



S P E C I M E N
N O V A E M E T H O D I
C U R V A R V M Q V A D R A T V R A S E T R E C T I -
F I C A T I O N E S A L I A S Q V E Q U A N T I T A T E S
T R A N S C E N D E N T E S I N T E R S E
C O M P A R A N D I .

Auctore

L. E V L E R O .

Quae nuper occasione inuentorum Ill. Comitis Fa-
guani commentatus sum de comparatione arcuum
ellipticis hyperbolae et curuae lemniscatae, multo latius
mihi quidem patere statim sunt visa. Cum enim me-
thodis adhuc consuetis eiusmodi tantum curuarum arcus
inter se comparari possent, quarum rectificatio vel a
quadratura circuli, vel a logarithmis penderet, quippe
quae quantitates, etsi sunt transcendentes, tamen ita
iam in Analyti prae ceteris ius quoddam ciuitatis sunt
adeptae, vt perinde atque algebraicae tractari queant :
maxima certe attentione erat dignum, quod a Fagnano
in hyperbola et ellipsi arcus sint assignati, quorum
differentia sit algebraica; in lemniscata autem eiusmodi
arcus, qui adeo inter se sint aequales, vel certam te-
neant rationem, propterea quod harum curuarum recti-
ficatio neque ad quadraturam circuli, neque ad logarith-
mos reduci queat. Hinc certe Theoriae quantitarum

L 2

transcen-

transcendentium insigne lumen accenderetur, si modo via, qua Fagnanus est usus, certam methodum suppeditaret in huiusmodi inuestigationibus ulterius progrediendi; sed quia tota substitutionibus precario factis et quasi casu fortuito adhibitis niritur, parum inde utilitatis in Analyfin redundat. Deinde iam notavi integrationes, quas operatio Fagnaniana complectitur, tantum esse particulares, neque idcirco methodum certam, a qua plura expectari liceat, suppeditare. Interim tamen ea amplissimum campum aperuisse est visa, in quo ulterius excolendo Geometrae vires suas summo cum fructu exercent, ad insigne Analyseos incrementum.

Res autem huc redit, ut propositis duabus formulis integralibus $\int X dx$ et $\int Y dy$, non integrabilibus, ubi X sit functio quaequam ipsius x , et Y ipsius y , eiusmodi relatio inter variables x et y definiatur, ut illae formulae vel inter se fiant aequales, vel datam rationem teneant, vel ut differentiam algebraice assignabilem obtineant. Quae inuestigatio cum latissime pateat, tum etiam insignes in se continet casus iam pridem non sine maximo Analyseos incremento euolutos; huc enim referenda sunt, quae de comparatione arcuum circularium, de lunulis quadrabilibus, de zonis cycloidibus quadrabilibus; tum vero arcubus parabolicis, qui vel datam inter se teneant rationem vel differentiam algebraicam habeant, a geometris sunt tradita: quin etiam haec inuestigatio a *Cel. Iob. Bernoulli* ad parabolas cubicales altiorisque ordinis est extensa, sed quia ratio, qua usus est, nulla certa methodo nitebatur, ulteriori vsu fere penitus caruit. Huc quoque pertinet, quod

quod multo ante iam acutissimus Hugenius in Horologio oscillatorio exposuerat, vbi proposito sphaeroide elliptico compresso, seu reuolutione circa axem minorem genito, inuenire docuit conoides hyperbolicum, ita vt summa vtriusque superficiei circulo exhiberi posset; cum tamen neutra superficies seorsim cum circulo comparari queat. Quae inuentio iam tum summis geometricis maxime memorabilis visa est; atque *Bernoullius* in litteris ad *Leibnizium* datis dolet, hanc inuentionem nulla certa methodo inniti, ex qua plura huius generis inuenta deriuare liceat; interim quia superficies tam illius sphaeroidis elliptici, quam conoidis hyperbolici a logarithmis pender, reductio vtriusque iunctim sumtae ad circulum, simili modo perfici potest, quo in parabola arcus algebraicam habentes differentiam assignari solent. Inprimis autem hoc loco non est praetereundum, *Tschirnhausium* quondam methodum a se inuentam iactasse, cuius beneficio curuarum quarumcunque non rectificabilium arcus ita inter se comparare posset, vt differentia fiat algebraica, sed praeterquam quod methodum suam nunquam aperuerit, manifestum est, eum paralogismo quodam fuisse deceptum, vt saepius alias, cum certum sit, rem ita generaliter omnino expediri non posse: neque ergo *Tschirnhausius* putandus est quicquam eorum habuisse, quae vel tum circa comparationem curuarum suae inuenta, vel adhuc forte eliciuntur.

Specimen igitur quoddam methodi huiusmodi quaestiones soluendi hic exhibere constitui, quod non obscure maiores progressus in hac re promittere videtur;

tar; atque cum non solum difficillimum sit, propositis in genere eiusmodi formulis integralibus, quaesitam inter variables relationem eruere, sed etiam hoc saepissime omnino ne fieri quidem possit; ordine inuerso ita tentavi, vt assumpta binarum variabilium relatione, inde ipsas formulas integrales inuestigarem, quae per hanc relationem inter se comparari possent. Quae methodus, cum facillime procedat, ad multo sublimiora perducere posse videtur, quae aliis methodis plane sint imperuia: hac enim methodo non solum ea, quae habet Fagnanus, facili negotio, ac sine taedioso calculo, sum affecutus, sed etiam multo ampliora atque illustriora reddidi, vt quae ille nimis particulariter definiuerat, ego satis vniuersaliter expediuerim: atque calculus, quo sum vsus, ita comparatus est, vt, quoniam operationes prorsus singulares complectitur, viam ad multo sublimiora sternere videatur.

Tum vero quaequam variabilium mutua relatio per methodos consuetas definiiri potest, quoties integratio vtriusque formulae $\int X dx$ et $\int Y dy$, vel a quadratura circuli, vel a logarithmis pendet; tamen et hoc plerumque non sine molesto calculo perficitur; dum partes, vel arcus circulares, vel logarithmos, continentes, se mutuo destruere debent: quemadmodum hoc in comparatione arcuum parabolicorum abunde perspicitur. Per meam autem methodum hae difficultates cunctae penitus euanescent, ac fere sine vilo negotio istae comparationes, tam in circulo, quam in parabola, absoluuntur: in quo sine dubio non exigua vis huius methodi sita esse censenda est, quod non solum multo facilius ea, quae

quae aliis methodis iam sunt eruta, praebent; sed etiam ad eiusmodi inuestigationes manuducat, in quibus aliae methodi nihil essent praestiturae. Quam ob rem hoc quidem loco istam methodum tantum ad eos casus applicabo, qui etiam aliis methodis, sed multo operosius, expediri solent, quo cum principia, quibus innititur, hac occasione exposuero, deinceps facilius eius applicationem ad quaestiones sublimiores suscipere possim. Quoniam igitur mihi a relatione inter binas variables, quam pro lubitu constituo, ordiendum est, a simplicioribus incipiam, ac primo quidem ab eiusmodi, quae ad similes formulas integrales perducant, seu in quibus X et Y similes sint proditurae functiones ipsarum x et y. Formulae ergo integrales hinc natae ob similitudinem quantitatum transcendentium exhibebunt, ad eandem lineam curuam pertinentes, deinceps autem ad formulas quoque dissimiles, quae ad diuersas curuas pertineant, sum progressurus.

RELATIO PRIMA inter binas variables x et y.

$$0 = \alpha + \gamma(xx + yy) + 2\delta xy.$$

x. Si hinc seorsim valores x et y extrahantur, reperietur:

$$y = \frac{-\delta x \pm \sqrt{(\delta\delta - \gamma\gamma)xx - \alpha\gamma}}{\gamma}$$

$$x = \frac{-\delta y \pm \sqrt{(\delta\delta - \gamma\gamma)yy - \alpha\gamma}}{\gamma}$$

vbi quouis casu dispiciendum est, vtrum signum quantitatum radicali sit praefigendum? Fieri enim potest, vt in vtraque formula, vel signa paria, vel disparia, locum habeant,

beant, dum alterutrum arbitrio nostro plane relinquitur: in quo iudicio imprimis natura variabilium x et y , vtrum affirmatiue, an negatiue accipiantur, spectari debet.

2. Ponantur breuitatis gratia membra irrationalia:

$\pm \sqrt{((\delta\delta - \gamma\gamma)xx - \alpha\gamma)} = P$, et $\pm \sqrt{((\delta\delta - \gamma\gamma)yy - \alpha\gamma)} = Q$
vt fit:

$$y = \frac{\delta x + P}{\gamma}, \text{ et } x = \frac{\delta y + Q}{\gamma}.$$

ficquè erit:

$$P = \gamma y + \delta x, \text{ et } Q = \gamma x + \delta y$$

vnde quouis casu facile colligere licet, vtrum quantitates P et Q habiturae sint valores affirmatiuos, an negatiuos.

3. Differentietur iam aequatio assumpta, eritque:

$$dx(\gamma x + \delta y) + dy(\gamma y + \delta x) = 0$$

atque ob $\gamma x + \delta y = Q$, et $\gamma y + \delta x = P$, habebitur haec aequatio:

$$Qdx + Pdy = 0, \text{ siue } \frac{dx}{P} + \frac{dy}{Q} = 0.$$

Restitutis ergo pro P et Q valoribus, huic aequationi integrali:

$$\int \frac{dx}{\sqrt{((\delta\delta - \gamma\gamma)xx - \alpha\gamma)}} + \int \frac{dy}{\sqrt{((\delta\delta - \gamma\gamma)yy - \alpha\gamma)}} = \text{Const.}$$

satisfacit relatio inter variables x et y assumpta.

4. Euoluamus haec accuratius, et quo facilius applicatio fieri queat, ponamus:

$$-\alpha\gamma = Ap \text{ et } \delta\delta - \gamma\gamma = Cp$$

vt fit:

$$\int \frac{dx}{\sqrt{(A + Cxx)}} + \int \frac{dy}{\sqrt{(A + Cy^2)}} = \text{Const.}$$

eritque

eritque $\alpha = -\frac{Ap}{\gamma}$ et $\delta = \sqrt{Cp + \gamma\gamma}$;
 sicque quantitates p et γ arbitrio nostro relinquuntur.

5. Statuatur ergo $\gamma = A$, et $p = Akk$, ita ut k
 sit noua quantitas constans, a nostro arbitrio pendens;
 eritque

$\alpha = -Akk$, $\gamma = A$, et $\delta = \sqrt{A(A + Ckk)}$
 et aequatio canonica, nostrae aequationi integrali satis-
 faciens, erit:

$$0 = -Akk + A(xx + yy) - 2xy\sqrt{A(A + Ckk)}$$

$$\text{Seu } y = \frac{-x\sqrt{A + Ckk} + k\sqrt{A + Cxx}}{\sqrt{A}}$$

$$\text{et } x = \frac{-y\sqrt{A + Ckk} + k\sqrt{A + Cyy}}{\sqrt{A}}$$

6. Si $\sqrt{A + Cyy}$ negative capiatur, itemque
 \sqrt{A} , tum huius aequationis differentialis

$$\frac{dx}{\sqrt{A + Cxx}} = \frac{dy}{\sqrt{A + Cyy}}$$

integralis erit:

$$0 = -Akk + A(xx + yy) - 2xy\sqrt{A(A + Ckk)},$$

ideoque

$$\text{vel } y = \frac{x\sqrt{A + Ckk} - k\sqrt{A + Cxx}}{\sqrt{A}}$$

$$\text{vel } x = \frac{y\sqrt{A + Ckk} + k\sqrt{A + Cyy}}{\sqrt{A}}$$

7. Quia ergo aequatio integralis constantem in
 se complectitur k , quae in differentiali non inest, in-
 dicio hoc est, integram esse completam; sicque diffe-
 rentiali nulla alia satisfacit integralis, nisi quae in for-
 ma inuenta comprehendatur. Atque haec est inte-
 gratio principalis, ad quam relatio inter x et y assumpta
 perducit.

