cdot \frac{\int x^{n-1} dx (1-x)^{e-n-1}}{\int x^{n-1} dx (1-x)^{e-n-1}}$$

$$P = \frac{\int x^{n+b-1} dx (1-x)^{e-b-n-1}}{\int x^{n+a-1} dx (1-x)^{e-b-n-1}} \cdot \frac{\int x^{n-1} dx (1-x)^{b-1}}{\int x^{n-1} dx (1-x)^{a-1}}$$

$$P = \frac{\int x^{e-1} dx (1-x)^{n+b-e-1}}{\int x^{c-1} dx (1-x)^{n+b-e-1}} \cdot \frac{\int x^{n-1} dx (1-x)^{b-1}}{\int x^{n-1} dx (1-x)^{a-1}}$$

$$P = \frac{\int x^{n-1} dx (1-x)^{a-1}}{\int x^{c-1} dx (1-x)^{a-1}} \cdot \frac{\int x^{b-1} dx (1-x)^{e-1}}{\int x^{a-1} dx (1-x)^{n-1}} \cdot \frac{\int x^{b-1} dx (1-x)^{e-1}}{\int x^{c-1} dx (1-x)^{a-1}}$$

quae postrema iam in praecedentibus continetur. Hic autem monendum est, superfluum esse hic rationem exponentium definire, uti supra factum est. Cum enim valor P certo sit finitus, si $a+c=b+e$; si quae-
piam formularum integralium habeat exponentes negati-
vos infra -1 ; tum eam ad exponentes maiores redu-
cere licet, ac tum verus valor ipsius P obtinebitur.
Formulae autem simpliciores continentur in hoc Theo-
remate:

Theorema 4.

40. Si fuerit $a+c=b+e=s$; tum erit

$$\frac{\int x^{a-1} dx (1-x)^{c-1}}{\int x^{b-1} dx (1-x)^{e-1}} = \frac{\int x^{a-1} dx (1-x)^{s-a-b-1}}{\int x^{b-1} dx (1-x)^{s-a-b-1}}$$

si quidem post integrationem statuatur $x=1$.

Demon-

Demonstratio.

Est enim ex praecedentibus formulis

$$\frac{\int x^{a-1} dx (1-x)^{c-1}}{\int x^{b-1} dx (1-x)^{e-1}} = \frac{\int x^{a-1} dx (1-x)^{c-b-1}}{\int x^{b-1} dx (1-x)^{e-a-1}}$$

At ob $a+c=b+e=s$, est $c=s-a$; et $e=s-b$; unde erit $c-b=e-a=s-a+b$, unde forma proposita conficitur. Q. E. D.

Coroll. 1.

41. Hic licet tam numeros a et c , quam b et e inter se permutare, unde quatuor obtinentur formulae primae aequales; scilicet singulae harum formularum:

$$\frac{\int x^{a-1} dx (1-x)^{s-a-b-1}}{\int x^{b-1} dx (1-x)^{s-a-b-1}}; \frac{\int x^{a-1} dx (1-x)^{s-a-e-1}}{\int x^{e-1} dx (1-x)^{s-a-e-1}}$$

$$\frac{\int x^{c-1} dx (1-x)^{s-b-e-1}}{\int x^{b-1} dx (1-x)^{s-b-e-1}}; \frac{\int x^{c-1} dx (1-x)^{s-c-e-1}}{\int x^{e-1} dx (1-x)^{s-c-e-1}}$$

aequales sunt huic formae $\frac{\int x^{a-1} dx (1-x)^{c-1}}{\int x^{b-1} dx (1-x)^{e-1}}$

Coroll. 2.

42. Valor autem uniuscuiusque harum formularum aequalis est huic producto ex factoribus infinitis constanti:

$$\frac{be}{ac} \cdot \frac{(b+1)(e+1)}{(a+1)(c+1)} \cdot \frac{(b+2)(e+2)}{(a+2)(c+2)} \cdot \text{etc.}$$

Coroll. 3.

