DE

INTEGRATIONE
AEQVATIONVM DIFFERENTIALIVM.

Auctore

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1.

Considero hic aequationes differentiales primi gradus, quae duas tantum variabiles inuoluunt, quas propterea sub hac forma generali $M\,dx + N\,dy = 0$, repraesentare licet, si quidem M et N denotent functiones quascunque binarum variabilium x et y. Demonstratum autem est, huiusmodi aequationem semper certam relationem inter variabiles x et y exprimere, qua pro quouis valore vnius certi valores pro altera definiantur. Cum autem per integrationem ista relatio finita inter ambas variabiles inueniri debeat, aequatio integralis, si quidem ad omnem amplitudinem extendatur, nouam quantitatem constantem recipiet, quae, dum penitus ab arbitrio nostro pendet, infinitas quasi aequationes integrales complectitur, quae omnes aequationi differentiali aeque conueniant.

2. Proposita igitur huiusmodi aequatione differentiali quacunque $M\,dx + N\,dy = 0$, tota vis Analyseos in hoc consistit, vt aequatio finita inter easdem varia-

biles x et y eliciatur, quae eandem inter illas relationem exprimat, atque ipsa differentialis, et quidem latissimo sensu, ita vt constantem quampiam arbitrariam, quae in differentiali non inest, contineat. Verum si haec quaestio ita generalissime proponatur, nulla plane adhuc inuenta est via ad eius solutionem perueniendi; atque omnes casus, quos adhuc resoluere licuit, ad numerum perquam exiguum reduci possunt, ita vt in hac Analyseos parte, perinde ac in reliquis, maxima adhuc incrementa desiderentur; neque ob hanc caussam vnquam plena omnium huius scientiae arcanorum cognitio expectari queat.

3. Quae quidem adhuc in hoc negotio sunt praestita, ea fere omnia ad hos casus referri possunt, quibus aequatio differentialis $Mdx + Ndy = 0$, vel sponte separationem variabilium admittit, vel per idoneas substitutiones ad talem formam reduci potest. Quodsi enim introducendis loco x et y binis nouis variabilibus v et z, aequatio differentialis proposita in huiusmodi formam $Vdv + Zdz = 0$ transmutari queat, in qua V sit functio ipsius v tantum, et Z ipsius z tantum, totum negotium erit confectum, dum aequatio integralis completa erit:

$$\int Vdv + \int Zdz = \text{Const.}$$

quae manifesto illam constantem arbitrariam, per generalem integrationem inuectam complectitur. Atque huc fere redeunt omnia artificia, quibus Analystae adhuc in resolutione huiusmodi aequationum sunt vsi.

4. Nisi

4. Nisi igitur aequatio proposita differentialis sponte separationem variabilium admittat, totum negotium in hoc consumi est solitum, vt idoneae substitutiones, quae ad separationem viam parent, inuestigantur, in quo etiam saepius summam sagacitatem, quam Geometrae ad scopum obtinendum adhibuerunt, admirari oportet. Interim tamen cum nulla certa via pateat, huiusmodi substitutiones inuestigandi, haec methodus minus ad rei naturam videtur accommodata, ex quo constitui, aliam methodum non nouam quidem, verum tamen etiamnunc non satis excultam, accuratius perpendere, quae vti substitutionibus non eget, ita etiam naturae aequationum magis consentanea videtur, dum eius ratio indoli differentialium innititur, tum vero etiam priorem methodum, velut partem, in se complectitur.

5. Aequatione differentiali ad hanc formam $M\,dx + N\,dy = 0$ perducta, consideretur formula $M\,dx + N\,dy$ sine respectu habito, quod ea euanescere debeat, et examinetur, vtrum ea sit differentiale cuiuspiam functionis ipsarum x et y, nec ne? Quemadmodum hoc examen sit instituendum, iam passim abunde est explicatum; vtramque scilicet functionem M et N differentiari oportet, et cum earum differentialia huiusmodi formam sint habitura:

$$dM = p\,dx + q\,dy \text{ et } dN = r\,dx + s\,dy$$

dispiciatur, vtrum sit $q = r$, nec ne? Quodsi enim fuerit $q = r$, hoc infallibile est criterium, formulam $M\,dx + N\,dy$ esse integrabilem: at si non fuerit $q = r$, aeque certum est, istam formulam ex nullius finitae functionis ipsarum x et y differentiatione esse ortam. Ex quo tota quaestio ad duos casus reducitur, quorum alter locum habet, si fuerit $q = r$, alter vero, si hae quantitates q et r non fuerint inter se aequales.

6. Ad aequalitatem igitur, vel inaequalitatem, quantitatum q et r agnoscendam, ne opus quidem est, vt functiones M et N penitus per differentiationem euoluantur, sed sufficit in functione M, quae cum dx est coniuncta, quantitatem x vt constantem spectare, eamque tantum eius differentialis partem quaerere, quae ex variabilitate ipsius y tantum nascitur, si quidem hoc modo membrum $q dy$ obtinetur, valorem autem ipsius q sic erutum hac scriptione $(\frac{dM}{dy})$ denotare soleo. Simili modo altera functio N, quae cum dy est coniuncta, ita differentietur, vt y pro constante tractetur, et ex variabilitate solius x impetretur differentialis pars $r dx$, vbi valorem ipsius r pariter per $(\frac{dN}{dx})$ exprimo. Quodsi ergo formula $M dx + N dy$ ita fuerit comparata, vt sit $(\frac{dM}{dy}) = (\frac{dN}{dx})$, ea est integrabilis, eiusque integrale sequenti modo inueniri poterit. Quo facto, si hoc criterium non locum habeat, videamus quomodo sit procedendum.

Problema I.

7. Si aequatio differentialis $M dx + N dy = 0$ ita fuerit comparata, vt sit $(\frac{dM}{dy}) = (\frac{dN}{dx})$, inuenire eius aequationem integralem.

Solutio.

Si fuerit $(\frac{dM}{dy}) = (\frac{dN}{dx})$, tunc datur functio finita binarum variabilium x et y, quae differentiata praebet

bet $M dx + N dy$. Sit V ista functio, et cum sit $dV = M dx + N dy$, erit $M dx$ differentiale ipsius V, si tantum x variabile sumatur, et $N dy$ eius differentiale, si tantum y variabile capiatur. Hinc ergo vicissim V reperietur, si vel $M dx$ integretur, spectata y vt constante, vel $N dy$ integretur, spectata x vt constante: sicque haec operatio reducitur ad integrationem formulae differentialis vnicam variabilem inuoluentis, quae in hoc negotio, siue algebraice succedat, siue quadraturas curuarum requirat, concedi postulatur. Cum autem hac ratione quantitas V duplici modo inueniatur, et altera integratio vice constantis functionem quamcunque ipsius y, altera vero ipsius x assumat, ita vt sit

$$\text{et } V = \int M dx + Y, \text{ et } V = \int N dy + X,$$

semper has functiones Y ipsius y, et X ipsius x, ita definire licet, vt fiat $\int M dx + Y = \int N dy + X$, id quod quouis casu facile praestatur. Quo facto cum quantitas V sit integrale formulae $M dx + N dy$, euidens est, aequationis propositae $M xd + N dx = 0$ integralem aequationem fore $V = \text{Const.}$ eamque completam, propterea quod inuoluit constantem quantitatem ab arbitrio nostro pendentem.

Coroll. 1.

8. In hoc problemate statim continetur casus aequationum separatarum. Si enim fuerit M functio ipsius x tantum, et N functio ipsius y tantum, erit vtique $\left(\frac{dM}{dy}\right) = 0$ et $\left(\frac{dN}{dx}\right) = 0$; ideoque $\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right)$; qui est ergo casus simplicissimus, quem problema in se complectitur.

Coroll.

Coroll. 2.

9. Quodsi autem in aequatione differentiali $M\,dx + N\,dy = 0$ fuerit M functio solius x, et N solius y, vtraque pars seorsim integrabilis existit, atque aequatio integralis erit:

$$\int M\,dx + \int N\,dy = \text{Const.}$$

Coroll. 3.

10. Praeterea vero nostrum problema resolutionem infinitarum aliarum aequationum differentialium largitur, quarum omnium character communis in hoc consistit, vt sit $\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right)$, earumque resolutio per integrationem formularum, vnicam variabilem continentium, expediri potest.

Scholion 1.

11. Quoties ergo in aequatione differentiali $M\,dx + N\,dy = 0$ fuerit $\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right)$, eius resolutio nullam habet difficultatem, dummodo integratio formularum vnicam variabilem inuoluentium concedatur; quam quidem iure postulare licet. Interim tamen determinatio functionum illarum X et Y, quae loco constantium introduci debent, molestiam quandam creare videri posset, quae autem singulis casibus mox euanescere reperietur. Verum quo magis et haec operatio contrahatur, ne duplici quidem integratione est opus. Postquam enim altera pars $M\,dx$, spectata y tanquam constanti, fuerit integrata, quod integrale sit $= Q$, statuatur $V = Q + Y$, posito tantisper Y pro functione inde-

indefinita ipſius y, in quam altera variabilis x prorſus non ingrediatur. Tum differentietur denuo haec quantitas $Q+Y$, tractando x tanquam conſtantem, et quia differentiale prodire debet $= N\,dy$, ex hac conditione functio Y facillime definietur, quandoquidem ex rei natura hinc ſponte eliminabitur quantitas x. Inuenta autem iſta functione Y, aequatio integralis erit $Q+Y=$ Conſt. quam operationem ſequentibus exemplis illuſtrari conueniet.

Exemplum I.

12. *Integrare hanc aequationem differentialem:*

$$2axy\,dx + axx\,dy - y^3\,dx - 3xyy\,dy = 0.$$

Comparata hac aequatione cum forma $M\,dx + N\,dy = 0$, erit:

$$M = 2axy - y^3 \text{ et } N = axx - 3xyy.$$

Primum igitur diſpiciendum eſt, vtrum hic caſus in problemate contineatur? quem in finem quaeramus valores:

$$\left(\tfrac{dM}{dy}\right) = 2ax - 3yy, \text{ et } \left(\tfrac{dN}{dx}\right) = 2ax - 3yy,$$

qui cum ſint aequales, operatio praeſcripta neceſſario ſuccedet. Reperietur autem, ſumta y pro conſtante:

$$\int M\,dx = axxy - y^3x + Y;$$

cuius formae ſi differentiale ſumatur, poſita x conſtante, prodibit:

$$axx\,dy - 3yyx\,dy + dY = N\,dy,$$

et pro N valore ſuo $axx - 3xyy$ reſtituto, fiet $dY = 0$, ex quo naſcitur $Y = 0$, vel $Y =$ conſt. Quare aequatio integralis quaeſita habebitur:

$$axxy - y^3x = \text{Conſt.}$$

Exemplum 2.

13. *Integrare hanc aequationem differentialem:*

$$\frac{y\,dy + x\,dx - 2y\,dx}{(y-x)^2} = 0$$

Comparata hac aequatione cum forma $M\,dx + N\,dy = 0$, erit:

$$M = \frac{x - 2y}{(y-x)^2} \text{ et } N = \frac{y}{(y-x)^2}$$

Iam vt pateat, num haec aequatio in casu problematis contineatur, quaerantur valores differentiales:

$$\left(\frac{dM}{dy}\right) = \frac{2y}{(y-x)^3} \text{ et } \left(\frac{dN}{dx}\right) = \frac{2y}{(y-x)^3}$$

qui cum sint aequales, negotium succedet. Quare secundum regulam colligatur, sumto y constante, integrale:

$$\int M\,dx = \int \frac{x\,dx - 2y\,dx}{(y-x)^2} = -\int \frac{dx}{y-x} - \int \frac{y\,dx}{(y-x)^2}$$

ac reperietur:

$$\int M\,dx = l(y-x) - \frac{y}{y-x} + Y$$

cuius differentiale, sumto x constante, producere debet alteram aequationis propositae partem $N\,dy$; vnde habebitur:

$$N\,dy = \frac{dy}{y-x} + \frac{x\,dy}{(y-x)^2} + dY = \frac{y\,dy}{(y-x)^2} + dY.$$

Cum igitur sit $N\,dy = \frac{y\,dy}{(y-x)^2}$ et $dY = 0$, et $Y = 0$, constantem enim in Y negligere licet, quia iam in aequationem integralem introducitur, quippe quae erit:

$$l(y-x) - \frac{y}{y-x} = \text{Const.}$$

Exemplum 3.

14. *Integrare hanc aequationem differentialem:*

$$\frac{dx}{x} + \frac{yy\,dx}{x^3} - \frac{y\,dy}{xx} + \frac{(y\,dx - x\,dy)\sqrt{(xx+yy)}}{x^3} = 0$$

Com-

Comparata hac aequatione cum forma $M\,dx + N\,dy = 0$, habebimus:

$$M = \frac{xx + yy + y\sqrt{(xx+yy)}}{x^3} \text{ et } N = \frac{-y - \sqrt{(xx+yy)}}{xx}$$

vnde pro criterio explorando quaeratur:

$$\left(\frac{dM}{dy}\right) = \frac{2y}{x^3} + \frac{\sqrt{(xx+yy)}}{x^3} + \frac{yy}{x^3\sqrt{(xx+yy)}} \text{ et}$$

$$\left(\frac{dN}{dx}\right) = \frac{2y}{x^3} + \frac{2\sqrt{(xx+yy)}}{x^3} - \frac{x}{xx\sqrt{(xx+yy)}}$$

qui valores reducti cum fiant aequales, ſcilicet

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right) = \frac{2y}{x^3} + \frac{xx + 2yy}{x^3\sqrt{(xx+yy)}}$$

reſolutio erit in poteſtate. Inueſtigetur ergo, ſumto y conſtante:

$$\int M\,dx = lx - \frac{yy}{2xx} + y\int \frac{dx}{x^3}\sqrt{(xx+yy)}.$$

At per regulas integrandi, formulas vnicam variabilem inuoluentes, quia hic y pro conſtante habetur, reperitur:

$$\int \frac{y\,dx}{x^3}\sqrt{(xx+yy)} = \frac{-y\sqrt{(xx+yy)}}{2xx} + \tfrac{1}{2}l\frac{\sqrt{(xx+yy)} - y}{y}$$

ita vt ſit:

$$\int M\,dx = lx - \frac{yy}{2xx} - \frac{y\sqrt{(xx+yy)}}{2xx} + \tfrac{1}{2}l\frac{\sqrt{(xx+yy)} - y}{y} + Y$$

At huius quantitatis differentiale, aſſumto x pro conſtante, quia praebere debet $N\,dy = -\frac{y\,dy - dy\sqrt{(xx+yy)}}{xx}$, nanciſcemur:

$$N\,dy = \frac{-y\,dy}{xx} - \frac{dy\sqrt{(xx+yy)}}{2xx} - \frac{yy\,dy}{2xx\sqrt{(xx+yy)}} - \frac{dy}{2y} - \frac{dy}{2\sqrt{(xx+yy)}} + dY$$

qua forma cum illa comparata fiet:

$$dY = -\frac{dy\sqrt{(xx+yy)}}{2xx} + \frac{yy\,dy}{2xx\sqrt{(xx+yy)}} + \frac{dy}{2y} + \frac{dy}{2\sqrt{(xx+yy)}}$$

vbi termini, qui adhuc continent x, ſponte ſe deſtruunt, ita vt ſit $dY = \frac{dy}{2y}$ et $Y = \frac{1}{2}ly$. Quo valore pro Y inuento, obtinebitur aequatio integralis quaeſita:

$$lx - \frac{yy}{2xx} - \frac{y\sqrt{(xx+yy)}}{2xx} + \tfrac{1}{2}l(\sqrt{(xx+yy)} - y) = \text{Conſt.}$$

 Scho-

Scholion 2.