8. Hinc autem deriuari possunt innumerabiles aliae integrationes. Si enim sint X et Y eiusmodi functiones ipsarum x et y , vt vi relationis assumtae sit $X = Y$, eadem relatio satisfaciet quoque huic aequationi differentiali:

$$\frac{X dx}{\sqrt{(A+Cxx)}} = \frac{Y dy}{\sqrt{(A+Cy y)}}$$

Infinitis autem modis huiusmodi functiones aequales exhiberi possunt ex formulis pro x et y inuentis.

9. Quo autem haec inuestigatio latius pateat, et X et Y sint functiones similes, eas non assumo inter se aequales, eiusmodi autem pro iis valores indago, vt sit:

$$\frac{X dx}{\sqrt{(A+Cxx)}} - \frac{Y dy}{\sqrt{(A+Cy y)}} = dV$$

atque quantitas V prodeat algebraica, si scilicet relatio §. 6. tradita locum habeat.

10. Cum igitur sit $\frac{dy}{\sqrt{(A+Cy y)}} = \frac{dx}{\sqrt{(A+Cxx)}}$, erit

$$\frac{(X-Y) dx}{\sqrt{(A+Cxx)}} = dV$$

et ob $P = k\sqrt{A(A+Cxx)} = \gamma y + \delta x = Ay + x\sqrt{A(A+Ckk)}$ sumto per §. 6. \sqrt{A} negatiuo, erit $\sqrt{(A+Cxx)} = \frac{x}{k} \sqrt{(A+Ckk)} - \frac{\gamma}{k} \sqrt{A}$, vnde fiet:

$$\frac{(X-Y)k dx}{x\sqrt{(A+Ckk)} - \gamma\sqrt{A}} = dV.$$

11. Cum sit porro ex aequatione differentiata

$$dx(Ax - y\sqrt{A(A+Ckk)}) = dy(x\sqrt{A(A+Ckk)} - Ay)$$

ponatur $xy = u$, erit $dy = \frac{du}{x} - \frac{y dx}{x}$, quo valore substituto fiet

$$dx\left(Ax - \frac{Ay y}{x}\right) = \frac{du}{x}(x\sqrt{A(A+Cxx)} - Ay)$$

seu

Seu $\frac{dx}{x\sqrt{(A+Cxx)}-y\sqrt{A}} = \frac{du}{(xx-yy)\sqrt{A}}$;
 sicque erit: $dV = \frac{kdu}{\sqrt{A} \cdot \frac{x-y}{xx-yy}}$

12. Quoties ergo $\frac{x-y}{xx+yy}$ eiusmodi functio ipsius u , quae ducta in du fiat integrabilis, toties valor quantitatis V algebraice exhiberi poterit: hoc autem evenit, quoties X et Y fuerint potestates parium exponentium ipsarum x et y , propterea cum sit ex aequatione assumpta

$$xx + yy = kk + \frac{2u}{A}VA(A + Ckk)$$

13. Ponatur ergo $X = x^n$ et $Y = y^n$; eritposito $n = 2$; $\frac{x-y}{xx-yy} = 1$; et $dV = \frac{kdu}{\sqrt{A}}$ ideoque $V = \frac{ku}{\sqrt{A}} + \text{Const.} = \frac{kxy}{\sqrt{A}} + \text{Const.}$
 Quam ob rem habebitur:

$$\int \frac{xx dx}{\sqrt{(A+Cxx)}} - \int \frac{yy dy}{\sqrt{(A+Cy)}} = \text{Const.} + \frac{kxy}{\sqrt{A}}$$

14. Sit iam $n = 4$; eritque $\frac{x-y}{xx-yy} = \frac{xx+yy}{xx+yy}$
 $= kk + \frac{2u}{A}VA(A + Ckk)$
 unde $dV = \frac{kdu}{A}(kk\sqrt{A} + 2uV(A + Ckk))$
 ergo $V = \frac{ku}{A}(kk\sqrt{A} + uV(A + Ckk))$.
 Hoc igitur casu erit:

$$\int \frac{x^4 dx}{\sqrt{(A+Cxx)}} - \int \frac{y^4 dy}{\sqrt{(A+Cy)}} = \text{Const.} + \frac{kxy}{A}(kk\sqrt{A} + xyV(A + Ckk))$$

similique modo ulterius progredi licet.

15. His igitur coniungendis si fuerit

$$xx + yy = kk + 2xyV(1 + \frac{C}{A}kk), \text{ siue}$$

$$y = \frac{x\sqrt{(A+Ckk)} - k\sqrt{(A+Cxx)}}{\sqrt{A}}$$

$$x = \frac{y\sqrt{(A+Ckk)} + k\sqrt{(A+Cy)}}{\sqrt{A}}$$

haec ratio inter x et y satisfacet huic aequationi integrali :

$$\int \frac{dx(A + Bxx + Cx^2)}{\sqrt{(A + Cx^2)}} - \int \frac{dy(A + Byy + Cy^2)}{\sqrt{(A + Cy^2)}} = \text{Const.}$$

$$+ \frac{Bkxy}{\sqrt{A}} + \frac{Ckxy}{\sqrt{A}} (kk + xy\sqrt{(1 + \frac{C}{A}kk)})$$

feu differentia istarum formularum integralium algebraice assignari potest.

RELATIO SECUNDA

inter binas variables x et y

$$0 = a + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy$$

16. Quoniam, uti in praecedentibus deprehendimus, ambiguitas signorum radicalium ab arbitrio nostro pendet, dummodò eius ratio in conclusionibus finalibus debite habeatur, si ad differentiam binarum formularum integralium peruenire velimus, extrahendo radices, habebimus :

$$y = \frac{-\beta - \delta x - \sqrt{(\beta\beta - a\gamma + 2\beta(\delta - \gamma)x + (\delta\delta - \gamma\gamma)xx)}}{\gamma}$$

$$x = \frac{-\beta - \delta y + \sqrt{(\beta\beta - a\gamma + 2\beta(\delta - \gamma)y + (\delta\delta - \gamma\gamma)yy)}}{\gamma}$$

17. Statuamus breuitatis gratia has formulas irrationales :

$$\sqrt{(\beta\beta - a\gamma + 2\beta(\delta - \gamma)x + (\delta\delta - \gamma\gamma)xx)} = P$$

$$\sqrt{(\beta\beta - a\gamma + 2\beta(\delta - \gamma)y + (\delta\delta - \gamma\gamma)yy)} = Q$$

eritque

$$-P = \beta + \gamma y + \delta x \text{ et } Q = \beta + \gamma x + \delta y$$

vnde eliciuntur istae relationes

$$P + Q = (\gamma - \delta)(x - y)$$

$$\gamma P + \delta Q = \beta(\delta - \gamma) + (\delta\delta - \gamma\gamma)y$$

$$\delta P + \gamma Q = \beta(\gamma - \delta) - (\delta\delta - \gamma\gamma)x$$

18. Ac-

18. Aequatio autem proposita differentiata dat:

$$dx(\beta + \gamma x + \delta y) + dy(\beta + \gamma y + \delta x) = 0$$

siue $Q dx - P dy = 0$,

unde oritur:

$$\frac{dx}{P} = \frac{dy}{Q} \text{ seu } \int \frac{dx}{P} - \int \frac{dy}{Q} = \text{Const.}$$

cui ergo aequationi integrali satisfacit ratio proposita, indeque valores pro x et y extracti.

19. Ut hinc simili modo alias integrationes obtineamus, sint iterum X et Y functiones similes ipsarum x et y ; ac posito:

$$\frac{X dx}{P} - \frac{Y dy}{Q} = dV$$

definiantur hae functiones ita, ut V prodeat quantitas algebraica, sicque habeatur:

$$\int \frac{X dx}{P} - \int \frac{Y dy}{Q} = V + \text{Const.}$$

20. Cum igitur fit $\frac{dy}{Q} = \frac{dx}{P}$, erit $dV = \frac{(X - Y) dx}{P}$, seu $dV = \frac{-dx(x - Y)}{\beta + \gamma y + \delta x}$. Sit iterum $xy = u$, ideoque $dy = \frac{du}{x} - \frac{y dx}{x}$, erit pro aequatione differentiali

$$dx(\beta + \gamma x + \delta y) + \frac{du}{x}(\beta + \gamma y + \delta x) - \frac{y dx}{x}(\beta + \gamma y + \delta x) = 0$$

seu $dx(\beta x - \beta y + \gamma x x - \gamma y y) + du(\beta + \gamma y + \delta x) = 0$.

21. Valore hinc pro dx substituto habebitur:

$$dV = \frac{du(x - y)}{(x - y)(\beta + \gamma(x + y))}$$

Ponatur autem viterius $x + y = t$; erit $xx + yy = tt - 2ut$ et quia aequatio assumpta in hanc formam abibit:

$$0 = \alpha + 2\beta t + \gamma t t + 2(\delta - \gamma)u$$

ex qua differentiando fit $dt(\beta + \gamma t) = (\gamma - \delta) du$,

seu $\frac{du}{\beta + \gamma t} = \frac{dt}{\gamma - \delta}$.

22. Hinc igitur simpliciori modo obtinetur

$$dV = \frac{dt(x-y)}{(\gamma-\delta)(x-y)}$$

vnde patet, si X et Y fuerint potestates ipsarum x et y , tum fractionem $\frac{x-y}{x-y}$ per t et u , ideoque et per solum t , ob $u = \frac{+\alpha+2\beta t+\gamma t^2}{2(\gamma-\delta)}$, commode exprimi posse.

23. Sit ergo $X=x^n$, et $Y=y^n$; ac ponatur primo $n=1$, erit $\frac{x-y}{x-y}=1$, et $dV = \frac{dt}{\gamma-\delta}$; vnde fit $V = \frac{t}{\gamma-\delta}$. Quocirca pro hoc casu erit

$$\int \frac{x dx}{P} - \int \frac{y dy}{Q} = \text{Const.} + \frac{(x+y)}{\gamma-\delta}.$$

cui ergo aequationi integrali satisfit per relationem inter x et y assumtam.

24. Sit $n=2$; eritque $\frac{x-y}{x-y} = x+y=t$; vnde fit

$$dV = \frac{t dt}{\gamma-\delta} \text{ et } V = \frac{t^2}{2(\gamma-\delta)} = \frac{(x+y)^2}{2(\gamma-\delta)}$$

Hoc ergo casu habebitur;

$$\int \frac{x^2 dx}{P} - \int \frac{y^2 dy}{Q} = \text{Const.} + \frac{(x+y)^2}{2(\gamma-\delta)}.$$

25. Si vterius progredi lubeat, ponatur $n=3$, eritque:

$$\frac{x^3-y^3}{x-y} = xx+xy+yy = tt-u = \frac{(\gamma-2\delta)tt-2\beta t-\alpha}{2(\gamma-\delta)}$$

et $V = \frac{\frac{1}{2}(\gamma-2\delta)t^2 - \beta tt - \alpha t}{2(\gamma-\delta)^2}$; ficque erit

$$\int \frac{x^3 dx}{P} - \int \frac{y^3 dy}{Q} = \text{Const.} + \frac{(\gamma-2\delta)(x+y)^2 - 3\beta(x+y) - 3\alpha(x+y)}{6(\gamma-\delta)^2}$$

26. His igitur formulis coniungendis, sequenti aequationi integrali

$\int dx$

$$\int \frac{dx(A + Bx + Cxx + Dx^2)}{\sqrt{(\beta\beta - \alpha\gamma + \beta\delta - \gamma\gamma)x + (\delta\delta - \gamma\gamma)xx}} - \int \frac{dy(A + By + Cy + Dy^2)}{\sqrt{(\beta\beta - \alpha\gamma + \beta(\delta - \gamma)y + (\delta\delta - \gamma\gamma)yy)}} \\ = \text{Const.} + \frac{B(x+y)}{\gamma - \delta} + \frac{B(x+y)^2}{2(\gamma - \delta)^2} + \frac{2((\gamma - \delta)(x+y)^2 - \beta(x+y)^2 - \alpha(x+y))}{6(\gamma - \delta)^2}$$

satisfacit relatio assumta

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy$$

indeque valores pro x et y initio eruti.

27. Quo applicatio ad casus particulares facilius fieri possit, ponamus $\beta\beta - \alpha\gamma = Ap$; $\beta(\delta - \gamma) = Bp$; et $\delta\delta - \gamma\gamma = Cp$; ut sit $P = \sqrt{p(A + 2Bx + Cxx)}$ et $Q = \sqrt{p(A + 2By + Cy^2)}$ fiatque :

$$\gamma = A + Bk, \text{ et } \delta = \sqrt{A(A + 2Bk + Ckk)}$$

$$\text{erit } p = \frac{(AC - BB)kk}{C}, \text{ et } \beta = \frac{B}{C}(\delta + \gamma)$$

$$\text{atque } \alpha = \frac{2BB}{CC}(\gamma + \delta) - \frac{(AC - BB)kk}{CC(A + Bk)}$$

RELATIO TERTIA

inter binas variables x et y

$$0 = \alpha + mxx + nyy + 2\delta xy$$

28. Extrahendo utramque radicem habebitur

$$y = \frac{-\delta x + \sqrt{((\delta\delta - mn)xx - \alpha n)}}{n}$$

$$x = \frac{-\delta y - \sqrt{((\delta\delta - mn)yy - \alpha m)}}{m}$$

hinc posito :

$$P = \sqrt{(\delta\delta - mn)xx - \alpha n} \text{ et } Q = \sqrt{((\delta\delta - mn)yy - \alpha m)}$$

$$\text{erit } P = \delta x + ny \text{ et } -Q = \delta y + mx.$$

29. Per differentiationem vero obtinemus :

$$dx(mx + \delta y) + dy(ny + \delta x) = 0$$

$$\text{seu } -Qdx + Pdy = 0, \text{ ideoque } \frac{dy}{Q} = \frac{dx}{P}$$

vnde

unde aequatio assumpta huic aequationi integrali

$$\int \frac{dy}{Q} = \int \frac{dx}{P} \text{ satisfacit.}$$

30. Sint iam X et Y functiones ipsarum x et y singulatim, ac ponatur

$$\int \frac{X dx}{P} - \int \frac{Y dy}{Q} = V$$

ita ut fiat V quantitas algebraica: eritque

$$\frac{(X-Y) dx}{P} = dV + \frac{(X-Y) dx}{\delta x + ny}$$

31. Posito $xy = u$, ut sit $dy = \frac{du}{x} - \frac{y dx}{x}$, eritque

$$dx(mxx - nyy) + du(ny + \delta x) = 0$$

unde, cum fiat $\frac{dx}{\delta x + ny} = \frac{-du}{mxx - nyy}$, erit

$$dV = \frac{-du(X-Y)}{mxx - nyy},$$

hincque non difficulter casus integrabiles eliciuntur.

32. Sit enim primo $X = mxx$, et $Y = nyy$, erit

$$dV = -du, \text{ et } V = -u = -xy$$

Hinc relatio inter x et y assumpta satisfacit huic aequationi integrali:

$$\int \frac{mxx dx}{P} - \int \frac{nyy dy}{Q} = \text{Const.} - xy.$$

33. Sit secundo $X = mmx^2$, et $Y = nny^2$, erit

$$dV = -du(mxx + nyy) = +du(\alpha + 2\delta u)$$

unde fit $V = u(\alpha + \delta u) = xy(\alpha + \delta xy)$

Ergo huic aequationi integrali

$$\int \frac{m mx^2 dx}{P} - \int \frac{n ny^2 dy}{Q} = \text{Const.} + xy(\alpha + \delta xy)$$

satisfacit relatio assumpta inter x et y .

34. His igitur colligendis relatio inter x et y assumpta satisfacet huic aequationi integrali:

$$\int \frac{dx(\mathcal{A} + \mathcal{B}mxx + \mathcal{C}m^2x^2)}{\sqrt{(\delta\delta - mn)xx - \alpha n}} - \int \frac{dy(\mathcal{A} + \mathcal{B}myy + \mathcal{C}m^2y^2)}{\sqrt{(\delta\delta - mn)yy - \alpha m}}$$

$$= \text{Const.} - \mathcal{B}xy + \mathcal{C}xy(\alpha + \delta xy).$$

35. Ponamus ad faciliorem applicationem:

$\delta\delta - mn = Cp$; $\alpha n = -Ap$ et $\alpha m = -Bp$,
 ut sit $P = \sqrt{p(A + Cxx)}$, et $Q = \sqrt{p(B + Cyy)}$,
 erit $\frac{m}{n} = \frac{B}{A}$. Sit ergo $m = B$, et $n = A$, erit $\alpha = -p$,
 et $\delta = \sqrt{AB + Cp}$. Sit ergo $p = Ckk$, ut sit $\alpha = -Ckk$,
 et aequatio relationem inter x et y definiens erit:

$$0 = -Ckk + Bxx + Ayy + 2xy\sqrt{AB + CCkk}$$

36. Quam ob rem valores ipsius x et y hinc erunt:

$$y = \frac{-xx(AB + CCkk) + k\sqrt{C(A + Cxx)}}{A}$$

$$x = \frac{-yy(AB + CCkk) - k\sqrt{C(B + Cyy)}}{B}$$

existente:

$$P = k\sqrt{C(A + Cxx)} \text{ et } Q = k\sqrt{C(B + Cyy)}.$$

37. Hi igitur valores conueniunt huic aequationi integrali:

$$\int \frac{dx(\mathcal{A} + \mathcal{B}Bxx + \mathcal{C}B^2x^2)}{\sqrt{(A + Cxx)}} - \int \frac{dy(\mathcal{A} + \mathcal{B}Ayy + \mathcal{C}A^2y^2)}{\sqrt{(B + Cyy)}}$$

$$= \text{Const.} - \mathcal{B}kxy\sqrt{C} + \mathcal{C}kxy(-Ckk + xy\sqrt{AB + C^2kk})\sqrt{C}.$$

38. Ponatur $B = \frac{CF}{F}$, quae aequatio latius patere videatur, atque, constantibus mutatis, prodibit ista aequatio integralis:

$$\int \frac{dx(\mathcal{A} + \frac{C}{A}\mathcal{B}xx + \frac{CC}{AA}\mathcal{C}x^2)\sqrt{C}}{\sqrt{(A + Cxx)}} - \int \frac{dy(\mathcal{A} + \frac{F}{E}\mathcal{B}yy + \frac{FF}{EE}\mathcal{C}y^2)\sqrt{F}}{\sqrt{(E + Fyy)}}$$

$$= \text{Const.} - \frac{CF}{AE}\mathcal{B}kxy - \frac{CCFF}{AAEE}\mathcal{C}k^2xy + \frac{CCFF}{AAEE}\mathcal{C}kxxyy\sqrt{\frac{AE}{CF} + kk}$$

Tom. VII. Nou. Com.

N

cui

cui satisfaciunt isti valores :

$$\frac{Ax}{C} = k\sqrt{\left(\frac{A}{C} + xx\right)} - x\sqrt{\left(\frac{AE}{CF} + kk\right)}$$

$$\frac{Ey}{F} = -k\sqrt{\left(\frac{E}{F} + yy\right)} - y\sqrt{\left(\frac{AE}{CF} + kk\right)}$$

qui oriuntur ex hac aequatione :

$$kk = \frac{E}{F}xx + \frac{A}{C}yy + 2xy\sqrt{\left(\frac{AE}{CF} + kk\right)}.$$

39. Hae formulae ratione signorum vtcunque transmutari possunt. Primo enim in formulis integralibus nihil mutando, tam k , quam $\sqrt{\left(\frac{AE}{CF} + kk\right)}$ pro lubitu vel affirmatiue, vel negatiue, accipi possunt, dummodo eadem signi ratio vbique obseruetur. Deinde etiam tam \sqrt{C} , quam \sqrt{F} , negatiue sumi potest; illo autem casu quoque $\sqrt{\left(\frac{A}{C} + xx\right)}$, quippe $\frac{\sqrt{(A + Cxx)}}{\sqrt{C}}$, hoc vero $\sqrt{\left(\frac{E}{F} + yy\right)}$ negatiue est accipiendum.

40. Denique patet, si C sit quantitas positia, tum quoque F quantitatem positiam esse oportere, quia alioquin altera formula integralis fieret imaginaria. Sin autem C sit quantitas negatiua, tum etiam F talis sit necesse est; et quo hoc casu imaginaria se destruant, pro kk quantitas negatiua accipienda erit; quo k et k^2 fiant quoque imaginariae.

41. Hoc ergo casu sequens habebitur aequatio integralis :

$$\int \frac{dx\left(A + \frac{C}{A}Bxx + \frac{CC}{AA}Cx^2\right)\sqrt{C}}{\sqrt{(A - Cxx)}} - \int \frac{dy\left(A + \frac{F}{E}Byy + \frac{FF}{EE}Cy^2\right)\sqrt{F}}{\sqrt{(E - Fyy)}} \\ = \text{Const.} + \frac{CF}{AE}Bkxy + \frac{CCFF}{AAEE}Ck^2xy + \frac{CCFF}{AAEE}Ckxyy\sqrt{\left(\frac{AE}{CF} - kk\right)}$$

cui

cui satisfaciunt isti valores :

$$\frac{A y}{C} = x \sqrt{\left(\frac{A E}{C F} - k k\right)} - k \sqrt{\left(\frac{A}{C} - x x\right)}$$

$$\frac{E x}{F} = y \sqrt{\left(\frac{A E}{C F} - k k\right)} + k \sqrt{\left(\frac{E}{F} - y y\right)}$$

ex hac aequatione oriundi :

$$k k = \frac{E}{F} x x + \frac{A}{C} y y - 2 x y \sqrt{\left(\frac{A E}{C F} - k k\right)}.$$

42. Hae formulae etiam eas, quae ex hypothesi prima sunt erutae, in se complectuntur, ponendo scilicet $E = A$, et $F = C$, quin etiam formulae secundae hypothesi his non latius patent. Si enim in relatione secundo loco assumpta pro $x + \frac{\beta}{\gamma + \delta}$ et $y + \frac{\beta}{\gamma + \delta}$ scribatur x et y , aequatio omnino primae formae oritur, similibusque modo si hanc relationem constituere velimus :

$$0 = \alpha + 2 b x + 2 \beta y + \gamma x x + c y y + 2 \delta x y$$

ea facile ad relationem tertiam reduceretur, unde eius evolutionem praetermitto.

43. Perspicuum nunc est, ex his formulis infinitas comparationes institui posse circa quantitates transcendentes, tam ratione spatiorum, quam arcuum, qui quidem vel a quadratura circuli pendent, vel a logarithmis. Et si autem hae comparationes etiam vulgari calculo institui possunt, tamen non inutile erit ostendere, quemadmodum eaedem multo facilius ex his formulis derivari queant; quod eo maius notatu dignum videtur, cum hic neque naturae circuli, neque logarithmorum, ratio peculiaris habeatur. Ex quo facilius intelligetur, quemadmodum haec methodus etiam pari

successu ad eiusmodi formulas integrales se extendat, quae neque ad circuli, neque hyperbolae quadraturam reuocari possunt.

I.

De comparatione arcuum circularium.

44. Sit radius circuli, seu sinus totus $= r$, ac posito sinu quocumque $= z$, sit arcus ei respondens $= \Pi. z$, sumto Π pro nota eius functionis, qua pendencia arcus a suo sinu denotatur. Erit ergo, uti constat, $\Pi. z = \int \frac{dz}{\sqrt{(1-zz)}}$; atque ut formulas integrales §. 41. erutas huc transferamus, poni oportet: $A = E = C = F = 1$; $H = 1$, $S = 0$, et $G = 0$.

45. Ex his autem valoribus emerget haec aequatio integralia complectens:

$$\int \frac{dx}{\sqrt{(1-xx)}} - \int \frac{dy}{\sqrt{(1-yy)}} = \text{Const.}$$

cui satisfacere inventae sunt hae formulae:

$$y = x\sqrt{(1-kk)} - k\sqrt{(1-xx)}$$

$$x = y\sqrt{(1-kk)} + k\sqrt{(1-yy)}$$

quae oriuntur ex hac aequatione:

$$kk = xx + yy - 2xy\sqrt{(1-kk)}.$$

46. Per has igitur determinationes satisfacit huic aequationi:

$$\Pi. x - \Pi. y = \text{Const.}$$

in qua constans ita determinabitur: ponatur $y = 0$ eritque $x = k$; ex quo casu prodit $\Pi. k - \Pi. 0 = \text{Const.}$

feu

seu ob $\Pi. 0 = 0$ erit $\text{Const.} = \Pi. k$, seu arcui cuius sinus $= k$. Hinc generatim habebimus :

$$\Pi. x - \Pi. y = \Pi. k.$$

47. Hinc ergo statim arcuum tam additio, quam subtractio colligitur. Si enim duo habeantur arcus $\Pi. k$ et $\Pi. y$, quorum sinus sint k et y , et summae arcuum sinus ponatur $= x$, ut sit $\Pi. x = \Pi. k + \Pi. y$, erit $x = y \sqrt{(1 - kk)} + k \sqrt{(1 - yy)}$. Porro si maioris arcus sinus sit $= x$, minoris $= k$, sinusque differentiae ponatur $= y$, ut sit $\Pi. y = \Pi. x - \Pi. k$, erit :

$$y = x \sqrt{(1 - kk)} - k \sqrt{(1 - xx)};$$

vti ex elementis est manifestum.

48. Perspicuum etiam est, quemadmodum hinc arcuum multiplicationem deduci oporteat. Posito enim $y = k$, ut sit $x = 2k \sqrt{(1 - kk)}$, erit $\Pi. x = 2 \Pi. k$. Ac si valor hic pro x inuentus loco y substituatur, in formula $x = y \sqrt{(1 - kk)} + k \sqrt{(1 - yy)}$, ob $\Pi. y = 2 \Pi. k$, prodibit $\Pi. x = 3 \Pi. k$.

49. In genere autem, si sit y sinus arcus nk , seu $\Pi. y = n \Pi. k$, et $\sqrt{(1 - yy)}$ sit cosinus arcus nk , vti $\sqrt{(1 - kk)}$ denotat cosinum arcus k , atque ponatur

$$x = y \sqrt{(1 - kk)} + k \sqrt{(1 - yy)} \text{ erit } \Pi. x = (n + 1) \Pi. k.$$

Ex sinu ergo cuiusvis multipli arcus k reperietur sinus multipli unitate altioris.

50. Quo autem haec facilius expediri queant, valorem quoque ipsius cosinus $\sqrt{(1 - xx)}$ nosse conueniet; quem in finem cum ex formula prima sit :

$$k \sqrt{(1 - xx)} = x \sqrt{(1 - kk)} - y$$

N 3

substi-

substituatur hic valor ipſus x ex altera formula
erit $k\sqrt{1-xx} = y(1-kk) + k\sqrt{1-kk}(1-yy) - y$, ideoque

$$\sqrt{1-xx} = \sqrt{1-kk}(1-yy) - ky$$

ſimilique modo erit:

$$\sqrt{1-yy} = \sqrt{1-kk}(1-xx) + kx.$$

51. Inuentis ergo valoribus, tam pro x , quam pro $\sqrt{1-xx}$, multiplicetur ille per λ et productum ad hunc addatur, eritque

$$\sqrt{1-xx} + \lambda x = \sqrt{1-kk}(1-yy) - ky + \lambda y\sqrt{1-kk} + \lambda k\sqrt{1-yy}$$

ſeu $\sqrt{1-xx} + \lambda x = (\sqrt{1-kk} + \lambda k)\sqrt{1-yy} + y(\lambda\sqrt{1-kk} - k)$

Quo igitur hi factores ſimiles reddantur, neceſſe eſt, vt ſit $\lambda = \sqrt{-1}$, eritque:

$$\sqrt{1-xx} + x\sqrt{-1} = (\sqrt{1-kk} + k\sqrt{-1})(\sqrt{1-yy} + y\sqrt{-1}).$$

52. Hanc ergo formulam loco ſuperioris adhibendo, ſtatim patet, vt ſit $\Pi. x = 2 \Pi. k$, ob $y = k$, eſſe oportere

$$\sqrt{1-xx} + x\sqrt{-1} = (\sqrt{1-kk} + k\sqrt{-1})^2$$

Ac ſi hic valor pro x inuentus loco y ſubſtituatur, vt ſit

$$\Pi. y = 2 \Pi. k, \text{ prodibit:}$$

$$\sqrt{1-xx} + x\sqrt{-1} = (\sqrt{1-kk} + k\sqrt{-1})^2 \text{ pro } \Pi. x = 3 \Pi. k$$

vnde in genere colligitur, vt ſit $\Pi. x = n \Pi. k$, debere eſſe:

$$\sqrt{1-xx} + x\sqrt{-1} = (\sqrt{1-kk} + k\sqrt{-1})^n.$$

53. Quia porro $\sqrt{-1}$ ob ſuam naturam tam negatiue, quam affirmatiue accipere licet, erit quoque pro eadem arcus multiplicatione: $\Pi. x = n \Pi. k$

$$\sqrt{1-xx} - x\sqrt{-1} = (\sqrt{1-kk} - k\sqrt{-1})^n$$

ideo-

ideoque vel

$$V(1-xx) = \frac{(V(1-kk)+kV-1)^n + (V(1-kk)-kV-1)^n}{2}$$

$$\text{vel } x = \frac{(V(1-kk)+kV-1)^n - (V(1-kk)-kV-1)^n}{2V-1}$$

quae formulae quoque valent pro valoribus fractis exponentis n .

ii.

De Comparatione arcuum parabolicorum.

54. Sit AB axis et A vertex parabolae, quem Tab II. tangat recta indefinita AV, super qua capiantur abscissae; Fig. 1. posito ergo parabolae latere recto $= z$, sit abscissa quaevis AP $= z$, erit applicata Pp $= \frac{1}{2}zz$, ex quo arcus parabolae huic abscissae respondens erit Ap $= \int dz V(1+zz)$: qui cum sit functio ipsius z , denotetur per $\Pi.z$, ita vt $\Pi.z$ significet arcum parabolae abscissae z conuenientem, seu sit

$$\Pi.z = \int dz V(1+zz).$$

55 Irrationalitate in denominatorem translata erit: $\Pi.z = \int \frac{dz(1+zz)}{\sqrt{1+zz}}$. Ad hanc ergo formam vt formulae integrales §. 38. reuocentur, erit A=E=1; C=F=1, M=1 et N=1 atque O=0. Vnde aequatio illa integralis in hanc abit formam:

$$\int \frac{dx(1+xx)}{\sqrt{1+xx}} - \int \frac{dy(1+yy)}{\sqrt{1+yy}} = \text{Const.} + kxy$$

cui satisfaciunt hi valores:

$y = -kV(1+xx) + xV(1+kk)$ et $x = kV(1+yy) + yV(1+kk)$ sumtis tam k quam $V(1+kk)$ negativis.

56. Hac

56. Hac igitur inter x et y relatione subsistente pro arcibus parabolae erit:

$$\Pi. x - \Pi. y = \text{Const.} + kxy$$

ad quam constantem determinandam ponatur $y=0$, et quia tum fit $x=k$, erit $\Pi. k = \text{Const.}$ Quocirca habebitur

$$\Pi. x - \Pi. y = \Pi. k + kxy.$$

57. Ut igitur haec aequatio locum habeat, relatio inter ternas abscissas k, x , et y eiusmodi erit:

$$x = k\sqrt{1+yy} + y\sqrt{1+kk}, \text{ seu } y = x\sqrt{1+kk} - k\sqrt{1+xx}$$

vnde praeterea eruuntur istae determinationes:

$$\begin{aligned} \sqrt{1+xx} &= \sqrt{1+kk}(\sqrt{1+yy}) + ky \text{ et } \sqrt{1+yy} \\ &= \sqrt{1+kk}(\sqrt{1+xx}) - kx \end{aligned}$$

ex quibus porro elicitur:

$$x + \sqrt{1+xx} = (k + \sqrt{1+kk})(y + \sqrt{1+yy}).$$

58. Si manente eadem abscissa k , capiantur aliae duae abscissae q et p , ut fit

$$q = k\sqrt{1+pp} + p\sqrt{1+kk} \text{ et } p = q\sqrt{1+kk} - k\sqrt{1+qq}$$

$$\text{seu } q + \sqrt{1+qq} = (k + \sqrt{1+kk})(p + \sqrt{1+pp})$$

$$\text{erit } \Pi. q - \Pi. p = \Pi. k + kpq.$$

Ideoque hanc aequationem ab illa subtrahendo habebitur:

$$(\Pi. x - \Pi. y) - (\Pi. q - \Pi. p) = k(xy - pq).$$

59. Pro hoc igitur casu erit

$$\frac{x + \sqrt{1+xx}}{y + \sqrt{1+yy}} = \frac{q + \sqrt{1+qq}}{p + \sqrt{1+pp}}$$

vnde relatio inter p, q, x et y siue k obtinetur: Erit autem

$$k = x\sqrt{1+yy} - y\sqrt{1+xx} = q\sqrt{1+pp} - p\sqrt{1+qq} \text{ et}$$

$$\sqrt{1+kk} = \sqrt{1+xx}(\sqrt{1+yy}) - xy = \sqrt{1+pp}(\sqrt{1+qq}) - pq$$

60. iam

60. Iam ob $\frac{1}{p + \sqrt{1 + pp}} = \sqrt{1 + pp} - p$ erit :

$$\sqrt{1 + xx} + x = \sqrt{1 + yy} + y)(\sqrt{1 + qq} + q)(\sqrt{1 + pp} - p)$$

vnde reperitur :

$$x = y\sqrt{1 + pp}(\sqrt{1 + qq} + q) - q\sqrt{1 + pp}(\sqrt{1 + yy} - p) - p\sqrt{1 + qq}(\sqrt{1 + yy} - p) + pqy$$

Quare erit :

$$(\Pi. x - \Pi. y) - (\Pi. q - \Pi. p) = (q\sqrt{1 + pp} - p\sqrt{1 + qq})(y\sqrt{1 + pp} - p\sqrt{1 + yy})(q\sqrt{1 + yy} + y\sqrt{1 + qq})$$

Problema 1.

61. Dato arcu parabolae quocunque Ak , in ver- Tab. II.
tice A terminato, ab alio quocunque puncto p arcum Fig. 1.
abscindere pq , qui arcum illum Ak superet quantitate
algebraice assignabili.

Solutio.

Posita parabolae parametro $= 2$, sit k abscissa
arculi Ak conveniens, abscissae autem punctis p et q
respondentes sint $AP = y$ et $AQ = x$; eritque arc. pq
 $= \Pi. x - \Pi. y$ et arc. $Ak = \Pi. k$; cum igitur data sit
abscissa $AP = y$, si capiatur altera

$$AQ = x = y\sqrt{1 + kk} + k\sqrt{1 + yy}$$

erit $\Pi. x - \Pi. y = \Pi. k + kxy$, ideoque

$$\text{Arc. } pq = \text{Arc. } Ak + kxy.$$

Superabit ergo arcus pq , qui in dato puncto p termina-
tur, arcum Ak quantitate algebraice assignabili kxy .

Poterit etiam a puncto p antrorsum abscindi ar-
cus pq' , qui pariter arcum Ak quantitate geometrica su-

peret; ad hoc ponatur $AP = x$ et $AQ' = y$, sitque $y = x\sqrt{(1+kk)} - k\sqrt{(1+xx)}$; et cum sit $\text{Arc. } pq' = \Pi.x - \Pi.y$, erit:

$$\text{Arc. } pq' = \text{Arc. } Ak + kxy$$

Vtraque igitur solutio ita coniungetur, vt posita abscissa data $AP = p$; capiendum sit:

$AQ = p\sqrt{(1+kk)} + k\sqrt{(1+pp)}$, et $AQ' = p\sqrt{(1+kk)} - k\sqrt{(1+pp)}$ quo facto erit:

$$\text{Arc. } pq = \text{Arc. } Ak + kp.AQ$$

$$\text{Arc. } pq' = \text{Arc. } Ak + kp.AQ'$$

sicque duplici modo problemati est satisfactum.

Coroll. 1.

62. Fieri autem nequit, vt excessus kxy , quo arcus pq arcum Ak superat, euanescat, deberet enim esse vel $x = 0$, vel $y = 0$. At casu $x = 0$ fieret $y = -k$, arcusque in ipso vertice A inciperet in altero ramo ipsi arcui Ak similis capiendus; altero autem casu, quo $y = 0$, fieret $x = k$, et arcus pq in arcum Ak abiret: vnde arcui Ak geometricæ in parabola abscindi nequit alius arcus ipsi æqualis, qui ipsi non simul futurus sit similis.

Coroll. 2.

63. Vicissim ergo dato arcu quocunque pq in parabola, semper a vertice arcus abscindi poterit Ak , qui ab illo deficiat quantitate geometrica. Cum enim nunc datae sint abscissæ $AP = y$ et $AQ = x$. erit $AK = k = x\sqrt{(1+yy)} - y\sqrt{(1+xx)}$, qua inuenta, erit $\text{Arc. } pq - \text{Arc. } Ak = kxy$.

Coroll.

Coroll. 3.

64. Quin etiam puncto p pro incognito habito, proposito arcu Ak , alius arcus pq assignari poterit, qui illum superet quantitate data, puta $=C$. Habebimus ergo has duas aequationes:

$$kxy = C \text{ et } xx + yy = kk + 2xy\sqrt{(1 + kk)}$$

seu $xx + yy = kk + \frac{2C}{k}\sqrt{(1 + kk)}$; ergo

$$x + y = \sqrt{(kk + \frac{2C}{k} + \frac{2C}{k}\sqrt{(1 + kk)})}$$

$$x - y = \sqrt{(kk - \frac{2C}{k} + \frac{2C}{k}\sqrt{(1 + kk)})}$$

Seu sint x et y binae radices huius aequationis quadraticae

$$zz - Pz + Q = 0; \text{ erit } Q = \frac{C}{k} \text{ et } P = \sqrt{(kk + \frac{2C}{k} + \frac{2C}{k}\sqrt{(1 + kk)})}$$

vnde $z = \frac{1}{2}\sqrt{(kk + \frac{2C}{k} + \frac{2C}{k}\sqrt{(1 + kk)})} \pm \frac{1}{2}\sqrt{(kk - \frac{2C}{k} + \frac{2C}{k}\sqrt{(1 + kk)})}$.

Coroll. 4.

65. Quantacunque sit haec quantitas C , modo sit affirmatiua, semper prodeunt pro x et y valores reales, iique affirmatiui: At si sit $C = 0$, fiet $x = k$, et $y = 0$. Quin etiam poni potest C negatiuum, quo casu y reperitur quoque negatiuum, et arcus quaesitus vtrinque circa verticem A erit dispositus. Verum si sit $C = -D$, necesse est, vt sit $D < \frac{k^2}{2(1 + \sqrt{(1 + kk)})}$, seu $D < \frac{1}{2}k(\sqrt{(1 + kk)} - 1)$; nam si D esset maius, vtraque abscissa fieret imaginaria.

Coroll. 5.

66. Casu autem $D = -C = \frac{1}{2}k(\sqrt{(1 + kk)} - 1)$, erit $zz = \frac{D}{k}$; ideoque $x = +\sqrt{\frac{1}{2}(\sqrt{(1 + kk)} - 1)}$

O 2

et

et $y = -\sqrt{\frac{1}{2}}(\sqrt{1+k^2}-1)$; hocque casu orietur arcus vtrunque a vertice aequae extensus, cuius defectus ab arcu Ak est minimus omnium, qui quidem geometricè construi possunt.

Problema 2.

Tab. II. 67. Dato arcu parabolae quocunque ef , a dato
Fig. 2. eius puncto quocunque p alium abscindere arcum pq ,
ita ut arcuum ef et pq differentia geometricè possit
assignari.

Solutio.

Posito parabolae latere recto $= 2$, tanget recta AV parabolam in vertice A , a quo capiantur abscissae, quae sint:

$$AE = e; AF = f; AP = p \text{ et } AQ = q$$

quarum tres priores e, f, p , sunt datae, haec vera q ita accipiat, ut sit per §. 59.

$$\frac{q + \sqrt{1+qq}}{p + \sqrt{1+pp}} = \frac{f + \sqrt{1+ff}}{e + \sqrt{1+ee}}$$

Tum vero sit $k = f\sqrt{1+ee} - e\sqrt{1+ff}$, scribendo e et f pro y et x , eritque $(\Pi. q - \Pi. p) - (\Pi. f - \Pi. e) = k(pq - ef)$.

Ideoque habebitur:

$$\text{Arc. } pq - \text{Arc. } ef = k(pq - ef).$$

Hinc etiam apparet, si punctum q fuerit datum, ex formula tradita simili modo punctum p antrosum procedendo definiri posse, ut arcuum differentia prodeat geometricè assignabilis.

Coroll.

Coroll. 1.

68. Ex reductione §. 60. facta patet esse

$$pq - eff = (p\sqrt{1+ee} - e\sqrt{1+pp})(p\sqrt{1+ff} + f\sqrt{1+pp})$$

ficque, sumta abscissa q , ex aequatione $\frac{q + \sqrt{1+qq}}{p + \sqrt{1+pp}} = \frac{f + \sqrt{1+ff}}{e + \sqrt{1+ee}}$ erit :

$$\text{Arc. } pq - \text{Arc. } ef = (f\sqrt{1+ee} - e\sqrt{1+ff})(p\sqrt{1+ee} - e\sqrt{1+pp})(p\sqrt{1+ff} + f\sqrt{1+pp}).$$

Coroll. 2.

69. Si velimus punctum p ita accipere, vt arcuum differentia euanescat, seu fiat $\text{Arc. } pq = \text{Arc. } ef$, oportet esse

$$\text{vel } p\sqrt{1+ee} - e\sqrt{1+pp} = 0, \text{ vel } p\sqrt{1+ff} + f\sqrt{1+pp} = 0$$

Priori casu fit $p = \frac{+}{-} e$; posteriori $p = \frac{+}{-} f$, utroque autem casu arcus pq vel cum arcu ef congruit, vel eius fit similis in altero parabolae ramo assumtus; ita vt geometricè duo arcus aequales exhiberi nequeant, quae non simul sibi futuri sunt similes.

Coroll. 3.

70. Cum fit $k = f\sqrt{1+ee} - e\sqrt{1+ff}$, erit $\sqrt{1+kk} = \sqrt{1+ee}\sqrt{1+ff} - ef$; hinc $k\sqrt{1+kk} = f\sqrt{1+ff} + 2eef\sqrt{1+ff} - 2eff\sqrt{1+ee} - e\sqrt{1+ff}$

sive

$$k\sqrt{1+kk} = f\sqrt{1+ff} - e\sqrt{1+ee} - 2ef(f\sqrt{1+ee} - e\sqrt{1+ff})$$

ideoque $k\sqrt{1+kk} = f\sqrt{1+ff} - e\sqrt{1+ee} - 2efk.$

Quo circa habebitur :

$$kef = \frac{1}{2}fV(1+ff) - \frac{1}{2}eV(1+ee) - \frac{1}{2}kV(1+kk).$$

Coroll. 4.

71. Quia igitur k simili quoque modo pendet a p et q , erit etiam

$$kpq = \frac{1}{2}qV(1+qq) - \frac{1}{2}pV(1+pp) - \frac{1}{2}kV(1+kk).$$

Quare cum arcuum differentia sit $= kpq - kef$; si quatuor parabolae puncta e, f, p, q ita a se inuicem pendent, ut sit :

$$\frac{q + \sqrt{1+qq}}{p + \sqrt{1+pp}} = \frac{f + \sqrt{1+ff}}{e + \sqrt{1+ee}}$$

erit

$$\text{Arc. } pq - \text{Arc. } ef = \frac{1}{2}qV(1+qq) - \frac{1}{2}pV(1+pp) - \frac{1}{2}fV(1+ff) + \frac{1}{2}eV(1+ee)$$

quae expressio, ob functiones quantitatum p, q, e, f a se inuicem separatas, est notatu digna.

Coroll. 5.

72. Relatio inter e, f, p, q etiam ita exprimi potest, ut sit

$$V(1+qq) + q = (V(1+ee) - e)(V(1+ff) + f)(V(1+pp) + p)$$

tum ob $\frac{1}{\sqrt{1+qq} + q} = V(1+qq) - q$ erit:

$$V(1+qq) - q = (V(1+ee) + e)(V(1+ff) - f)(V(1+pp) - p)$$

vnde datis e, f , et p , facile valor tam pro q , quam pro p , eruitur.

Coroll. 6.

73. Ex formula Coroll. 1. data apparet, arcum pq semper maiorem fore arcu ef , si punctum p a vertice

vertice parabolae A magis fuerit remotum, quam punctum e ; contra autem arcum pq proditurum esse minorem. Ac si quidem sit $p=0$, erit $\text{Arc. } ef - \text{Arc. } pq = ef(f\sqrt{1+ee} - e\sqrt{1+ff})$; minimus autem omnium arcus pq euadet, si capiatur $p = -\sqrt{\frac{1}{2}}(\sqrt{1+ee})(1+ff) - ef - 1$ et $q = +\sqrt{\frac{1}{2}}(\sqrt{1+ee})(1+ff) - ef - 1$ tumque erit:

$$\text{Arc. } ef - \text{Arc. } pq = \frac{1}{2}(e+f)(\sqrt{1+ff} - \sqrt{1+ee})$$

Arcusque pq vtrinque aequae circa verticem A erit dispositus.

Problema 3.

74. Dato arcu parabolae ef , a puncto dato p Tab. II. abscindere arcum pz , qui superet datum multipulum arcus ef quantitate geometricae assignabili. Fig. 3.

Solutio.

Posito parabolae latere recto $= z$, sint in verticis tangente abscissae datae $AE = e$, $AF = f$, et $AP = p$; tum capiantur abscissae $AQ = q$; $AR = r$; $AS = s$; $AT = t$; et vltima sit $AZ = z$; quae ita determinentur, vt sit:

$$\text{Primo } \frac{q + \sqrt{1+qq}}{p + \sqrt{1+pp}} = \frac{f + \sqrt{1+ff}}{e + \sqrt{1+ee}}$$

eritque ex §. 71.

$$\text{Arc } pq - \text{Arc. } ef = \frac{1}{2}q\sqrt{1+qq} - \frac{1}{2}p\sqrt{1+pp} - \frac{1}{2}f\sqrt{1+ff} + \frac{1}{2}e\sqrt{1+ee}.$$

Deinde ex puncto q simili modo definiatur punctum r , vt sit:

$$\frac{r + \sqrt{1+rr}}{q + \sqrt{1+qq}} = \frac{f + \sqrt{1+ff}}{e + \sqrt{1+ee}}, \text{ seu } \frac{r + \sqrt{1+rr}}{p + \sqrt{1+pp}} = \left(\frac{f + \sqrt{1+ff}}{e + \sqrt{1+ee}} \right)^2$$

eritque

eritque

$$\text{Arc. } qr - \text{Arc. } ef = \frac{1}{2}rV(1+rr) - \frac{1}{2}qV(1+qq) - \frac{1}{2}fV(1+ff) + \frac{1}{2}eV(1+ee)$$

qua aequatione ad illam addita prodibit:

$$\text{Arc. } pr - 2\text{Arc. } ef = \frac{1}{2}rV(1+rr) - \frac{1}{2}pV(1+pp) - \frac{2}{2}fV(1+ff) + \frac{2}{2}eV(1+ee)$$

Tertio ex puncto r capiatur punctum s , vt fit:

$$\frac{r+\sqrt{(1+ss)}}{r+\sqrt{(1+rr)}} = \frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}}, \text{ seu } \frac{s+\sqrt{(1+ss)}}{p+\sqrt{(1+pp)}} = \frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}}$$

eritque:

$$\text{Arc. } rs - \text{Arc. } ef = \frac{1}{2}sV(1+ss) - \frac{1}{2}rV(1+rr) - \frac{1}{2}fV(1+ff) + \frac{1}{2}eV(1+ee)$$

quae ad praecedentem addita praebet:

$$\text{Arc. } ps - 3\text{Arc. } ef = \frac{1}{2}sV(1+ss) - \frac{1}{2}pV(1+pp) - \frac{3}{2}fV(1+ff) + \frac{3}{2}eV(1+ee).$$

Atque hoc modo si vltius progrediamur, sitque z punctum post n huiusmodi operationes inuentum, erit:

$$\frac{z+\sqrt{(1+zz)}}{p+\sqrt{(1+pp)}} = \left(\frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}}\right)^n$$

vnde immediate punctum z reperietur, ita vt fit:

$$\text{Arc. } pz - n\text{Arc. } ef = \frac{1}{2}zV(1+zz) - \frac{1}{2}pV(1+pp) - \frac{n}{2}fV(1+ff) + \frac{n}{2}eV(1+ee)$$

sicque arcus pz est inuentus a dato puncto p abscissus, qui arcum ef vicibus n sumtum superat quantitate geometrica.

Coroll. I.

75. Quodcunque ergo multipulum arcus ef proponatur, cuius multipli exponens sit numerus n , siue is sit integer, siue fractus, a dato puncto p semper abscindi

scindi poterit arcus ps , qui hoc multipulum excedat
quantitate geometricae assignabili; erit enim:

$$\begin{aligned} \mathcal{V}(1+zs)+z &= (\mathcal{V}(1+pp)+p)(\mathcal{V}(1+ff)+f)^n (\mathcal{V}(1+ee)-e)^n \text{ et} \\ \mathcal{V}(1+zs)-z &= (\mathcal{V}(1+pp)-p)(\mathcal{V}(1+ff)-f)^n (\mathcal{V}(1+ee)+e)^n. \end{aligned}$$

Coroll. 2.

76. Quodsi ergo ad abbreviandum ponatur:

$$\begin{aligned} \mathcal{V}(1+ee)+e &= E; \quad \mathcal{V}(1+ff)+f = F; \quad \mathcal{V}(1+pp)+p = P \\ \text{erit } \mathcal{V}(1+zs)+z &= \frac{PF^n}{E^n} \text{ et } \mathcal{V}(1+zs)-z = \frac{E^n}{PF^n} \end{aligned}$$

unde oritur:

$$\mathcal{V}(1+zs) = \frac{P^2 F^{2n} + E^{2n}}{2 P E^n F^n} \text{ et } z = \frac{P^2 F^{2n} - E^{2n}}{2 P E^n F^n}.$$

Coroll. 3.

$$77. \text{ Hinc ergo fiet } \frac{1}{2} z \mathcal{V}(1+zs) = \frac{P^4 F^{4n} - E^{4n}}{8 P^2 E^{2n} F^{2n}}$$

Quia tum simili modo est

$$\frac{1}{2} e \mathcal{V}(1+ee) = \frac{E^4 - 1}{8 E E}; \quad \frac{1}{2} f \mathcal{V}(1+ff) = \frac{F^4 - 1}{8 F F}$$

$$\text{et } \frac{1}{2} p \mathcal{V}(1+pp) = \frac{P^4 - 1}{8 P P}$$

erit:

$$\text{Arc. } ps - n \text{ Arc. } ef = \frac{P^4 F^{4n} - E^{4n}}{8 P^2 E^{2n} F^{2n}} - \frac{P^4 - 1}{8 P P} - \frac{n(F^4 - 1)}{8 F F} + \frac{n(E^4 - 1)}{8 E E}$$

Coroll. 4.

78. Si huius expressionis partes binæ in vnam congregentur, reperietur ista differentia geometrica :

$$\text{Arc. } pz - n \text{Arc. } ef = \frac{(F^{2n} - E^{2n})(F^4 F^{2n} + E^{2n})}{8 P^2 E^{2n} F^{2n}} \frac{n(FE - EE)(EEFF + 1)}{8 EEFF} :$$

Coroll. 5.

79. Quemadmodum hic ex puncto dato p alterum punctum z determinauimus, ita vicissim, si punctum z pro dato accipiatur, antrosum progrediendo, simili modo punctum p ex eadem aequatione reperietur, ita vt Arc. pz superet arcum ef , n vicibus sumtum quantitate geometricè assignabili.

Problema 4.

80. Dato in parabola arcu quocunque ef , inuenire alium arcum pz , qui se habeat ad illum in data ratione $n : 1$, ita vt sit $\text{Arc. } pz = n \text{Arc. } ef$.

Solutio.

Retentis iisdem denominationibus, quibus in probl. praecedenti eiusque Coroll. 2. vsi sumus; quoniam fieri debet :

$$\text{Arc. } pz - n \text{Arc. } ef = 0$$

quantitas illa algebraica, cui haec arcum differentia aequalis est inuenta, in nihilum abire debet. Habebimus ergo ex Coroll. 4. hanc aequationem :

$$F^{2n} P^4 + E^{2n} = \frac{n E^{2n-2} F^{2n-2} (FE - EE)(EEFF + 1)}{F^{2n} - E^{2n}} P^2$$

Pona-

Ponamus brevitatis gratia $\frac{F}{E} = C$, eritque

$$C^{2n}P^4 + 1 = \frac{nC^{2n-2}(CC-1)(CCE^4+1)}{(C^{2n}-1)EE} PP$$

vnde fit:

$$C^{2n}P^2 = \frac{nC^{n-2}(CC-1)(CCE^4+1)}{2(C^{2n}-1)EE} \sqrt{\left(\frac{nnC^{n-4}(CC-1)^2(CCE^4+1)^2}{4(C^{2n}-1)^2E^4} - 1\right)}$$

ideoque

$$P = \sqrt{\left(\frac{n(CC-1)(CCE^4+1)}{2(C^{2n}-1)CCEE} \sqrt{\left(\frac{nn(CC-1)^2(CCE^4+1)^2}{4(C^{2n}-1)^2C^4E^4} - \frac{1}{C^{2n}}\right)}\right)}$$

sive

$$P = \sqrt{\left(\frac{n(CC-1)(CCE^4+1)}{4(C^{2n}-1)CCEE} + \frac{1}{2C^n}\right) \sqrt{\left(\frac{n(CC-1)(CCE^4+1)}{4(C^{2n}-1)CCEE} - \frac{1}{2C^n}\right)}}$$

Deinde si pari modo ponatur $\sqrt{(1+zz)} + z = Z$, erit $Z = C^n P$. Ex inuentis autem quantitibus P et Z ita eliciuntur ipsae abscissae p et z, vt sit:

$$p = \frac{PP-1}{2P} \quad \text{et} \quad z = \frac{ZZ-1}{2Z}$$

Restituito autem pro C valore $\frac{F}{E}$, si ponamus:

$$\sqrt{\left(\frac{n(FF-EE)(EEFF+1)}{4EEFF(F^{2n}-E^{2n})} + \frac{1}{2E^nF^n}\right)} = M$$

$$\sqrt{\left(\frac{n(FF-EE)(EEFF+1)}{4EEFF(F^{2n}-E^{2n})} - \frac{1}{2E^nF^n}\right)} = N$$

reperietur:

$$P = E^n(M-N) \quad \text{et} \quad \frac{1}{P} = F^n(M+N)$$

$$Z = F^n(M-N) \quad \text{et} \quad \frac{1}{Z} = E^n(M+N)$$

vnde concluduntur ipsae abscissae

$$p = -\frac{1}{2}M(F^n - E^n) - \frac{1}{2}N(F^n + E^n)$$

$$z = +\frac{1}{2}M(F^n - E^n) - \frac{1}{2}N(F^n + E^n)$$

P 2

Cum

Coroll. 6.

86. Vt ergo arcus *ef* triplum exhiberi possit, is non in vertice A terminari potest, seu E debet esse maius quam 1, atque adeo limes dabitur, infra quem accipi nequeat. Ad quem limitem inueniendum, resolvi oportet hanc aequationem

$$3E^3F^2 + 3EF = 2F^4 + 2EEFF + 2E^4.$$

In hunc finem ponatur $EF = S$, et $EE + FF = R$, erit:

$$3S^2 + 3S = 2RR - 2SS, \text{ ideoque } R = \sqrt{\left(\frac{2}{3}S^2 + SS + \frac{1}{3}S\right)}$$

vnde fit:

$$F + E = \sqrt{2S + \sqrt{\left(\frac{2}{3}S^2 + SS + \frac{1}{3}S\right)}}$$

$$F - E = \sqrt{-2S + \sqrt{\left(\frac{2}{3}S^2 + SS + \frac{1}{3}S\right)}}$$

Et cum fit $E > 1$, et $F > 1$, debet esse $R > 2$, et

$$3S^2 + 2SS + 3S > 8; \text{ ideoque } S > 1.$$

Coroll. 7.

87. Generatim ergo pro casu $n = 3$ oportet fit

$$3S^2 + 3S > 2RR - 2SS; \text{ ideoque } R < \sqrt{\left(\frac{2}{3}S^2 + SS + \frac{1}{3}S\right)}$$

quare si α sit numerus unitate minor: reperitur

$$F + E = \sqrt{2S + \alpha \sqrt{\left(\frac{2}{3}S^2 + SS + \frac{1}{3}S\right)}}$$

$$F - E = \sqrt{-2S + \alpha \sqrt{\left(\frac{2}{3}S^2 + SS + \frac{1}{3}S\right)}}$$

Debet ergo esse $\alpha\alpha > \frac{8}{3SS + 2S + 3}$ et $S > 1$.

Coroll. 8.

88. Ponamus $S = 2$; erit $\alpha\alpha > \frac{16}{19}$. Capiatur $\alpha = 1$, vt fit $EF = 2$, et $EE + FF = \sqrt{19}$; erit

$$F + E = \sqrt{\sqrt{19} + 4}; \quad E = \frac{1}{2}\sqrt{\sqrt{19} + 4} - \frac{1}{2}\sqrt{\sqrt{19} - 4}$$

$$F - E = \sqrt{\sqrt{19} - 4}; \quad F = \frac{1}{2}\sqrt{\sqrt{19} + 4} + \frac{1}{2}\sqrt{\sqrt{19} - 4}$$

ergo

ergo $e = \frac{1}{2} \sqrt{\sqrt{19} + 4} - \frac{1}{2} \sqrt{\sqrt{19} - 4}$
 et $f = \frac{1}{2} \sqrt{\sqrt{19} + 4} + \frac{1}{2} \sqrt{\sqrt{19} - 4}$

Porro reperitur :

$M = \frac{1}{2\sqrt{2}}$ et $N = 0$; vnde

$z = -p = \frac{1}{2\sqrt{2}} (2 + \sqrt{19}) \sqrt{\sqrt{19} - 4}$

hic ergo arcus triplus vtrinque circa verticem aequaliter extenditur.

III.

De Comparatione superficierum
 sphaeroidis elliptici compressi et
 conoidis hyperbolici.

89. Sit igitur primum propositum sphaeroides Tab. II
Fig. 4
 ellipticum genitum rotatione ellipsis BMA circa axem
 minorem AC. Ponatur semiaxis minor CA = a; et se-
 miaxis maior CB = a√m, existente m numero vnitatis
 maiori. Sumta iam in axe minore a centro C abscissa
 CP = x, erit applicata PM = √m(aa - xx), vnde ele-
 mentum ellipticum = dx √ $\frac{aa + (m-1)xx}{aa - xx}$.

90. Posita nunc ratione diametri ad peripheriam
 = 1 : π, erit portio superficierum sphaeroidicae, a reuolu-
 tione arcus AM genita, seu quae respondet abscissae
 CP = x, aequalis huic integrali 2π ∫ dx √m(aa + (m-1)xx).
 Indicetur hoc integrale, quod tanquam functio abscissae
 x spectetur, hoc modo

$\int dx \sqrt{m(aa + (m-1)xx)} = \Pi. x.$

91. Portio ergo, superficierum sphaeroidicae ellipticae
 abscissae CP = x respondens, erit = 2π. Π. x: vbi
 functio

functio Πx , vti perspicuum est, a logarithmis, seu rectificatione parabolae pendet, eritque $\Pi x = 0$, si $x = 0$; sin autem ponatur $x = a$, tum $2\pi \cdot \Pi a$ exhibebit semissem totius superficiei sphaeroidis.

92. Sit porro conoides hyperbolicum genitum reuolutione hyperbolae am circa suam axem cap , cuius centrum sit in c . Ponatur eius semiaxis transuersus $ca = c$, semiaxis autem coniugatus $= c\sqrt{n}$. Sumpta ergo in axe a centro c abscissa quacunque $cp = y$, quae quidem sit $> c$, erit applicata $pm = \sqrt{n(yy - cc)}$, et elementum hyperbolicum $= \int dy \sqrt{\frac{(n+1)yy - cc}{yy - cc}}$.

93. Hinc erit portio superficiei conoidis istius hyperbolici, ex arcu am genita, seu abscissae $cp = y$ respondens $= 2\pi \int dy \sqrt{n((n+1)yy - cc)}$. Quod integrale cum spectari possit tanquam functio ipsius y , ita indicetur:

$$\int dy \sqrt{n((n+1)yy - cc)} = \Theta y$$

fitque $\Theta y = 0$, si capiatur $y = c$. Erit ergo superficies conoidis hyperbolici abscissae $cp = y$ respondens $= 2\pi \cdot \Theta y$.

94. Comparentur hae binae formulae cum illis, quae supra §. 38. sunt expositae, et cum sit:

$$\Pi. x = \int \frac{dx (aa + (m-1)xx) \sqrt{m}}{\sqrt{(aa + (m-1)xx)}}$$

erit $A = aa$; $C = m-1$; $\mathcal{A} \sqrt{(m-1)} = aa \sqrt{m}$; et

$$\frac{m-1}{aa} \mathcal{B} \sqrt{(m-1)} = (m-1) \sqrt{m};$$

unde fit $\mathcal{A} = \frac{aa \sqrt{m}}{\sqrt{(m-1)}}$ et $\mathcal{B} = \frac{aa \sqrt{m}}{\sqrt{(m-1)}}$.

95. Deinde pro hyperbola cum sit

$$\Theta. y = \int \frac{dy (-cc + (n+1)yy) \sqrt{n}}{\sqrt{(-cc + (n+1)yy)}}$$

fiat

fiat $E = -cc$, et $F = n + 1$; eritque ob $\mathcal{E} = 0$

$$-\int \frac{dy(\mathcal{A} + \frac{F}{E} \mathcal{B}yy) \sqrt{F}}{\sqrt{(E + Fyy)}} = \frac{aa \sqrt{m(n+1)}}{\sqrt{(m-1)}} \int \frac{dy(-1 + \frac{(n+1)yy}{cc})}{\sqrt{(-cc + (n+1)yy)}}$$

ergo $-\int \frac{dy(\mathcal{A} + \frac{F}{E} \mathcal{B}yy) \sqrt{F}}{\sqrt{(E + Fyy)}} = \frac{aa \sqrt{m(n+1)}}{cc \sqrt{n(m-1)}} \ominus y$

96. His ergo substitutionibus factis habebimus hanc aequationem:

$\Pi x + \frac{aa \sqrt{m(n+1)}}{cc \sqrt{n(m-1)}} \ominus y = \text{Const.} + \frac{(n+1) \sqrt{m(m-1)}}{cc} kxy$
 cui satisfacit haec relatio inter x et y :

$$\frac{aay}{m-1} = k \sqrt{\left(\frac{aa}{m-1} + xx\right)} - x \sqrt{\left(kk - \frac{aacc}{(m-1)(n+1)}\right)} \text{ seu}$$

$$\frac{ccx}{n+1} = k \sqrt{\left(-\frac{cc}{n+1} + yy\right)} + y \sqrt{\left(kk - \frac{aacc}{(m-1)(n+1)}\right)}$$

vbi $\sqrt{\left(kk - \frac{aacc}{(m-1)(n+1)}\right)}$ negativè accipi conveniet.

97. Vel ponatur $k = \frac{ae}{\sqrt{(m-1)}}$, et si fuerit

$y = \frac{e}{a} \sqrt{(aa + (m-1)xx)} + \frac{x \sqrt{(m-1)}}{a \sqrt{(n+1)}} \sqrt{((n+1)ee - ac)}$
 seu $x = \frac{ae \sqrt{(n+1)}}{cc \sqrt{(m-1)}} \sqrt{((n+1)yy - cc)} - \frac{ay \sqrt{(n+1)}}{cc \sqrt{(m-1)}} \sqrt{((n+1)ee - ca)}$
 erit

$$\Pi x + \frac{aa \sqrt{m(n+1)}}{cc \sqrt{n(m-1)}} \ominus y = \text{Const.} + \frac{(n+1)ae \sqrt{m}}{cc} xy$$

98. Ad constantem autem definiendam ponatur $x = 0$, vt sit $\Pi x = 0$, eritque $y = e$, vnde prodit:

Const. = $\frac{aa \sqrt{m(n+1)}}{cc \sqrt{n(m-1)}} \ominus e$; sicque habebitur:

$$\Pi x + \frac{aa \sqrt{m(n+1)}}{cc \sqrt{n(m-1)}} (\ominus y - \ominus e) = \frac{(n+1)ae \sqrt{m}}{cc} xy$$

At si in hyperbola capiatur abscissa $cf = e$, erit superficies conoidis ex arcu em nata = $2\pi \cdot (\ominus y - \ominus e)$.

99. Quoniam igitur y per x determinatur, erit quoque

$$V((n+1)(yy-cc)) = \frac{e}{a} x V(m-1)(n+1) + \frac{1}{a} V(aa+(m-1)xx) \\ ((n+1)ee-cc)$$

vnde fit:

$$y + \delta V((n+1)yy-cc) = \left(\frac{e}{a} + \frac{\delta}{a}\right) V((n+1)ee-cc) V(aa+(m-1)xx) \\ + x \left(\frac{\delta e}{a} V(m-1)(n+1) + \frac{V(m-1)}{a\sqrt{(n+1)}} V((n+1)ee-cc)\right)$$

fit $\delta : \delta V(m-1)(n+1) = \delta : \frac{\sqrt{(n-1)}}{\sqrt{(n+1)}}$ erit $\delta = \frac{1}{\sqrt{(n+1)}}$
hincque obtinetur:

$$V((n+1)yy-cc) + y V(n+1) = \\ \left(\frac{e\sqrt{(n+1)}}{a} + \frac{1}{a}\right) V((n+1)ee-cc) (V(aa+(m-1)xx) + x V(m-1)).$$

100. Datis ergo abscissis $CP = x$, et $cf = e$, abscissa $cp = y$ ita definiri debet, vt fit

$$\frac{\sqrt{(n+1)yy-cc} + y\sqrt{(n+1)}}{\sqrt{(n+1)ee-cc} + e\sqrt{(n+1)}} = V(1 + \frac{(m-1)xx}{aa} + \frac{x}{a} V(m-1))$$

Deinde autem est

$$e xy = \frac{ay\sqrt{(n+1)yy-cc}}{a\sqrt{(m-1)(n+1)}} - \frac{ae\sqrt{(n+1)ee-cc}}{2\sqrt{(m-1)(n+1)}} + \frac{ccx\sqrt{aa+(m-1)xx}}{2a(n+1)}$$

Problema Hugenianum.

Dato sphaeroide elliptico lato ABC, inuenire conoides hyperbolicum apm , ita vt circulus describi possit geometricè, cuius area aequalis sit futura vtrique superficiei sphaeroidicae et conoidicae iunctim sumtae.

Solutio prima.

101. Manentibus pro vtroque corpore denomina-
tionibus, modo expositis, statuatur $\frac{aa\sqrt{m(n+1)}}{ec\sqrt{n(m-1)}} = 1$,
feu

feu $ec = \frac{aa\sqrt{m(n+1)}}{\sqrt{n(m-1)}}$, vnde femiaxis transuersus hyperbolae e determinatur, numero n feu eius specie arbitrio nostro relicta: eritque stabilita superiori relatione inter x et y

$$\text{II. } x + (\ominus y - \ominus e) = \frac{(n+1)ae\sqrt{m}}{ec} xy = \frac{exy\sqrt{n(m-1)(n+1)}}{a}$$

102 Cum nunc fit superficies sphaeroidis ex arcu BM nata, seu Sup. BM = $2\pi \cdot \Pi x$; et superficies conoidis ex arcu em nata, seu Sup. em = $2\pi(\ominus y - \ominus e)$; erit

$$\text{Sup. BM} + \text{Sup. em} = \frac{2\pi exy\sqrt{n(m-1)(n+1)}}{a}$$

Vnde si hae duae superficies iunctim sumtae aequentur circulo, cuius radius = r , ob eius aream = $\pi r r$ erit

$$r r = \frac{2exy\sqrt{n(m-1)(n+1)}}{a}$$

103. Hic iam continetur solutio problematis sensu multo latiori accepti. Casu enim Hugenio, quo integrum sphaeroides assumitur, seu, quod eodem redit, eius semissis, erit $x = a$; tum vero punctum e in vertice a capi oportet, vnde fit $e = c$. Erit ergo hoc casu:

$$y = c\sqrt{m} + \frac{c\sqrt{n(m-1)}}{\sqrt{(n+1)}} = cp,$$

fietque:

$$\text{Sup. BA} + \text{Sup. am} = 2\pi(n+1)aa\left(m + \frac{\sqrt{mn(m-1)}}{\sqrt{(n+1)}}\right)$$

104. Radio ergo circuli vtrique superficiei simul aequalis posito = r erit, $rr = 2aa(m(n+1) + \sqrt{mn(m-1)(n+1)})$

$$\text{siue } r = a\sqrt{2(\sqrt{m(n+1)} + \sqrt{n(m-1)})\sqrt{m(n+1)}}$$

Q 2

Atque

Atque erit $cp = y = \frac{c}{\sqrt{n+1}} (\sqrt{m(n+1)} + \sqrt{n(m-1)})$

tum vero accipi debet $c = a \sqrt{\frac{m(n+1)}{n(m-1)}}$.

Quae est solutio simplicissima Problematis Hugeniani.

Solutio secunda.

105. Cum relatio inter x et y sit ita comparata, vt sit

$$\frac{\sqrt{(n+1)yy-cc} + y\sqrt{n+1}}{\sqrt{(n+1)ee-cc} + e\sqrt{n+1}} = \sqrt{1 + \frac{(m-1)xx}{aa}} + \frac{x}{a} \sqrt{m-1}$$

fitque II. $x + \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}} (\ominus y - \ominus e) = \frac{\sqrt{m(n+1)}}{2cc}$.

$$\left(\frac{aay\sqrt{(n+1)yy-cc}}{\sqrt{m-1}} - \frac{aae\sqrt{(n+1)ee-cc}}{\sqrt{m-1}} + \frac{ccx\sqrt{aa+(m-1)xx}}{\sqrt{n+1}} \right)$$

Capiatur in conoide noua abscissa $cq = z$, et pro e iam sumatur y , vt sit

$$\frac{\sqrt{(n+1)zz-cc} + z\sqrt{n+1}}{\sqrt{(n+1)yy-cc} + y\sqrt{n+1}} = \sqrt{1 + \frac{(m-1)xx}{aa}} + \frac{x}{a} \sqrt{m-1}$$

erit pariter II. $x + \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}} (\ominus z - \ominus y) = \frac{\sqrt{m(n+1)}}{2cc} x$

$$\left(\frac{aaz\sqrt{(n+1)zz-cc}}{\sqrt{m-1}} - \frac{aay\sqrt{(n+1)yy-cc}}{\sqrt{m-1}} + \frac{ccx\sqrt{aa+(m-1)xx}}{\sqrt{n+1}} \right)$$

106. Addantur hae formulae inuicem, atque y prorsus eliminabitur; fiet enim

$$\frac{\sqrt{(n+1)zz-cc} + z\sqrt{n+1}}{\sqrt{(n+1)ee-cc} + e\sqrt{n+1}} = \left(\sqrt{1 + \frac{(m-1)xx}{aa}} + \frac{x}{a} \sqrt{m-1} \right)^2$$

eritque: 2 II. $x + \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}} (\ominus z - \ominus e) = \frac{\sqrt{m(n+1)}}{2cc} x$

$$\left(\frac{aaz\sqrt{(n+1)zz-cc}}{\sqrt{m-1}} - \frac{aae\sqrt{(n+1)ee-cc}}{\sqrt{m-1}} + \frac{2ccx\sqrt{aa+(m-1)xx}}{\sqrt{n+1}} \right)$$

107. Statuatur iam $\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}} = 2$, seu $cc = \frac{aa\sqrt{m(n+1)}}{2\sqrt{n(m-1)}}$
erit per $\frac{2\pi}{2}$ multiplicando

$$\text{Sup. BM} + \text{Sup. en} = \frac{\pi\sqrt{m(n+1)}}{2cc} x$$

$$\left(\frac{aaz\sqrt{(n+1)zz-cc}}{\sqrt{m-1}} - \frac{aae\sqrt{(n+1)ee-cc}}{\sqrt{m-1}} + \frac{2ccx\sqrt{aa+(m-1)xx}}{\sqrt{n+1}} \right)$$

vnde

vnde facile radius circuli aequalis definitur.

108. Sit nunc pro casu Hugenario $x = a$, et $e = c$, erit:

$$\frac{\sqrt{(n+1)zz - cc} + z\sqrt{n+1}}{c(\sqrt{n} + \sqrt{n+1})} = (\sqrt{m} + \sqrt{m-1})^2$$

Hincque inuento z , existenteque $cc = \frac{aa\sqrt{m(n+1)}}{2\sqrt{n(m-1)}}$, erit

$$\text{Sup. BA} + \text{Sup. an} = \frac{\pi\sqrt{m(n+1)}}{2cc} \left(\frac{aa\sqrt{(n+1)zz - cc}}{\sqrt{m-1}} - \frac{aacc\sqrt{n}}{\sqrt{m-1}} + \frac{2aacc\sqrt{m}}{\sqrt{n+1}} \right).$$

Solutio Generalis.

109. Si hac ratione continuo ulterius progrediamur, vt supra pro parabola est factum, reperietur, si abscissa $cq = z$ existente, $cf = e$ ita capiatur, vt fit

$$\frac{\sqrt{(n+1)zz - cc} + z\sqrt{n+1}}{\sqrt{(n+1)ee - cc} + e\sqrt{n+1}} = \left(\sqrt{1 + \frac{(m-1)xx}{aa}} + \frac{x}{a} \sqrt{m-1} \right)^\mu$$

fore $\mu \Pi x + \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}} (\ominus z - \ominus e) = \frac{\sqrt{m(n+1)}}{2cc} x$

$$\left(\frac{aa\sqrt{(n+1)zz - cc}}{\sqrt{m-1}} - \frac{aae\sqrt{(n+1)ee - cc}}{\sqrt{m-1}} + \frac{\mu ccx\sqrt{aa + (m-1)xx}}{\sqrt{n+1}} \right)$$

$$= \frac{\mu}{2\pi} \text{Sup. BM} + \frac{aa\sqrt{m(n+1)}}{2\pi cc\sqrt{n(m-1)}} \text{Sup. en.}$$

110. Pro casu ergo Hugonii, posito $x = a$, et $e = c$, fiat $\frac{aa\sqrt{m(n+1)}}{2cc\sqrt{n(m-1)}} = \mu$; et capiatur abscissa $cq = z$, ita vt fit:

$$\frac{\sqrt{(n+1)zz - cc} + z\sqrt{n+1}}{c(\sqrt{n} + \sqrt{n+1})} = (\sqrt{m} + \sqrt{m-1})^\mu$$

eritque

$$\text{Sup. BA} + \text{Sup. an} = \frac{\pi\sqrt{m(n+1)}}{\mu cc} \left(\frac{aa\sqrt{(n+1)zz - cc}}{\sqrt{m-1}} - \frac{aacc\sqrt{n}}{\sqrt{m-1}} + \frac{\mu aacc\sqrt{m}}{\sqrt{n+1}} \right)$$

sive

$$\text{Sup. BA} + \text{Sup. an} = \pi(z\sqrt{n((n+1)zz - cc)} - ncc + \frac{\mu cc\sqrt{mn(m-1)}}{\sqrt{n+1}})$$

$$= \pi(z\sqrt{n((n+1)zz - cc)} - ncc + maa).$$

111. Quaecunq;ue ergo fuerit hyperbola, ex qua conoides nascitur, dummodo fit $\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}} = \mu$ numerus rationalis, ab eo semper portio an abscindi poterit, cuius superficies ad superficiem sphaeroidis BMA addita, per circulum exhiberi potest, cuius radius r geometricae est assignabilis: erit enim

$$r = \sqrt{maa - ncc + z\sqrt{n((n+1)zz - cc)}}.$$

112. Quo autem facilius pateat, quomodo abscissa $cq = z$ reperiri debeat, cum sit

$$\sqrt{\left(\frac{(n+1)zz}{cc} - 1\right) + \frac{z}{c}\sqrt{(n+1)}} = (\sqrt{(n+1)} + \sqrt{n})(\sqrt{m} + \sqrt{(m-1)})^\mu$$

erit

$$\frac{z}{c}\sqrt{(n+1)} - \sqrt{\left(\frac{(n+1)zz}{cc} - 1\right)} = (\sqrt{(n+1)} - \sqrt{n})(\sqrt{m} - \sqrt{(m-1)})^\mu$$

hinc facile tam z , quam $\sqrt{(n+1)zz - cc}$ colligentur.

113. Hinc autem porro concluditur, fore

$$z\sqrt{n((n+1)zz - cc)} = \frac{cc\sqrt{n}}{4\sqrt{(n+1)}}(\sqrt{(n+1)} + \sqrt{n})^2(\sqrt{m} + \sqrt{(m-1)})^{2\mu} \\ - \frac{cc\sqrt{n}}{4\sqrt{(n+1)}}(\sqrt{(n+1)} - \sqrt{n})^2(\sqrt{m} - \sqrt{(m-1)})^{2\mu}$$

At si ponatur breuitatis gratia $\sqrt{(m+1)} + \sqrt{(m-1)} = M$, et $\sqrt{n} + \sqrt{(n+1)} = N$, erit $z = \frac{c}{2\sqrt{(n+1)}}(M^\mu N - M^{-\mu} N^{-1})$, et

$$r = \sqrt{maa + \frac{cc\sqrt{n}}{4\sqrt{(n+1)}}(M^\mu - M^{-\mu})(M^\mu N^2 + M^{-\mu} N^{-2})}$$

ficque problema non difficulter construitur, dummodo exponens μ fuerit rationalis.

114. Haec igitur exempla sufficiant vsu, nouae methodi, quam adumbraui, ostendisse; etsi enim haec eadem exempla methodo consueta iam sint soluta, tamen non solum ad calculos admodum intricatos deueniri

niri solet, sed etiam integratione, qua formulae differentiales, vel ad quadraturam circuli, vel ad logarithmos reducuntur, absolute est opus. Huius igitur nouae methodi insigne commodum in hoc consistit, quod eius beneficio eadem problemata, tam sine laborioso calculo, quam sine vlla integratione resolui queant; quam ob causam inde merito multo maiora ac sublimiora expectare licet, quae vim omnium consuetarum methodorum penitus superent.