43. Si fit $e = 1$; ideoque $b = s - 1$; $a = s - c$; erit, posito

$$P = \frac{1(s-1)}{c(s-c)} \cdot \frac{2s}{(c+1)(s-c+1)} \cdot \frac{3(s+1)}{(c+2)(s-c+2)} \cdot \text{etc.}$$

ob $\int x^{b-1} dx (1-x)^{e-1} = \int x^{s-2} dx = \frac{x}{s-1}$,

$$P = (s-1) \int x^{s-c-1} dx (1-x)^{c-1}$$

$$P = (c-1) \int x^{s-c-1} dx (1-x)^{c-2} = (s-1)$$

$$P = (s-c-1) \int x^{c-1} dx (1-x)^{s-c-2}$$

$$P = \frac{\int x^{s-c-1} dx (1-x)^{c-1} \int x^{c-1} dx (1-x)^{-c}}{\int x^{s-2} dx (1-x)^{c-1} \int x^{s-2} dx (1-x)^{-c}} = (s-1) \int x^{s-c-1} dx (1-x)^{c-1}$$

Scholion.

44. Quoniam vero huiusmodi formularum integralium comparationes iam plures exposui, hic imprimis nonnullos valores prae reliquis notabiles persequar, et quemadmodum ii per formulas integrales exprimi queant, ostendam. Notatu autem potissimum digna sunt illa producta infinita, quibus sinus et cosinus cuiusque anguli exprimitur. Denotante enim ξ angulum rectum, et Φ angulum quemcunque, constat esse

$$\sin. \Phi = \Phi \left(1 - \frac{\Phi\Phi}{4\xi\xi}\right) \left(1 - \frac{\Phi\Phi}{16\xi\xi}\right) \left(1 - \frac{\Phi\Phi}{36\xi\xi}\right) \left(1 - \frac{\Phi\Phi}{64\xi\xi}\right) \text{etc.}$$

$$\text{et } \cos. \Phi = \left(1 - \frac{\Phi\Phi}{\xi\xi}\right) \left(1 - \frac{\Phi\Phi}{9\xi\xi}\right) \left(1 - \frac{\Phi\Phi}{25\xi\xi}\right) \left(1 - \frac{\Phi\Phi}{49\xi\xi}\right) \text{etc.}$$

Quodsi iam ponatur $\Phi = \frac{m}{n} \xi$, erit

$$\left(1 - \frac{m m}{4 n n}\right) \left(1 - \frac{m m}{16 n n}\right) \left(1 - \frac{m m}{36 n n}\right) \text{etc.} = \frac{n}{m \xi} \sin. \frac{m}{n} \xi$$

$$\left(1 - \frac{m m}{n n}\right) \left(1 - \frac{m m}{9 n n}\right) \left(1 - \frac{m m}{25 n n}\right) \text{etc.} = \cos. \frac{m}{n} \xi$$

Vel

Vel si angulus duobus rectis aequalis π introducatur, et ob $\varrho = \frac{1}{2}\pi$ pro m scribatur $2m$, erit factores euol-
vendo

$$\frac{(n-m)(n+m)}{n \cdot n} \cdot \frac{(2n-m)(2n+m)}{2n \cdot 2n} \cdot \frac{(3n-m)(3n+m)}{3n \cdot 3n} \text{ etc.} = \frac{n}{m\pi} \sin. \frac{m}{n} \pi$$

$$\frac{(n-2m)(n+2m)}{n \cdot n} \cdot \frac{(3n-2m)(3n+2m)}{3n \cdot 3n} \cdot \frac{(5n-2m)(5n+2m)}{5n \cdot 5n} \text{ etc.} = \text{cof.} \frac{m}{n} \pi.$$

Ucendo autem differentias ad unitatem, erit

$$\frac{(1-\frac{m}{n})(1+\frac{m}{n})}{1 \cdot 1} \cdot \frac{(2-\frac{m}{n})(2+\frac{m}{n})}{2 \cdot 2} \cdot \frac{(3-\frac{m}{n})(3+\frac{m}{n})}{3 \cdot 3} \text{ etc.} = \frac{n}{m\pi} \sin. \frac{m}{n} \pi$$

$$\frac{(1-\frac{m}{n})(1+\frac{m}{n})}{\frac{1}{2} \cdot \frac{1}{2}} \cdot \frac{(3-\frac{m}{n})(3+\frac{m}{n})}{\frac{3}{2} \cdot \frac{3}{2}} \text{ etc.} = \text{cof.} \frac{m}{n} \pi.$$

Problema 6.

45. Inuenire formulam integram, cuius valor casu $x=1$ praebeat $\sin. \frac{m}{n} \pi$.

Solutio.

