ANNOTATIONES IN LOCVM QVENDAM CARTESII AD CIRCVLI QVADRATVRAM SPECTANTEM.

Auctore

L. EVLERO.

In excerptis ex Manuscriptis *Cartesii* paucis quidem verbis refertur constructio quaedam geometrica promtissime ad circuli veram dimensionem appropinquans, sed quae siue *Cartesius* ipse eam inuenerit, siue ab alio habuerit communicatam, acutissimum inuentoris ingenium, illo praesertim tempore, luculenter declarat. Qui deinceps hoc idem argumentum pertractarunt, quantum equidem memini, nullam huius eximiae constructionis mentionem faciunt, vt periculum sit, ne tandem penitus obliuione obruatur. Demonstratio quidem, quae non adiuncta reperitur, haud difficulter suppletur; verum non solum elegantia constructionis vberiorem explicationem meretur, sed etiam tam insignes conclusiones inde deduci possunt, quae per se omni attentione dignae videntur. Pulcherrima autem haec constructio ipsis *Cartesii* verbis ita est proposita:

„Ad quadrandum circulum nihil aptius inuenio, Tab. I.
„quam si dato quadrato *bf* adiungatur rectangulum *cg* Fig. 1.
„comprehensum sub lineis *ac* et *bc*, quod sit aequale
„quartae parti quadrati *bf*: item rectangulum *db* fa-
„ctum ex lineis *da*, *dc*, aequale quartae parti praece-

 „dentis;

„dentis; et eodem modo rectangulum *ei*, atque alia „infinita vsque ad *x*: et erit haec linea *ax* diameter „circuli, cuius circumferentia aequalis est circumferen„tiae quadrati *bf*.

Vis igitur huius constructionis in hoc consistit, vt continua appositione istiusmodi rectangulorum *cg*, *db*, *ei*, etc. quorum anguli superiores dextri in diagonalem quadrati productam cadunt, tandem ad punctum *x* perueniatur, quo terminatur diameter circuli *ax*, cuius peripheria aequalis est perimetro quadrati *bf*, seu quadruplo rectae *ab*.

Cum horum rectangulorum quodque aequetur parti quartae praecedentis, iam ipse *Cartesius* obseruat, summam omnium horum rectangulorum aequalem fore parti tertiae quadrati *bf*; quod quidem manifestum est, cum huius seriei $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+$ etc. in infinitum continuatae summa sit $=\frac{1}{3}$.

Praeterea etiam *Cartesius* indicat rationem, cui haec constructio innititur; concipit scilicet polygona regularia 8, 16, 32, 64 etc. laterum, quorum perimetri sint inter se aequales simulque perimetro quadrati *bf*. Iam cum *ab* sit diameter circuli quadrato inscripti, ita affirmat fore *ac* diametrum circuli octogono inscripti, tum vero *ad* diametrum circuli 16gono, *ae* 32gono inscripti, et ita porro. Vnde liquet *ax* fore diametrum circuli polygono infinitorum laterum regulari inscripti, ideoque eius peripheriam aequari perimetro quadrati.

Quo facilius demonstrationem huius constructionis adornem, obseruo, quae hic de diametris circulorum dicuntur, etiam valere pro radiis, ita vt *ab*, *ac*, *ad*, *ae* *ae* etc. ſpectari poſſint tanquam radii circulorum, quibus ſi circumſcribantur polygona regularia 4, 8, 16, 32 etc. laterum, eorum perimetri futurae ſint inter ſe aequales.

Problema.

Dato circulo, cui polygonum [illegible] que ſit circumſcriptum, inuenire circulum alium, cui ſi polygonum regulare duplo plurium laterum circumſcribatur, perimeter huius polygoni aequalis ſit futura perimetro illius polygoni.

Solutio.

Sit ENM circulus datus et EP ſemilatus polygoni ipſi circumſcripti, centro exiſtente in C; CF autem ſit radius circuli quaeſiti, et FQ ſemilatus polygoni ipſi circumſcribendi. Neceſſe ergo eſt, vt ſit FQ ſemiſſis ipſius EP, et angulus FCQ ſemiſſis anguli ECP. Quare recta CQ angulum ECP, et recta QO ipſi CE parallela lineam EP biſecabit. Cum nunc Fig. 2.

