

CONSIDERATIO FORMVLARVM,
 QVARVM INTEGRATIO PER ARCVS
 SECTIONVM CONICARVM ABSOLVI
 POTEST.

Auctore

L. EULER O.

L e m m a t a.

I. $\int dz \sqrt{\frac{f+gzz}{b+kzz}} = \frac{1}{k} \int dx \sqrt{\frac{fk-gb+gxx}{xx-b}}$
 posito $x = \sqrt{(b+kzz)}$

II. $\int \frac{zz dz}{\sqrt{(f+gzz)(b+kzz)}} = \frac{1}{b} \int dx \sqrt{\frac{xx-f}{gb-fk+kxx}}$
 $= \frac{1}{k} \int dy \sqrt{\frac{yy-b}{fk-gb+gyy}}$
 posito $x = \sqrt{(f+gzz)}$ et $y = \sqrt{(b+kzz)}$

III. $\int \frac{dz \sqrt{(f+gzz)}}{(b+kzz)^{\frac{3}{2}}} = -\frac{1}{k} \int dx \sqrt{\frac{g+(fk-gb)xx}{1-bxx}}$
 $= \frac{1}{b} \int dy \sqrt{\frac{f+(gb-fk)yy}{1-kyy}}$

posito $x = \frac{1}{\sqrt{(b+kzz)}}$ et $y = \frac{z}{\sqrt{(b+kzz)}}$

IV. $\int \frac{dz \sqrt{(b+kzz)}}{(f+gzz)^{\frac{3}{2}}} = -\frac{1}{g} \int dx \sqrt{\frac{k+(gb-fk)xx}{1-fxx}}$
 $= \frac{1}{j} \int dy \sqrt{\frac{b+(fk-gb)yy}{1-gyy}}$

Tom. VIII. Nou. Comm.

R

posito

$$\text{posito } x = \frac{1}{\sqrt{(f+gzz)}} \text{ et } y = \frac{z}{\sqrt{(f+gzz)}}$$

$$\text{V. } \int \frac{dz}{(f+gzz)^{\frac{1}{2}} \sqrt{(b+kzz)}} = \int dx \sqrt{\frac{1-gxx}{b+(fk-gb)xx}}$$

$$= \frac{1}{fk-gb} \int dy \sqrt{\frac{k-gyy}{fyy-b}}$$

$$\text{posito } x = \frac{z}{\sqrt{(f+gzz)}} \text{ et } y = \sqrt{\frac{b+kzz}{f+gzz}}$$

$$\text{VI. } \int \frac{dz}{(b+kzz)^{\frac{1}{2}} \sqrt{(f+gzz)}} = \int dx \sqrt{\frac{1-kxx}{f+(gb-fk)xx}}$$

$$= \frac{1}{gb-fk} \int dy \sqrt{\frac{g-kyy}{byy-f}}$$

$$\text{posito } x = \frac{z}{\sqrt{(b+kzz)}} \text{ et } y = \sqrt{\frac{f+gzz}{b+kzz}}$$

$$\text{VII. } \int \frac{z dz}{(f+gzz)^{\frac{1}{2}} \sqrt{(b+kzz)}} = -\int dx \sqrt{\frac{1-fxx}{k+(gb-fk)xx}}$$

$$= \frac{1}{fk-gb} \int dy \sqrt{\frac{fyy-b}{k-gyy}}$$

$$\text{posito } x = \frac{1}{\sqrt{(f+gzz)}} \text{ et } y = \sqrt{\frac{b+kzz}{f+gzz}}$$

$$\text{VIII. } \int \frac{z dz}{(b+kzz)^{\frac{1}{2}} \sqrt{(f+gzz)}} = -\int dx \sqrt{\frac{1-bxx}{g+(fk-gb)xx}}$$

$$= \frac{1}{gb-fk} \int dy \sqrt{\frac{byy-f}{g-kyy}}$$

$$\text{posito } x = \frac{1}{\sqrt{(b+kzz)}} \text{ et } y = \sqrt{\frac{f+gzz}{b+kzz}}$$

Theore-

Theoremata.

I. $\int dz \sqrt{\frac{f+gz}{b+kz}} = \frac{1}{k} \int dx \sqrt{\frac{fk-gb+gxx}{xx-b}}$
 posito $x = \sqrt{(b+kz)}$

II. $\int dz \sqrt{\frac{f+gz}{b+kz}} = z \sqrt{\frac{f+gz}{b+kz}} - \int dx \sqrt{\frac{bxx-f}{g-kxx}}$
 posito $x = \sqrt{\frac{f+gz}{b+kz}}$

III. $\int dz \sqrt{\frac{f+gz}{b+kz}} = z \sqrt{\frac{f+gz}{b+kz}} + \frac{gb-fk}{k} \int dx \sqrt{\frac{1-bxx}{g+(fk-gb)xx}}$
 posito $x = \frac{1}{\sqrt{(b+kz)}}$

IV. $\int dz \sqrt{\frac{f+gz}{b+kz}} = \frac{g}{k} z \sqrt{\frac{b+kz}{f+gz}} + \frac{fk-gb}{k} \int dx \sqrt{\frac{1-gxx}{b+(fk-gb)xx}}$
 posito $x = \frac{z}{\sqrt{(f+gz)}}$

V. $\int dz \sqrt{\frac{f+gz}{b+kz}} = \frac{g}{k} z \sqrt{\frac{b+kz}{f+gz}} + \frac{f}{k} \int dx \sqrt{\frac{k-gxx}{fxx-b}}$
 posito $x = \sqrt{\frac{b+kz}{f+gz}}$

VI. $\int dz \sqrt{\frac{f+gz}{b+kz}} = \frac{f}{b} \int dz \sqrt{\frac{b+kz}{f+gz}} + \frac{gb-fk}{gb} \int dx \sqrt{\frac{xx-f}{gb-fk+kxx}}$
 posito $x = \sqrt{(f+gz)}$

VII. $\int dz \sqrt{\frac{f+gz}{b+kz}} = \frac{f}{b} \int dz \sqrt{\frac{b+kz}{f+gz}} + \frac{gb-fk}{bk} \int dx \sqrt{\frac{xx-b}{fk-gb+gxx}}$
 posito $x = \sqrt{(b+kz)}$