Cum fit

$$\frac{n}{m\pi} \sin \frac{m}{n} \pi = \frac{(1-\frac{m}{n})(1+\frac{m}{n})}{1 \cdot 1} \cdot \frac{(2-\frac{m}{n})(2+\frac{m}{n})}{2 \cdot 2} \text{ etc.}$$

comparetur hoc productum infinitum cum forma generali

$$P = \frac{be}{ac} \cdot \frac{(b+1)(e+1)}{(a+1)(c+1)} \cdot \frac{(b+2)(e+2)}{(a+2)(c+2)} \text{ etc.}$$

cuius valor ante §. 41. pluribus modis in formulis integralibus est exhibitus. Statui ergo oportet $a=1$; $c=1$; $b=1-\frac{m}{n}$; et $e=1+\frac{m}{n}$; eritque $s=a+c=b+e=2$; tum vero

$$s-a-b-1 = -1 + \frac{m}{n}; \quad s-a-e-1 = -1 - \frac{m}{n}$$

$$s-b-c-1 = -1 + \frac{m}{n}; \quad s-c-e-1 = -1 - \frac{m}{n}$$

Hinc

Hinc ergo pro P prodit sequens expressio :

$$P = 1 \frac{\int dx (1-x)^{\frac{m}{n}}}{\int x^{-\frac{m}{n}} dx (1-x)^{\frac{m}{n}}} = \frac{1}{\int x^{-\frac{m}{n}} dx (1-x)^{\frac{m}{n}}}$$

ad quam reliquae omnes facile reducuntur. Haec igitur forma dat

$$\int x^{-\frac{m}{n}} dx (1-x)^{\frac{m}{n}} = \int \frac{x^{\frac{m}{n}} dx}{(1-x)^{\frac{m}{n}}} = \frac{m \pi}{n \sin \frac{m}{n} \pi}$$

et posito $x = y^n$, habebitur

$$\int \frac{y^{m+n-1} dy}{(1-y^n)^{\frac{m}{n}}} = \frac{m \pi}{n \sin \frac{m}{n} \pi} \text{ seu } \int \frac{y^{m-1} dy}{(1-y^n)^{\frac{m}{n}}} = \frac{\pi}{n \sin \frac{m}{n} \pi}$$

Inuenimus ergo $\sin \frac{m}{n} \pi = \frac{\pi}{n} : \int \frac{y^{m-1} dy}{(1-y^n)^{\frac{m}{n}}}$ Q. E. I.

Coroll. 1.

46. Per Theorema ergo primum haec forma $\int y^{m-1} dy (1-y^n)^{-\frac{m}{n}}$ ob $k = -\frac{m}{n}$ conuertitur in hanc $\int \frac{y^{m-1} dy}{1+y^n}$

ideoque habebitur

$$\int \frac{y^{m-1} dy}{1+y^n} = \frac{\pi}{n \sin \frac{m}{n} \pi} \text{ casu } y = \infty,$$

quae forma ob simplicitatem imprimis est notatu digna.

Coroll. 2.

47. Habemus ergo has duas aequalitates admodum notabiles :

$$\frac{m \pi}{n \sin \frac{m}{n} \pi} = \int \frac{m y^{m-1} dy}{(1-y^n)^{\frac{m}{n}}} \text{ posito } y = 1$$

$$\text{et } \frac{m \pi}{n \sin \frac{m}{n} \pi} = \int \frac{m y^{m-1} dy}{1+y^n} \text{ posito } y = \infty$$

quibus

quibus igitur casibus vtriusque formulae integrale satis commode exhiberi potest.

Coroll. 3.

48. Cum ergoposito $x=1$, et $y=\infty$, fit

$$\frac{\pi}{n \sin \frac{m}{n} \pi} = \int \frac{x^{m-1} dx}{(1-x^n)^{\frac{m}{n}}} = \int \frac{y^{m-1} dy}{1+y^n}$$

si pro m scribatur $2in+m$, ob $\sin. \frac{2in+m}{n} \pi = \sin. \frac{m}{n} \pi$, erit quoque

$$\int \frac{x^{2in+m-1} dx}{(1-x^n)^{\frac{2in+m}{n}}} = \int \frac{x^{m-1} dx}{(1-x^n)^{\frac{m}{n}}} = \int \frac{\pi}{n \sin. \frac{m}{n} \pi} \text{ et}$$

$$\int \frac{y^{2in+m-1} dy}{1+y^n} = \int \frac{y^{m-1} dy}{1+y^n} = \frac{\pi}{n \sin. \frac{m}{n} \pi}$$

denotante i numerum integrum quemcunque.