ſit $EV : CE = FQ : CF$

et $EV : CE = EP : CE + CP$

erit $FQ : CF = EP : CE + CP$

ſed quia $FQ = \frac{1}{2} EP$, erit etiam $CF = \frac{1}{2}(CE + CP)$. Hinc auferatur CF, et habebitur $EF = \frac{1}{2}(CP - CE)$ ex quo erit rectangulum $CF . EF = \frac{1}{4}(CP^2 - CE^2) = \frac{1}{4} EP^2$ ideoque punctum F ita definiri debet, vt ſit rectangulum, ſub CF et EF comprehenſum, aequale parti quartae quadrati rectae EP, ſeu ipſi quadrato rectae FQ.

Coroll. 1.

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Cum sit $CF.EF = FQ^2$ erit $CF : FQ = FQ : EF$, vnde ducta recta QE, fiet triangulum FQE simile triangulo FCQ, vel ECV, ideoque angulus FQE aequalis angulo ECV.

Coroll. 2.

Cum sit $CE : EV = EO : EF$, punctum F etiam ita definiri poterit: ex O ducatur recta ad CV productam normalis, eaque basi CE in F occurret.

Coroll. 3.

Si polygonum circulo ENM circumscriptum sit n laterum, erit angulus $ECP = \frac{\pi}{n}$, denotante π mensuram duorum angulorum rectorum; et angulus $FCQ = \frac{\pi}{2n}$. Hinc si radius $CE = r$, erit $EP = r\,\text{tang.}\frac{\pi}{n}$ et $FQ = \frac{1}{2}r\,\text{tang.}\frac{\pi}{n}$.

Coroll. 4.

Iam quia angulus $FQE = \frac{\pi}{2n}$ erit $EF = FQ\,\text{tang.}\frac{\pi}{2n} = \frac{1}{2}r\,\text{tang.}\frac{\pi}{n}\,\text{tang.}\frac{\pi}{2n}$. Verum si vocemus $CF = s$, erit $FQ = s\,\text{tang.}\frac{\pi}{2n}$, vnde ob $FQ = \frac{1}{2}r\,\text{tang.}\frac{\pi}{n}$ fiet $s = \frac{1}{2}r\,\text{tang.}\frac{\pi}{n}\,\text{cot.}\frac{\pi}{2n}$.

Demonstratio Constructionis Cartesianae.

Fig. 3. Sit iam CE radius circuli quadrato inscripti, CF octogono inscripti, CG polygono regulari 16 laterum, CH

CH polygono 32 laterum et ita porro. Sit porro EP semilatus quadrati, FQ semilatus octogoni, GR polygoni 16, HS polygoni 32 laterum, etc. et quia haec polygona eiusdem perimetri assumuntur, erit $FQ = \frac{1}{2}EP$; $GR = \frac{1}{2}FQ = \frac{1}{4}EP$; $HS = \frac{1}{2}GR = \frac{1}{4}FQ = \frac{1}{8}EP$, etc. Iam ex problemate praemisso est $CF.EF = \frac{1}{4}EP^2 = FQ^2$; tum vero ex eodem simili modo

$$CG.FG = \tfrac{1}{4}FQ^2 = \tfrac{1}{4}CF.EF = GR^2$$

$$CH.GH = \tfrac{1}{4}GR^2 = \tfrac{1}{4}CG.FG = HS^2 \text{ etc.}$$

sicque puncta F, G, H etc. eodem plane modo determinantur, vti habet constructio *Cartesiana*; et quia interualla EF, FG, GH etc. continuo fiunt minora, satis promte ad punctum vltimum x appropinquatur, eritque Cx radius circuli, cuius peripheria aequatur perimetro polygonorum praecedentium, ideoque rectae EP octies sumtae. Q. E. D.

Coroll. 1.

Si ponatur $CE = a$, $CF = b$, $CG = c$, $CH = d$, etc. progressio harum quantitatum ita est comparata, vt sit ob $EP = a$

$$b(b-a) = \tfrac{1}{4}aa;\; c(c-b) = \tfrac{1}{4}b(b-a);\; d(d-c) = \tfrac{1}{4}c(c-b) \text{ etc.}$$

ideoque

$$b = \frac{a + \sqrt{2aa}}{2};\; c = \frac{b + \sqrt{(2bb - ab)}}{2};\; d = \frac{c + \sqrt{(2cc - bc)}}{2} \text{ etc.}$$

et harum quantitatum infinitesima est radius circuli cuius peripheria est $= 8a$.

Coroll. 2.

Cum ſit angulus ECP ſemirectus, ſeu $ECP = \frac{\pi}{4}$, erunt anguli $FCQ = \frac{\pi}{8}$; $GCR = \frac{\pi}{16}$; $HCS = \frac{\pi}{32}$, etc. Quare ob $EP = a$; $FQ = \frac{1}{2}a$; $GR = \frac{1}{4}a$; $HS = \frac{1}{8}a$ etc. erit per cotangentes

$$CE = a\cot.\tfrac{\pi}{4};\ CF = \tfrac{1}{2}a\cot.\tfrac{\pi}{8};\ CG = \tfrac{1}{4}a\cot.\tfrac{\pi}{16};\ CH = \tfrac{1}{8}a\cot.\tfrac{\pi}{32} \text{ etc.}$$

Vnde denotante n numerum infinitum, fit harum linearum vltima $= \frac{1}{n}a\cot.\frac{\pi}{4n}$.

Coroll. 3.

Sed $\cot.\frac{\pi}{4n} = 1 : \text{tang}.\frac{\pi}{4n}$; et quia angulus $\frac{\pi}{4n}$ eſt infinite paruus, erit $\text{tang}.\frac{\pi}{4n} = \frac{\pi}{4n}$, ideoque $\cot.\frac{\pi}{4n} = \frac{4n}{\pi}$. Quare linearum illarum vltima fit $= \frac{4a}{\pi}$, quo radio ſi circulus deſcribatur, erit eius peripheria $= 2\pi . \frac{4a}{\pi} = 8a$.

Coroll. 4.

Deinde quia ex coroll. 4. praec. probl. eſt $EF = FQ\,\text{tang}.\,FCQ$, erit ob eandem rationem:

$$FG = GR.\,\text{tang}.\,GCR;\ GH = HS.\,\text{tang}.\,HCS \text{ etc.}$$

vnde haec interualla ſequenti modo exprimentur:

$$EF = \tfrac{1}{2}a\,\text{tang}.\tfrac{\pi}{8};\ FG = \tfrac{1}{4}a\,\text{tang}.\tfrac{\pi}{16};\ GH = \tfrac{1}{8}a\,\text{tang}.\tfrac{\pi}{32} \text{ etc.}$$

antecedens vero ad analogiam $CE = a\,\text{tang}.\frac{\pi}{4} = a$.

Coroll. 5.

His cum praecedentibus collatis nanciſcemur:

$$CF = a(\text{tang}.\tfrac{\pi}{4} + \tfrac{1}{2}\text{tang}.\tfrac{\pi}{8}) = \tfrac{1}{2}a\cot.\tfrac{\pi}{8}$$

$$CG = a(\text{tang}.\tfrac{\pi}{4} + \tfrac{1}{2}\text{tang}.\tfrac{\pi}{8} + \tfrac{1}{4}\text{tang}.\tfrac{\pi}{16}) = \tfrac{1}{4}a\cot.\tfrac{\pi}{16}$$

$$CH = a(\text{tang}.\tfrac{\pi}{4} + \tfrac{1}{2}\text{tang}.\tfrac{\pi}{8} + \tfrac{1}{4}\text{tang}.\tfrac{\pi}{16} + \tfrac{1}{8}\text{tang}.\tfrac{\pi}{32}) = \tfrac{1}{8}a\cot.\tfrac{\pi}{32}$$

etc.

ſicque

ſicque omnium huiusmodi progreſſionum ſummae expedite aſſignari poſſunt.

Coroll. 6.

In infinitum ergo progrediendo obtinebimus ſummationem huius ſeriei:

$$\text{tang.}\,\tfrac{\pi}{4}+\tfrac{1}{2}\,\text{tang.}\,\tfrac{\pi}{8}+\tfrac{1}{4}\,\text{tang.}\,\tfrac{\pi}{16}+\tfrac{1}{8}\,\text{tang.}\,\tfrac{\pi}{32}+\text{etc.}=\tfrac{4}{\pi}$$

quae ergo per quadraturam circuli determinatur. Hinc occaſionem arripio ſequens problema ſoluendi.

Problema.

Denotante Φ arcum quemcunque circuli cuius radius $=1$, inuenire ſummam huius ſeriei infinitae:

$$\text{tang.}\,\Phi+\tfrac{1}{2}\,\text{tang.}\,\tfrac{1}{2}\Phi+\tfrac{1}{4}\,\text{tang.}\,\tfrac{1}{4}\Phi+\tfrac{1}{8}\,\text{tang}\,\tfrac{1}{8}\Phi+\tfrac{1}{16}\,\text{tang.}\,\tfrac{1}{16}\Phi\ \text{etc.}$$

Solutio.

Si in fig. 2. vti ſupra eſt conſtructa, ponatur an- Tab. I.
gulus $ECP=\Phi$, erit $FCQ=\frac{1}{2}\Phi$: iam poſito $FQ=1$ Fig. 2.
erit $EP=2$, hincque $CE=2\,\text{cot.}\,\Phi$; $CF=\text{cot.}\,\frac{1}{2}\Phi$ et $EF=\text{tang.}\,\frac{1}{2}\Phi$, ex quo habetur:

$$2\,\text{cot.}\,\Phi=\text{cot.}\,\tfrac{1}{2}\Phi-\text{tang.}\,\tfrac{1}{2}\Phi \quad \text{et} \quad \text{tang.}\,\tfrac{1}{2}\Phi=\text{cot.}\,\tfrac{1}{2}\Phi-2\,\text{cot.}\,\Phi$$

eodemque modo $\text{tang.}\,\Phi=\text{cot.}\,\Phi-2\,\text{cot.}\,2\Phi$. Collocentur hi valores tangentium per cotangentes expreſſi in ſerie propoſita

$$\begin{aligned}
\text{tang.}\,\Phi&=\text{cot.}\,\Phi-2\,\text{cot.}\,2\Phi\\
\tfrac{1}{2}\,\text{tang.}\,\tfrac{1}{2}\Phi&=\tfrac{1}{2}\,\text{cot}\,\tfrac{1}{2}\Phi-\text{cot.}\,\Phi\\
\tfrac{1}{4}\,\text{tang.}\,\tfrac{1}{4}\Phi&=\tfrac{1}{4}\,\text{cot}\,\tfrac{1}{4}\Phi-\tfrac{1}{2}\,\text{cot.}\,\tfrac{1}{2}\Phi\\
\tfrac{1}{8}\,\text{tang.}\,\tfrac{1}{8}\Phi&=\tfrac{1}{8}\,\text{cot.}\,\tfrac{1}{8}\Phi-\tfrac{1}{4}\,\text{cot.}\,\tfrac{1}{4}\Phi\\
&\text{etc.}
\end{aligned}$$

et colligendo consequemur:

$$\mathrm{tang.}\,\Phi = \mathrm{cot.}\,\Phi - 2\,\mathrm{cot.}\,2\Phi$$

$$\mathrm{tang.}\,\Phi + \tfrac{1}{2}\mathrm{tang}\,\tfrac{1}{2}\Phi = \tfrac{1}{2}\mathrm{cot.}\,\tfrac{1}{2}\Phi - 2\,\mathrm{cot.}\,2\Phi$$

$$\mathrm{tang}\,\Phi + \tfrac{1}{2}\mathrm{tang}\,\tfrac{1}{2}\Phi + \tfrac{1}{4}\mathrm{tang}\,\tfrac{1}{4}\Phi = \tfrac{1}{4}\mathrm{cot.}\,\tfrac{1}{4}\Phi - 2\,\mathrm{cot.}\,2\Phi$$

$$\mathrm{tang.}\,\Phi + \tfrac{1}{2}\mathrm{tang}\,\tfrac{1}{2}\Phi + \tfrac{1}{4}\mathrm{tang.}\,\tfrac{1}{4}\Phi + \tfrac{1}{8}\mathrm{tang.}\,\tfrac{1}{8}\Phi = \tfrac{1}{8}\mathrm{cot.}\,\tfrac{1}{8}\Phi - 2\,\mathrm{cot.}\,2\Phi$$

etc.

vnde in infinitum progrediendo, si n denotet numerum infinitum, quia $\mathrm{tang.}\,\frac{1}{n}\Phi = \frac{\Phi}{n}$, hincque $\mathrm{cot.}\,\frac{1}{n}\Phi = \frac{n}{\Phi}$, erit: $\frac{1}{n}\mathrm{cot}\,\frac{1}{n}\Phi = \frac{1}{\Phi}$, ideoque summa seriei propositae:

$$\mathrm{tang}\,\Phi + \tfrac{1}{2}\mathrm{tang}\,\tfrac{1}{2}\Phi + \tfrac{1}{4}\mathrm{tang}\,\tfrac{1}{4}\Phi + \tfrac{1}{8}\mathrm{tang.}\,\tfrac{1}{8}\Phi + \mathrm{etc.} = \tfrac{1}{\Phi} - 2\,\mathrm{cot.}\,2\Phi$$

Vnde si 2Φ est angulus rectus, seu $\Phi = \frac{\pi}{4}$, ob $\mathrm{cot}\,\frac{\pi}{2} = 0$ fit seriei summa $= \frac{1}{\Phi} = \frac{4}{\pi}$, qui est casus supra tractatus.

Ex hac serie plures aliae deriuari possunt non minus notitu dignae.

I. Ex eius differentiatione adipiscimur:

$$\frac{1}{\mathrm{cos.}\,\Phi^2} + \frac{1}{4\,\mathrm{cos.}\,\frac{1}{2}\Phi^2} + \frac{1}{4^2\,\mathrm{cos.}\,\frac{1}{4}\Phi^2} + \frac{1}{4^3\,\mathrm{cos}\,\frac{1}{8}\Phi^2} + \frac{1}{4^4\,\mathrm{cos}\,\frac{1}{16}\Phi}$$
$$+ \mathrm{et} = -\frac{1}{\Phi\Phi} + \frac{4}{\mathrm{sin.}\,2\Phi^2}$$

vel cum sit $\frac{1}{\mathrm{cos.}\,\Phi} = \mathrm{sec.}\,\Phi$ erit quoque

$$(\mathrm{sec.}\,\Phi)^2 + \tfrac{1}{4}(\mathrm{sec.}\,\tfrac{1}{2}\Phi)^2 + \tfrac{1}{4^2}(\mathrm{sec}\,\tfrac{1}{4}\Phi)^2 + \tfrac{1}{4^3}(\mathrm{sec.}\,\tfrac{1}{8}\Phi)^2 + \mathrm{etc.}$$
$$= \frac{1}{\mathrm{sin.}\,\Phi^2\,\mathrm{cos.}\,\Phi^2} - \frac{1}{\Phi\Phi}$$

II. Deinde ob $\mathrm{cos.}\,\Phi^2 = \frac{1+\mathrm{cos.}\,2\Phi}{2}$, et $\mathrm{sin.}\,2\Phi^2 = \frac{1-\mathrm{cos.}\,4\Phi}{2}$ erit vbique per 2 diuidendo:

$$\frac{1}{1+\mathrm{cos.}\,2\Phi} + \frac{1}{4(1+\mathrm{cos.}\,\Phi)} + \frac{1}{4^2(1+\mathrm{cos.}\,\frac{1}{2}\Phi)}$$
$$+ \frac{1}{4^3(1+\mathrm{cos.}\,\frac{1}{8}\Phi)} + \mathrm{etc.} = \frac{2}{1-\mathrm{cos.}\,4\Phi} - \frac{1}{2\Phi\Phi}.$$

seu

seu pro Φ scribendo $\frac{1}{2}\Phi$

$$\frac{1}{1+\operatorname{cos}.\Phi}+\frac{1}{4(1+\operatorname{cos}.\frac{1}{2}\Phi)}+\frac{1}{4^2(1+\operatorname{cos}.\frac{1}{4}\Phi)}$$

$$+\frac{1}{4^3(1+\operatorname{cos}.\frac{1}{8}\Phi)}+\text{etc.}=\frac{2}{1-\operatorname{cos}.2\Phi}-\frac{2}{\Phi\Phi}.$$

III. Si series inuenta per $d\Phi$ multiplicetur et integretur, ob $\int d\Phi \operatorname{tang}.\Phi=\int\frac{d\Phi \operatorname{sin}.\Phi}{\operatorname{cos}.\Phi}=-l\operatorname{cos}.\Phi$, et $\int 2d\Phi\operatorname{cot}.2\Phi=l\operatorname{sin}.2\Phi$, habebitur

$$-l\operatorname{cos}.\Phi-l\operatorname{cos}.\tfrac{1}{2}\Phi-l\operatorname{cos}.\tfrac{1}{4}\Phi-l\operatorname{cos}.\tfrac{1}{8}\Phi-l\operatorname{cos}.\tfrac{1}{16}\Phi-\text{etc.}$$
$$=l\Phi-l\operatorname{sin}.2\Phi+\text{Const.}$$

ad quam constantem definiendam ponamus $\Phi=0$, et quia $l\operatorname{cos}.0=l\,1=0$, ex priori parte habemus 0, ex posteriori vero ob $\operatorname{sin}.2\Phi=2\Phi$, habemus $l\Phi-l2\Phi+\text{Const.}=-l2+\text{Const.}$ vnde $\text{Const.}=l2$. Hinc ad numeros progrediendo erit:

$$\frac{1}{\operatorname{cos}.\Phi\operatorname{cos}.\frac{1}{2}\Phi\operatorname{cos}.\frac{1}{4}\Phi\operatorname{cos}.\frac{1}{8}\Phi\operatorname{cos}.\frac{1}{16}\Phi\text{ etc.}}=\frac{2\Phi}{\operatorname{sin}.2\Phi}$$

IV. Cum sit $\frac{1}{\operatorname{cos}.\Phi}=\operatorname{sec}.\Phi$, habebitur etiam hoc Theorema pro secantibus:

$$\operatorname{sec}.\Phi\operatorname{sec}.\tfrac{1}{2}\Phi\operatorname{sec}.\tfrac{1}{4}\Phi\operatorname{sec}.\tfrac{1}{8}\Phi\operatorname{sec}.\tfrac{1}{16}\Phi\text{ etc.}=\frac{2\Phi}{\operatorname{sin}.2\Phi}$$

vnde si ratio diametri ad peripheriam ponatur $=1:\pi$ et q denotet angulum rectum, si statuamus $2\Phi=q$, $=\frac{\pi}{2}$ erit:

$$\operatorname{sec}.\tfrac{1}{2}q\operatorname{sec}.\tfrac{1}{4}q\operatorname{sec}.\tfrac{1}{8}q\operatorname{sec}.\tfrac{1}{16}q\operatorname{sec}.\tfrac{1}{32}q\text{ etc.}=\tfrac{\pi}{2}.$$

Problema.

Innenire feriem quantitatum: a, b, c, d, e, f; etc. cuius haec fit proprietas, vt fit: $c(c-b)=\frac{1}{4}b(b-a)$; $d(d-c)=\frac{1}{4}c(-b)$; $e(e-d)=\frac{1}{4}d(d-c)$ etc. feu vt quantitates inde deriuatae

$$b(b-a);\ c(c-b);\ d(d-c);\ e(e-d);\ f(f-e),\ \text{etc.}$$

decrefcant fecundum rationem quadruplam.

Solutio.

Cum fit $\text{tang.}\,\frac{1}{2}\Phi=\cot.\frac{1}{2}\Phi-2\cot.\Phi$, fi multiplicemus vtrinque per $\cot.\frac{1}{2}\Phi$, ob $\text{tang}\,\frac{1}{2}\Phi\,\cot.\frac{1}{2}\Phi=1$ erit $\cot.\frac{1}{2}\Phi(\cot.\frac{1}{2}\Phi-2\cot.\Phi)=1$. Statuatur ergo $a=r\cot.\Phi$; $b=\frac{1}{2}r\cot.\frac{1}{2}\Phi$; $c=\frac{1}{4}r\cot.\frac{1}{4}\Phi$; $d=\frac{1}{8}r\cot.\frac{1}{8}\Phi$, etc. eritque

$$\frac{2b}{r}\left(\frac{2b}{r}-\frac{2a}{r}\right)=1 \quad \text{hinc } b(b-a)=\frac{rr}{4}$$
$$\frac{4c}{r}\left(\frac{ac}{r}-\frac{4b}{r}\right)=1 \quad \text{hinc } c(c-b)=\frac{rr}{4^2}$$
$$\frac{8d}{r}\left(\frac{8d}{r}-\frac{8c}{r}\right)=1 \quad \text{hinc } d(d-c)=\frac{rr}{4^3}$$

etc.

Quare haec feries

$$a=r\cot\Phi;\ b=\tfrac{1}{2}r\cot.\tfrac{1}{2}\Phi;\ c=\tfrac{1}{4}r\cot.\tfrac{1}{4}\Phi;\ d=\tfrac{1}{8}r\cot.\tfrac{1}{8}\Phi;\ \text{etc.}$$

hanc habet proprietatem, vt quantitates inde formatae

$$b(b-a);\ c(c-b);\ d(d-c);\ e(e-d);\ \text{etc.}$$

in ratione quadrupla decrefcant.

Coroll. 1.

Coroll. 1.

Datis duobus terminis primis a et b reliqui c, d, e, f inde ſucceſſiue ita determinantur, vt ſit $c = \frac{b + \sqrt{(2bb - ab)}}{2}$; $d = \frac{c + \sqrt{(2cc - bc)}}{2}$; $e = \frac{d + \sqrt{(2dd - cd)}}{2}$ etc. ideoque binis terminis initialibus pro lubitu aſſumtis, tota ſeries ope harum formularum exhiberi poteſt.

Coroll. 2.

Datis autem terminis a et b, inde angulus Φ cum quantitate r ita definitur, vt ſit:

$$\text{tang.}\,\Phi = \frac{2\sqrt{(bb - ab)}}{a} \text{ et } r = 2\sqrt{(bb - ab)}$$

vnde inuento angulo Φ reliqui termini etiam ita exprimuntur, vt ſit:

$$c = \tfrac{1}{4} r \cot. \tfrac{1}{4}\Phi;\ d = \tfrac{1}{8} r \cot. \tfrac{1}{8}\Phi;\ e = \tfrac{1}{16} r \cot. \tfrac{1}{16}\Phi \text{ etc.}$$

Coroll. 3.

Hinc iſtius ſeriei termini infiniteſimi fient $= \frac{r}{\Phi}$, ad quem valorem termini ſeriei ſatis cito conuergunt. Quaeratur ſcilicet in circulo radii $= 1$, arcus cuius tangens $= \frac{2\sqrt{(bb - ab)}}{a}$, qui arcus ſit $= \Phi$, et ſeriei noſtrae termini infiniteſimi erunt $= \frac{2\sqrt{(bb - ab)}}{\Phi}$.

Scholion.

Caeterum hic monuiſſe iuuabit puncta P, Q, R, S, x ſita eſſe in curua quadratrice veterum, propterea quod applicatae EP, FQ, GR, HS eandem inter ſe rationem tenent, quam anguli ECP, FCQ, Tab. I. Fig. 3.

FCQ, GCR, HCS etc. Et quoniam x, vbi haec curua in basin incidit, iam olim circuli quadraturam indicare eſt inuentum, vnde ei iſtud nomen eſt inditum, conſtructio Carteſii cum hac veterum quadratura egregie quidem conuenit; ſed multo commodius et accuratius puncta E, F, G, H etc. ſucceſſiue praebet, quam a continua biſectione angulorum expectari queat.

DEMON-