VIII. $\int dz \sqrt{\frac{f+gz}{b+kz}} = z \sqrt{\frac{f+gz}{b+kz}} + P + Q$

$$\text{vbi } P = \frac{gb-fk}{gk} \int dx \sqrt{\frac{g+(fk-gb)xx}{1-bxx}} = \frac{fk-gb}{gb} \int dy \sqrt{\frac{f+(gb-fk)yy}{1-kyy}}$$

$$\text{posito } x = \frac{z}{\sqrt{b+kzz}} \text{ et } y = \frac{z}{\sqrt{b+kzz}}$$

$$\text{et } Q = \frac{f(fk-gb)}{gb} \int dx \sqrt{\frac{1-kxx}{f+(gb-fk)xx}} = \frac{f}{g} \int dy \sqrt{\frac{g-kyy}{kyy-f}}$$

$$\text{posito } x = \frac{z}{\sqrt{b+kzz}} \text{ et } y = \sqrt{\frac{f+gzz}{b+kzz}}$$

$$\text{IX. } \int dz \sqrt{\frac{f+gzz}{b+kzz}} = \frac{fk}{gb} z \sqrt{\frac{f+gzz}{b+kzz}} + P + Q$$

$$\text{vbi } P = \frac{gb-fk}{gb} \int dx \sqrt{\frac{xx-f}{gb-jk+kxx}} = \frac{gb-jk}{bk} \int dy \sqrt{\frac{yy-b}{jk-gb+gyy}}$$

$$\text{posito } x = \sqrt{f+gzz} \text{ et } y = \sqrt{b+kzz}$$

$$\text{atque } Q = \frac{f(gb-fk)}{gb} \int dx \sqrt{\frac{1-kxx}{f+(gb-fk)xx}} = \frac{f}{g} \int dy \sqrt{\frac{g-kyy}{byy-f}}$$

$$\text{posito } x = \frac{z}{\sqrt{b+kzz}} \text{ et } y = \sqrt{\frac{f+gzz}{b+kzz}}$$

$$\text{X. } \int dz \sqrt{\frac{f+gzz}{b+kzz}} = \frac{gb-fk}{gb} z \sqrt{\frac{f+gzz}{b+kzz}} + \frac{f}{b} \int dz \sqrt{\frac{b+kzz}{f+gzz}} + P$$

$$\text{vbi } P = \frac{gb-fk}{gk} \int dx \sqrt{\frac{g+(fk-gb)xx}{1-bxx}} = \frac{fk-gb}{gb} \int dy \sqrt{\frac{f+(gb-fk)yy}{1-kyy}}$$

$$\text{posito } x = \frac{z}{\sqrt{b+kzz}} \text{ et } y = \frac{z}{\sqrt{b+kzz}}$$

$$\text{XI. } \int dz \sqrt{\frac{f+gzz}{b+kzz}} = \frac{f}{b} z \sqrt{\frac{b+kzz}{f+gzz}} + P + Q$$

$$\text{vbi } P = \frac{gb-fk}{gb} \int dx \sqrt{\frac{xx-f}{gb-jk+kxx}} = \frac{gb-fk}{bk} \int dy \sqrt{\frac{yy-b}{jk-gb+gyy}}$$

$$\text{posito } x = \sqrt{f+gzz} \text{ et } y = \sqrt{b+kzz}$$

atque

atque $Q = \frac{f(fk - gb)}{gb} \int dx \sqrt{\frac{r - fxx}{z + (gb - fk)xx}} = \frac{-f}{b} \int dy \sqrt{\frac{fyy - b}{k - gyy}}$
 posito $x = \frac{r}{V(j + gzz)}$ et $y = V \frac{b + kzz}{j + gzz}$

XII. $\int dz \sqrt{\frac{f + gzz}{b + kzz}} = \frac{g}{k} z \sqrt{\frac{b + kzz}{j + gzz}} + P + Q$

vbi $P = \frac{f(gb - fk)}{gbk} \int dx \sqrt{\frac{k + (gb - fk)xx}{1 - jxx}} = \frac{fk - gb}{bk} \int dy \sqrt{\frac{b + (fk - gb)yy}{1 - gyy}}$

posito $x = \frac{r}{V(j + gzz)}$ et $y = V \frac{z}{j + gzz}$

atque $Q = \frac{f(fk - gb)}{gb} \int dx \sqrt{\frac{r - fxx}{k + (gb - fk)xx}} = \frac{-f}{b} \int dy \sqrt{\frac{fyy - b}{k - gyy}}$

posito $x = V \frac{r}{j + gzz}$ et $y = V \frac{b + kzz}{j + gzz}$

XIII. $\int dz \sqrt{\frac{f + gzz}{b + kzz}} = \frac{gb - fk}{bk} z \sqrt{\frac{b + kzz}{f + gzz}} + \frac{f}{b} \int dz \sqrt{\frac{b + kzz}{f + gzz}} + P$

vbi $P = \frac{f(gb - fk)}{gk} \int dx \sqrt{\frac{k + (gb - fk)xx}{1 - fxx}} = \frac{fk - gb}{bk} \int dy \sqrt{\frac{b + (fk - gb)yy}{1 - gyy}}$

posito: $x = \frac{r}{V(j + gzz)}$ et $y = \frac{z}{V(j + gzz)}$

Theorema. Singulare.

$\int dz \sqrt{\frac{f + gzz}{b + kzz}} = \frac{-gxx}{Vp} - \int dx \sqrt{\frac{f + gxx}{b + kxx}}$ vbi p denotat constantem arbitrariam, posita inter x et z hac relatione:

R 3 $gkxx$

$$gkx^2z^2 - px^2 - pzz - 2xz\sqrt{(p+fk)(p+gb)} + fb = 0 \text{ siue}$$

$$x = \frac{-z\sqrt{(p+fk)(p+gb)} + \sqrt{p(f+gzz)(b+kzz)}}{p-gkzz}$$

Hypothesis.

Haec scribendi formula $\Pi x [a]$ denotet sectionis conicae, cuius semiparameter $= 1$, et semiaxis transversus $= a$, arcum a vertice sumtum, cui in axe transverso conueniat abscissa $= x$.

Corollarium.

Si a sit quantitas positua, hoc modo designatur arcus ellipsis; siu negatiua, arcus hyperbolae. Si modo x fuerit quantitas positua et minor quam $2a$.

Integrationes formulae $\int dz \sqrt{\frac{f+gzz}{b+kzz}}$ in 12 casus distributae.

Casus I. $\int dz \sqrt{\frac{f+gzz}{b+kzz}}$:

Integrale est immedie:

$$C - \frac{(fk+gb)}{k\sqrt{fk}} \Pi \frac{fk}{fk+gb} \left(1 - z\sqrt{\frac{k}{b}}\right) \left[\frac{fk}{fk+gb}\right]$$

vel etiam per Theor. I.

$$C + \frac{f}{\sqrt{(fk+gb)}} \Pi \frac{fk+gb}{fk} \left(1 - \frac{\sqrt{(b-kzz)}}{\sqrt{k}}\right) \left[\frac{fk+gb}{fk}\right]$$

Casus II. $\int dz \sqrt{\frac{f-gzz}{b-kzz}}$, existente $fk > gb$

Integrale est immedie:

$$C - \frac{(fk-gb)}{k\sqrt{fk}} \Pi \frac{fk}{fk-gb} \left(1 - z\sqrt{\frac{k}{b}}\right) \left[\frac{fk}{fk-gb}\right]$$

vel

vel etiam per Theor. I.

$$C + \frac{f}{\sqrt{fk-gb}} \Pi \frac{fk-gb}{fk} \left(1 - \frac{\sqrt{(b-kzz)}}{\sqrt{b}}\right) \left[\frac{fk-gb}{fk} \right]$$

Casus III. $\int dz \sqrt{\frac{-f+gzz}{-b+kzz}}$, existente $fk < gb$

Integrale est immedie:

$$C + \frac{gb-fk}{k\sqrt{fk}} \Pi \frac{fk}{gb-fk} \left(z \sqrt{\frac{k}{b}} - 1 \right) \left[\frac{-fk}{gb-fk} \right]$$

Casus IV. $\int dz \sqrt{\frac{f+gzz}{b+kzz}}$, existente $fk < gb$

Integrale est per Theor. I.

$$C + \frac{f}{\sqrt{gb-fk}} \Pi \frac{gb-fk}{fk} \left(\frac{\sqrt{(b+kzz)}}{\sqrt{b}} - 1 \right) \left[\frac{-gb+fk}{fk} \right]$$

Casus V. $\int dz \sqrt{\frac{-f+gzz}{b+kzz}}$

Integrale est per Theor. III.

$$C + z \sqrt{\frac{-f+gzz}{b+kzz}} - \frac{f}{\sqrt{(fk+gb)}} \Pi \frac{fk+gb}{fk} \left(1 - \frac{\sqrt{(fk+gb)}}{\sqrt{g(b+kzz)}} \right) \left[\frac{fk+gb}{fk} \right]$$

vel etiam per Theor. II.

$$C + z \sqrt{\frac{-f+gzz}{b+kzz}} + \frac{fk+gb}{k\sqrt{fk}} \Pi \frac{fk}{fk+gb} \left(1 - \frac{\sqrt{k(-f+gzz)}}{\sqrt{g(b+kzz)}} \right) \left[\frac{fk}{fk+gb} \right]$$

Casus VI. $\int dz \sqrt{\frac{-f+gzz}{-b+kzz}}$, existente $fk > gb$

Integrale est per Theor. III.

$$C + z \sqrt{\frac{-f+gzz}{-b+kzz}} - \frac{f}{\sqrt{(fk-gb)}} \Pi \frac{fk-gb}{fk} \left(1 - \frac{\sqrt{(fk-gb)}}{\sqrt{g(-b+kzz)}} \right) \left[\frac{fk-gb}{fk} \right]$$

vel

vel etiam per Theor. II.

$$C + z \sqrt{\frac{-f+gzz}{-b+kzz}} + \frac{fk-gb}{k\sqrt{fk}} \Pi \frac{fk}{fk-gb} \left(\frac{\sqrt{k(-f+gzz)}}{\sqrt{g(-b+kzz)}} - 1 \right) \left[\frac{fk}{fk-gb} \right]$$

Casus VII. $\int dz \sqrt{\frac{f-gzz}{b-kzz}}$, existente $fk < gb$

Integrale est per Theor. III.

$$C + \frac{gz}{k} \sqrt{\frac{b-kzz}{f-gzz}} - \frac{(gb-fk)}{k\sqrt{fk}} \Pi \frac{fk}{gb-fk} \left(\frac{\sqrt{f(b-kzz)}}{\sqrt{b(f-gzz)}} - 1 \right) \left[\frac{-fk}{gb-fk} \right]$$

Casus VIII. $\int dz \sqrt{\frac{-f+gzz}{b-kzz}}$, existente $fk < gb$

Integrale est per Theor. II.

$$C + z \sqrt{\frac{-f+gzz}{b-kzz}} - \frac{f}{\sqrt{gb-fk}} \Pi \frac{gb-fk}{fk} \left(\frac{\sqrt{gb-fk}}{\sqrt{g(b-kzz)}} - 1 \right) \left[\frac{-gb+fk}{fk} \right]$$

vel etiam per Theor. V.

$$C - \frac{gz}{k} \sqrt{\frac{b-kzz}{-f+gzz}} + \frac{f}{\sqrt{gb-fk}} \Pi \frac{gb-fk}{fk} \left(\frac{z\sqrt{gb-fk}}{\sqrt{b(-f+gzz)}} - 1 \right) \left[\frac{-gb+fk}{fk} \right]$$

Casus IX. $\int dz \sqrt{\frac{f+gzz}{b+kzz}}$, existente $fk > gb$

Integrale est per Theor. X.

$$C - \frac{(fk-gb)z}{gb} \sqrt{\frac{f+gzz}{b+kzz}} - \frac{(fk-gb)}{k\sqrt{fk}} \Pi \frac{fk}{gb} \left(1 - \frac{z\sqrt{k}}{\sqrt{b+kzz}} \right) \left[\frac{fk}{gb} \right]$$

$$+ \frac{f}{\sqrt{(fk-gb)}} \Pi \frac{fk-gb}{gb} \left(\frac{\sqrt{f+gzz}}{\sqrt{f}} - 1 \right) \left[\frac{-fk+gb}{gb} \right]$$

vel etiam per Theor. XIII.

$$C - \frac{(fk-gb)z}{bk} \sqrt{\frac{b+kzz}{f+gzz}} + \frac{(fk-gb)}{k\sqrt{fk}} \Pi \frac{fk}{gk} \left(1 - \frac{\sqrt{f}}{\sqrt{f+gzz}} \right) \left[\frac{fk}{gb} \right]$$

$$+ \frac{f}{\sqrt{(fb-gb)}} \Pi \frac{fk-gb}{gb} \left(\frac{\sqrt{b+kzz}}{\sqrt{b}} - 1 \right) \left[\frac{-fk+gb}{gb} \right]$$

Casus X.

Cafus X. $\int dz \sqrt{\frac{f-gzz}{-b+kzz}}$, existente $fk > gb$

Integrale est per Theor. IX.

$$C + \frac{fkz}{gb} \sqrt{\frac{f-gzz}{-b+kzz}} + \frac{(fk-gb)}{k\sqrt{fk}} \Pi \frac{fk}{gb} \left(1 - \frac{\sqrt{k(f-gzz)}}{\sqrt{(fk-gb)}} \right) \left[\frac{fk}{gb} \right] \\ - \frac{f}{\sqrt{(fk-gb)}} \Pi \frac{fk-gb}{gb} \left(\frac{z\sqrt{(fk-gb)}}{\sqrt{(-b+kzz)}} - 1 \right) \left[\frac{-fk+gb}{gb} \right]$$

vel etiam per Theor. XI.

$$C - \frac{fz}{b} \sqrt{\frac{-b+kzz}{f-gzz}} + \frac{(fk-gb)}{k\sqrt{fk}} \Pi \frac{fk}{gb} \left(1 - \frac{\sqrt{k(f+gzz)}}{\sqrt{(fk-gb)}} \right) \left[\frac{fk}{gb} \right] \\ + \frac{f}{\sqrt{(fk-gb)}} \Pi \frac{fk-gb}{gb} \left(\frac{\sqrt{(fk-gb)}}{\sqrt{k(f-gzz)}} - 1 \right) \left[\frac{-fk+gb}{gb} \right]$$

Cafus XI. $\int dz \sqrt{\frac{f+gzz}{-b+kzz}}$

Integrale est per Theor. XI.

$$C - \frac{fz}{b} \sqrt{\frac{-b+kzz}{f+gzz}} + \frac{f}{\sqrt{(fk+gb)}} \Pi \frac{fk+gb}{gb} \left(1 - \frac{\sqrt{k(f+gzz)}}{\sqrt{(fk+gb)}} \right) \left[\frac{fk+gb}{gb} \right] \\ + \frac{(fk+gb)}{k\sqrt{fk}} \Pi \frac{fk}{gb} \left(\frac{\sqrt{k(f+gzz)}}{\sqrt{(fk+gb)}} - 1 \right) \left[\frac{-fk}{gb} \right]$$

vel etiam per Theor. XII.

$$C + \frac{gz}{k} \sqrt{\frac{-b+kzz}{f+gzz}} + \frac{f}{\sqrt{(fk+gb)}} \Pi \frac{fk+gb}{gb} \left(1 - \frac{\sqrt{k(f+gzz)}}{\sqrt{k(f+gzz)}} \right) \left[\frac{-fk+gb}{gb} \right] \\ + \frac{(fk+gb)}{k\sqrt{fk}} \Pi \frac{fk}{gb} \left(\frac{\sqrt{f}}{\sqrt{(f+gzz)}} - 1 \right) \left[\frac{-fk}{gb} \right]$$

Cafus XII. $\int dz \sqrt{\frac{f-gzz}{b+kzz}}$

Integrale est per Theor. XIII.

$$C - \frac{(fk+gb)}{bk} \sqrt{\frac{b+kzz}{f-gzz}} + \frac{f}{\sqrt{(fk+gb)}} \Pi \frac{fk+gb}{gb} \left(1 - \frac{\sqrt{(f-gzz)}}{\sqrt{f}} \right) \left[\frac{fk+gb}{gb} \right] \\ + \frac{(fk+gb)}{k\sqrt{fk}} \Pi \frac{fk}{gb} \left(\frac{\sqrt{f}}{\sqrt{(f-gzz)}} - 1 \right) \left[\frac{-fk}{gb} \right]$$

Omnes ergo casus formulæ $\int dz \sqrt{\frac{\alpha + \beta zz}{\gamma + \delta zz}}$, quomodo-
cunque litteræ α , β , γ , δ fuerint comparatæ, per
arcus sectionum conicarum integrari possunt.

Non solum igitur formulae initio commemoratae integrationem per arcus sectionum conicarum admittunt, sed etiam innumerabiles aliae, quae per substitutionem ad formam $\int dx \sqrt{\frac{\alpha + \beta x x}{\gamma + \delta x x}}$ se reduci patiuntur, cuiusmodi sunt

$$1^{\circ} \int \frac{dz}{zz} \sqrt{\frac{f+gzx}{b+kzx}} = -\int dx \sqrt{\frac{fxx+g}{bxx+k}} = -\int dy \sqrt{\frac{fyy-k}{yy-k}}$$

posito $x = \frac{z}{z}$ et $y = \frac{\sqrt{(b+kzx)}}{z}$

$$2^{\circ} \int \frac{dz}{zz \sqrt{(f+gzx)(b+kzx)}} = -\int dx \sqrt{\frac{xx-g}{bxx+jk-gb}} = -\int dy \sqrt{\frac{yy-k}{fyy-jk+gb}}$$

posito $x = \frac{\sqrt{(f+gzx)}}{z}$ et $y = \frac{\sqrt{(b+kzx)}}{z}$

$$3^{\circ} \int \frac{dz}{\sqrt{(f+gzx)(b+kzx)}} = \frac{k}{jk+gb} \int dz \sqrt{\frac{f+gzx}{b+kzx}} - \frac{g}{fk-gb} \int dz \sqrt{\frac{b+kzx}{f+gzx}}$$

cuius formulae reductio etiam ita instituitur:

$$\int \frac{dz}{\sqrt{(f+gzx)(b+kzx)}} = \frac{f}{fk-gb} \int dx \sqrt{\frac{k-gxx}{fxx-h}} + \frac{g}{fk-gb} \int dx \sqrt{\frac{fxx-h}{k-gxx}}$$

posito $x = \sqrt{\frac{b+kzx}{f+gzx}}$

vel etiam sic:

$$\int \frac{dz}{\sqrt{(f+gzx)(b+kzx)}} = \int dx \sqrt{\frac{r-gxx}{b+(jk-gb)xx}} - \int dy \sqrt{\frac{r-fyy}{k+(gb-jk)yy}}$$

posito $x = \frac{z}{\sqrt{(f+gzx)}}$ et $y = \frac{r}{\sqrt{(f+gzx)}}$

Ponamus $z:z = v$ atque obtinebimus sequentes formulas, quae pariter per arcus sectionum conicarum construi poterunt:

$$1^{\circ} \int \frac{dv \sqrt{(f+gv)}}{v \sqrt{(b+kv)}}$$

$$2^{\circ} \int \frac{dv \sqrt{(f+gv)}}{v \sqrt{v(b+kv)}}$$

$$3^{\circ} \int \frac{dv \sqrt{v}}{\sqrt{(f+gv)(b+kv)}}$$

$$4^{\circ} \int \frac{dv}{\sqrt{v(f+gv)(b+kv)}}$$

$$5^{\circ} \int \frac{dv V(f+gv)}{(b+kv)^{\frac{3}{2}} Vv} ; 6^{\circ} \int \frac{dv}{v Vv(f+gv)(b+kv)}$$

$$7^{\circ} \int \frac{dv}{(f+gv)^{\frac{3}{2}} Vv(b+kv)} ; 8^{\circ} \int \frac{dv}{(f+gv)^{\frac{3}{2}} V(b+kv)}$$

hae enim vicissim, posito $v=zz$, ad formas praecedentes reducuntur.

Hinc patet, istam formulam satis late patentem ad arcus sectionum conicarum reduci posse

$$\int \frac{(A+Bv) du}{\sqrt{(a+\beta u)(\gamma+\delta u)(\epsilon+\zeta u)}}$$

quae imprimis notari meretur. Ponatur enim $\alpha+\beta u=v$, ut sit $u=\frac{v-\alpha}{\beta}$, haecque formula transmutabitur in hanc:

$$\int \frac{dv(\Lambda\beta - B\alpha + Bv)}{\beta\gamma v(\beta\gamma - \alpha\delta + \delta v)(\beta\epsilon - \alpha\zeta + \zeta v)}$$

quae ad binas formulas, sub n^o. 3 et 4 allatas, reuocatur.

Quare, si $\alpha + \beta x + \gamma x^2 + \delta x^3$ habeat tres factores reales, haec formula

$$\int \frac{dx(A+Bx)}{\sqrt{(a+\beta x+\gamma x^2+\delta x^3)}}$$

modo exposito integrari poterit: semper autem vnum factorem certe habet realem. Sin autem bini sint imaginarii, formula $\alpha + \beta x + \gamma x^2 + \delta x^3$ ita referri potest $y(pp + 2npqy + qqy^2)$, existente $nn < 1$, ut definiendum sit integrale harum formularum:

$$\int \frac{Cdy}{\sqrt{y(pp + 2npqy + qqy^2)}} + \int \frac{Ddy\sqrt{y}}{\sqrt{(pp + 2npqy + qqy^2)}}$$

Ponatur $V(pp + 2npqy + qqy^2) = p + qyz$, fietque

$$y = \frac{2p(z-n)}{q\sqrt{(1-zz)}}$$

qua substitutione prior formula abit in

$$\frac{C\sqrt{2q}}{\sqrt{p}} \int \frac{dz}{\sqrt{(z-n)(1-z)(1+z)}}$$

S 2

in

in hanc $\frac{2DV_2p}{Vq} \int \frac{dzV(z-n)}{(1-zz)^{\frac{3}{2}}}$; cum vero sit $\int \frac{dzV(z-n)}{(1-zz)^{\frac{3}{2}}}$
 $= \frac{z\sqrt{(z-n)}}{\sqrt{(1-zz)}} - \frac{1}{2} \int \frac{z dz}{\sqrt{(z-n)(1-z)(1+z)}}$ etiam haec per superi-
 oria construi potest. Sicque in genere habetur con-
 structio huius formulae $\int \frac{dx(A+Bx)}{\sqrt{(a+\beta x+\gamma xx+\delta xx^2)}}$

Problema 1.

Integrationem huius formulae $\int \frac{dx}{\sqrt{(a+bx+cx^2+dx^3+ex^4)}}$
 per arcus sectionum conicarum perficere.

Solutio.

Quantitatem $a+bx+cx^2+dx^3+ex^4$ semper
 in duos factores trinomiales reales resolvere licet, qui
 sint $(\alpha+2\beta x+\gamma xx)$ et $(\delta+2\epsilon x+\zeta xx)$, ita ut
 habeatur haec formula integranda: $\int \frac{dx}{\sqrt{(\alpha+2\beta x+\gamma xx)(\delta+2\epsilon x+\zeta xx)}}$
 Ponatur $\delta+2\epsilon x+\zeta xx = (\alpha+2\beta x+\gamma xx)y$, ut
 formula proposita fiat $\int \frac{dx}{(\alpha+2\beta x+\gamma xx)\sqrt{y}}$. At aequatio
 assumpta per radicis extractionem praebet

$\epsilon+\zeta x-\beta y-\gamma xy \sqrt{(py+qy+r)}$,
 posito $p = \beta\beta - \alpha\gamma$; $q = \alpha\zeta - 2\beta\epsilon + \gamma\delta$; et $r = \epsilon\epsilon - \beta\zeta$.
 Tum vero eadem differentiata dat:

$$dx(\epsilon+\zeta x-\beta y-\gamma xy) = \frac{1}{2} dy(\alpha+2\beta x+\gamma xx)$$

seu $\frac{dx}{\alpha+2\beta x+\gamma xx} = \frac{\frac{1}{2} dy}{\epsilon+\zeta x-\beta y-\gamma xy}$

Quare.

Quare si pro hoc postremo denominatore valorem irrationalem modo inuentum substituamus, formula proposita abit in hanc:

$$\int \frac{dy}{\sqrt{y(py+qy+r)}}$$

cuius integratio per arcus sectionum conicarum supra est ostensa.

Hic igitur nascitur quaestio, quid tenendum sit de hac formula:

$$\int \frac{dx(A+Bx+Cxx)}{\sqrt{(a+bx+cy+dxx+ex^2)}}$$

Euidens enim est, non necesse esse, ut numeratori altiores potestates ipsius x tribuantur; quam etiam *Cell. d'Alambert* fatetur, se in genere ad rectificationem sectionum conicarum perducere non posse. Considerat quidem in Vol. IV. Mem. Acad. R. Berol. pag. 254 casum, quo $A=0$, $C=0$ et $a=0$, ita ut formula sit

$\int \frac{dx \sqrt{x}}{\sqrt{(b+cx+ax^2+ex^3)}}$ conaturque ostendere (pag. 257.) eius integrationem casu $dd=4ce$ per arcus sectionum conicarum absolui posse: verum methodus, qua vitur, negotium minime conficere videtur, uti rem accuratius perpendenti mox patebit. Transformationes autem, quas demceps tradit, casus nonnunquam hoc modo tractabiles suppeditant. Quocirca haec inuestigatio, uti est difficillima, merito omni attentione digna est censenda: vnde etiam mea tentamina super hac quaestione proposuisse iuuabit.

Problema 2.

Inueffigare conditiones, sub quibus integrationem huius formulae $\int \frac{dy(\mathcal{A} + \mathcal{D}y + \mathcal{H}yy)}{\sqrt{\mathcal{A}y^2 + 2\mathcal{B}y + \mathcal{C}y^2 + 2\mathcal{D}y + \mathcal{E}}}$ ad hanc simpliciore $\int \frac{dx(P + Qx + Rxx)}{\sqrt{(Ax^2 + Cxx + D)}}$ reducere liceat.

Solutio.

Statuatur inter variables x et y talis relatio:
 $axxy + 2xy(\beta x + \gamma y) + \delta xx + \epsilon yy + 2\zeta xy + 2\eta x + 2\theta y + \kappa = 0$,
 cuius coefficientes ita determinentur, vt sit

$$\begin{aligned} \beta\zeta - \alpha\eta - \gamma\delta &= 0; & \zeta\theta - \gamma\kappa - \epsilon\eta &= 0 \\ \gamma\gamma - \alpha\epsilon &= \mathcal{A}; & \gamma\zeta - \alpha\theta - \beta\epsilon &= \mathcal{B} \\ \eta\eta - \delta\kappa &= \mathcal{C}; & \zeta\eta - \beta\kappa - \delta\theta &= \mathcal{D} \end{aligned}$$

et $\zeta\zeta + 2\gamma\eta - \alpha\kappa - \delta\epsilon - 4\beta\theta = \mathcal{E}$

hincque erit pro denominatore transformatae:

$$\begin{aligned} \mathcal{A} &= \beta\beta - \alpha\delta; & \text{et } \mathcal{C} &= \zeta\zeta + 2\beta\theta - \alpha\kappa - \delta\epsilon - 4\gamma\eta \\ \mathcal{E} &= \theta\theta - \epsilon\kappa; \end{aligned}$$

Cum autem nouem habeantur litterae $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \kappa$, his septem conditionibus praescriptis vtique satisfieri poterit, relinqueturque adhuc vna arbitrio nostro determinanda. Si iam breuitatis gratia ponamus:

$\mathcal{A}y^2 + 2\mathcal{B}y + \mathcal{C}y^2 + 2\mathcal{D}y + \mathcal{E} = Y$ et $Ax^2 + Cxx + E = X$,
 resolutio aequationis assumtae praebet:

$$\begin{aligned} axxy + 2\beta xy + \delta x + \gamma yy + \zeta y + \eta &= \sqrt{Y} \\ axxy + 2\gamma xy + \epsilon y + \beta xx + \zeta x + \theta &= \sqrt{X} \end{aligned}$$

eius-

eiusque differentiatio ducit ad hanc aequationem :

$$\frac{dy}{\sqrt{y}} + \frac{dx}{\sqrt{x}} = 0. \text{ Ponamus ergo :}$$

$$\int \frac{d(\mathcal{P} + \mathcal{Q}y + \mathcal{R}yy)}{\sqrt{\mathcal{A}y^2 + 2\mathcal{B}y + \mathcal{C}}} = \mathcal{V} - \int \frac{dx(\mathcal{P} + \mathcal{Q}x + \mathcal{R}xx)}{\sqrt{\mathcal{A}x^2 + \mathcal{C}x^2 + \mathcal{E}}},$$

ac fit \mathcal{V} talis functio algebraica :

$$\mathcal{V} = mx + ny + pxy + \frac{1}{2}qxx + \frac{1}{2}ryy + txyy.$$

Hinc sumtis differentialibus terminisque homogeneis seorsim aequatis, reperientur sequentes determinaciones :

$$m = \frac{\beta \mathcal{R}}{\mathcal{A}}; n = \frac{\gamma \mathcal{R}}{\mathcal{A}}; p = \frac{\alpha \mathcal{R}}{\mathcal{A}}; q = 0, r = 0 \text{ et } t = 0,$$

praeterea vero haec determinatio accedit, vt fit $\mathcal{A}\mathcal{D} = \mathcal{B}\mathcal{R}$.

Deinde vero fit :

$$\mathcal{P} = \mathcal{P} + \frac{(\beta\theta - \gamma\eta)\mathcal{R}}{\mathcal{A}}; \mathcal{Q} = 0; \text{ et } \mathcal{R} = \frac{\mathcal{A}\mathcal{R}}{\mathcal{A}}$$

Definitis ergo coefficientibus $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \kappa$, quibus constat relatio inter x et y , ex iis innotescunt quantitates $\mathcal{A}, \mathcal{C}, \mathcal{E}$, quibus inuentis, si fuerit $\mathcal{A}\mathcal{D} = \mathcal{B}\mathcal{R}$, erit :

$$\int \frac{d(\mathcal{P} + \mathcal{Q}y + \mathcal{R}yy)}{\sqrt{\mathcal{A}y^2 + 2\mathcal{B}y + \mathcal{C}}} = \text{Const.} + \frac{\mathcal{R}}{\mathcal{A}}(\beta x + \gamma y + \alpha xy) - \int \frac{dx(\mathcal{P} + \frac{(\beta\delta + \gamma\eta)\mathcal{R}}{\mathcal{A}} + \frac{\mathcal{A}\mathcal{R}}{\mathcal{A}}xx)}{\sqrt{\mathcal{A}x^2 + \mathcal{C}x^2 + \mathcal{E}}}$$

Dummodo ergo fuerit $\mathcal{D} = \frac{\mathcal{B}\mathcal{R}}{\mathcal{A}}$, formulae propositae integratio reducta est ad hanc simpliciore : $\int \frac{dx(\mathcal{P} - \mathcal{R}xx)}{\sqrt{\mathcal{A}x^2 + \mathcal{C}x^2 + \mathcal{E}}}$

Corollarium I.

Determinatio coefficientium α, β, γ , etc. commodissime hoc modo instituetur : Primo quaeratur valor

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lor ipsius s ex hac aequatione:

$$\mathcal{C} = \frac{\mathfrak{B}\mathfrak{B} - \mathfrak{D}\mathfrak{D}s}{\mathfrak{A} - \mathfrak{E}s} + \frac{2\mathfrak{A}\mathfrak{D} - 2\mathfrak{B}\mathfrak{E}s}{\mathfrak{B} - \mathfrak{D}s}$$

quae, cum sit cubica, certe valorem realem pro s fuggerit: quo inuento, sumtaque ad arbitrium quantitate t , sit breuitatis gratia $\frac{\mathfrak{A} - \mathfrak{E}s}{\mathfrak{B} - \mathfrak{D}s} = u$, tum autem valores omnium ϑ coefficientium ita se habebunt:

$$\zeta = u \sqrt{\frac{\mathfrak{A}\mathfrak{B} - s\mathfrak{A}\mathfrak{D}s + 2\mathfrak{B}\mathfrak{E}s - \mathfrak{D}\mathfrak{E}s^2}{s - uu}}$$

$$\gamma = \frac{\zeta}{2u}; \quad \alpha = \frac{u}{2t(s - uu)}$$

$$\eta = \frac{\zeta}{2u}; \quad \delta = \frac{u}{2st(s - uu)}$$

$$\beta = \frac{1}{2t(s - uu)}; \quad \vartheta = \frac{1}{2}t(2(\mathfrak{A} + \mathfrak{E}ss) - \frac{1}{u}(\mathfrak{B} + \mathfrak{D}s))$$

$$e = \frac{1}{2}t(4\mathfrak{A}u - 3\mathfrak{B}s + \mathfrak{D}ss); \quad \kappa = \frac{1}{2}t(4\mathfrak{E}su + \mathfrak{B} - 3\mathfrak{D}s).$$

Coroll. 2.

Alio adhuc modo idem praestari potest. Extracto scilicet, ut ante, valore s ex hac aequatione:

$$\mathcal{C} = \frac{\mathfrak{B}\mathfrak{B} - \mathfrak{D}\mathfrak{D}s}{\mathfrak{A} - \mathfrak{E}s} + \frac{2\mathfrak{A}\mathfrak{D} - 2\mathfrak{B}\mathfrak{E}s}{\mathfrak{B} - \mathfrak{D}s}$$

positoque breuitatis gratia $\frac{\mathfrak{A} - \mathfrak{E}s}{\mathfrak{B} - \mathfrak{D}s} = u$, et, sumto t pro arbitrio, erit:

$$\alpha = -\frac{1}{4tu}; \quad \beta = 0; \quad \gamma = \frac{1}{2}\sqrt{\frac{\mathfrak{B} + \mathfrak{D}s}{u}}; \quad \delta = \frac{1}{4tsu}$$

$$e = t(4\mathfrak{A}u - \mathfrak{B}s - \mathfrak{D}ss); \quad \zeta = \sqrt{u\frac{\mathfrak{B} + \mathfrak{D}s}{s}}; \quad \eta = \frac{1}{2}\sqrt{\frac{\mathfrak{B} + \mathfrak{D}s}{u}}$$

$$\theta = 2tu; \quad \kappa = t(\mathfrak{B} + \mathfrak{D}s - 4\mathfrak{E}su).$$

Coroll.

Coroll. 3.

Si fuerit $\mathcal{A} : \mathcal{C} = \mathcal{B}\mathcal{B} : \mathcal{D}\mathcal{D}$, aequatio cubica valori s definiendo fit inepta. Hoc autem incommodum facile tollitur, transformanda formula differentiali per positionem $y = y + a$; qua etiam forma numerationis non turbatur.

Scholion.

Posito $\mathcal{M} = n\mathcal{A}$, et $\mathcal{D} = n\mathcal{B}$, integratio huius formulae :

$$\int \frac{dy (\mathcal{B} + n\mathcal{B}y + n\mathcal{A}yy)}{\sqrt{\mathcal{A}y^4 + 2\mathcal{B}y^3 + \mathcal{C}y^2 + 2\mathcal{D}y + \mathcal{E}}},$$

semper reduci potest ad integrationem talis :

$$\int \frac{dx (P + Rxx)}{\sqrt{(Ax^2 + Cxx + E)}}$$

quae, si denominator $Ax^2 + Cxx + E$ in huiusmodi duos factores reales $(f + gxx)(b + kxx)$ se resolui patitur; per rectificationem sectionum conicarum conficitur; at, si talis resolutio non succedit, sequenti artificio negotium absolui poterit.

Problema 3.

Si in formula $\int \frac{dx (P + Rxx)}{\sqrt{(Ax^2 + Cxx + E)}}$ quantitas $Ax^2 + Cxx + E$ in factores reales huiusmodi $(f + gxx)(b + kxx)$ resolui nequeat, eam in aliam transformare, quae per arcus sectionum conicarum certo integrari queat.

Solutio.

Inducatur alia variabilis z , cuius ratio ad x hac aequatione exprimatur:

$$4Exxz^2 - 4xxzz\sqrt{AE} - 4Ezz^2 + 2\sqrt{AE} - C = 0$$

ubi \sqrt{AE} erit utique quantitas realis, si quidem $Ax^2 + Cxx + E$ non habeat factores binomios reales. Hanc autem fiet:

$$\int \frac{dx(P + Rxx)}{\sqrt{(Ax^2 + Cxx + E)}} = \text{Const.} + \frac{Rx}{\sqrt{A}} - \frac{2R\sqrt{E}}{A} xz - 2 \int \frac{dz(P - \frac{R\sqrt{E}}{\sqrt{A}} + \frac{2ER}{A} z^2)}{\sqrt{(4Ez^2 + (C - 6\sqrt{AE})zz + 2A - \frac{C\sqrt{A}}{\sqrt{E}})}}$$

in qua noua formula quantitas, in denominatore contenta, certe in duos factores binomios reales est resoluibilis, cum sit $(C - 6\sqrt{AE})^2 > 16E(2A - \frac{C\sqrt{A}}{\sqrt{E}})$; propterea quod hinc sequitur $CC + 4C\sqrt{AE} + 4AE = (C + 2\sqrt{AE})^2 > 0$.

Aliter.

Habeat noua variabilis z ad x talem relationem:

$$2Exxz^2 - Cxxzz + \frac{CC - 4AE}{4E} xx - 2Ezz = 0$$

eritque:

$$\int \frac{dx(P + Rxx)}{\sqrt{(Ax^2 + Cxx + E)}} = \frac{CR}{2A\sqrt{E}} x^2 - \frac{2R\sqrt{E}}{A} xz - 2 \int \frac{dz(P - \frac{CR}{A} + \frac{2ER}{A} z^2)}{\sqrt{(4Ez^2 - 2Czz + \frac{CC - 4AE}{4E})}}$$

cuius

cuius denominator pariter certe in factores reales binomios est resolubilis.

Conclusio.

His demonstratis manifestum est, hanc formulam:

$$\int \frac{dy (\mathcal{P} + n \mathcal{B}y + n^2 \mathcal{A}yy)}{\sqrt{(\mathcal{A}y^2 + 2\mathcal{B}y + \mathcal{C})^2 + \mathcal{D}y + \mathcal{E}}}$$

semper per arcus sectionum conicarum construi posse. Cum igitur denominator semper in duos factores trinomiales reales resolui possit, hac formula ita exhiberi potest:

$$\int \frac{dy (\mathcal{A} + n(\alpha\epsilon + \beta\delta)y + n^2\delta yy)}{\sqrt{(\alpha yy + 2\beta y + \gamma)(\delta yy + 2\epsilon y + \zeta)}}$$

cuius ergo eadem datur constructio. Porro augendo vel diminuendo y quantitate constante, formula nostra etiam ita repraesentari potest:

$$\int \frac{dy (M + Nyy)}{\sqrt{(Ay^2 + Cy + 2Dy + E)}}$$

In his autem fere omnes casus, quos quidem per rectificationem sectionum conicarum integrale licet, contineri videntur. Sed in medium afferamus adhuc aliam reductionem.

Problema IV.

Inuestigare conditiones, sub quibus integrationem huius formulae:

$$\int \frac{dy (\mathcal{P} + \mathcal{Q}y + \mathcal{R}yy)}{\sqrt{(\mathcal{A}y^2 + 2\mathcal{B}y + \mathcal{C})^2 + \mathcal{D}y + \mathcal{E}}} \text{ ad hanc simpliciolem}$$

$$\int \frac{dx (P + Qx + Rxx)}{\sqrt{(2Bx^2 + Cx^2 + 2Dx)}} \text{ perducere liceat:}$$

T 2

Solu-

Solutio.

Statuatur inter variables x et y talis relatio :

$$\alpha xxyy + 2xy(\beta x + \gamma y) + \delta xx + \varepsilon yy + 2\zeta xy + 2\eta x + 2\theta y + \kappa = 0,$$

cuius coefficientes ita determinentur, ut sit :

$$\beta\beta - \alpha\delta = 0; \quad \gamma\gamma - \alpha\varepsilon = \mathcal{A}; \quad \gamma\zeta - \alpha\theta - \beta\varepsilon = \mathcal{B}$$

$$\theta\theta - \varepsilon\kappa = 0; \quad \eta\eta - \delta\kappa = \mathcal{C}; \quad \zeta\eta - \beta\kappa - \delta\theta = \mathcal{D}$$

$$\text{atque } \zeta\zeta + 2\gamma\eta - \alpha\kappa - \delta\varepsilon - 4\beta\theta = \mathcal{E},$$

quem in finem definiatur primo p ex hac aequatione cubica :

$$p^3 - \mathcal{E}pp - (\mathcal{A}\mathcal{C} - \mathcal{B}\mathcal{D})p + \frac{1}{2}(\mathcal{C}\mathcal{A}\mathcal{E} - \mathcal{A}\mathcal{D}\mathcal{D} - \mathcal{B}\mathcal{B}\mathcal{E}) = 0$$

Deinde, pro lubitu sumto numero m , definiatur q ex hac aequatione quadratica : $qq - q(\mathcal{D}m - \mathcal{B}) + (m\mathcal{C} - p)(mp - \mathcal{A}) = 0$, quo facto, si denuo numerus arbitarius accipiatur n , erit :

$$\beta = \frac{n(m\mathcal{E} - p)}{\sqrt{(2mp - \mathcal{A} - mm\mathcal{E})}}; \quad \theta = \frac{mp - \mathcal{A}}{n\sqrt{(2mp - \mathcal{A} - mm\mathcal{E})}}$$

$$\alpha = \frac{nq}{\sqrt{(2mp - \mathcal{A} - mm\mathcal{E})}}; \quad \kappa = \frac{q}{n\sqrt{(2mp - \mathcal{A} - mm\mathcal{E})}}$$

$$\delta = \frac{n(m\mathcal{E} + p)^2}{q\sqrt{(2mp - \mathcal{A} - mm\mathcal{E})}}; \quad \varepsilon = \frac{(mp - \mathcal{A})^2}{nq\sqrt{(2mp - \mathcal{A} - mm\mathcal{E})}}$$

$$\gamma = \frac{m\sqrt{(p - \mathcal{A}\mathcal{E})}}{\sqrt{(2mp - \mathcal{A} - mm\mathcal{E})}}; \quad \eta = \frac{\sqrt{(p - \mathcal{A}\mathcal{E})}}{\sqrt{(2mp - \mathcal{A} - mm\mathcal{E})}}$$

$$\text{et } \zeta = \frac{\mathcal{D}(mp - \mathcal{A}) - \mathcal{B}(m\mathcal{E} - p)}{\sqrt{(pp - \mathcal{A}\mathcal{E})(2mp - \mathcal{A} - mm\mathcal{E})}}$$

Quibus inuentis erit :

$$\mathcal{B} = \beta\zeta - \alpha\eta - \gamma\delta; \quad \mathcal{D} = \zeta\theta - \gamma\kappa - \varepsilon\eta$$

$$\text{et } \mathcal{C} = \zeta\zeta + 2\beta\theta - \alpha\kappa - \delta\varepsilon - 4\gamma\eta.$$

Ponatur iam :

$$\int \frac{dy(\mathcal{P} + \mathcal{D}y + \mathcal{R}yy)}{\sqrt{(\mathcal{A}y^2 + 2\mathcal{B}y + \mathcal{C})^2 + (\mathcal{E}y^2 + 2\mathcal{D}y + \mathcal{E})}} = \text{Const.} + mx + ny + pxy - \int \frac{dx(\mathcal{P} + \mathcal{Q}x + \mathcal{R}xx)}{\sqrt{(2\mathcal{B}x^2 + \mathcal{C}x^2 + 2\mathcal{D}x)}}$$

atque reperitur, vt ante,

$$m = \frac{\beta\mathcal{R}}{\mathcal{A}}; n = \frac{\gamma\mathcal{R}}{\mathcal{A}}; \text{ et } p = \frac{\alpha\mathcal{R}}{\mathcal{A}}$$

deinde $\mathcal{P} = \mathcal{P} + \frac{(\beta\theta - \gamma\eta)\mathcal{R}}{\mathcal{A}}; \mathcal{D} = \frac{\mathcal{B}\mathcal{R}}{\mathcal{A}}$ et $\mathcal{R} = 0$.

Necesse autem est, vt in formula proposita sit $\mathcal{A}\mathcal{D} = \mathcal{B}\mathcal{R}$, neque ergo haec reductio nouos casus suppeditat. At posito $x = zz$, formula transformata abit in hanc :

$$- 2 \int \frac{dz(\mathcal{P} + \mathcal{Q}zz)}{\sqrt{(2\mathcal{B}zz^2 + \mathcal{C}zz^2 + 2\mathcal{D})}}$$

quae reductio saepe facilius succedit, quam praecedens.