Coroll. 4.

49. Quia porro denotante i numerum integrum quemcunque, si pro m scribatur $2in-m$, est $\sin. \frac{2in-m}{n} \pi = -\sin. \frac{m}{n} \pi$, erit

$$\int \frac{x^{2in-m-1} dx}{(1-x^n)^{\frac{2in-m}{n}}} = -\int \frac{x^{m-1} dx}{(1-x^n)^{\frac{m}{n}}} = -\frac{\pi}{n \sin. \frac{m}{n} \pi} \text{ et}$$

$$\int \frac{y^{2in-m-1} dy}{1+y^n} = -\int \frac{y^{m-1} dy}{1+y^n} = -\frac{\pi}{n \sin. \frac{m}{n} \pi}$$

Tom. VI. Nou. Com.

T

Deinde

Deinde si pro m scribatur $(2i-1)n-m$, ob fin. $\frac{(2i-1)n-m}{n}\pi$
 $= \sin \frac{m}{n}\pi$, erit

$$\int \frac{x^{(2i-1)n-m-1} dx}{(1-x^n)^{\frac{(2i-1)n-m}{n}}} = \int \frac{x^{m-1} dx}{(1-x^n)^{\frac{m}{n}}} = \frac{\pi}{n \sin \frac{m}{n}\pi}$$

$$\int \frac{y^{(2i-1)n-m-1} dy}{1+y^n} = \int \frac{y^{m-1} dy}{1+y^n} = \frac{\pi}{n \sin \frac{m}{n}\pi}$$

Denique erit eodem modo

$$\int \frac{x^{(2i-1)n+m-1} dx}{(1-x^n)^{\frac{(2i-1)n+m}{n}}} = -\int \frac{x^{m-1} dx}{(1-x^n)^{\frac{m}{n}}} = -\frac{\pi}{n \sin \frac{m}{n}\pi}$$

$$\int \frac{y^{(2i-1)n+m-1} dy}{1+y^n} = -\int \frac{y^{m-1} dy}{1+y^n} = -\frac{\pi}{n \sin \frac{m}{n}\pi}$$

Coroll. 5.

50. Cum formula integralis $\int \frac{y^{m-1} dy}{1+y^n}$ saepius occurrat, operae pretium erit, eius valores pro praecipuis casibus exponere, posito $y = \infty$. Erit ergo

$$\int \frac{dy}{1+y^2} = \frac{\pi}{2 \sin \frac{\pi}{2}} = \frac{\pi}{2} \quad \text{ob fin. } \frac{\pi}{2} = 1$$

$$\int \frac{dy}{1+y^3} = \frac{\pi}{3 \sin \frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}} \quad \text{ob fin. } \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\int \frac{y dy}{1+y^3} = \frac{\pi}{3 \sin \frac{2\pi}{3}} = \frac{2\pi}{3\sqrt{3}} \quad \text{ob fin. } \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

-f

$$\int \frac{dy}{1+y^2} = \frac{\pi}{4 \sin. \frac{\pi}{4}} = \frac{\pi}{2\sqrt{2}}$$

$$\int \frac{y^2 dy}{1+y^4} = \frac{\pi}{4 \sin. \frac{3\pi}{4}} = \frac{\pi}{2\sqrt{2}}$$

$$\int \frac{dy}{1+y^5} = \int \frac{y^3 dy}{1+y^5} = \frac{\pi}{5 \sin. \frac{\pi}{5}}$$

$$\int \frac{y dy}{1+y^5} = \int \frac{y^2 dy}{1+y^5} = \frac{\pi}{5 \sin. \frac{2\pi}{5}}$$

$$\int \frac{dy}{1+y^6} = \int \frac{y^2 dy}{1+y^6} = \frac{\pi}{6 \sin. \frac{\pi}{6}} = \frac{\pi}{3}$$

et ita porro.

Problema 2.

51. Inuenire formulam integram, cuius valor casu $x=1$ praebeat $\text{cof. } \frac{m}{n} \pi$.

Solutio.