15. Ex his exemplis ſatis perſpicitur, quemadmodum perpetuo operatio praeſcripta ſit inſtituenda, ita vt hinc nulla amplius difficultas moleſtiam faceſſat, niſi quae ex integratione formularum, vnicam variabilem involuentium, quandoque relinquitur, dum integratio neque algebraice abſolui, neque ad circuli hyperbolaeue quadraturam reduci patitur. Verum tum ſuperiores quadraturas ſimili modo tractari oportet, et ſi quae difficultates relinquantur, eae non huic methodo ſunt adſcribendae. Quam ob rem hic aſſumere licet, quoties aequatio differentialis $Mdx + Ndy = 0$ ita fuerit comparata, vt in ea ſit $\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right)$, toties integrationem eſſe in noſtra poteſtate; vnde ad eas aequationes pergo, in quibus hoc criterium non habet locum.

Theorema.

16. Si in aequatione differentiali $Mdx + Ndy = 0$ non fuerit $\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right)$, ſemper datur multiplicator, per quem formula $Mdx + Ndy$ multiplicata fiat integrabilis.

Demonſtratio.

Cum non ſit $\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right)$, etiam formula $Mdx + Ndy$ non erit integrabilis, ſeu nulla exiſtit functio ipſarum x et y, cuius differentiale ſit $Mdx + Ndy$. Verum hic non tam formulae $Mdx + Ndy$, quam aequationis $Mdx + Ndy = 0$, quaeritur integrale; et cum eadem aequatio ſubſiſtat, ſi per functionem quam-

quamcunque L ipsarum x et y multiplicetur, ita vt sit $LMdx + LNdy = 0$, demonstrandum est, semper eiusmodi dari functionem L, vt formula $LMdx + LNdy$ fiat integrabilis. Quo enim hoc eueniat, necesse est, vt sit:

$$\left(\frac{d.LM}{dy}\right) = \left(\frac{d.LN}{dx}\right)$$

vel si ponatur $dL = Pdx + Qdy$, cum sit $\left(\frac{dL}{dy}\right) = Q$, et $\left(\frac{dL}{dx}\right) = P$, functio L ita debet esse comparata, vt sit:

$$L\left(\frac{dM}{dy}\right) + MQ = L\left(\frac{dN}{dx}\right) + NP.$$

Euidens autem est, hanc conditionem sufficere ad definiendam functionem L, per quam si formula $Mdx + Ndy$ multiplicetur, fiat integrabilis.

Coroll. 1.

17. Inuento ergo tali multiplicatore L, qui reddat formulam $Mdx + Ndy$ integrabilem, aequatio $Mdx + Ndy = 0$, in formam $LMdx + LNdy = 0$ translata, integrari poterit methodo in problemate praecedente exposita.

Coroll. 2.

18. Quaeratur scilicet, spectata y tanquam constante, integrale $\int LMdx$, ad quod adiiciatur talis functio Y ipsius y, vt si aggregatum $\int LMdx + Y$ denuo differentietur, spectata iam x vt constante, prodeat $LNdy$. Quo facto erit aequatio integralis $\int LMdx + Y = Const.$

Coroll. 3.

19. Multiplicator igitur L ita debet esse comparatus, vt posito $dL = Pdx + Qdy$, satisfiat huic aequationi:

$$L\left(\frac{dM}{dy}\right) + MQ = L\left(\frac{dN}{dx}\right) + NP$$

vel huic:

$$\frac{NP - MQ}{L} = \left(\frac{dM}{dy}\right) - \left(\frac{dN}{dx}\right)$$

vnde manifestum est, si esset $\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right)$, pro L sumi posse vnitatem, vel quantitatem constantem quamcunque, dum sit $P = 0$, et $Q = 0$.

Scholion.

20. Si ergo hinc in genere multiplicator L inveniri posset, haberetur vniuersalis resolutio omnium aequationum differentialium primi gradus; id quod ne sperare quidem licet. Contentos ergo nos esse oportet, si pro variis casibus, pluribusque aequationum differentialium generibus, huiusmodi factores inuestigare valeamus. Sunt autem duo aequationum genera, pro quibus tales factores commode erui possunt, quorum alterum eas comprehendit aequationes, in quibus altera variabilis nusquam vltra vnam dimensionem exsurgit; alterum vero genus est aequationum homogenearum. Praeter haec vero duo genera plures alii existunt casus, quibus inuentio talis factoris absolui potest, quos diligentius examinasse, vsu non carebit, cum haec sola via patere videatur ad eam Analyseos partem, quae adhuc desideratur, excolendam ac perficiendam. Quam ob

ob rem hic constitui, plura aequationum genera colligere, quae per huiusmodi multiplicatorem ad integrabilitatem perduci possunt.

Problema 2.

21. Cognito vno multiplicatore L, qui formulam $Mdx + Ndy$ integrabilem reddit, inuenire infinitos alios multiplicatores, qui idem officium praestent.

Solutio.

Cum formula $L(Mdx + Ndy)$ per hypothesin sit integrabilis, sit eius integrale $= z$, ita vt sit $dz = L(Mdx + Ndy)$, existente z quapiam functione ipsarum x et y. Denotet iam Z functionem quamcunque ipsius z, et quia formula Zdz est etiam integrabilis, ob $Zdz = LZ(Mdx + Ndy)$, manifestum est formulam propositam $Mdx + Ndy$ quoque fieri integrabilem, si per LZ multiplicetur. Dato ergo vno multiplicatore L, qui formulam $Mdx + Ndy$ integrabilem reddat, ex eo innumerabiles alii factores LZ inveniri possunt, qui idem sint praestituri, sumendo pro Z functionem quamcunque integralis $\int L(Mdx + Ndy)$.

Coroll. 1.

22. Proposita igitur formula differentiali quacunque $Mdx + Ndy$, non solum vnus, sed etiam infiniti dantur multiplicatores, qui eam integrabilem reddant. Quorum autem vnum inuenisse sufficit, cum reliqui omnes per hunc determinentur.

Coroll.

Coroll. 2.

23. Si ergo habeatur aequatio differentialis $M dx + N dy = 0$, ea infinitis modis ad integrabilitatem perduci poteſt. Siue autem capiatur multiplicator L, ſiue alius quicunque LZ, aequatio integralis inuenta eodem redit; ſiquidem ille factor L praebet $z =$ Conſt. hic vero $\int Z dz =$ Conſt. id quod conuenit cum $\int Z dz$ et ſit functio ipſius z.

Exemplum 1.

24. *Inuenire omnes multiplicatores, qui reddant hanc formulam* $\alpha y dx + \beta x dy$ *integrabilem.*

Vnus multiplicator hoc praeſtans in promtu eſt, ſcilicet $\frac{1}{xy}$. Sit ergo $L = \frac{1}{xy}$, fiatque $dz = \frac{\alpha y dx + \beta x dy}{xy} = \frac{\alpha dx}{x} + \frac{\beta dy}{y}$, vnde integrando prodit $z = \alpha lx + \beta ly = lx^\alpha y^\beta$. Denotet iam Z functionem quamcunque ipſius $z = lx^\alpha y^\beta$, hoc eſt ipſius $x^\alpha y^\beta$, atque omnes multiplicatores quaeſiti in hac forma generali $\frac{1}{xy}$ funct. $x^\alpha y^\beta$ continebuntur.

Simpliciores ergo multiplicatores reperientur, ſi loco functionis poteſtas quaecunque ipſius $x^\alpha y^\beta$ capiatur; ſicque formula $\alpha y dx + \beta x dy$ integrabilis redditur per hunc multiplicatorem latius patentem $x^{\alpha n - 1} y^{\beta n - 1}$. Si magis compoſiti deſiderentur, plures huiusmodi vtcunque inter ſe combinari poterunt, vt habeatur $A x^{\alpha n - 1} y^{\beta n - 1} + B x^{\alpha m - 1} y^{\beta m - 1}$ etc.

Exem-

Exemplum 2.

25. *Inuenire omnes multiplicatores, qui reddant hanc formulam differentialem $\alpha x^{\mu-1} y^{\nu} dx + \beta x^{\mu} y^{\nu-1} dy$ integrabilem.*

Hic iterum ſtatim ſe offert vnus multiplicator $L = \frac{1}{x^{\mu} y^{\nu}}$, qui praebet $dz = \frac{\alpha dx}{x} + \frac{\beta dy}{y}$, vnde fit $z = \alpha lx - \beta ly = lx^{\alpha} y^{\beta}$ Poſito igitur Z pro functione quacunque ipſius $x^{\alpha} y^{\beta}$, omnes multiplicatores continebuntur in hac expreſſione generali $\frac{Z}{x^{\mu} y^{\nu}} = \frac{1}{x^{\mu} y^{\nu}}$ funct. $x^{\alpha} y^{\beta}$. Si loco iſtius functionis ſumatur poteſtas quaecunque $x^{\alpha n} y^{\beta n}$, innumeri hinc obtinebuntur multiplicatores, vnico termino conſtantes $x^{\alpha n-\mu} y^{\beta n-\nu}$, ſumendo pro n numeros quoscunque.

Scholion.

26. Fieri igitur poteſt, vt duae pluresue huiusmodi formulae differentiales $\alpha x^{\mu-1} y^{\nu} dx + \beta x^{\mu} y^{\nu-1} dy$ communem recipiant multiplicatorem: quod ſi eueniat aequatio differentialis, ex huiusmodi formulis, tanquam membris, compoſita, integrabilis reddi poterit, dum multiplicator iſte communis adhibetur. Quem caſum iam olim tractatum euoluamus.

Problema 3.

27. Propoſita ſit iſta aequatio differentialis:

$$\alpha y dx + \beta x dy + \gamma x^{\mu-1} y^{\nu} dx + \delta x^{\mu} y^{\nu-1} dy = 0$$

cuius integralem inueniri oporteat.

 Solutio.

Solutio.

Ad multiplicatorem idoneum inueniendum, quo haec aequatio reddatur integrabilis, consideretur vtrumque membrum seorsim. Ac prius quidem membrum $\alpha y dx + \beta x dy$ vidimus integrabile reddi hoc multiplicatore $x^{\alpha n - 1} y^{\beta n - 1}$, posterius vero membrum $\gamma x^{\mu - 1} y^{\nu} dx + \delta x^{\mu} y^{\nu - 1} dy$ hoc $x^{\gamma m - \mu} y^{\delta m - \nu}$. Quia nunc pro n et m numeros quoscunque accipere licet, hi duo factores ad aequalitatem reduci poterunt; vnde fit

$$\alpha n - 1 = \gamma m - \mu \text{ et } \beta n - 1 = \delta m - \nu$$

ideoque $n = \frac{\gamma m - \mu + 1}{\alpha} = \frac{\delta m - \nu + 1}{\beta}$, hincque obtinetur

$$m = \frac{\alpha \nu - \beta \mu - \alpha + \beta}{\alpha \delta - \beta \gamma} \text{ et } n = \frac{\gamma \nu - \delta \mu - \gamma + \delta}{\alpha \delta - \beta \gamma}$$

His valoribus pro m et n inuentis, iste multiplicator communis dabit hanc aequationem integralem:

$$\tfrac{1}{n} x^{\alpha n} y^{\beta n} + \tfrac{1}{m} x^{\gamma m} y^{\delta m} = \text{Const.}$$

Coroll. 1.

28. Haec ergo aequatio integralis semper est algebraica, siquidem pro m et n valores veri reperiantur. Ii igitur tantum casus singulari reductione indigent, quibus numeri m et n vel in infinitum abeunt, vel euanescunt.

Coroll. 2.

29. Infiniti autem euadunt ambo numeri m et n, si fuerit $\alpha \delta = \beta \gamma$. Verum hoc casu ipsa aequatio differentialis in duos factores resoluitur, hancque formam acquirit

$$(\alpha y dx + \beta x dy)(1 + \tfrac{\gamma}{\alpha} x^{\mu - 1} y^{\nu - 1}) = 0$$

ideoque

ideoque erit vel $\alpha y dx + \beta x dy = 0$, vel $1 + \frac{\gamma}{\alpha} x^{\mu-1} y^{\nu-1} = 0$, quarum reſolutionum neutra difficultate laborat.

Coroll. 3.

30. At ſi fiat $n = 0$, ſeu $\gamma(\nu - 1) = \delta(\mu - 1)$, conſideretur numerus n, vt valde paruus, et cum ſit per ſeriem conuergentem

$$x^{\alpha n} = 1 + \alpha n l x + \tfrac{1}{2}\alpha^2 n^2 (lx)^2 + \text{etc. et } y^{\beta n} = 1 + \beta n l y + \tfrac{1}{2}\beta^2 n^2 (ly)^2 + \text{etc.}$$

erit

$$\tfrac{1}{n} x^{\alpha n} y^{\beta n} = \tfrac{1}{n} + \alpha l x + \beta l y = l x^\alpha y^\beta$$

prima parte $\frac{1}{n}$ in conſtantem inuoluta. Hoc ergo caſu erit aequatio integralis:

$$l x^\alpha y^\beta + \tfrac{1}{m} x^{\gamma m} y^{\delta m} = \text{Conſt.}$$

Coroll. 4.

31. Statuatur ergo pro hoc caſu $\mu = \gamma k + 1$ et $\nu = \delta k + 1$, vt habeatur iſta aequatio differentialis:

$$\alpha y dx + \beta x dy + \gamma x^{\gamma k} y^{\delta k + 1} dx + \delta x^{\gamma k + 1} y^{\delta k} dy = 0$$

et cum ſit $m = \frac{\alpha \delta k - \beta \gamma k}{\alpha \delta - \beta \gamma} = k$, erit aequatio integralis

$$l x^\alpha y^\beta + \tfrac{1}{k} x^{\gamma k} y^{\delta k} = \text{Conſt.}$$

Coroll. 5.

32. Simili modo ſi fuerit $m = 0$, ſeu $\alpha(\nu - 1) = \beta(\mu - 1)$ ob $\frac{1}{m} x^{\gamma m} y^{\delta m} = l x^\gamma y^\delta$, ſi ponatur $\mu = \alpha k + 1$ et $\nu = \beta k + 1$, vnde fit $n = \frac{\gamma \beta k - \delta \alpha k}{\alpha \delta - \beta \gamma} = -k$; erit huius aequationis

$$\alpha y dx + \beta x dy + \gamma x^{\alpha k} y^{\beta k + 1} dx + \delta x^{\alpha k + 1} y^{\beta k} dy = 0$$

 inte-

integralis

$$-\tfrac{1}{k}x^{-\alpha k}y^{-\beta k}+lx^{\gamma}y^{\delta}=\text{Const.}$$

Scholion.

33. Neque vero huiusmodi resolutio in membra, quae per eundem multiplicatorem reddantur integrabilia, ad omnis generis aequationes patet. Euenire enim vtique potest, vt tota aequatio per quampiam quantitatem multiplicata integrabilis euadat, cum tamen nulla eius pars inde seorsim integrabilis existat, ex quo huic tractationi, qua hic sum vsus, non nimis tribui oportet.

Problema 4.

34. Si proposita sit aequatio differentialis

$$Pdx+Qydx+Rdy=0$$

vbi P, Q et R denotant functiones quascunque ipsius x, ita vt altera variabilis y plus vna dimensione non habeat, inuenire multiplicatorem, qui eam reddat integrabilem.

Solutio.

Comparata hac aequatione cum forma $Mdx+Ndy=0$ erit $M=P+Qy$ et $N=R$, vnde fiet

$$\left(\frac{dM}{dy}\right)=Q \text{ et } \left(\frac{dN}{dx}\right)=\frac{dR}{dx}$$

Statuatur iam L pro multiplicatore quaesito, sitque $dL=pdx+qdy$, atque huic aequationi satisfieri oportet

$$\frac{Np-Mq}{L}=Q-\frac{dR}{dx}=\frac{Rp-(P+Qy)q}{L}$$

Cum

Cum iam sit $Q-\frac{dR}{dx}$ functio ipsius x tantum, pro L quoque functio ipsius x tantum accipi poterit, ita vt sit $q=0$, et $dL=pdx$; vnde erit:

$$Q-\frac{dR}{dx}=\frac{Rp}{L}, \text{ seu } Qdx-dR=\frac{RdL}{L}$$

ideoque $\frac{dL}{L}=\frac{Qdx}{R}-\frac{dR}{R}$. Quare integrando habebitur $lL=\int\frac{Qdx}{R}-lR$, et sumto e pro numero, cuius logarithmus hyperbolicus est vnitas, prodit

$$L=\frac{1}{R}e^{\int\frac{Qdx}{R}}$$

Inuento autem hoc multiplicatore erit aequatio integralis:

$$\int\frac{Pdx}{R}e^{\int\frac{Qdx}{R}}+ye^{\int\frac{Qdx}{R}}=\text{Const.}$$

Coroll. 1.

35. Si aequatio habeat formam propositam, ea, antequam hoc modo tractetur, diuidi poterit per R, vt hanc formam induat $Pdx+Qydx+dy=0$, seu statim assumere licet $R=1$, quo facto multiplicator erit $e^{\int Qdx}$, et aequatio integralis $\int e^{\int Qdx}Pdx+e^{\int Qdx}y=\text{Const.}$

Coroll. 2.

36. Si ponatur hoc integrale $\int e^{\int Qdx}Pdx+e^{\int Qdx}y=z$, ita vt z sit functio quaepiam ambarum variabilium, tum vero Z denotet functionem quamcunque ipsius z; omnes multiplicatores, qui formulam $Pdx+Qydx+dy$ reddunt integrabilem, in hac forma generali $e^{\int Qdx}Z$ continentur.

Problema 5.

37. Si propofita fit aequatio differentialis:

$$Py^n dx + Qy dx + R dy = 0$$

vbi P, Q et R denotent functiones quascunque ipfius x, inuenire multiplicatorem, qui eam reddat integrabilem.

Solutio.

Erit ergo $M = Py^n + Qy$ et $N = R$, hincque

$$\left(\frac{dM}{dy}\right) = nPy^{n-1} + Q, \text{ et } \left(\frac{dN}{dx}\right) = \frac{dR}{dx}$$

Quare pofito multiplicatore quaefito L et $dL = p dx + q dy$, erit ex ante inuentis:

$$\frac{Rp - Py^n q - Qyq}{L} = nPy^{n-1} + Q - \frac{dR}{dx}.$$

Fingatur $L = Sy^m$, exiftente S functione ipfius x tantum, erit $p = \frac{y^m dS}{dx}$, et $q = mSy^{m-1}$, quibus valoribus fubftitutis, prodibit:

$$\frac{RdS}{Sdx} - mPy^{n-1} - mQ = nPy^{n-1} + Q - \frac{dR}{dx}$$

Quae aequatio vt fubfiftere poffit, fumi debet $m = -n$, ac fiet

$$\frac{RdS}{Sdx} = (1-n)Q - \frac{dR}{dx}, \text{ feu } \frac{dS}{S} = \frac{(1-n)Q dx}{R} - \frac{dR}{R}$$

Vnde cum integrando proueniat $S = \frac{1}{R} e^{(1-n)\int \frac{Q dx}{R}}$, erit, ob $m = -n$, multiplicator quaefitus:

$$L = \frac{y^{-n}}{R} e^{(1-n)\int \frac{Q dx}{R}}$$

et aequatio integralis erit

$$\frac{y^{1-n}}{1-n} e^{(1-n)\int \frac{Q dx}{R}} + \int \frac{P dx}{R} e^{(1-n)\int \frac{Q dx}{R}} = \text{Conft.}$$

Coroll. 1.

Coroll. 1.

38. Si $n=0$, habemus casum ante tractatum aequationis $P\,dx+Q\,y\,dx+R\,dy=0$, quae per multiplicatorem $\frac{1}{R}e^{\int\frac{Q\,dx}{R}}$ integrabilis redditur; et cuius aequatio integralis est

$$y\,e^{\int\frac{Q\,dx}{R}}+\int\frac{P\,dx}{R}e^{\int\frac{Q\,dx}{R}}=\text{Const.}$$

Coroll. 2.

39. At sit $n=1$, vt aequatio differentialis sit:

$$P\,y\,dx+Q\,y\,dx+R\,dy=0$$

multiplicator, ob $1-n=0$, erit $\frac{1}{R\,y}$; quo aequatio reducitur ad hanc formam $\frac{P\,dx+Q\,dx}{R}+\frac{dy}{y}=0$, cuius integralis manifesto est $\int\frac{(P+Q)\,dx}{R}+ly=\text{Const.}$

Scholion.

40. Caeterum hoc problema ex antecedente facile deducitur. Diuidatur enim aequatio differentialis proposita per y^n, et habebitur:

$$P\,dx+Q\,y^{1-n}\,dx+R\,y^{-n}\,dy=0$$

Ponatur $y^{1-n}=z$, erit $(1-n)\,y^{-n}\,dy=dz$, sicque aequatio transit in hanc:

$$P\,dx+Q\,z\,dx+\frac{1}{1-n}R\,dz=0$$

quae cum aequatione problematis praecedentis conuenit. Cum igitur hae duae aequationes referendae sint ad casum, quo altera variabilis nusquam vltra vnam dimensionem ascendit, hunc methodo hac per multiplicatores

res expediuimus. Pergo itaque ad alterum genus aequationum differentialium homogenearum, quas etiam hac methodo tractari poſſe conſtat. Ad hoc autem lemma, quo natura functionum homogenearum continetur, praemitti neceſſe eſt, ſi quidem operationem ex primis principiis petere velimus.

Lemma.

41. Si V fuerit functio homogenea, in qua binae variabiles x et y vbique n dimenſiones conſtituant, eius differentiale $dV = Pdx + Qdy$ ita erit comparatum, vt ſit $Px + Qy = nV$.

Demonſtratio.

Ponatur $y = xz$, et functio V induet huiusmodi formam $x^n Z$, exiſtente Z quapiam functione ipſius z tantum. Hinc ergo erit $dV = nx^{n-1} Z dx + x^n dZ$. Ad has duas variabiles x et z etiam differentiale propoſitum $dV = Pdx + Qdy$ reducatur, et cum ſit $dy = zdx + xdz$, erit

$$dV = (P + Qz)dx + Qxdz$$

neceſſe igitur eſt, vt ſit $nx^{n-1} Z = P + Qz$, et per x vtrinque multiplicando: $nx^n Z = nV = Px + Qxz = Px + Qy$: ita vt ſit $R + Qy = nV$.

Coroll. 1.

42. Quia ergo habemus duas aequationes:

$$dV = Pdx + Qdy, \text{ et } nV = Px + Qy$$

hinc

hinc ambae functiones P et Q definiri poterunt; reperietur enim:

$$P = \frac{y\,dV - nV\,dy}{y\,dx - x\,dy} \text{ et } Q = \frac{nV\,dx - x\,dV}{y\,dx - x\,dy}.$$

Coroll. 2.

43. Quoties ergo V est functio homogenea n dimensionum, toties ob $P = (\frac{dV}{dx})$ et $Q = (\frac{dV}{dy})$ erit

$$(\frac{dV}{dx}) = \frac{y\,dV - nV\,dy}{y\,dx - x\,dy} \text{ et } (\frac{dV}{dy}) = \frac{nV\,dx - x\,dV}{y\,dx - x\,dy}$$

vbi notandum est, in his fractionibus differentialia se mutuo tollere, seu vtrumque numeratorem fore per $y\,dx - x\,dy$ diuisibilem.

Problema 6.

44. Proposita aequatione differentiali $M\,dx + N\,dy = 0$, in qua M et N sint functiones homogeneae ipsarum x et y eiusdem ambae dimensionum numeri, inuenire multiplicatorem, qui eam aequationem reddat integrabilem.

Solutio.

Sit n numerus dimensionum, vtrique functioni M et N conueniens, eritque per §. praec.

$$(\frac{dM}{dy}) = \frac{nM\,dx - x\,dM}{y\,dx - x\,dy} \text{ et } (\frac{dN}{dx}) = \frac{y\,dN - nN\,dy}{y\,dx - x\,dy}$$

ideoque

$$(\frac{dM}{dy}) - (\frac{dN}{dx}) = \frac{n(M\,dx + N\,dy) - x\,dM - y\,dN}{y\,dx - x\,dy}.$$

Iam facile colligere licet, dari multiplicatorem, qui etiam sit functio homogenea ipsarum x et y. Sit ergo L talis

 functio

functio homogenea m dimensionum. Quare si in §. 19 ponatur $dL = Pdx + Qdy$, erit (42.)

$$P = \frac{ydL - mLdy}{ydx - xdy}, \quad \text{et } Q = \frac{mLdx - xdL}{ydx - xdy}$$

hincque, cum esse oporteat per §. 19.

$$\frac{NP - MQ}{L} = \left(\frac{dM}{dy}\right) - \left(\frac{dN}{dx}\right)$$

obtinebitur vtrinque per $ydx - xdy$ multiplicando:

$$\frac{NydL - mLNdy - mLMdx + MxdL}{L} = n(Mdx + Ndy) - xdM - ydN$$

vnde elicitur:

$$\frac{dL}{L} = \frac{(m+n)(Mdx + Ndy) - xdM - ydN}{Mx + Ny}$$

quae formula manifesto fit integrabilis posito $m + n = -1$, quo facto erit $lL = -l(Mx + Ny)$. Quam ob rem multiplicator quaesitus habebitur $L = \frac{1}{Mx + Ny}$.

Coroll. 1.

45. Proposita igitur aequatione differentiali homogenea $Mdx + Ndy = 0$, ea facillime ad integrabilitatem reducetur, propterea quod formula $\frac{Mdx + Ndy}{Mx + Ny}$ est integrabilis, cuius integrale, per methodum supra traditam inuentum, dabit aequationem integralem quaesitam.

Coroll. 2.

46. Eo casu tantum incommodum oritur, vbi fit $Mx + Ny = 0$, veluti euenit in aequatione $ydx - xdy = 0$, quae diuidi deberet per $xy - xy = 0xy$. Sed quia huius diuisoris multiplum quodcunque aeque satisfacit, diuisor xy negotium conficiet, quemadmodum per se est perspicuum.

Scholion.

Scholion.

47. Notissima est methodus, qua sagacissimus *Ioh. Bernoullius* olim omnes aequationes differentiales homogeneas ad separabilitatem variabilium perducere docuit. Proposita scilicet huiusmodi aequatione $Mdx + Ndy = 0$, in qua M et N sint functiones homogeneae n dimensionum, ponere iubet $y = ux$, quo facto functiones M et N huiusmodi formas induent, vt sit $M = x^n U$, et $N = x^n V$, existentibus U et V functionibus ipsius u tantum. Aequatio ergo proposita per x^n diuisa abibit in hanc: $Udx + Vdy = 0$. Cum autem sit $dy = udx + xdu$, habebimus $Udx + Vudx + Vxdu = 0$, quae per $x(U + Vu)$ diuisa fit separabilis, seu haec forma

$$\frac{(U + Vu)dx + Vxdu}{x(U + Vu)} \text{ integrabilis.}$$

At est $(U + Vu)dx + Vxdu = \frac{1}{x^n}(Mdx + Ndy)$

et $x^n(U + Vu) = M + Nu$. Integrabilis ergo erit haec formula:

$$\frac{Mdx + Ndy}{x(M + Nu)} = \frac{Mdx + Ndy}{Mx + Ny} \text{ ob } ux = y.$$

Expositis igitur his duobus aequationum generibus, quae per idoneos multiplicatores integrabiles reddi possunt, videamus, ad quaenam alia genera eadem methodus extendi possit: ac primo quidem obseruo, omnes aequationes differentiales, quae aliis methodis integrari possunt, etiam hac methodo per idoneum multiplicatorem tractari posse, id quod in sequente problemate clarius explicabitur.

Problema 7.

48. Propofita aequatione differentiali $Mdx+Ndy=0$, fi inuenta fuerit eius integralis aequatio completa, affignare omnes multiplicationes, qui aequatorem differentialem reddant integrabilem.

Solutio.

Cum aequatio integralis completa inuoluat quantitatem conftantem arbitrariam C, quae in aequatione differentiali non ineft, vtcunque ea fit implicata, quaeratur eius valor per refolutionem aequationis, qui fit $C=V$, eritque V functio ipfarum x et y, quae infuper conftantes aequationis differentialis in fe complectetur. Tum ifta aequatio $C=V$ differentietur, ficque prodibit $0=dV$. Ac iam neceffe eft, vt dV diuiforem habeat ipfam formulam differentialem propofitam. Sit itaque $dV=L(Mdx+Ndy)$, eritque L multiplicator idoneus, qui aequationem differentialem propofitam reddit integrabilem. Deinde cum, denotante Z functionem quamcunque ipfius V, fit etiam formula $ZdV=LZ(Mdx+Ndy)$ integrabilis, expreffio LZ omnes multiplicatores includet, quibus aequatio differentialis propofita $Mdx+Ndy=0$ fit integrabilis.

Coroll. 1.

49. Quoties ergo aequationis differentialis $Mdx+Ndy=0$ integrale completum affignari poteft, toties non folum vnus, fed plane omnes multiplicatores definire licet, quibus ea aequatio integrabilis reddatur.

Coroll. 2.

50. Cum ergo aliis methodis plurium aequationum differentialium integralia completa sint inuenta, hinc methodus hactenus tradita, quae ad duo tantum aequationum genera adhuc est applicata, non mediocriter amplificari poterit.

Scholion.

51. Interim tamen, nisi ad specialissima exempla descendere velimus, aequationes differentiales, quarum integralia completa assignare licet, ad exiguum numerum reducuntur. Ac primo quidem occurrunt aequationes differentiales primi gradus in hac forma contentae

$$dx(\alpha+\beta x+\gamma y)+dy(\delta+\varepsilon x+\zeta y)=0$$

quae quia facile ad homogeneas reuocantur, etiam hac methodo per multiplicatores tractari poterunt. Deinde memoratu digna est haec forma $dy+Pydx+Qyydx=Rdx$, cuius si constet vnus valor singularis satisfaciens, ex eo integrale completum elici potest, ex quo his casibus multiplicatores idoneos assignare licebit. Tertio etiam perpendi merentur casus huius aequationis $dy+yydx=ax^m dx$, ab inuentore Riccatiana dictae, quibus ea ad separabilitatem reduci potest. Denique existunt casus huius aequationis $ydy+Pydx=Qdx$, qui cum sint integrabiles, ad multiplicatorum inuestigationem sunt accommodati. Hinc noua patefiet via ex data multiplicatorum forma eas aequationes inueniendi, quae per eos fiunt integrabiles, vnde fortasse haud spernenda analyseos incrementa haurire licebit.

Problema 8.

52. Propofita aequatione differentiali primi gradus:

$$(\alpha+\beta x+\gamma y)dx+(\delta+\varepsilon x+\zeta y)dy=0$$

inuenire multiplicatores, qui eam reddant integrabilem.

Solutio.

Reducatur haec aequatio ad homogeneitatem ponendo:

$x=t+f$ et $y=u+g$, vt prodeat

$$(\alpha+\beta f+\gamma g+\beta t+\gamma u)dt+(\delta+\varepsilon f+\zeta g+\varepsilon t+\zeta u)du=0$$

quae pofito $\alpha+\beta f+\gamma g=0$ et $\delta+\varepsilon f+\zeta g=0$, vnde quantitates f et g determinantur, vtique fit homogenea, fcilicet

$$(\beta t+\gamma u)dt+(\varepsilon t+\zeta u)du=0;$$

ideoque per multiplicatorem $\frac{1}{\beta tt+(\gamma+\varepsilon)tu+\zeta uu}$ integrabilis redditur. Hinc inuentis litteris f et g aequatio propofita integrabilis euadet, fi diuidatur per

$$\beta(x-f)^2+(\gamma+\varepsilon)(x-f)(y-g)+\zeta(y-g)^2,$$

feu per

$$\beta xx+(\gamma+\varepsilon)xy+\zeta yy-(2\beta f+\gamma g+\varepsilon g)x$$
$$-(2\zeta g+\gamma f+\varepsilon f)y+\beta ff+(\gamma+\varepsilon)fg+\zeta gg$$

Cum autem fit $f=\frac{\alpha\zeta-\gamma\delta}{\gamma\varepsilon-\beta\zeta}$, et $g=\frac{\beta\delta-\alpha\varepsilon}{\gamma\varepsilon-\beta\zeta}$,

prodibit diuifor quaefitus:

$$\beta xx+(\gamma+\varepsilon)xy+\zeta yy+\frac{\alpha\gamma\delta-\alpha\alpha\zeta+\alpha\delta\varepsilon-\beta\delta\delta}{\gamma\varepsilon-\beta\zeta}$$
$$\frac{-2\alpha\beta\zeta+\beta\gamma\delta-\beta\delta\varepsilon+\alpha\gamma\varepsilon+\alpha\varepsilon\varepsilon}{\gamma\varepsilon-\beta\zeta}x$$
$$\frac{-2\beta\delta\zeta+\alpha\varepsilon\zeta-\alpha\gamma\zeta+\gamma\delta\varepsilon+\gamma\gamma\delta}{\gamma\varepsilon-\beta\zeta}y.$$

In-

Inuento autem vno diuifore, feu multiplicatore, ex eo reperientur facile omnes poffibiles.

Coroll. 1.

53. Forma ergo diuiforis, per quem aequatio differentialis

$$(\alpha+\beta x+\gamma y)dx+(\delta+\varepsilon x+\zeta y)dy=0$$

redditur integrabilis, eft

$$\beta xx+(\gamma+\varepsilon)yx+\zeta yy+Ax+By+C$$

vbi conftantes A, B, C fupra funt definitae.

Coroll. 2.

54. Cum diuifor inuentus etiam fatisfaciat, fi per $\gamma\varepsilon-\beta\zeta$ multiplicetur, patet, cafu, quo $\beta\zeta=\gamma\varepsilon$, diuiforem fore:

$$(\alpha\varepsilon\varepsilon-\beta\delta\varepsilon+\beta\gamma\delta-\alpha\beta\zeta)x+(\gamma\gamma\delta-\alpha\gamma\zeta+\alpha\varepsilon\zeta-\beta\delta\zeta)y+\alpha\gamma\delta-\alpha\alpha\zeta+\alpha\delta\varepsilon-\beta\delta\delta$$

qui pofito $\beta=mf$; $\gamma=nf$; $\varepsilon=mg$; $\zeta=ng$, abit in

$$m(\alpha g-\delta f)(mg-nf)x+n(\alpha g-\delta f)(mg-nf)y+(\alpha g-\delta f)(\delta m-\alpha n)$$

Coroll. 3.

55. Quare fi aequatio propofita fuerit huiusmodi:

$$(\alpha+f(mx+ny))dx+(\delta+g(mx+ny))dy=0$$

ea reddetur integrabilis, fi diuidatur per

$$(mg-nf)(mx+ny)+\delta m-\alpha n$$

fiue per $mx+ny+\frac{\delta m-\alpha n}{mg-nf}$. At fi fuerit $mg-nf=0$, aequatio propofita iam ipfa eft integrabilis.

Prob-

Problema 9.

56. Proposita hac aequatione differentiali:

$$dy + Pydx + Qyydx + Rdx = 0$$

vbi P, Q et R sint functiones ipsius x tantum, si constet, huic aequationi satisfacere $y = v$, existente v functione ipsius x, inuenire multiplicatores, qui istam aequationem reddant integrabilem.

Solutio.

Cum aequationi satisfaciat valor $y = v$, erit

$$dv + Pvdx + Qvvdx + Rdx = 0;$$

si ergo ponatur $y = v + \frac{1}{z}$, habebitur

$$-\frac{dz}{zz} + \frac{Pdx}{z} + \frac{2Qvdx}{z} + \frac{Qdx}{zz} = 0$$

siue:

$$dz - (P + 2Qv)zdx - Qdx = 0$$

quae integrabilis redditur per multiplicatorem

$$e^{-\int(P + 2Qv)dx}.$$

Hic ergo multiplicator per zz multiplicatus conueniet aequationi propositae. Cum ergo sit $z = \frac{1}{y - v}$, multiplicator aequationem propositam integrabilem reddens erit:

$$\frac{1}{(y - v)^2} e^{-\int(P + 2Qv)dx}$$

Sit breuitatis gratia $e^{-\int(P + 2Qv)dx} = S$. Quia aequationis $dz - (P + 2Qv)zdx - Qdx = 0$ integrale est

$$Sz - \int QSdx = \text{Const.}$$

omnes multiplicatores quaesiti continebuntur in hac forma:

$$\frac{S}{(y - v)^2} \text{ funct. } \left(\frac{S}{y - v} - \int QSdx\right)$$

vbi

vbi per hypothesin v est functio cognita ipsius x, ideoque etiam $S = e^{-\int (P + 2Qv)dx}$

Coroll. 1.

57. Multiplicator ergo, qui primum se obtulit, est $\frac{S}{(y-v)^2}$, tum vero etiam multiplicator erit $\frac{S}{S(y-v)-(y-v)^2\int QSdx}$ qui etsi continet formulam integralem $\int QS\,dx$, saepe numero illo simplicior euadere potest.

Coroll. 2.

58. Si enim S est quantitas exponentialis, fieri potest, vt $\int QS\,dx$ huiusmodi formam S T induat, existente T functione algebraica, quo casu multiplicator erit

$$\frac{1}{y-v-(y-v)^2 T} = \frac{1}{(y-v)(1-Ty+Tv)}$$

ideoque algebraicus, quod in priori forma fieri nequit.

Coroll. 3.

59. Cum his duobus casibus multiplicator sit fractio, in cuius solum denominatorem variabilis y ingreditur, ibique vltra quadratum non ascendat, innumerabiles alii huiusmodi multiplicatores exhiberi possunt: Sit enim $\int QS\,dx = V$, et fractionis $\frac{S}{(y-v)^2}$ denominatorem multiplicare licebit per $A + B(\frac{S}{y-v} - V) + C(\frac{S}{y-v} - V)^2$, sicque erit generalior multiplicatoris forma:

$$\frac{S}{A(y-v)^2 + BS(y-v) - BV(y-v)^2 + CSS - 2CSV(y-v) + CVV(y-v)^2}$$

siue:

$$\frac{S}{(A - BV + CVV)y^2 - (2Av - BS - 2BVv + 2CSV + 2CVVv)y + Avv - BSv - BVvv + CSS + 2CSVv + CV^2v^2}.$$

Coroll. 4.

60. Quodsi ergo haec formula $\frac{dy+Pydx+Qyydx+Rdx}{Lyy+My+N}$ fuerit integrabilis, denominator ita debet esse comparatus, vt sit

$$SL=A-BV+CVV,\ SM=S(B-2CV)-2v(A-BV+CVV)$$

et $SN=CSS-Sv(B-2CV)+vv(A-BV+CVV)$ existente $dv+Pvdx+Qvvdx+Rdx=0$, $S=e^{-\int(P+2Qv)dx}$ et $V=\int QSdx$.

Problema 10.

61. Proposita aequatione differentiali praecedente:

$$dy+Pydx+Qyydx+Rdx=0$$

inuenire functiones L, M et N ipsius x, vt ea per formulam $Lyy+My+N$ diuisa fiat integrabilis.

Solutio.

Cum igitur integrabilis esse debeat haec formula:

$$\frac{dy+dx(Py+Qyy+R)}{Lyy+My+N}$$

per proprietatem generalem esse opportet, postquam per $(Lyy+My+N)^2$ multiplicauerimus:

$$-\frac{yydL}{dx}-\frac{ydM}{dx}-\frac{dN}{dx}=\begin{matrix}+QMyy-2RLy+NP\\ -PLyy+2QNy-RM\end{matrix}$$

Vnde pro determinatione functionum L, M et N has consequimur aequationes:

I. $dL=PLdx-QMdx$

II. $dM=2RLdx-2QNdx$

III. $dN=RMdx-PNdx$,

ex

ex quarum prima deducimus:

$$M = \frac{PL}{Q} - \frac{dL}{Q\,dx}$$

et ex secunda: $N = \frac{RL}{Q} - \frac{dM}{2Q\,dx}$,

qui valores pro M et N in tertia substituti, dant:

$$dN = \frac{P\,dM}{2Q} - \frac{R\,dL}{Q},$$

Cum autem sit, sumto differentiali dx constante,

$$dM = \frac{P\,dL + L\,dP}{Q} - \frac{PL\,dQ}{QQ} - \frac{ddL}{Q\,dx} + \frac{dQ\,dL}{QQ\,dx}, \text{ erit}$$

$$N = \frac{RL}{Q} - \frac{P\,dL}{2QQ\,dx} - \frac{L\,dP}{2QQ\,dx} + \frac{PL\,dQ}{2Q^3\,dx} + \frac{ddL}{2QQ\,dx^2} - \frac{dQ\,dL}{2Q^3\,dx^2}$$

$$\text{et } dN = \frac{PP\,dL}{2QQ} + \frac{PL\,dP}{2QQ} - \frac{PPL\,dQ}{2Q^3} - \frac{P\,ddL}{2QQ\,dx} + \frac{P\,dQ\,dL}{2Q^3\,dx} - \frac{R\,dL}{Q}$$

quod ergo illius differentiali debet aequari, vnde fit:

$$\begin{aligned} 0 = {} & QQ\,d^3L - 3Q\,dQ\,ddL - PPQQ\,dL\,dx^2 - 2QQ\,dP\,dL\,dx \\ & + 3\,dQ^2\,dL + 2PQ\,dQ\,dL\,dx - Q\,dL\,ddQ + 4Q^3R\,dL\,dx^2 \\ & - PQQL\,dP\,dx^2 + PPQL\,dQ\,dx^2 - QQL\,dx\,ddP \\ & \qquad + PQL\,dx\,ddQ \\ & + 3QL\,dP\,dQ\,dx - 3PL\,dQ^2\,dx + 2Q^3L\,dR\,dx^2 \\ & \qquad - 2Q^2RL\,dQ\,dx^2 \end{aligned}$$

Haec autem aequatio si per $\frac{L}{Q^4}$ multiplicetur, integrari poterit, eritque eius integralis

$$\text{Const.} = \frac{L\,ddL}{QQ} - \frac{L\,dL\,dQ}{Q^3} - \frac{dL^2}{2QQ} - \frac{PPLL\,dx^2}{2QQ} - \frac{LL\,dP\,dx}{QQ} + \frac{PLL\,dQ\,dx}{Q^3} + \frac{2RLL\,dx^2}{Q}$$

quae in hanc formam abit:

$$\begin{aligned} 2EQ^3\,dx^2 = {} & 2QL\,ddL - 2L\,dL\,dQ - Q\,dL^2 - PPQLL\,dx^2 \\ & - 2QLL\,dP\,dx + 2PLL\,dQ\,dx + 4QQRLL\,dx^2. \end{aligned}$$

Quodsi ponatur $L = zz$, aequatio induet hanc formam:

$$\begin{aligned} \frac{2EQ^3\,dx^2}{z^3} = {} & 4Q\,ddz - 4\,dQ\,dz - z(PPQ\,dx^2 + 2Q\,dP\,dx \\ & - 2P\,dQ\,dx - 4QQR\,dx^2). \end{aligned}$$

 Coroll. I.

Coroll. 1.

62. Quoties ergo per problema praecedens, valor ipsius L assignari potest, toties aequatio differentialis tertii ordinis hic inuenta, et ea secundi ordinis, ad quam illam reduxi, generaliter resolui poterit: quae resolutio, cum alias foret difficillima, probe est notanda.

Coroll. 2.

63. Scilicet si v fuerit eiusmodi functio ipsius x, quae loco y posita, satisfaciat aequationi $dy + Py\,dx + Qyy\,dx + R\,dx = 0$, capiatur $S = e^{-\int(P+2Qv)dx}$, statuaturque $V = \int QS\,dx$, quo facto erit pro nostra aequatione differentiali tertii ordinis $L = \frac{A - BV + CVV}{S}$, qui valor cum tres constantes arbitrarias complectatur, adeo erit eius aequationis differentiale completum.

Coroll. 3.

63. Si sit $P = 0$, $Q = 1$ et R functio quaecunque ipsius x, aequatio differentialis tertii gradus hanc accipiet formam:

$$0 = d^3L + 4R\,dL\,dx^2 + 2L\,dR\,dx^2$$

pro cuius ergo differentiali completo inueniendo, quaeratur primo functio ipsius x, quae sit $= v$, quae satisfaciat huic aequationi $dv + vv\,dx + R\,dx = 0$: tum ponatur $V = \int e^{-2\int v\,dx}\,dx$, eritque $L = (A - BV + CVV)\,e^{+2\int v\,dx}$.

Coroll. 4.

Coroll. 4.

64. Idem ergo integrale ſatisfaciet huic aequationi differentiali ſecundi gradus:

$$E\,dx^2 = 2\,L\,ddL - dL^2 + 4\,R\,L\,L\,dx^2$$

et, poſito $L = zz$, etiam huic:

$$\frac{E\,dx^2}{z^3} = ddz + R\,z\,dx^2$$

pro qua itaque eſt $z = e^{+\int v\,dx}\sqrt{(A - BV + CVV)}$.

Scholion.

65. Omnino animaduerti meretur haec integratio, quippe quae ex aliis principiis vix quidem praeſtari poteſt. Hinc autem adipiſcimur integrationem completam ſequentis aequationis differentio-differentialis ſatis late patentis:

$$ddz + S\,dx\,dz + T\,z\,dx^2 = \frac{E\,dx^2}{z^3}\,e^{-2\int S\,dx}$$

Primo nempe quaeratur valor ipſius v ex hac aequatione differentiali primi gradus:

$$dv + vv\,dx + S\,v\,dx + T\,dx = 0$$

quo inuento ponatur breuitatis ergo $V = \int e^{-2\int v\,dx - \int S\,dx}\,dx$ eritque:

$$z = e^{\int v\,dx}\sqrt{(A + BV + CVV)},$$

ſi modo conſtantes arbitrariae A, B. C ita accipiantur, vt ſit $AC - \frac{1}{4}BB = E$, ſicque adhuc duae conſtantes arbitrio noſtro relinquuntur, vti natura integrationis completae poſtulat.

Exemplum 1.

66. *Proposita sit haec aequatio differentialis*

$$dy + y\,dx + yy\,dx - \frac{dx}{x} = 0,$$

cuius multiplicatores, qui eam reddant integrabilem, investigari oporteat.

Erit ergo, Problema 9. huc transferendo, $P = 1$, $Q = 1$ et $R = -\frac{1}{x}$, et quia aequationi satisfacit valor $y = \frac{1}{x}$, erit $v = \frac{1}{x}$. Quare fiet $S = e^{-\int(1+\frac{2}{x})dx} = \frac{1}{xx}e^{-x}$ et multiplicator, qui primum se offert, habebitur $= e^{-x}\frac{1}{(xy-1)^2}$. Hunc autem porro multiplicare licet per functionem quamcunque huius formae $e^{-x}\frac{1}{x(xy-1)} - \int e^{-x}\frac{dx}{xx}$; cum vero haec forma integrari nequeat, alii multiplicatores idonei assignari nequeunt. Ob primum ergo integrabilis est haec forma:

$$e^{-x}\frac{1}{(xy-1)^2}\left(dy + y\,dx + yy\,dx - \frac{dx}{x}\right)$$

cuius, si x capitur constans, integrale est.

$$\frac{-e^{-x}}{x(xy-1)} + X$$

quae differentiata, posito y constante, praebet

$$\frac{e^{-x}dx(xxy + 2xy - x - 1)}{xx(xy-1)^2} + dX$$

quod aequari debet alteri membro $\frac{e^{-x}}{(xy-1)^2}\left(y\,dx + yy\,dx - \frac{dx}{x}\right)$ vnde fit $dX = \frac{e^{-x}dx}{xx(xy-1)^2}(xxyy - 2xy + 1) = e^{-x}\frac{dx}{xx}$;

sicque

ſicque integrale completum noſtrae aequationis eſt

$$\frac{-e^{-x}}{x(xy-1)}+\int e^{-x}\frac{dx}{xx}=\text{Conſt.}$$

Exemplum 2.

67. Inuenire multiplicatores idoneos, qui reddant hanc aequationem integrabilem:

$$dy+yy\,dx-\frac{a\,dx}{(\alpha+\beta x+\gamma xx)^2}=0.$$

Caſus ſingularis huic aequationi ſatisfaciens eſt

$$y=\frac{k+\gamma x}{\alpha+\beta x+\gamma xx}=v$$

exiſtente $k=\frac{1}{2}\beta\pm\sqrt{(\frac{1}{4}\beta\beta-\alpha\gamma+a)}$.

Cum nunc ſit $P=0$, et $Q=1$, erit

$$S=e^{-\int\frac{2k\,dx+2\gamma x\,dx}{\alpha+\beta x+\gamma xx}}$$

vel poſito breuitatis gratia $\pm\sqrt{(\frac{1}{4}\beta\beta-\alpha\gamma+a)}=\frac{1}{2}n$ erit

$$S=\frac{1}{\alpha+\beta x+\gamma xx}\,e^{-\int\frac{n\,dx}{\alpha+\beta x+\gamma xx}}$$

et $\int S\,dx=-\frac{1}{n}\,e^{-\int\frac{n\,dx}{\alpha+\beta x+\gamma xx}}$

Multiplicator ergo primum inuentus eſt

$$e^{-\int\frac{n\,dx}{\alpha+\beta x+\gamma xx}}\cdot\frac{\alpha+\beta x+\gamma xx}{((\alpha+\beta x+\gamma xx)y-k-\gamma x)^2}$$

qui porro duci poteſt in functionem quamcunque huius quantitatis

$$e^{-\int\frac{n\,dx}{\alpha+\beta x+\gamma xx}}\left(\frac{1}{(\alpha+\beta x+\gamma xx)y-k-\gamma x}+\frac{1}{n}\right).$$

Ducatur ergo in

$$e^{\int\frac{n\,dx}{\alpha+\beta x+\gamma xx}}\cdot\frac{(\alpha+\beta x+\gamma xx)y-k-\gamma x}{(\alpha+\beta x+\gamma xx)y+n-k-\gamma x}$$

ac

ac prodibit multiplicator algebraicus:

$$\frac{\alpha+\beta x+\gamma xx}{((\alpha+\beta x+\gamma xx)y-k-\gamma x)((\alpha+\beta x+\gamma xx)y+n-k-\gamma x)}$$

qui reducitur ad hanc formam:

$$\frac{1}{(\alpha+\beta x+\gamma xx)\left(y-\frac{-2\gamma x-\beta-\sqrt{(\beta\beta-4\alpha\gamma+4a)}}{2(\alpha+\beta x+\gamma xx)}\right)\left(y-\frac{-2\gamma x-\beta+\sqrt{(\beta\beta-4\alpha\gamma+4a)}}{2(\alpha+\beta x+\gamma xx)}\right)}$$

Aequationis autem integrale completum eſt

$$e^{-\int\frac{n\,dx}{\alpha+\beta x+\gamma xx}}\;\frac{(\alpha+\beta x+\gamma xx)y+n-k-\gamma x}{(\alpha+\beta x+\gamma xx)y-k-\gamma x}=\text{Conſt.}$$

exiſtente $n=\sqrt{(\beta\beta-4a\gamma+4a)}$ et $k=\frac{\beta+n}{2}$.

Ex quo aequatio integralis completa erit

$$e^{-\int\frac{n\,dx}{\alpha+\beta x+\gamma xx}}\cdot\frac{2(\alpha+\beta x+\gamma xx)+n-\beta-2\gamma x}{2(\alpha+\beta x+\gamma xx)y-n-\beta-2\gamma x}=\text{Conſt.}$$

cuius indoles eſt manifeſta, dummodo $n=\sqrt{(\beta\beta-4\alpha\gamma+4a)}$ ſit numerus realis.

Quodſi autem valor ipſius n ſit imaginarius, puta $n=m\sqrt{-1}$, ob $e^{p\sqrt{-1}}=\text{coſ.}\,p+\sqrt{-1}\,\text{ſin.}\,p$, aequatio integralis ita ad realitatem perduci poteſt. Sit $-m\int\frac{dx}{\alpha+\beta x+\gamma xx}=p$, et $2(\alpha+\beta x+\gamma xx)y-\beta-2\gamma x=q$, eritque ea:

$$(\text{coſ.}\,p+\sqrt{-1}\,\text{ſin.}\,p).\frac{q+m\sqrt{-1}}{q-m\sqrt{-1}}=\text{Conſt.}=A+B\sqrt{-1}$$

hinc fit:

$$q\,\text{coſ.}\,p-m\,\text{ſin.}\,p+(m\,\text{coſ.}\,p+q\,\text{ſin.}\,p)\sqrt{-1}=Aq+Bm+(Bq-Am)\sqrt{-1}$$

aequentur ſeorſim membra realia et imaginaria:

$$q\,\text{coſ}\,p-m\,\text{ſin.}\,p=Aq+Bm;\quad m\,\text{coſ}\,p+q\,\text{ſin.}\,p=Bq-Am$$

quae duae aequationes congruunt, ſi capiatur $AA+BB=1$.

Sit

Sit itaque constans arbitraria $A = \cos.\theta$, vt sit $B = \sin.\theta$ et casu, quo $\sqrt{(\beta\beta - 4\alpha\gamma + 4a)} = m\sqrt{-1}$, aequatio realis erit

$$q\cos.p - m\sin.p = q\cos.\theta + m\sin.\theta \text{ seu } q = \frac{m(\sin.p + \sin.\theta)}{\cos.p - \cos.\theta} = m\cot.\frac{\theta - p}{2}$$

Quare aequationis differentialis

$$dy + yy\,dx + \frac{(mm + \beta\beta - 4\alpha\gamma)\,dx}{4(\alpha + \beta x + \gamma xx)^2} = 0$$

posito $p = \int \frac{-m\,dx}{\alpha + \beta x + \gamma xx}$, aequatio integralis completa est

$$2(\alpha + \beta x + \gamma xx)y = \beta + 2\gamma y + m\cot.\frac{\theta - p}{2}$$

seu $$y = \frac{\frac{1}{2}\beta + \gamma x + \frac{1}{2}m\cot.\frac{\theta - p}{2}}{\alpha + \beta x + \gamma xx}$$

Vel sit $\theta = 180° - \zeta$, et habebitur $y = \frac{\frac{1}{2}\beta + \gamma x + \frac{1}{2}m\,\mathrm{tang}.\frac{\zeta + p}{2}}{\alpha + \beta x + \gamma xx}$

Hoc autem casu notandum est, integrale speciale, ex quo haec omnia deduximus, fieri imaginarium, quo tamen non obstante inde integrale completum in forma reali exhibere licuit.

Exemplum 3.

68. *Proposita aequatione Riccatiana* $dy + yy\,dx - ax^m\,dx = 0$, *pro casibus exponentis* m, *quibus eam separare licet, inuenire multiplicatores idoneos.*

Sit $y = v$ valor aequationi satisfaciens, et cum sit $P = 0$, $Q = 1$, et $R = -ax^m$, erit primus multiplicator, aequationem integrabilem reddens,

$$e^{-2\int v\,dx} \frac{1}{(y - v)^2}$$

per quem si aequatio multiplicetur, cum integrale completum sit

$$e^{-2\int v\,dx}\frac{1}{y-v}-\int e^{-2\int v\,dx}dx = \text{Const.}$$

Quare si Z denotet functionem quamcunque huius quantitatis, omnes multiplicatores continebuntur in hac forma:

$$e^{-2\int v\,dx}\frac{Z}{(y-v)^2}.$$

Hinc si ponatur $\int e^{-2\int v\,dx}dx = V$, omnes multiplicatores in hac forma $\frac{1}{Lyy+My+N}$ contenti obtinebuntur, si capiatur:

$$L = e^{2\int v\,dx}(A-BV+CVV)$$
$$M = B-2CV-2ve^{2\int v\,dx}(A-BV+CVV)$$
$$N = Ce^{-2\int v\,dx}-v(B-2CV)+vve^{2\int v\,dx}(A-BV+CVV).$$

Verum hic valor ipsius L simul est integrale completum huius aequationis differentialis tertii gradus:

$$0 = d^3L - 4ax^m dL\,dx^2 - 2maLx^{m-1}dx^2$$

hincque etiam huius secundi gradus:

$$E\,dx^2 = 2L\,ddL - dL^2 - 4aLLx^m dx^2$$

existente $E = 4AC-BB$.

Scholion.

69. Re attentius perpensa aequationem differentialem tertii ordinis etiam methodo directa resolui, eiusque integrale completum idem, quod hic est assignatum, elici posse deprehendi. Sit enim proposita haec aequatio:

$$d^3L + 4R\,dL\,dx^2 + 2L\,dR\,dx^2 = 0$$

vbi

vbi R sit functio quaecunque ipsius x, sumto differentiali dx constante. Iam quaero functionem ipsius x, per quam ista aequatio multiplicata euadat integrabilis. Sit S ista functio, et aequationis

$$S\,d^3L + 4SR\,dL\,dx^2 + 2SL\,dR\,dx^2 = 0$$

integrale erit

$$S\,ddL - dS\,dL + L(ddS + 4SR\,dx^2) = 2C\,dx^2$$

dummodo sit

$$d^3S + 2S\,dR\,dx^2 + 4R\,dS\,dx^2 = 0.$$

Sufficit scilicet quemuis valorem particulariter satisfacientem sumsisse. At haec aequatio, per S multiplicata, neglecta constante, dat integrale:

$$S\,ddS - \tfrac{1}{2}dS^2 + 2SSR\,dx^2 = 0.$$

Ponatur $S = e^{2\int v\,dx}$, eritque

$$2\,dv + 2vv\,dx + 2R\,dx = 0$$

vnde negotium huc redit, vt pro v saltem valor particularis inuestigetur, qui satisfaciat huic aequationi differentiali primi gradus: $dv + vv\,dx + R\,dx = 0$, quem igitur tanquam concessum assumo. Hinc nostra aequatio semel integrata erit, ob $S = e^{2\int v\,dx}$,

$$ddL - 2v\,dx\,dL + L(2dv\,dx + 4vv\,dx^2 + 4R\,dx^2) = 2Ce^{-2\int v\,dx}dx^2$$

Cum igitur, ob $R\,dx = -dv - vv\,dx$, habeamus

$$ddL - 2v\,dx\,dL - 2L\,dx\,dv = 2Ce^{-2\int v\,dx}dx^2$$

eius integrale manifesto est:

$$dL - 2Lv\,dx = B\,dx + 2C\,dx\int e^{-2\int v\,dx}dx$$

et per $e^{-2\int v\,dx}$, denuo multiplicando integrale, prodibit

$$e^{-2\int v dx}L = A + B\int e^{-2v dx}\,dx + 2C\int e^{-2\int v dx}dx\int e^{-2\int v dx}dx$$

Quare si breuitatis gratia ponatur $\int e^{-2\int v\,dx}dx = V$, habebimus

$$L = e^{2\int v\,dx}(A + BV + CVV)$$

prorsus vti ante inuenimus.

Problema 2.

70. Proposita aequatione Riccatiana $dy + yydx = ax^m dx$, inuenire eius integralia particularia, casibus, quibus ea separabilis existit.

Solutio.

Ponendo $a = cc$, et $m = -4n$, tribuatur aequationi ista forma:

$$dy + yydx - ccx^{-4n}dx = 0.$$

Cum enim quaestio circa integralia particularia versetur, nihil interest, vtrum ea sint realia, nec ne. Quo autem facilius, et vna quasi operatione, hos casus, quibus y per functionem ipsius x exprimere licet, eliciamus: statuamus $y = cx^{-2n} + \frac{dz}{zdx}$, et sumto dx constante, nanciscemur hanc aequationem differentialem secundi gradus:

$$-2ncx^{-2n-1}dx + \frac{ddz}{zdx} + \frac{2cx^{-2n}dz}{z} = 0, \text{ seu}$$

$$\frac{ddz}{dx^2} + \frac{2c\,dz}{x^{2n}dx} - \frac{2ncz}{x^{2n+1}} = 0$$

cuius valor fingatur:

$$z = Ax^n + Bx^{3n-1} + Cx^{5n-2} + Dx^{7n-3} + Ex^{9n-4} + \text{etc.}$$

quo

quo debite ſubſtituto obtinebimus:

$$0 = n(n-1)Ax^{n-2} + (3n-1)(3n-2)Bx^{3n-3} + (5n-2)(5n-3)Cx^{5n-4} + \text{etc.}$$
$$+ 2ncAx^{-n-1} + 2(3n-1)cB + 2(5n-2)cC + 2(7n-3)cD$$
$$- 2ncA - 2ncB - 2ncC - 2ncD$$

vnde coefficientes ficti ita determinantur:

$$2(2n-1)cB + n(n-1)A = 0; \quad B = \frac{-n(n-1)A}{2(2n-1)c}$$
$$2(4n-2)cC + (3n-1)(3n-2)B = 0; \quad C = \frac{-(3n-1)(3n-2)B}{4(2n-1)c}$$
$$2(6n-3)cD + (5n-2)(5n-3)C = 0; \quad D = \frac{-(5n-2)(5n-3)C}{6(2n-1)c}$$

Statim igitur atque vnus coefficiens euaneſcit, ſequentes ſimul omnes euaneſcunt, id quod euenit his caſibus:

$n=0$; $n=\frac{1}{3}$; $n=\frac{2}{5}$; $n=\frac{3}{7}$; etc.
$n=1$; $n=\frac{2}{3}$; $n=\frac{3}{5}$; $n=\frac{4}{7}$; etc.

Denotante igitur i numerum integrum quemcunque, quoties fuerit $n=\frac{i}{2i\pm 1}$, toties reſolutio aequationis exhiberi poteſt. Erit enim $y = cx^{-2n} + \frac{dz}{z\,dx}$, exiſtente

$$z = Ax^n + Bx^{3n-1} + Cx^{5n-2} + Dx^{7n-3} + Ex^{9n-4} + \text{etc.}$$

Proueniet ergo hic valor particularis ipſius y:

$$y = cx^{-n2} + \frac{nAx^{n-1} + (3n-1)Bx^{3n-2} + (5n-2)Cx^{5n-3}}{Ax^n + Bx^{3n-1} + Cx^{5n-2}} + \text{etc.}$$

Coroll. 1.

71. Quodſi ergo iſte valor particularis ipſius y vocetur $=v$, erit aequationis propoſitae multiplicator idoneus $= e^{-2\int v\,dx} \cdot \frac{1}{(y-v)^2}$. Ac ſi ponatur $\int e^{-2\int v\,dx}dx$

 $=V$,

$= V$, sumtis $A = 0$, et $C = 0$, erit alius factor simplicior

$$\frac{1}{e^{2\int v\,dx}Vyy-(1+2ve^{2\int v\,dx}V)y+v+vve^{2\int v\,dx}V}.$$

Coroll. 2.

72. At est $\int v\,dx = \frac{-c}{(2n-1)x^{2n-1}} + l(Ax^n + Bx^{3n-1} + Cx^{5n-2} + \text{etc.}$

vnde fit $e^{-2\int v dx} = \frac{2c}{e^{(2n-1)x^{2n-1}}} \cdot \frac{1}{(Ax^n + Bx^{3n-1} + Cx^{5n-2} + \text{etc.})^2}$

ex quo porro inueniri potest valor ipsius $V = \int e^{-2\int v dx}dx$ qui si fuerit huiusmodi $e^{-2\int v\,dx}T$, existente T functione algebraica, erit superior multiplicator algebraicus.

Coroll. 3.

73. Inuento valore v, seu integrali particulari aequationis propositae, inde statim habebitur integrale completum eiusdem, quippe quod erit:

$$\frac{e^{-2\int v\,dx}}{y-v} - \int e^{-2\int v\,dx}dx = \text{Const.}$$

Casus 1. quo $n = 0$.

74. Pro hac ergo aequatione $dy + yydx = ccdx$, ob $B = 0$, $C = 0$ etc. erit valor particularis $y = c$; Quare

Quare posito $v = c$, erit $e^{-2\int v\,dx} = e^{-2cx}$ et $V = \int e^{-2\int v dx} dx = -\frac{1}{2c} e^{-2cx}$; vnde integrale completum est

$$\frac{e^{-2cx}}{y-c} + \frac{1}{2c} e^{-2cx} = \text{Const.}$$

seu $$\frac{e^{-2cx}(y+c)}{y-c} = \text{Const.}$$

Porro, ob $e^{2\int v\,dx} V = -\frac{1}{2c}$, et $v = c$, erit multiplicator algebraicus:

$$\frac{1}{-\frac{1}{2c}yy + \frac{1}{2}c}$$

qui reducitur ad $\frac{1}{yy - cc}$, vti per se est perspicuum.

Casus 2. quo $n = 1$.

75. Pro hac ergo aequatione $dy + yydx = \frac{ccdx}{x^4}$ ob $B = 0$, $C = 0$ etc. erit valor particularis $y = \frac{c}{xx} + \frac{1}{x}$. Quare posito $v = \frac{c}{xx} + \frac{1}{x}$, erit $e^{-2\int v\,dx} = \frac{e^{\frac{2c}{x}}}{xx}$ et $V = -\frac{1}{2c} e^{\frac{2c}{x}}$. Hinc integrale completum est

$$\frac{e^{\frac{2c}{x}}}{xxy - x - c} + \frac{1}{2c} e^{\frac{2c}{x}} = \text{Const.}$$

seu $e^{\frac{2c}{x}}, \frac{xxy - x + c}{xxy - x - c} = \text{Const.}$

Porro, ob $e^{2\int v\,dx} V = -\frac{xx}{2c}$, et $v = \frac{x+c}{xx}$, habebitur multiplicator algebraicus:

$$\frac{1}{xxyy - 2xy + 1 - \frac{cc}{xx}} = \frac{1}{(xy-1)^2 - \frac{cc}{xx}}$$

siue

siue aequatio proposita $dy+yydx-\frac{ccdx}{x^4}=0$ fit integrabilis, si diuidatur per $(xy-1)^2-\frac{cc}{xx}$

Casus 3. quo $n=\frac{1}{3}$.

76. Pro hac ergo aequatione $dy+yydx-ccx^{-\frac{4}{3}}dx=0$ est $B=-\frac{A}{3c}$, $C=0$, etc. vnde integrale particulare

$$y=cx^{-\frac{2}{3}}+\frac{cx^{-\frac{2}{3}}}{3cx^{\frac{1}{3}}-1}=\frac{3ccx^{1-\frac{1}{3}}}{3cx^{\frac{1}{3}}-1}=v$$

et $e^{-2\int v dx}=e^{-6cx^{\frac{1}{3}}}\frac{\text{Const.}}{(x^{\frac{1}{3}}-\frac{1}{3c})^2}=e^{-6cx^{\frac{1}{3}}}\frac{1}{(3cx^{\frac{1}{3}}-1)^2}$

hincque $V=\int e^{-6cx^{\frac{1}{3}}}\frac{dx}{(3cx^{\frac{1}{3}}-1)^2}=-e^{-6cx^{\frac{1}{3}}}\frac{3cx^{\frac{2}{3}}+1}{18c^3(3cx^{\frac{1}{3}}-1)}$

Quare integrale completum est

$$\frac{e^{-6cx^{\frac{1}{3}}}}{(3cx^{\frac{1}{3}}-1)^2y-3ccx^{-\frac{1}{3}}(3cx^{\frac{1}{3}}-1)}+\frac{e^{-6cx^{\frac{1}{3}}}(3cx^{\frac{1}{3}}+1)}{18c^3(3cx^{\frac{1}{3}}-1)}=\text{Const.}$$

siue $e^{-6cx^{\frac{1}{3}}}\frac{y(1+3cx^{\frac{1}{3}})+3ccx^{-\frac{1}{3}}}{y(1-3c3^{\frac{1}{3}})+3ccx^{-\frac{1}{3}}}=\text{Const.}$

Tum, ob $e^{2\int v dx}V=\frac{1-9ccx^{\frac{2}{3}}}{18c^3}$, prodibit diuisor aequationem integrabilem reddens;

$$(y+3ccx^{-\frac{1}{3}})^2-9ccx^{\frac{2}{3}}yy$$

Casus

Casus 4. quo $n = \frac{2}{3}$.

77. Pro hac ergo aequatione $dy + yydx - ccx^{-\frac{8}{3}}dx = 0$ est $B = +\frac{A}{3c}$, $C = 0$ etc. vnde integrale particulare:

$$y = cx^{-\frac{4}{3}} + \frac{2cx^{-\frac{1}{3}} + 1}{3cx^{\frac{2}{3}} + x} = \frac{3ccx^{-\frac{2}{3}} + 3cx^{-\frac{1}{3}} + 1}{3cx^{\frac{2}{3}} + x} = v$$

et $e^{-2\int v dx} = e^{6cx^{-\frac{1}{3}}} \cdot \frac{1}{(3cx^{\frac{2}{3}} + x)^2}$: ex quo porro elicitur:

$$V = \int \frac{e^{6cx^{-\frac{1}{3}}} dx}{(3cx^{\frac{2}{3}} + x)^2} = \frac{-e^{6cx^{-\frac{1}{3}}}(3cx^{\frac{2}{3}} - x)}{18c^3(3cx^{\frac{2}{3}} + x)}.$$

Quare integrale completum erit:

$$e^{6cx^{-\frac{1}{3}}} \cdot \frac{(x - 3cx^{\frac{2}{3}})y - 1 + 3cx^{-\frac{1}{3}} - 3ccx^{-\frac{2}{3}}}{(x + 3cx^{\frac{2}{3}})y - 1 - 3cx^{-\frac{1}{3}} - 3ccx^{-\frac{2}{3}}} = \text{Const.}$$

Tum ob $e^{2\int v dx} V = \frac{xx - 9ccx^{\frac{4}{3}}}{18c^3}$ prodit diuisor algebraicus aequationem propositam integrabilem reddens:

$$((x + 3cx^{\frac{2}{3}})y - 1 - 3cx^{-\frac{1}{3}} - 3ccx^{-\frac{2}{3}})((x - 3cx^{\frac{2}{3}})y - 1 + 3cx^{-\frac{1}{3}} - 3ccx^{-\frac{2}{3}}).$$

Casus 5. quo $n = \frac{2}{5}$.

78. Pro hac ergo aequatione $dy + yydx - ccx^{-\frac{8}{5}}dx = 0$ erit $B = -\frac{3A}{5c}$; $C = -\frac{B}{5c} = +\frac{3A}{25cc}$; $D = 0$ etc. ideo-

que integrale particulare:

$$y = cx^{-\frac{4}{5}} + \frac{\frac{2}{5}x^{-\frac{2}{5}} - \frac{1}{5}\cdot\frac{3}{5c}x^{-\frac{4}{5}}}{x^{\frac{2}{5}} - \frac{3}{5c}x^{\frac{1}{5}} + \frac{3}{25cc}} = cx^{-\frac{4}{5}} + \frac{10ccx^{-\frac{3}{5}} - 3cx^{-\frac{4}{5}}}{25ccx^{\frac{2}{5}} - 15cx^{\frac{1}{5}} + 3}$$

seu $y = \frac{25c^3x^{-\frac{2}{5}} - 5ccx^{-\frac{3}{5}}}{25ccx^{\frac{2}{5}} - 15cx^{\frac{1}{5}} + 3} = v.$ Vnde integrale completum oritur:

$$e^{-10cx^{\frac{1}{5}}}\cdot\frac{(3+15cx^{\frac{1}{5}}+25ccx^{\frac{2}{5}})y+5ccx^{-\frac{2}{5}}+25c^3x^{-\frac{2}{5}}}{(3-15cx^{\frac{1}{5}}+25ccx^{\frac{2}{5}})y+5ccx^{-\frac{3}{5}}-25c^3x^{-\frac{2}{5}}} = \text{Const.}$$

Et si huius fractionis ponatur

numerator $(3+15cx^{\frac{1}{5}}+25ccx^{\frac{2}{5}})y+5ccx^{-\frac{3}{5}}+25c^3x^{-\frac{2}{5}} = P$, et

denominator $(3-15cx^{\frac{1}{5}}+25ccx^{\frac{2}{5}})y+5ccx^{-\frac{2}{5}}-25c^3x^{-\frac{2}{5}} = Q$,

erit diuisor, aequationem propositam integrabilem reddens $= PQ$.

Casus 6. quo $n = \frac{3}{5}$.

79. Pro hac ergo aequatione $dy + yydx - ccx^{-\frac{12}{5}}dx = 0$, erit $B = \frac{3A}{5c}$; et $C = \frac{B}{5c} = \frac{3A}{25cc}$; $D = 0$ etc. hincque integrale particulare prodit:

$$y = cx^{-\frac{6}{5}} + \frac{15ccx^{-\frac{2}{5}} + 12cx^{-\frac{1}{5}} + 3}{25ccx^{\frac{3}{5}} + 15cx^{\frac{4}{5}} + 3x} \text{ seu}$$

$$y = \frac{25c^3x^{-\frac{2}{5}} + 30ccx^{-\frac{3}{5}} + 15cx^{-\frac{1}{5}} + 3}{25ccx^{\frac{3}{5}} + 15cx^{\frac{4}{5}} + 3x} = v$$

vnde

vnde integrale completum obtinetur:

$$e^{10cx^{\frac{-1}{5}}}\cdot\frac{(3x-15cx^{\frac{4}{5}}+25ccx^{\frac{3}{5}})y-3+15cx^{\frac{-1}{5}}-30ccx^{\frac{-2}{5}}+25c^3x^{\frac{-3}{5}}}{(3x+15cx^{\frac{4}{5}}+25ccx^{\frac{3}{5}})y-3-15cx^{\frac{-1}{5}}-30ccx^{\frac{-2}{5}}-25c^3x^{\frac{-3}{5}}}=\text{Const.}$$

Ac neglecto factore exponentiali $e^{10cx^{\frac{-1}{5}}}$, productum ex numeratore et denominatore praebebit diuisorem, per quem aequatio proposita diuisa euadit integrabilis.

Problema 12.

80. Denotante i numerum quemcunque integrum, exhibere resolutionem huius aequationis:

$$dy+yydx-ccx^{\frac{-4i}{2i+1}}dx=0.$$

Solutio.

Cum igitur sit $n=\frac{i}{2i+1}$, reperietur

$$B=-\frac{(i+1)i}{2(2i+1)c}A$$

$$C=+\frac{(i+2)(i+1)i(i-1)}{2\cdot4(2i+1)^2c^2}A$$

$$D=-\frac{(i+3)(i+2)(i+1)i(i-1)(i-2)}{2\cdot4\cdot6(2i+1)^3c^3}A$$

$$E=+\frac{(i+4)(i+3)(i+2)(i+1)(i-1)(i-2)(i-3)}{2\cdot4\cdot6\cdot8(2i+1)^4c^4}A$$

etc.

tum vero integrale particulare erit:

$$y=cx^{\frac{-2i}{2i+1}}+\frac{\frac{i}{2i+1}Ax^{\frac{-i-1}{2i+1}}+\frac{i-1}{2i+1}Bx^{\frac{-i-2}{2i+1}}+\frac{i-2}{2i+1}Cx^{\frac{-i-3}{2i+1}}+\frac{i-3}{2i+1}Dx^{\frac{-i-4}{2i+1}}+\text{etc.}}{Ax^{\frac{i}{2i+1}}+Bx^{\frac{i-1}{2i+1}}+Cx^{\frac{i-2}{2i+1}}+Dx^{\frac{i-3}{2i+1}}+\text{etc.}}$$

quod vt ad eundem denominatorem reducatur, ſtatuamus:

$$\mathfrak{A} = cA$$
$$\mathfrak{B} = -\frac{i(i-1)}{2(2i+1)}A$$
$$\mathfrak{C} = +\frac{(i+1)i(i-1)(i-2)}{2\cdot 4(2i+1)^2 c}A$$
$$\mathfrak{D} = -\frac{(i+2)(i+1)i(i-1)(i-2)(i-3)}{2\cdot 4\cdot 6(2i+1)^3 c^2}A$$
etc.

vnde fiet:

$$y = \frac{\mathfrak{A}x^{\frac{-i}{2i+1}} + \mathfrak{B}x^{\frac{-i-1}{2i+1}} + \mathfrak{C}x^{\frac{-i-2}{2i+1}} + \mathfrak{D}x^{\frac{-i-3}{2i+1}} + \text{etc.}}{\mathfrak{A}x^{\frac{i}{2i+1}} + \mathfrak{B}x^{\frac{i-1}{2i+1}} + \mathfrak{C}x^{\frac{i-2}{2i+1}} + \mathfrak{D}x^{\frac{i-3}{2i+1}} + \text{etc.}}$$

Ponamus porro breuitatis gratia:

$$Ax^{\frac{i}{2i+1}} + Bx^{\frac{i-1}{2i+1}} + Cx^{\frac{i-2}{2i+1}} + Dx^{\frac{i-3}{2i+1}} + \text{etc.} = P$$
$$Ax^{\frac{i}{2i+1}} - Bx^{\frac{i-1}{2i+1}} + Cx^{\frac{i-2}{2i+1}} - Dx^{\frac{i-3}{2i+1}} + \text{etc.} = Q$$
$$\mathfrak{A}x^{\frac{-i}{2i+1}} + \mathfrak{B}x^{\frac{-i-1}{2i+1}} + \mathfrak{C}x^{\frac{-i-2}{2i+1}} + \mathfrak{D}x^{\frac{-i-3}{2i+1}} + \text{etc.} = \mathfrak{P}$$
$$-\mathfrak{A}x^{\frac{-i}{2i+1}} + \mathfrak{B}x^{\frac{-i-1}{2i+1}} - \mathfrak{C}x^{\frac{-i-2}{2i+1}} + \mathfrak{D}x^{\frac{-i-3}{2i+1}} - \text{etc.} = \mathfrak{Q}$$

atque integrale completum erit:

$$e^{-2(2i+1)cx^{\frac{+1}{2i+1}}}\,\frac{Qy-\mathfrak{Q}}{Py-\mathfrak{P}} = \text{Conſt.}$$

Tum vero diuiſor, aequationem propoſitam reddens integrabilem, erit $=(Py-\mathfrak{P})(Qy-\mathfrak{Q})$.

Coroll. 1.

81. Quodſi ergo in aequatione $dy + yydx + ax^{\frac{-4i}{2i+1}}dx = 0$ coefficiens a fuerit quantitas negatiua, vt poſito $a =$

$\alpha = -cc$, ſit c quantitas realis, integrale completum hic inuentum formam habet realem, et quouis caſu facile exhiberi poteſt, pariter ac diuiſor, qui aequationem integrabilem reddit.

Coroll. 2.

82. At ſi α fuerit quantitas poſitiua, puta $\alpha = aa$, vt habeatur haec aequatio: $dy + yy\,dx + aa\,x^{\frac{-4i}{2i+1}}\,dx = 0$, erit $c = a\sqrt{-1}$, et coefficientes B, D, F etc. et $\mathfrak{A}$, $\mathfrak{C}$, $\mathfrak{E}$ etc. fient imaginarii; vnde valores particulares $y = \frac{\mathfrak{P}}{P}$ et $y = \frac{\mathfrak{Q}}{Q}$ prodibunt imaginarii.

Coroll. 3.

83. Hoc tamen caſu, quo $c = a\sqrt{-1}$ et $cc = -aa$, fient $P + Q$ et $\mathfrak{P} + \mathfrak{Q}$ quantitates reales, at $P - Q$ et $\mathfrak{P} - \mathfrak{Q}$ imaginariae. Quodſi ergo ponatur

$$P + Q = 2R;\quad P - Q = 2S\sqrt{-1};\quad \mathfrak{P} + \mathfrak{Q} = 2\mathfrak{R}$$
$$\text{et } \mathfrak{P} - \mathfrak{Q} = 2\mathfrak{S}\sqrt{-1}$$

erunt R, S, $\mathfrak{R}$ et $\mathfrak{S}$ quantitates reales, et ob

$$P = R + S\sqrt{-1};\quad Q = R - S\sqrt{-1};\quad \mathfrak{P} = \mathfrak{R} + \mathfrak{S}\sqrt{-1};$$
$$\mathfrak{Q} = \mathfrak{R} - \mathfrak{S}\sqrt{-1}$$

fiet diuiſor, reddens aequationem integrabilem,

$$(RR + SS)yy - 2(R\mathfrak{R} + S\mathfrak{S})y + \mathfrak{R}\mathfrak{R} + \mathfrak{S}\mathfrak{S}$$

ideoque realis.

Coroll. 4.

84. At eodem casu $c = a\sqrt{-1}$, ob $e^{-p\sqrt{-1}} = \operatorname{cos}. p - \sqrt{-1} \operatorname{sin}. p$, erit $e^{-2(2i+1)ax^{\frac{1}{2i+1}}\sqrt{-1}} = \operatorname{cos}. 2(2i+1)ax^{\frac{1}{2i+1}} - \sqrt{-1}. \operatorname{sin}. 2(2i+1)ax^{\frac{1}{2i+1}}$; vnde posito breuitatis gratia $2(2i+1)ax^{\frac{1}{2i+1}} = p$, erit integrale completum:

$$(\operatorname{cos}. p - \sqrt{-1} \operatorname{sin}. p). \frac{(R - S\sqrt{-1})y - \mathfrak{R} + \mathfrak{S}\sqrt{-1}}{(R + S\sqrt{-1})y - \mathfrak{R} - \mathfrak{S}\sqrt{-1}} = \text{Const.}$$

quae forma est imaginaria.

Coroll. 5.

85. Tribuatur autem constanti talis forma: $\alpha - \beta\sqrt{-1}$, et aequatione integrali euoluta, erit:

$$(Ry - \mathfrak{R})\operatorname{cos}. p - (Ry - \mathfrak{R})\operatorname{sin}. p\sqrt{-1} - (Sy - \mathfrak{S}) \operatorname{cos}. p\sqrt{-1} - (Sy - \mathfrak{S})\operatorname{sin}. p = (Ry - \mathfrak{R})\alpha - (Ry - \mathfrak{R})\beta\sqrt{-1} + (Sy - \mathfrak{S})\alpha\sqrt{-1} + (Sy - \mathfrak{S})\beta.$$

Iam aequentur seorsim partes reales et imaginariae:

$$(Ry - \mathfrak{R})\operatorname{cos}. p - (Sy - \mathfrak{S})\operatorname{sin}. p = \alpha(Ry - \mathfrak{R}) + \beta(Sy - \mathfrak{S})$$
$$(Ry - \mathfrak{R})\operatorname{sin}. p + (Sy - \mathfrak{S})\operatorname{cos}. p = \beta(Ry - \mathfrak{R}) - \alpha(Sy - \mathfrak{S})$$

quae duae aequationes conueniunt, si modo sit $\alpha\alpha + \beta\beta = 1$. Sit ergo $\alpha = \operatorname{cos}. \zeta$, et $\beta = \operatorname{sin}. \zeta$, prodibitque ex vtraque

$$\frac{Ry - \mathfrak{R}}{Sy - \mathfrak{S}} = \frac{\operatorname{sin}. p + \operatorname{sin}. \zeta}{\operatorname{cos}. p - \operatorname{cos}. \zeta} = \operatorname{cot}. \frac{\zeta - p}{2}.$$

Coroll. 6.

Coroll. 6.

86. Sumto ergo pro ζ angulo quocunque, si sit $c = a\sqrt{-1}$, erit integrale completum aequationis propositae

$$\frac{Ry - \mathfrak{R}}{Sy - \mathfrak{S}} = \cot. \frac{\zeta - p}{2}$$

seu $y = \dfrac{\mathfrak{R} \sin. \frac{\zeta - p}{2} - \mathfrak{S} \cos. \frac{\zeta - p}{2}}{R \sin. \frac{\zeta - p}{2} - S \cos. \frac{\zeta - p}{2}}$

existente $p = 2(2i + 1) a x^{\frac{1}{2i+1}}$.

Problema 13.

87. Denotante i numerum quemcunque integrum, exhibere resolutionem huius aequationis:

$$dy + yydx - ccx^{\frac{-4i}{2i-1}} dx = 0.$$

Solutio.

Quia est $n = \frac{i}{2i-1}$, haec resolutio deriuari potest ex solutione praecedentis problematis, ponendo $-i$ loco i. Quare tribuantur litteris B, C, D, etc. sequentes valores:

$$B = + \frac{i(i-1)}{2(2i-1)c} A$$
$$C = + \frac{(i+1)i(i-1)(i-2)}{2 \cdot 4 (2i-1)^2 c^2} A$$
$$D = + \frac{(i+2)(i+1)i(i-1)(i-2)(i-3)}{2 \cdot 4 \cdot 6 (2i-1)^3 c^3} A$$
etc.

Tum

Tum vero alterarum litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. determinatio ita ſe habebit:

$$\mathfrak{A} = c\,A$$

$$\mathfrak{B} = + \frac{(i+1)i}{2(2i-1)}A$$

$$\mathfrak{C} = + \frac{(i+2)(i+1)i(i-1)}{2.4\,(2i-1)^2 c}A$$

$$\mathfrak{D} = + \frac{(i+3)(i+2)(i+1)i(i-1)(i-2)}{2.4.6\,(2i-1)^3 c^2}A$$

etc.

Quibus valoribus conſtitutis, ponatur breuitatis gratia:

$$A x^{\frac{+i}{2i-1}} + B x^{\frac{+i+1}{2i-1}} + C x^{\frac{+i+2}{2i-1}} + D x^{\frac{+i+3}{2i-1}} + \text{etc.} = P$$

$$A x^{\frac{+i}{2i-1}} - B x^{\frac{+i+1}{2i-1}} + C x^{\frac{+i+2}{2i-1}} + D x^{\frac{+i+3}{2i-1}} + \text{etc.} = Q$$

$$\mathfrak{A} x^{\frac{-i}{2i-1}} + \mathfrak{B} x^{\frac{-i+1}{2i-1}} + \mathfrak{C} x^{\frac{-i+2}{2i-1}} + \mathfrak{D} x^{\frac{-i+3}{2i-1}} + \text{etc.} = \mathfrak{P}$$

$$-\mathfrak{A} x^{\frac{-i}{2i-1}} + \mathfrak{B} x^{\frac{-i-1}{2i-1}} - \mathfrak{C} x^{\frac{-i+2}{2i-1}} + \mathfrak{D} x^{\frac{-i+3}{2i-1}} - \text{etc.} = \mathfrak{Q}$$

atque hinc ſtatim habentur duae integrationes particulares:

$$\text{I. } y = \frac{\mathfrak{P}}{P}, \text{ et II. } y = \frac{\mathfrak{Q}}{Q}.$$

Tum vero aequatio integralis completa erit:

$$e^{2(2i-1)c\,x^{\frac{-1}{2i-1}}}\,\frac{Qy-\mathfrak{Q}}{Py-\mathfrak{P}} = \text{Conſt.}$$

et diuiſor aequationem propoſitam integrabilem reddens, fiet $= (Py-\mathfrak{P})(Qy-\mathfrak{Q})$.

Coroll.

Coroll. 1.

88. Quodsi autem aequatio proposita fuerit huiusmodi:

$$dy + yy\,dx + aax^{\frac{-4i}{2i-1}}dx = 0$$

vt sit $cc = -aa$, et $c = a\sqrt{-1}$, integrationes particulares exhibitae fient imaginariae, ob B, D, F, etc. item $\mathfrak{A}$, $\mathfrak{C}$, $\mathfrak{E}$ etc. imaginarias, dum reliquarum litterarum valores sunt reales.

Coroll. 2.

89. At si ponatur:

$$P + Q = 2R;\ P - Q = 2S\sqrt{-1};\ \mathfrak{P} + \mathfrak{Q} = 2\mathfrak{R}$$
$$\text{et } \mathfrak{P} - \mathfrak{Q} = 2\mathfrak{S}\sqrt{-1}$$

quantitates R, S, $\mathfrak{R}$ et $\mathfrak{S}$ nihilo minus fient, vt ante, reales, et diuisor aequationem reddens integrabilem erit:

$$(RR + SS)yy - 2(R\mathfrak{R} + S\mathfrak{S})y + \mathfrak{R}\mathfrak{R} + \mathfrak{S}\mathfrak{S}.$$

Coroll. 3.

90. Tum vero, si ponatur breuitatis causa $2(2i-1)$ $ax^{\frac{-1}{2i-1}} = p$, aequatio integralis completa erit:

$$\frac{Ry - \mathfrak{R}}{Sy - \mathfrak{S}} = \cot.\frac{\zeta + p}{2}$$

vnde elicitur:

$$y = \frac{\mathfrak{R}\sin.\frac{\zeta+p}{2} - \mathfrak{S}\cos.\frac{\zeta+p}{2}}{R\sin.\frac{\zeta+p}{2} - S.\cos.\frac{\zeta+p}{2}}$$

vbi angulus ζ vicem gerit constantis arbitrariae.

Scholion.

91. Solutiones horum duorum postremorum problematum non tam per accuratam analysin sunt euolutae, quam per inductionem ex casibus particularibus supra expeditis deriuatae, quandoquidem progressio ab his casibus ad sequentes satis erat manifesta. Fundamentum autem harum solutionum in hoc potissimum est situm, quod solutio particularis, vnde omnia sunt deducta, re vera est geminata, cum quantitas c, cuius quadratum tantum in aequatione differentiali occurrit, aeque negatiue, ac positiue, accipi possit. Quoties autem huiusmodi aequationum binae solutiones particulares sunt cognitae, ex iis multo facilius solutio generalis, indeque multiplicatores, eas integrabiles reddentes, erui possunt, id quod operae pretium erit clarius exposuisse.

Problema 14.

92. Datis duabus solutionibus particularibus huiusmodi aequationis:

$$dy + Pydx + Qyydx + Rdx = 0$$

inuenire eius solutionem generalem, et multiplicatorem, qui eam integrabilem reddat.

Solutio.

Sint M et N huiusmodi functiones ipsius x, quae loco y substitutae, ambae aequationi propositae satisfaciant, ita vt sit:

$$dM + PMdx + QM^2dx + Rdx = 0$$

et $$dN + PNdx + QN^2dx + Rdx = 0.$$

Ponatur

Ponatur $\frac{y-M}{y-N}=z$, seu $y=\frac{M-Nz}{1-z}$, erit

$$dy=\frac{dM-zdM+Mdz-Ndz-zdN+zzdN}{(1-z)^2}$$

quibus valoribus in aequatione proposita substitutis, et tota aequatione per $(1-z)^2$ multiplicata, prodibit:

$$(1-z)dM-z(1-z)dN+(M-N)dz+P(1-z)Mdx-P(1-z)Nzdx$$
$$+QMMdx-2QMNzdx+QNNzzdx+R(1-z)^2dx=0.$$

Iam pro dM et dN substituantur valores ex binis superioribus aequationibus differentialibus oriundi:

$$\begin{array}{l} -P(1-z)Mdx-Q(1-z)M^2dx-R(1-z)dx \\ +Pz(1-z)Ndx+Qz(1-z)N^2dx+Rz(1-z)dx+(M-N)dz=0 \\ +P(1-z)Mdx+QM^2dx \qquad +R(1-z)^2dx \\ -Pz(1-z)Ndx-2QMNzdx \\ \qquad\qquad +QN^2zzdx \end{array}$$

qua aequatione in ordinem redacta, orietur:

$$QzM^2dx+QzN^2dx-2QMNzdx+(M-N)dz=0$$

seu $Q(M-N)dx+\frac{dz}{z}=0$, ita vt sit:

$$z=Ce^{-\int Q(M-N)dx}$$

vnde aequatio integrata generalis erit:

$$e^{\int Q(M-N)dx}\frac{y-M}{y-N}=\text{Const.}$$

Pro multiplicatore autem inueniendo, notetur, aequationem propositam, facta substitutione primum per $(1-z)^2$, esse multiplicatam, tum vero diuisam per $z(M-N)$, euasisse integrabilem. Statim ergo per $\frac{(1-z)^2}{(M-N)z}$ multiplicata fiet integrabilis: ex quo factor erit $\frac{(1-z)^2}{(M-N)z}$, qui ob $z=\frac{y-M}{y-N}$ hanc induet formam:

$$\frac{M-N}{(y-M)(y-N)}.$$

Problema 15.

93. Proposita aequatione $ydy + Pydx + Qdx = 0$, inuenire conditiones functionum P et Q, vt huiusmodi multiplicator $(y+M)^n$ eam reddat integrabilem.

Solutio.

Ex natura ergo differentialium esse oportet:

$$\frac{1}{dx} d. y(y+M)^n = \frac{1}{dy} d(Py+Q)(y+M)^n$$

vnde cum M sit functio ipsius x tantum, erit

$$ny(y+M)^{n-1}\frac{dM}{dx} = P(y+M)^n + n(Py+Q)(y+M)^{n-1}$$

quae diuisa per $(y+M)^{n-1}$ abit in hanc:

$$\frac{ny\,dM}{dx} = (n+1)Py + PM + nQ$$

vnde necesse est sit:

$$P = \frac{n\,dM}{(n+1)dx} \quad \text{et} \quad Q = \frac{-PM}{n} = -\frac{M\,dM}{(n+1)dx}$$

His igitur valoribus substitutis aequatio

$$y\,dy + \frac{ny\,dM}{n+1} - \frac{M\,dM}{n+1} = 0$$

fit integrabilis, si multiplicetur per $(y+M)^n$.

Coroll. 1.

94. Quia haec aequatio est homogenea, ea quoque fit integrabilis, si diuidatur per $(n+1)yy + nyM - MM = (y+M)((n+1)y - M)$. Neque ergo hinc nouae aequationes methodo hac tractabiles obtinentur.

Coroll. 2.

95. Quoniam autem habemus duos multiplicatores $(y+M)^n$ et $\frac{1}{(y+M)((n+1)y-M)}$: si alter per alterum

rum diuidatur, quoties conſtanti arbitrariae aequatus dabit integrale completum. Quare aequatio $y\,dy + \frac{ny\,dM}{n+1} - \frac{M\,dM}{n+1} = 0$ generaliter integrata praebet:

$$(y+M)^{n+1}((n+1)y-M) = \text{Conſt.}$$

Problema 16.

96. Propoſita aequatione $y\,dy + Py\,dx + Q\,dx = 0$, inuenire conditiones functionum P et Q, vt huiusmodi multiplicator $(yy+My+N)^n$ eam reddat integrabilem.

Solutio.

Ex natura differentialium ſit neceſſe eſt:

$$\frac{1}{dx}d.y(yy+My+N)^n = \frac{1}{dy}d.(Py+Q)(yy+My+N)^n$$

Cum igitur M, N, P et Q ſint per hypotheſin functiones ipſius x, erit, facta euolutione:

$$ny(yy+My+N)^{n-1}\left(y\frac{dM}{dx}+\frac{dN}{dx}\right) = P(yy+My+N)^n$$
$$+\,n(Py+Q)(2y+M)(yy+My+N)^{n-1}$$

et poſt diuiſionem per $(yy+My+N)^{n-1}$

$$nyy\frac{dM}{dx} + \frac{ny\,dN}{dx} = (2n+1)Pyy + (n+1)PMy + PN$$
$$+\,2nQy \qquad + nQM$$

Hinc fiere oportet:

I. $n\,dM = (2n+1)P\,dx$

II. $n\,dN = (n+1)PM\,dx + 2nQ\,dx$

III. $0 = PN + nQM$

Prima dat $P = \frac{n\,dM}{(2n+1)dx}$, et vltima $Q = \frac{-PN}{nM}$, seu $Q = \frac{-N\,dM}{(2n+1)M\,dx}$, qui valores in media substituti praebent:

$$n\,dN = \frac{n(n+1)M\,dM}{2n+1} - \frac{2nN\,dM}{(2n+1)M} \text{ seu}$$

$$(2n+1)M\,dN + 2N\,dM = (n+1)MM\,dM$$

quae multiplicata per $M^{\frac{-2n+1}{2n+1}}$ et integrata praebet:

$$(2n+1)M^{\frac{2}{2n+1}}N = \text{Const.} + (n+1)\int M^{\frac{2n+3}{2n+1}}dM$$

seu $(2n+1)M^{\frac{2}{2n+1}}N = \text{Const.} + \frac{2n+1}{4}M^{\frac{4n+4}{2n+1}}$,

vnde fit $N = \alpha M^{\frac{-2}{2n+1}} + \frac{1}{4}M^2$.

Cum ergo sit

$$P\,dx = \frac{n\,dM}{2n+1} \text{ et } Q\,dx = -\frac{\alpha M^{\frac{-2n-3}{2n+1}}dM}{2n+1} - \frac{M\,dM}{4(2n+1)}$$

ista aequatio differentialis;

$$y\,dy + \frac{ny\,dM}{2n+1} - \frac{M\,dM}{4(2n+1)} - \frac{\alpha}{2n+1}M^{\frac{-2n-3}{2n+1}}dM = 0$$

integrabilis redditur, si multiplicetur per

$$(yy + My + \tfrac{1}{4}M^2 + \alpha M^{\frac{-2}{2n+1}})^n.$$

Coroll. 1.

97. Si fuerit $-\frac{2n+3}{2n+1} = 1$, seu $n = -1$; aequatio differentialis est homogenea, et si $-\frac{2n+3}{2n+1} = 0$ seu $n = -\frac{3}{2}$, primi gradus. Vtroque autem casu nulla est difficultas, cum aequatio facile tractari possit.

Coroll.

Coroll. 2.

98. Magis ergo abſtruſi erunt caſus, quibus exponens $-\frac{2n-3}{2n+1}$ neque eſt o, neque 1. Sit ergo $-\frac{2n-3}{2n+1}=m$, vnde fit $2n=-\frac{m-3}{m+1}$ et aequatio differentialis
$$ydy+\tfrac{1}{4}(m+3)ydM+\tfrac{1}{8}(m+1)MdM+\tfrac{1}{2}\alpha(m+1)M^m dM=0$$
integrabilis reddetur per multiplicatorem
$$(yy+My+\tfrac{1}{4}MM+\alpha M^{m+1})^{\frac{-m-3}{2(m+1)}}$$

Coroll. 3.

99. Quod ſi iam pro M functiones quaecunque ipſius x ſubſtituantur, aequationes tam complicatae formari poterunt, quas quomodo aliis methodis tractari oporteat, vix liquet, cum tamen hac methodo earum reſolutio ſit in promtu.

Scholion.

100. Si quis haec veſtigia vlterius proſequi voluerit, dubium eſt nullum, quin haec methodus mox multo maiora ſit acceptura incrementa, quibus vniuerſa Analyſis non mediocriter promoueatur. Specimina etiam hic euoluta ita ſunt comparata, vt viam ad inueſtigationes profundiores parare videantur, praecipue ſi inſuper alia aequationum differentialium genera ſimili modo pertractentur. Verum haec, quae hactenus protuli, ſufficere videntur, animis Geometrarum ad ampliorem huius methodi enucleationem incitandis, quem ſcopum mihi equidem potiſſimum propoſueram.

SOLVTIO