Cum fit

$$\text{cof. } \frac{m}{n} \pi = \frac{(\frac{1}{2} - \frac{m}{n})(\frac{1}{2} + \frac{m}{n})}{\frac{1}{2} \cdot \frac{1}{2}} \cdot \frac{(\frac{3}{2} - \frac{m}{n})(\frac{3}{2} + \frac{m}{n})}{\frac{3}{2} \cdot \frac{3}{2}} \cdot \text{etc.}$$

comparetur cum hoc productio infinito forma generalis

$$P = \frac{be}{ac} \cdot \frac{(b+1)(c+1)}{(a+1)(c+1)} \cdot \text{etc.}$$

T 2

hinc

hincque statuat $a = \frac{1}{2}$; $c = \frac{1}{2}$; $b = \frac{1}{2} - \frac{m}{n}$; $e = \frac{1}{2} + \frac{m}{n}$ ita,
 ut fit $s = a + c = b + e = 1$, atque

$$\begin{aligned} s - a - b - 1 &= -1 + \frac{m}{n}; & s - a - e - 1 &= -1 - \frac{m}{n}; \\ s - b - c - 1 &= -1 + \frac{m}{n}; & s - c - e - 1 &= -1 - \frac{m}{n}; \end{aligned}$$

Eritque idcirco:

$$P = \frac{\int x^{-\frac{1}{2}} dx (1-x)^{-\frac{m}{n}}}{\int x^{-\frac{1}{2}-\frac{m}{n}} dx (1-x)^{-\frac{1}{2}+\frac{m}{n}}} = \frac{\int dx \sqrt{x-xx}}{\int \frac{x^{\frac{m}{n}-\frac{1}{2}} dx}{(1-x)^{\frac{1}{2}+\frac{m}{n}}}}$$

At est $\int \frac{dx}{\sqrt{x-xx}} = \pi$, posito $x = 1$.

$$\text{Vnde fit } P = \text{cof. } \frac{m}{n} \pi = \frac{\pi}{\int \frac{x^{\frac{m}{n}-\frac{1}{2}} dx}{(1-x)^{\frac{1}{2}+\frac{m}{n}}}}$$

Per reliquas vero formulas ipsius P habebitur:

$$P = \text{cof. } \frac{m}{n} \pi = \frac{\int x^{-\frac{1}{2}} dx (1-x)^{-1+\frac{m}{n}}}{\int x^{-\frac{1}{2}-\frac{m}{n}} dx (1-x)^{-1+\frac{m}{n}}} = \frac{\int x^{-\frac{1}{2}} dx (1-x)^{-1-\frac{m}{n}}}{\int x^{-\frac{1}{2}+\frac{m}{n}} dx (1-x)^{-1-\frac{m}{n}}}$$

Q. E. I.

Coroll. I.

52. Ponatur $x = y^2$, et prior forma abit in hanc:

$$\text{cof. } \frac{m}{n} \pi = \frac{\pi}{2 \int \frac{y^{\frac{m}{n}} dy}{(1-yy)^{\frac{1}{2}+\frac{m}{n}}}}$$

ita

ata vt fit
$$\int \frac{x^{\frac{m}{n}} dx}{(1-xx)^{\frac{1}{2} + \frac{m}{n}}} = \frac{\pi}{2 \operatorname{col.} \frac{m}{n} \pi}$$

Coroll. 2.

53. Est vero etiam per Theorema primum

$$\int x^{\frac{m}{n} - \frac{1}{2}} dx (1-x)^{-\frac{1}{2} - \frac{m}{n}} = \int \frac{y^{\frac{m}{n} - \frac{1}{2}} dy}{1+y}, \text{posito } y = \infty,$$

Cum igitur sit $\int \frac{y^{\frac{m}{n} - \frac{1}{2}} dy}{1+y} = \frac{\pi}{\operatorname{col.} \frac{m}{n} \pi}$ ponatur y^n pro y et erit

$$\int \frac{y^{m + \frac{1}{2}n - 1} dy}{1+y^n} = \frac{\pi}{n \operatorname{col.} \frac{m}{n} \pi} = \int \frac{y^{\frac{1}{2}n - m - 1} dy}{1+y^n}.$$

Coroll. 3.

54. Si et reliquae formulae per Theorema primum conuertantur, prodibit :

$$\int x^{-\frac{1}{2}} dx (1-x)^{-1 + \frac{m}{n}} = \int \frac{y^{-\frac{1}{2}} dy}{(1+y)^{\frac{1}{2} + \frac{m}{n}}} = \int \frac{y^{\frac{m}{n} - 1} dy}{(1+y)^{\frac{1}{2} + \frac{m}{n}}}$$

$$\int x^{-\frac{1}{2} - \frac{m}{n}} dx (1-x)^{-1 + \frac{m}{n}} = \int \frac{y^{-\frac{1}{2} - \frac{m}{n}} dy}{\sqrt{1+y}} = \int \frac{y^{\frac{m}{n} - 1} dy}{\sqrt{1+y}}$$

posito $y = \infty$. Posito ergo y^n pro y , erit

$$\cos. \frac{m}{n} \pi = \frac{\int y^{m-1} dy}{(1+y^n)^{\frac{1}{n} + \frac{m}{n}}} = \frac{\int y^{\frac{1}{2}n-1} dy}{(1+y^n)^{\frac{1}{2} + \frac{m}{n}}} \\ = \frac{\int y^{m-1} dy}{V(1+y^n)} = \frac{\int y^{\frac{1}{2}n-m-1} dy}{V(1+y^n)}.$$

Coroll. 4.

§4. Si pro m scribatur $\frac{1}{2}n - m$, ob $\cos. (\frac{1}{2}n - m) \frac{\pi}{n} = \sin. \frac{m}{n} \pi$, obtinebitur primum $\frac{\pi}{n \sin \frac{m}{n} \pi} = \int \frac{y^{m-1} dy}{1+y^n}$ ut ante; reliquae vero formulae dabunt:

$$\sin. \frac{m}{n} \pi = \frac{\int y^{\frac{1}{2}n-m-1} dy}{(1+y^n)^{\frac{1}{n} + \frac{m}{n}}} = \frac{\int y^{\frac{1}{2}n-1} dy}{(1+y^n)^{\frac{1}{n} + \frac{m}{n}}} \\ = \frac{\int y^{\frac{1}{2}n-m-1} dy}{V(1+y^n)} = \frac{\int y^{m-1} dy}{V(1+y^n)}$$

et quia pro cosinu licet m negativae sumere erit quoque

$$\sin. \frac{m}{n} \pi = \frac{\int y^{m-\frac{1}{2}n-1} dy}{(1+y^n)^{\frac{m}{n}}} = \frac{\int y^{\frac{1}{2}n-1} dy}{(1+y^n)^{\frac{m}{n}}} \\ = \frac{\int y^{m-\frac{1}{2}n-1} dy}{V(1+y^n)} = \frac{\int y^{n-m-1} dy}{V(1+y^n)}$$

Coroll.

Coroll. 5.

55. At vero etiam ex praeced. problemate aliam formulam pro cosinu licet elicere: Cum enim, posito $2m$ pro m , fit

$$\frac{\pi}{n \sin. \frac{2m}{n} \pi} = \frac{\pi}{2n \sin. \frac{m}{n} \pi \cos. \frac{m}{n} \pi} = \int \frac{y^{2m-1} dy}{1+y^n}.$$

et $\int \frac{y^{m-1} dy}{1+y^n} = \frac{\pi}{n \sin. \frac{m}{n} \pi}$ si haec forma per illam dividatur, habebimus:

$$2 \cos. \frac{m}{n} \pi = \frac{\int \frac{y^{m-1} dy}{1+y^n}}{\int \frac{y^{2m-1} dy}{1+y^n}} \text{ seu } \cos. \frac{m}{n} \pi = \frac{\frac{1}{2} \int \frac{y^{m-1} dy}{1+y^n}}{\int \frac{y^{2m-1} dy}{1+y^n}}$$

Coroll. 6.

56 En ergo plures formas integrales, quae casu $y = \infty$ praebent $\sin. \frac{m}{n} \pi$

I. $\frac{\pi}{n \int \frac{y^{m-1} dy}{1+y^n}};$

II. $\frac{\int \frac{y^{\frac{1}{2}n-m-1} dy}{1+y^n}}{2 \int \frac{y^{n-2m-1} dy}{1+y^n}};$

III. $\frac{\int \frac{y^{\frac{1}{2}n-1} dy}{(1+y^n)^{\frac{m}{n}}}}{\int \frac{y^{n-m-1} dy}{\sqrt{1+y^n}}};$

IV. $\frac{\int \frac{y^{m-\frac{1}{2}n-1} dy}{(1+y^n)^{\frac{m}{n}}}}{\int \frac{y^{m-\frac{1}{2}n-1} dy}{\sqrt{1+y^n}}};$
V.

$$\text{V. } \int \frac{y^{\frac{1}{2}n-1} dy}{(1+y^n)^{\frac{1-m}{n}}}$$

$$\int \frac{y^{m-1} dy}{\sqrt[1]{(1+y^n)}}$$

$$\text{VI. } \int \frac{y^{\frac{1}{2}n-m-1} dy}{(1+y^n)^{\frac{1-m}{n}}}$$

$$\int \frac{y^{\frac{1}{2}n-m-1} dy}{\sqrt[1]{(1+y^n)}}$$

vbi notandum est, in formis III et IV, item in V et VI, seorsim numeratores et denominatores inter se esse aequales.

Coroll. 7.

57. Simili modo totidem habebimus formulas pro cof. $\frac{m}{n} \pi$ quae sunt:

$$\text{I. } \frac{\pi}{n \int \frac{y^{\frac{1}{2}n-m-1} dy}{1+y^n}};$$

$$\text{II. } \frac{\int \frac{y^{m-1} dy}{1+y^n}}{2 \int \frac{y^{2m-1} dy}{1+y^n}}$$

$$\text{III. } \frac{\int \frac{y^{m-1} dy}{(1+y^n)^{\frac{1}{2}+\frac{m}{n}}}}$$

$$\text{IV. } \frac{\int \frac{y^{\frac{1}{2}n-1} dy}{(1+y^n)^{\frac{1}{2}+\frac{m}{n}}}}{\int \frac{y^{\frac{1}{2}n-m-1} dy}{\sqrt[1]{(1+y^n)}}}$$

$$\text{V. } \frac{\int \frac{y^{-m-1} dy}{(1+y^n)^{\frac{1}{2}+\frac{m}{n}}}}$$

$$\int \frac{y^{\frac{1}{2}n-1} dy}{(1+y^n)^{\frac{1}{2}+\frac{m}{n}}}$$

$$\int \frac{y^{-m-1} dy}{\sqrt[1]{(1+y^n)}}$$

$$\text{VI. } \frac{\int \frac{y^{\frac{1}{2}n+m-1} dy}{\sqrt[1]{(1+y^n)}}}$$

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Scholion.

58. Hinc vero etiam plures formulas pro tangente anguli $\frac{m}{n}\pi$ deducere licet, quarum quæ sunt simpliciores, hic exhibebo.

$$\text{tang. } -\frac{m}{n}\pi = \frac{\int y^{\frac{1}{2}n-m-1} dy}{\int \frac{y^{m-1} dy}{1+y^n}};$$

$$\text{tang. } \frac{m}{n}\pi = \frac{\int \frac{y^{\frac{1}{2}n-1} dy}{(1+y^n)^{\frac{1-m}{n}}} \int \frac{y^{\frac{1}{2}n-m-1} dy}{(1+y^n)^{\frac{1-m}{n}}}}{\int \frac{y^{m-1} dy}{(1+y^n)^{\frac{1}{2}+\frac{m}{n}}} \int \frac{y^{\frac{1}{2}n-1} dy}{(1+y^n)^{\frac{1}{2}+\frac{m}{n}}}}$$

Deinde vero ex combinatione harum formularum insignes proprietates innotescunt, veluti si $n=4$ et $m=1$, erit

$$\frac{1}{\sqrt{2}} = \frac{\pi}{4 \int \frac{dy}{1+y^4}} = \frac{\int \frac{dy}{1+y^4}}{2 \int \frac{y dy}{1+y^4}} = \frac{\int \frac{y dy}{\sqrt{1+y^4}}}{\int \frac{y y dy}{\sqrt{1+y^4}}}$$

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$$\begin{aligned} & \int \frac{y dy}{\sqrt{(1+y^2)^3}} = \int \frac{dy}{\sqrt{(1+y^2)^3}} \\ & = \frac{\int dy}{\int \sqrt{(1+y^2)}} = \frac{\int dy}{\int \sqrt{(1+y^2)}} \end{aligned}$$

unde colligitur fore $\int \frac{y dy}{\sqrt{(1+y^2)^3}} = \int \frac{dy}{\sqrt{(1+y^2)^3}}$ casu $y=\infty$,

feu esse $\int \frac{(1-y) dy}{\sqrt{(1+y^2)^3}} = 0$; talibus autem proprietatibus

eruedis hic non immoror.

SOLV: