DE MOTV CORPORIS AD DVO CENTRA VIRIVM FIXA ATTRACTI.

Auctore

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1.

Cum nunc quidem nullum amplius dubium superſit, quin corpora coeleſtia perinde moueantur, ac ſi ſe mutuo attraherent in ratione reciproca duplicata diſtantiarum theoria Aſtronomiae ad ſummum faſtigium eueheretur, ſi motum quotcunque corporum ſe mutuo in iſta ratione attrahentium definire liceret. Hinc Aſtronomiae perfectio a Mechanica eſt expectanda, ex cuius principiis cum motus illi facile ad aequationes differentiales reuocentur, totum negotium ab Analyſi pendet, eaque eius parte, quae in reſolutione aequationum differentialium conſumitur. Quae ergo in Aſtronomia nondum ſatis ſunt explorata, eorum cognitio ex ſola Analyſi eſt haurienda, cuius adhuc inſignis promotio deſideratur, antequam vel vnici phaenomeni perfectam explicationem reddere valeamus.

2. Quod autem in hoc negotio adhuc praeſtare licuit, tam exiguam vniuerſae theoriae particulam complectitur, quae fere pro nihilo ſit reputanda; plus enim ab Auctoribus, qui hoc argumentum tractauerunt, non eſt effectum, quam vt motum tantum duorum

duorum corporum, quae se mutuo in ratione reciproca distantiarum duplicata attrahunt, accurate definire docuerint. Statim ac tria corpora se hac lege attrahentia proponuntur, qui tamen casus a scopo praefixo adhuc longissime abest, cum numerus corporum in mundo se mutuo attrahentium sit maximus; omnia artificia, quae quidem adhuc in Analysi sunt inuenta, ei enodando minime sufficiunt. Et qui hoc problema sunt aggressi, plus non praestiterunt, quam vt casu, quo vnius corporis vis prae duobus reliquis valde est exigua, eorum motus vero tantum proxime assignauerint.

3. Ex hoc fonte omnia sunt hausta, quae adhuc de motu Lunae, ac de perturbationibus, quibus motus planetarum afficiuntur, sunt explorata, vbi commode vsu venit, vt vis, qua Luna ad Terram vrgetur, plurimum superet vim ad Solem directam, in planetis autem vis ad Solem tendens multo maior sit viribus, quibus in se inuicem agunt. Quae circumstantia nisi accederet, omnis opera in motuum determinatione frustra impenderetur. Quod ad Lunam attinet, cuius motum adhuc per approximationes satis exacte definire licuit, si longius a Terra esset remota, non video, quomodo vix vllam eius motus notitiam adipisci possemus, si scilicet tam longe a Terra esset remota, vt sortem satellitis Terrae esset amissura, iam in ordinem planetarum primariorum transitura. Tum nimirum eius motus mediam quandam legem sequeretur inter motum satelliti telluris et motum planetae primarii, cuius autem rationem vix vllo modo perspicere liceret, quandoquidem approximationibus nullus amplius locus relinqueretur.

4.

4. Quod si Luna Terrae esset vicinior, quam re vera est, vis perturbans a Sole profecta minueretur, ideoque Luna in motu suo circa Terram exactius leges *Keplerianas* sequeretur; aberrationes autem facilius certiusque definirentur. Quo longius autem Lunam a Terra remoueri fingamus, eo maiores aberrationes eius motum inquinabunt, quoad in eiusmodi regionem perueniat, vbi vis ad Solem tendens multum superet vim Terrae, ibique quasi Terram deferens, incipiet rationem motus planetarum primariorum sequi, verum tamen quasdam adhuc perturbationes a vi Terrae patietur, quas denuo, sed alio modo, per approximationes inuestigare licebit, perinde ac perturbationes in motibus planetarum primariorum repraesentari solent.

5. Motus autem Lunae maxime foret indeterminabilis, si quater vel quinquies a Terra magis esset remota, quam re vera est, ac si creatori libuisset, Lunae motum in tali regione assignare, Astronomi certe miris modis in eius inuestigatione, ac fortasse in cassum, defatigarentur, qui cum nunc sine auxilio Theoriae locum Lunae ad datum tempus sine notabili errore definire haud potuerint, eo casu semper in Lunae locis assignandis enormes errores essent commissuri, etiamsi fortasse innumerabiles obseruationes collegissent. Quin etiam ne suspicari quidem licet, qualem formam tum tabulis Astronomicis induci conueniret; neque patet, quomodo tabula mediorum motuum construi queat, cum eos neque ad Terram, neque ad Solem, referre liceat, multo minus intelligitur, quibusnam argumentis pro anomaliis definiendis esset vtendum. Ita tanquam eximium Astro-

 nomiae

nomiae commodum spectare debemus, quod in systemate saltem nostro solari non eiusmodi dentur corpora, de quibus dubium sit, vtrum planetis primariis an secundariis annumerari debeant.

6. In crassissima autem ignorantia circa motus coelestes versaremur, si ipsa Terra ita inter reliqua corpora fuisset disposita, vt neque legem planetarum primariorum neque secundariorum sequeretur; quoniam tum motus Solis apparens, cui cognitio reliquorum motuum innititur, nobis omnino esset inexplicabilis, quamuis plurium saeculorum obseruationes collegissemus. Vnica via ad Astronomiae notitiam perueniendi vtique per Analysin pateret, cuius beneficio problema de motu trium pluriumue corporum se mutuo attrahentium resolui deberet, neque hoc subsidio destituti quicquam in hac scientia praestare possemus. Verum etiamnum solutio huius problematis summam esset allatura vtilitatem, dum motus coelestes, quos iam proxime tantum agnoscere datur, accurate assignare valeremus; ita vt tum demum Astronomiae studium ad summum perfectionis gradum euehi sit censendum.

7. Cum igitur euolutio casus plurium corporum nequicquam ante sit expectanda, quam casus trium fuerit expeditus, hic tanquam fundamentum plenioris cognitionis astronomicae spectari debet, qui propterea omnino dignus est iudicandus, ad cuius resolutionem omnes Geometrae vires suas coniunctim impendant. Maximae quidem occurrunt difficultates, quas frustra adhuc superare conati sunt Geometrae, verum fructu inde

sperandi nimis sunt pretiosi, quam vt ab vlteriori inuestigatione deterreri quemquam conueniat. Ac si hoc ipsum problema tentantes omnes vias penetrandi occlusas offendimus, quod in aliis laboribus saepe vsui fuit adminiculum, dum tractatio aliarum quaestionum affinium tandem ad quaesitum scopum perduxit, eodem et hic vtamur, atque vires in aliis quaestionibus similibus, etiamsi per se nullum vsum habiturae videantur, exerceamus, certa spe freti, inde quicquam luminis ad tenebras illas dissipandas esse affulsurum.

8. Hoc igitur institutum sequens, istud problema tractandum suscepi, vt propositis duobus corporibus fixis motum tertii cuiusdam corporis, quod ad vtrumque attrahatur, inuestigarem. Sint scilicet corpora fixa in A et B, quorum massae iisdem litteris A et B indicentur, tertium autem corpus, cuius motum in eodem plano cum punctis A et B absolui assumo, iam elapso aliquo tempore t versetur in M, cuius motus assignari debeat. Quod problema, etsi in mundo casus similis non occurrat, similibus tamen difficultatibus, atque id cui vniuersa Astronomia innititur, implicatum deprehenditur, quae autem propterea, quod hic duo corpora A et B immota finguntur, facilius superari posse videntur; casus enim continet aliquos per se perspicuos, quorum consideratio ad euolutionem generalem perducere videatur. Tab. III. Fig. 1.

9. Primum enim obseruo, si massa alterius corporis A vel B euanescat, motum corporis M leges *Keplerianas* esse secuturum, ita vt sectionem conicam

 circa

circa alterutrum punctum B vel A esset descripturum. Quod idem proxime eueniet, si corpus M ita fuerit proiectum, vt alteri corpori maneat valde vicinum, ab altero autem tantopere semper distet, vt vis eo tendens prae altera sit minima. Vnde patet motum eo magis a regulis *Keplerianis* abhorrere; quominus distantiae a punctis A et B futurae sint inaequales; hocque casu motus corporis M non adeo dissimilis videtur ei, quem secuturus esset, si corpora A et B non forent fixa, vt inde nihil luminis sperari posset. Tum vero etiam hic notari meretur casus, quo ambo corpora A et B inter se sunt aequalia, corpusque M ita mouetur, vt eius orbita ad ambo aequaliter referatur; hoc enim casu motus quoque in sectione conica fieri deprehendetur.

10. His igitur obseruatis statuamus distantiam constantem $AB=a$, et variabiles $AM=v$, $BM=u$; tum vero demisso ab M in AB perpendiculo MP, sit $AP=x$, et $PM=y$, hincque $BP=a-x$ et:

$$v=\sqrt{(xx+yy)} \text{ atque } u=\sqrt{((a-x)^2+yy)}.$$

Cum iam vis acceleratrix, qua corpus M ad A attrahitur, sit $\frac{A}{vv}$, et qua ad B attrahitur vt $\frac{B}{uu}$, hinc nascetur vis secundum directionem $PA=\frac{Ax}{v^3}-\frac{B(a-x)}{u^3}$ et secundum directionem $MP=\frac{Ay}{v^3}+\frac{By}{u^3}$; ex quibus sumto elemento temporis dt constante principia mechanica has suppeditant formulas:

$$\text{I. } ddx=-2g\,dt^2\left(\frac{Ax}{v^3}-\frac{B(a-x)}{u^3}\right)$$

$$\text{II. } ddy=-2g\,dt^2\left(\frac{Ay}{v^3}+\frac{By}{u^3}\right)$$

quae

quae motus determinationem continent, vbi g est certa quaedam quantitas constans pro mensuris absolutis introducta.

11. Cum neutra harum aequationum integrationem admittat, videndum est, num eas ita inter se combinare liceat, vt inde aequatio integrabilis exsurgat, hocque duplici modo praestari necesse est. Atque vna quidem huiusmodi combinatio in promtu est; priore enim per dx et altera per dy multiplicata summa prodit

$$dx\,ddx+dy\,ddy=2g\,dt^2\left(\frac{A(x\,dx+y\,dy)}{v^3}+\frac{B(y\,dy-(a-x)\,dx)}{u^3}\right)$$

quae ob $v\,dv=x\,dx+y\,dy$ et $u\,du=y\,dy-(o-x)\,dx$ abit in hanc

$$dx\,ddx+dy\,ddy=-2g\,dt^2\left(\frac{A\,dv}{vv}+\frac{B\,du}{uu}\right)$$

cuius integralis, introducta noua constante, est

$$dx^2+dy^2=4g\,dt^2\left(\frac{A}{v}+\frac{B}{u}+\frac{C}{a}\right)$$

vbi cum $\sqrt{(dx^2+dy^2)}$ elementum curuae a corpore M tempusculo dt descriptae exprimat, erit $\frac{\sqrt{(dx^2+dy^2)}}{dt}$ vera corporis M celeritas, quae ergo per distantias v et u commode determinatur.

12. Vna integratione expedita, vt aliam insuper exploremus, elidamus ex formulis primo inuentis alteram massam, sicque obtinebimus has aequationes:

$$(a-x)\,ddy+y\,ddx=-2g\,dt^2.\frac{Aay}{v^3}$$

$$x\,ddy-y\,ddx=-2g\,dt^2.\frac{Bay}{u^3},$$

vnde quidem parum lucri consecuturi videmur. Verum si perpendamus esse

$$d.\frac{x}{v}=\frac{(xx+yy)\,dx-x(x\,dx+y\,dy)}{v^3}=\frac{y(y\,dx-x\,dy)}{v^3} \text{ et}$$

$$d.\frac{a-u}{u}=-\frac{dx((a-x)^2+yy)-(a-x)(y\,dy-(a-x)dx)}{u^3}=-\frac{y(y\,dx+(a-x)dy)}{u^3}$$

multi-

multiplicemus priorem per $xdy - ydx$ et posteriorem per $(a-x)dy + ydx$, habebimusque

$$(xdy - ydx)((a-x)ddy + yddx) = 2g\,\mathrm{A}\,adt^2 . d.\tfrac{x}{v}$$

$$((a-x)dy + ydx)(xddy - yadx) = 2g\,\mathrm{B}\,adt^2 . d.\tfrac{a-x}{u}.$$

13. Cum iam sit $(a-x)ddy + yddx = d.((a-x)dy + ydx)$ et $xddy - yddx = d.(xdy - ydx)$, commode euenit, vt summa harum aequationum sit integrabilis, integrali prodeunte:

$$(xdy - ydx)((a-x)dy + ydx) = 2gadt^2\left(\tfrac{\mathrm{A}x}{v} + \tfrac{\mathrm{B}(a-x)}{u} + \mathrm{D}\right)$$

sicque problema iam perduximus ad resolutionem aequationum differentialium primi gradus, quousque in solutione problematis de tribus corporibus mobilibus pertingere adhuc non licuit. Quodsi iam elementum temporis dt hinc elidamus, peruenimus ad hanc aequationem simpliciter differentialem:

$$a(dx^2 + dy^2)\left(\tfrac{\mathrm{A}x}{v} + \tfrac{\mathrm{B}(a-x)}{u} + \mathrm{D}\right) = 2(xdy - ydx)((a-x)dy + ydx)\left(\tfrac{\mathrm{A}}{v} + \tfrac{\mathrm{B}}{u} + \tfrac{\mathrm{C}}{a}\right)$$

inter binas variabiles x et y, qua natura curuae quaesitae determinatur; ita vt nunc quidem totum negotium ad resolutionem aequationis differentialis primi gradus sit perductum, quo in genere Analysis iam eximiis adminiculis est instructa.

14. Duo autem hic obstacula occurrunt, alterum quod differentialia dx et dy ad duas dimensiones ascendunt, alterum vero in quantitatibus irrationalibus v et u consistit. Quo haec obstacula facilius vincere queamus, pona-

ponamus angulos $\mathrm{BAM} = \zeta$, $\mathrm{ABM} = \eta$, eritque $x = v\,\mathrm{cos}.\zeta$; $y = \mathrm{sin}.\zeta = u\,\mathrm{sin}\,\eta$; et $a - x = u\,\mathrm{cos}.\eta$, vnde colligitur

$$dx^2 + dy^2 = dv^2 + vvd\zeta^2 = du^2 + uud\eta^2$$

$$xdy - ydx = vvd\zeta \text{ et } (a-x)dy + ydx = uud\eta$$

quibus valoribus aequatio nostra ad hanc formam simpliciorem reducitur:

$$a(dv^2 + vvd\zeta^2)\mathrm{A}\,\mathrm{cos}.\zeta + \mathrm{B}\,\mathrm{cos}.\eta + \mathrm{D}) = 2vvuud\zeta d\eta\left(\tfrac{\mathrm{A}}{v} + \tfrac{\mathrm{B}}{u} + \tfrac{\mathrm{C}}{a}\right).$$

Porro autem ob $v = \frac{a\,\mathrm{sin}.\eta}{\mathrm{sin}.(\zeta+\eta)}$ et $u = \frac{a\,\mathrm{sin}.\zeta}{\mathrm{sin}.(\zeta+\eta)}$ erit $x = \frac{a\,\mathrm{cos}.\zeta\,\mathrm{sin}.\eta}{\mathrm{sin}.(\zeta+\eta)}$ et $y = \frac{a\,\mathrm{sin}.\zeta\,\mathrm{sin}.\eta}{\mathrm{sin}.(\zeta+\eta)}$, hincque $dx = -\frac{ad\zeta\,\mathrm{sin}.\eta\,\mathrm{cos}.\eta + ad\eta\,\mathrm{sin}.\zeta\,\mathrm{cos}.\zeta}{\mathrm{sin}.(\zeta+\eta)^2}$ et $dy = \frac{ad\zeta\,\mathrm{sin}.\eta^2 + ad\eta\,\mathrm{sin}.\zeta^2}{\mathrm{sin}.(\zeta+\eta)^2}$, vnde fit $dx^2 + dy^2$

$$= \frac{aa(d\zeta^2\,\mathrm{sin}.\eta^2 + d\eta^2\,\mathrm{sin}.\zeta^2 - 2d\zeta d\eta\,\mathrm{sin}.\zeta\,\mathrm{sin}.\eta\,\mathrm{cos}.(\zeta+\eta))}{\mathrm{sin}.(\zeta+\eta)^4} = dv^2 + vvd\zeta^2.$$

15. Si ope horum valorum aequationem nostram ad solos binos angulos ζ et η reducamus, nanciscemur:

$$(d\zeta^2\mathrm{sin}.\eta^2 d\eta^2\mathrm{sin}.\zeta^2 - 2d\zeta d\eta\,\mathrm{sin}\,\zeta\,\mathrm{sin}.\eta\,\mathrm{cos}.(\zeta+\eta))(\mathrm{A}\,\mathrm{cos}.\zeta\ \mathrm{B}\,\mathrm{cos}.\eta + \mathrm{D})$$

$$= 2d\zeta d\eta\,\mathrm{sin}.\zeta^2\,\mathrm{sin}.\eta^2\left(\tfrac{\mathrm{A}\,\mathrm{sin}.(\zeta+\eta)}{\mathrm{sin}.\eta} + \tfrac{\mathrm{B}\,\mathrm{sin}.(\zeta+\eta)}{\mathrm{sin}.\zeta} + \mathrm{C}\right)$$

$$= 2d\zeta d\eta\,\mathrm{sin}.\zeta\,\mathrm{sin}.\eta(\mathrm{A}\,\mathrm{sin}.\zeta\,\mathrm{sin}.(\zeta+\eta) + \mathrm{B}\,\mathrm{sin}.\eta\,\mathrm{sin}\,(\zeta+\eta) + \mathrm{C}\,\mathrm{sin}.\zeta\,\mathrm{sin}.\eta)$$

quae reuocatur ad hanc formam multo simpliciorem:

$$(d\zeta^2\mathrm{sin}.\eta^2 + d\eta^2\mathrm{sin}.\zeta^2)(\mathrm{A}\,\mathrm{cos}.\zeta + \mathrm{B}\,\mathrm{cos}.\eta + \mathrm{D}) = 2d\zeta d\eta\,\mathrm{sin}\,\zeta\,\mathrm{sin}.\eta\,(\mathrm{A}\,\mathrm{cos}.\eta + \mathrm{B}\,\mathrm{cos}.\zeta + \mathrm{C}\,\mathrm{sin}.\zeta\,\mathrm{sin}.\eta + \mathrm{D}\,\mathrm{cos}.(\zeta+\eta)).$$

Vel

Vel ob $\cos.(\zeta+\eta)=\cos.\zeta\cos.\eta-\sin.\zeta\sin.\eta$ statuamus $C-D=E$, vt habeamus:

$$d\zeta^2\sin.\eta^2+d\eta^2\sin.\zeta^2=\frac{2d\zeta d\eta\sin.\zeta\sin.\eta(A\cos.\eta+B\cos.\zeta+D\cos.\zeta\cos.\eta+E\sin.\zeta\sin.\eta)}{A\cos.\zeta+B\cos.\eta+D}$$

vnde si ponamus breuitatis gratia

$$A\cos\eta+B\cos.\zeta+D\cos\zeta\cos\eta+E\sin.\zeta\sin.\eta=P \text{ et}$$
$$A\cos.\zeta+B\cos.\eta+D=Q$$

deducimus radicem extrahendo:

$$\frac{d\zeta\sin.\eta}{d\eta\sin.\zeta}=\frac{P\pm\surd(PP-QQ)}{Q}.$$

16. Cum nulla via pateat huiusmodi aequationes resoluendi, contemplemur casus, quibus resolutio est in potestate, qui sunt, quando vel $A=0$, vel $B=0$; tametsi enim etiam his casibus aequatio postrema parum tractabilis videtur, tamen ex formulis principalibus solutio facile deducitur. Si enim ponamus $B=0$, prior integratio praebet:

$$dx^2+dy^2=4g\,dt\,(\tfrac{A}{v}+\tfrac{C}{a})$$

tum vero ex §. 12. ob $B=0$ impetramus

$$x\,ddy-y\,ddx=0 \text{ hincque } x\,dy-y\,dx=\text{Const.}\,dt$$

ponamus ergo $(x\,dy-y\,dx)^2=4gFa\,dt^2$, fietque

$$Fa(dx^2+dy^2)^2=2(x\,dy-y\,dx)^2(\tfrac{A}{v}+\tfrac{C}{a})=v^4d\zeta^2(\tfrac{A}{v}+\tfrac{C}{a})$$

et factis substitutionibus supra indicatis:

$$F(d\zeta^2\sin.\eta^2+d\eta^2\sin.\zeta^2-2d\zeta d\eta\sin.\zeta\sin.\eta\cos.(\zeta+\eta))$$
$$=d\zeta^2\sin.\eta^4(\frac{A\sin.(\zeta+\eta)}{\sin.\eta}+C)$$

seu

$$d\zeta^2\sin.\eta^2(1-\frac{A\sin.\eta\sin.(\zeta+\eta)-C\sin.\eta^2}{F})+d\eta^2\sin.\zeta^2$$
$$=2d\zeta d\eta\sin.\zeta\sin.\eta\cos.(\zeta+\eta)$$

cuius

cuius quidem resolutio vix facilior videtur, quam praecedentis; at extracta radice quadrata satis fit manifesta.

17. Verum antequam ad hanc vltimam aequationem inter ζ et η pertigimus, iam affecuti eramus aequationem duabus tantum litteris v et ζ constantem, hanc:

$$Fa(dv^2+vvd\zeta^2)=v^4d\zeta^2(\tfrac{A}{v}+\tfrac{C}{a})$$

ex qua statim elicitur

$$Fa\,dv^2=vvd\zeta^2(Av+\tfrac{Cvv}{a}-Fa) \text{ seu}$$
$$Fa\,dv^2=v^4d\zeta^2(\tfrac{C}{a}+\tfrac{A}{v}-\tfrac{Fa}{vv})$$

vnde fit

$$\frac{dv}{vv}\surd Fa=d\zeta\surd(\tfrac{C}{a}+\tfrac{A}{v}-\tfrac{Fa}{vv})$$

ex qua indoles sectionum conicarum more solito elici solet. Posito nempe $\frac{1}{v}=\frac{z}{a}$, fit

$$-dz=d\zeta\surd(\tfrac{C+Az}{F}-zz), \text{ vnde sequitur}$$
$$\zeta+\alpha=\text{Arc. cos. }\frac{2Fz-A}{\surd(AA+4CF)}$$

hincque $2Fz=A+\cos.(\zeta+\alpha).\surd(AA+4CF)$

ita vt fit $v=\frac{2Fa}{A+\cos.(\zeta+\alpha).\surd(AA+4CF)}=\frac{a\sin.\eta}{\sin.(\zeta+\eta)}$.

18. Hinc ergo aequatio integralis inter angulos ζ et η ita exprimitur, vt sit

$$\frac{2F\sin.(\zeta+\eta)}{\sin.\eta}=A+\cos.(\zeta+\alpha).\surd(AA+4CF)$$

seu, quo ex dato angulo ζ angulus η facilius inueniri possit, ob $\sin.(\zeta+\eta)=\sin.\zeta\cos\eta+\cos.\zeta\sin.\eta$, erit

$$2F\sin.\zeta\cot.\eta+2F\cos.\zeta=A+\cos.(\zeta+\alpha).\surd(AA+4CF)$$

seu mutata forma constantium:

$$\cot.\eta+\cot.\zeta=\frac{A+M\cos.\zeta+N\sin.\zeta}{2F\sin.\zeta}.$$

Atque

Atque hinc ſimul intelligimus, ſi ponamus $A = 0$, fore

$$\cot.\zeta + \cot.\eta = \frac{B + M'\cos.\eta + N'\sin.\eta}{2F'\sin.\eta}.$$

Quare iam conſtant formae, ad quas integrale aequationis differentialis §. 15. datae caſibus vel $A = 0$ vel $B = 0$ perducatur, quo ipſo via ad haec integralia perveniendi inueſtigari poterit.

Pro caſu B = 0.

19. Hoc caſu aequatio noſtra principalis §. 15. inuenta abit in hanc formam:

$$d\zeta^2 \sin.\eta^2 + d\eta^2 \sin.\zeta^2 = \frac{2d\zeta\, d\eta \sin.\zeta \sin.\eta (A\cos.\eta + D\cos.\zeta\cos.\eta + E\sin.\zeta\sin.\eta)}{A\cos.\zeta + D}$$

cuius ergo nouimus integralem huiusmodi formam eſſe habituram:

$$\cot.\eta + \cot.\zeta = \frac{A + M\cos.\zeta + N\sin.\zeta}{2F\sin.\zeta}$$

ſeu breuius $\cot.\eta = \alpha + \frac{\beta + \gamma\cos.\zeta}{\sin.\zeta}$

quae igitur quomodo ex differentiali ſit eruenda, inveſtigari oportet. Quod etiamſi non difficulter per calculum ſtatim ab initio ad hunc caſum accommodatum perſpiciatur, tamen conſideratio corporis B calculum ita immutauit, vt haec concluſio non niſi per ambages inde colligi poſſe videatur. Primum autem intelligimus, loco anguli η non inutiliter eius cotangentem introductum iri; aequatione ergo per $\sin.\eta^4$ diuiſa habemus

$$\frac{d\zeta^2}{\sin.\eta^2} + \sin.\zeta^2 (d.\cot.\eta)^2 = -\frac{2d\zeta \sin.\zeta\, d.\cot.\eta (A\cot.\eta + D\cos.\zeta\cot.\eta + E\sin.\zeta)}{A\cos.\zeta + D}.$$

20. Ponamus $\cot.\eta = z$, ob $\sin.\eta = \frac{1}{\sqrt{(1+zz)}}$ erit:

$$d\zeta^2(1+zz) + dz^2 \sin.\zeta^2 = -\frac{2d\zeta\, dz \sin.\zeta (z(A + D\cos.\zeta) + E\sin.\zeta)}{A\cos.\zeta + D}$$

vnde

vnde radicem extrahendo fit

$$\frac{dz\sin.\zeta}{d\zeta} = -\frac{z(A + D\cos.\zeta) - E\sin.\zeta + \sqrt{\left\{(AA-DD)zz\sin.\zeta^2 + 2E(A + D\cos.\zeta)z\sin.\zeta + EE\sin.\zeta^2 - (A\cos.\zeta + D)^2\right\}}}{A\cos.\zeta + D}$$

vbi notetur, quantitatem ſigno radicali inuolutam ita referri poſſe:

$$\left(z\sin.\zeta.\sqrt{(AA-DD)} + \frac{E(A + D\cos.\zeta)}{\sqrt{(AA-DD)}}\right)^2 - \frac{(AA-DD+EE)(A\cos.\zeta + D)^2}{AA-DD}$$

Quare poſito

$$z\sin.\zeta.\sqrt{(AA-DD)} + \frac{E(A + D\cos.\zeta)}{\sqrt{(AA-DD)}} = \frac{s(A\cos.\zeta + D)\sqrt{(AA-DD+EE)}}{\sqrt{(AA-DD)}}$$

vt ſit

$$z\sin.\zeta = -\frac{E(A + D\cos.\zeta) + s(A\cos.\zeta + D)\sqrt{(AA-DD+EE)}}{AA-DD}$$

erit quantitas ſigno radicali implicata

$$\frac{(A\cos.\zeta + D)\sqrt{(AA-DD+EE)}}{\sqrt{(AA-DD)}}\sqrt{(ss-1)}.$$

21. Ponatur breuitatis gratia haec quantitas formulae irrationali aequalis $= V$, et cum noſtra aequatio ſit

$$dz\sin.\zeta(A\cos.\zeta + D) + zd\zeta(A + D\cos.\zeta) + Ed\zeta\sin.\zeta = Vd\zeta$$

diuidatur ea per $(A\cos.\zeta + D)^2$, et ita repraeſentari poterit

$$d.\frac{Az\sin.\zeta + E}{A(A\cos.\zeta + D)} = \frac{Vd\zeta}{(A\cos.\zeta + D)^2}.$$

At per noſtram ſubſtitutionem eſt

$$Az\sin.\zeta + E = -\frac{DE(A\cos.\zeta + D) + As(A\cos.\zeta + D)\sqrt{(AA-DD+EE)}}{AA-DD}$$

quo valore ſubſtituto, ſimulque valore ipſius V reſtituto, erit

$$d.\frac{-DE + As)\sqrt{(AA-DD+EE)}}{A(AA-DD)} = \frac{d\zeta\sqrt{(AA-DD+EE)}}{(A\cos.\zeta + D)\sqrt{(AA-DD)}}\sqrt{(ss-1)}$$

ſeu

$$\frac{ds}{\sqrt{(AA-DD)}} = \frac{d\zeta\sqrt{(ss-1)}}{A\cos.\zeta + D} \text{ vel } \frac{ds}{\sqrt{(ss-1)}} = \frac{d\zeta(AA-DD)}{A\cos.\zeta + D}$$

quae

quae etiam ita repraesentari potest:

$$\frac{-ds}{\sqrt{(1-ss)}} = \frac{d\zeta\sqrt{(DD-AA)}}{A\,\text{cos.}\,\zeta+D}$$

cuius integrale est

$$\text{Arc. cos.}\,s = \text{Arc. cos}\,\frac{A+D\,\text{cos.}\,\zeta}{A\,\text{cos.}\,\zeta+D} + \alpha.$$

22. Cum iam sit $\text{Arc. cos.}\,\frac{A+D\,\text{cos.}\,\zeta}{A\,\text{cos.}\,\zeta+D} = \text{Arc. sin.}\,\frac{\text{sin.}\,\zeta\sqrt{(DD-AA)}}{A\,\text{cos.}\,\zeta+D}$ fiet $s = \frac{(A+D\,\text{cos.}\,\zeta)\,\text{cos.}\,\alpha - \text{sin.}\,\alpha\,\text{sin.}\,\zeta.\sqrt{(DD-AA)}}{A\,\text{cos.}\,\zeta+D}$ siue hoc modo:

$$s = \frac{n(A+D\,\text{cos.}\,\zeta) - \text{sin.}\,\zeta\sqrt{(1-nn)(DD-AA)}}{A\,\text{cos.}\,\zeta+D}$$

vbi si fuerit $D < A$, numerum $n > 1$ capi conuenit.

Hoc itaque valore substituto aequatio integralis quaesita erit:

$$\text{sin.}\,\zeta\,\text{cot.}\,\eta = \frac{E(A+D\,\text{cos.}\,\zeta)}{DD-AA} - \frac{n(A+D\,\text{cos.}\,\zeta)+\text{sin.}\,\zeta\sqrt{(1-nn)(DD-AA)}}{DD-AA}\sqrt{(AA-DD+EE)}$$

et posito $n = \frac{E-F}{\sqrt{(AA-DD+EE)}}$ erit

$$\text{sin.}\,\zeta\,\text{cot.}\,\eta = \frac{F(A+D\,\text{cos.}\,\zeta)}{DD-AA} + \frac{\text{sin.}\,\zeta.\sqrt{(AA-DD+2EF-FF)}}{\sqrt{(DD-AA)}}$$

vbi F est quantitas constans arbitraria per nouam integrationem introducta. Ea autem mutata erit

$$\text{sin.}\,\zeta\,\text{cot.}\,\eta = \frac{A+D\,\text{cos.}\,\zeta}{G} + \text{sin.}\,\zeta.\sqrt{\left(\frac{2E}{G} + \frac{AA-DD}{GG} - 1\right)}.$$

Pro casu A=B et D=E=o.

23. Simili modo expeditur casus $A = o$, et aequatio integralis non differt a praecedente, nisi quod litterae A et B, item anguli ζ et η, inter se permutentur. Verum hoc casu, quo $A = B$, atque $D = E = o$, aequatio nostra fit

$$d\zeta^2\,\text{sin.}\,\eta^2 + d\eta^2\,\text{sin.}\,\zeta^2 = 2\,d\zeta\,d\eta\,\text{sin.}\,\zeta\,\text{sin.}\,\eta$$

quae

quae manifesto praebet $d\zeta\,\text{sin.}\,\eta = d\eta\,\text{sin.}\,\zeta$, hincque integrando

$$l\,\text{tang.}\,\tfrac{1}{2}\zeta = \text{Const.} + l\,\text{tang.}\,\tfrac{1}{2}\eta, \text{ vnde fit}$$

$m\,\text{tang.}\,\tfrac{1}{2}\zeta = n\,\text{tang.}\,\tfrac{1}{2}\eta$, seu $m(1-\text{cos.}\,\zeta)\,\text{sin.}\,\eta = n(1-\text{cos.}\,\eta)\,\text{sin.}\,\zeta$ ita vt tangentes semissium angulorum BAM et ABM perpetuo eandem rationem seruent. Cum iam coordinatis x et y introductis sit $\text{cos.}\,\zeta = \frac{x}{v}$; $\text{sin.}\,\zeta = \frac{y}{v}$; $\text{cos.}\,\eta = \frac{a-x}{u}$ et $\text{sin.}\,\eta = \frac{y}{u}$, erit

$$\frac{m(v-x)y}{vu} = \frac{n(u-a+x)y}{vu} \text{ seu } m(v-x) = n(u-a+x)$$

ita vt sit $m(AM-AP) = n(BM-BP)$.

24. Cum igitur sit $m(v-x) = n(u-a+x)$, notetur esse

$$x = \frac{aa+vv-uu}{2a} \text{ et } a-x = \frac{aa+uu-vv}{2a}, \text{ vnde fit}$$

$$m(uu-(a-v)^2) = n(vv-(a-u)^2) \text{ seu}$$

$$m(u+v-a)(u+a-v) = n(v+u-a)(v+a-u)$$

quae diuisa per $u+v-a)$ praebet

$$m(a+u-v) = n(a+v-u) \text{ seu } (m+n)(u-v) = (n-m)a$$

ita vt sit $u-v = \frac{(n-m)a}{m+n}$, quae comparetur cum hac: $nu - mv = na - (m+n)x$, vnde colligitur

$$(n-m)u = \frac{(mm+nn)a}{m+n} - (m+n)x, \text{ et}$$

$$(n-m)v = \frac{2mna}{m+n} - (m+n)x$$

quae quadrata suppeditat

$$(n-m)^2yy + (n-m)^2xx = \frac{4mmnnaa}{(m+n)^2} - 4mnax + (m+n)^2xx$$

seu $(n-m)^2yy = \frac{4mmnnaa}{(m+n)^2} - 4mnax + 4mnxx.$

25. Su-

25. Sumamus abſciſſas a puncto medio C, ſitque $CA = CB = b$, ideoque $a = 2b$; et ponatur $CP = z$; tum vero ſtatuatur $m + n = b$ et $n - m = c$, et habebimus, ob $x = b - z$,

$$cv = bz - cc \quad \text{et} \quad cu = bz + cc$$

hincque $yy = \frac{bb - cc}{cc}(zz - cc)$, vnde patet, curuam eſſe hyperbolam centro C deſcriptam, cuius ſemiaxis $= c$, et diſtantia focorum a centro $CA = CB = b$, foreque tang. $\frac{1}{2}\zeta$: tang. $\frac{1}{2}\eta = b + c : b - c$. Cum porro ſit $dy = \frac{zdz}{\sqrt{(zz - cc)}} \cdot \frac{\sqrt{(bb - cc)}}{c}$, erit $dx^2 + dy^2 = dy^2 + dz^2 = \frac{dz^2(bbzz - c^4)}{cc(zz - cc)}$; vnde quia ob $C = D + E = o$ et $B = A$ habemus $dx^2 + dy^2 = 4Agdt^2(\frac{c}{bz - cc} + \frac{c}{bz - cc})$ $= \frac{8Abcgzdt^2}{bbzz - c^4}$, erit celeritas in $M = \frac{\sqrt{(dx^2 + dy^2)}}{dt} = \frac{2\sqrt{2Abcgz}}{\sqrt{(bbzz - c^4)}}$; et poſito $z = c$, prodit celeritas in vertice hyperbolae $= \frac{2\sqrt{2Abg}}{\sqrt{(bb - cc)}}$. Etſi ergo hyperbola abeat in ellipſin ſumendo $c > b$, tamen euidens eſt, motum in ellipſi abſolui non poſſe, quia celeritas foret imaginaria, ita vt hoc caſu corpus M nonniſi in hyperbola moueri poſſit.

26. Quemadmodum autem iſte motus in hyperbola futurus ſit comparatus, ex temporis ratione colligetur. Scilicet cum

ſit $\sqrt{(dx^2 + dy^2)} = \frac{dz}{c}\sqrt{\frac{bbzz - c^4}{zz - cc}}$ erit

$$2dt\sqrt{2Abcg} = \frac{dz(bbzz - c^4)}{c\sqrt{z(zz - cc)}} \text{ ideoque}$$

$$2ct\sqrt{2Abcg} = \int\frac{dz(bbzz - c^4)}{\sqrt{z(zz - cc)}}.$$

Per reductionem autem integralium conſtat eſſe:

$$\int\frac{zzdz}{\sqrt{z(zz - cc)}} = \tfrac{2}{3}\sqrt{z(zz - cc)} + \tfrac{1}{3}cc\int\frac{dz}{\sqrt{z(zz - cc)}}$$

vnde

vnde tempus t ita determinatur, vt ſit

$$2ct\sqrt{2Abcg} = \tfrac{2}{3}bb\sqrt{z(zz - cc)} + \tfrac{1}{3}cc(bb - 3cc)\int\frac{dz}{\sqrt{z(zz - cc)}}.$$

Pendet ergo determinatio temporis ab integratione formulae $\int\frac{dz}{\sqrt{z(zz - cc)}}$, quam neque ad quadraturam circuli, neque hyperbolae, reduci poſſe conſtat.

27. Reducamus hanc determinationem quoque ad angulum $BAM = \zeta$, et cum ſit tang $\frac{1}{2}\zeta = \frac{v - x}{y} = \frac{v - b + z}{y}$, habebimus tang. $\frac{1}{2}\zeta = \sqrt{\frac{b + c}{b - c} \cdot \frac{z - c}{z + c}}$, hincque $z = \frac{c(b - c\,\text{coſ}.\zeta)}{c - b\,\text{coſ}.\zeta}$, vnde ſit $v = \frac{bb - cc}{c - b\,\text{coſ}.\zeta}$. Quare porro nanciſcimur $\sqrt{(zz - cc)} = \frac{c\,\text{ſin}.\zeta.\sqrt{(bb - cc)}}{c - b\,\text{coſ}.\zeta}$ et $dz = -\frac{c(bb - cc)d\zeta\,\text{ſin}.\zeta}{(c - b\,\text{coſ}.\zeta)^2}$. Ergo $\frac{dz}{\sqrt{(zz - cc)}} = -d\zeta.\sqrt{(bb - cc)}$ et

$$\frac{dz}{\sqrt{z(zz - cc)}} = -\frac{d\zeta\sqrt{(bb - cc)(c - b\,\text{coſ}.\zeta)}}{\sqrt{c(b - c\,\text{coſ}.\zeta)}}$$

vnde colligimus

$$2ct\sqrt{2Abcg} = \frac{2bbc\,\text{ſin}.\zeta\sqrt{(bb - cc)c(b - c\,\text{coſ}.\zeta)}}{3(c - b\,\text{coſ}.\zeta)\sqrt{(c - b\,\text{coſ}.\zeta)}}$$

$$- \tfrac{1}{3}cc(bb - 3cc)\int\frac{d\zeta\sqrt{(bb - cc)(c - b\,\text{coſ}.\zeta)}}{\sqrt{c(b - c\,\text{coſ}.\zeta)}} \text{ ſiue}$$

$$\frac{2t\sqrt{2Abg}}{\sqrt{(bb - cc)}} = \frac{2bb\,\text{ſin}.\zeta\sqrt{(b - c\,\text{coſ}.\zeta)}}{3(c - b\,\text{coſ}.\zeta)^{\frac{3}{2}}} - \tfrac{1}{3}(bb - 3cc)\int\frac{d\zeta\sqrt{(c - b\,\text{coſ}.\zeta)}}{\sqrt{(b - c\,\text{coſ}.\zeta)}}.$$

28. Caſus hic notatu dignus occurrit, quo $bb = 3cc$, quoniam eo tempus algebraice aſſignari poteſt. Tum autem erit celeritas in vertice hyperbolae $= 2\sqrt{\frac{Ag\sqrt{3}}{c}} = 2\sqrt{\frac{3Ag}{b}}$. Quae celeritas ſi dicatur $= k$, erit

$$kt = \frac{2cc\,\text{ſin}.\zeta}{c - b\,\text{coſ}.\zeta}\sqrt{\frac{b - c\,\text{coſ}.\zeta}{c - b\,\text{coſ}.\zeta}} = \frac{2c\,\text{ſin}.\zeta}{1 - \text{coſ}.\zeta.\sqrt{3}}\sqrt{\frac{\sqrt{3} - \text{coſ}.\zeta}{1 - \text{coſ}.\zeta.\sqrt{3}}},$$

vel breuius ita: $\frac{t}{c}\sqrt{2Abcg} = \sqrt{z(zz - cc)}$, vnde vt ad datum tempus t definiatur locus corporis M, reſolui oportet hanc aequationem cubicam:

$$z^3 - ccz = \frac{2Acgtt}{cc} = \frac{2Agtt\sqrt{3}}{c}.$$

Pro

Pro aliis autem casibus praeter tractatos vix quicquam circa motum definire licebit; hic vero occasio se obtulit eiusmodi artificia adhibendi, quae forte in vberiore huius argumenti tractatione vtilitatem afferre poterunt. Adiungamus tamen adhuc casum, quo corpus in ellipsi, cuius ambo foci sint in punctis A et B, mouebitur.

De motu corporis M in Ellipsi.

29 Quoniam casu praecedente vidimus corpus in hyperbola moueri posse, dubium est nullum, quin etiam certo quodam casu motus in ellipsi fieri queat, qui autem diuersus erit a praecedenti, quo erat $D=0$ et $E=0$, existente $A=B$. Vt autem ellipsis prodeat, necesse est, vt fiat $\text{tang.}\frac{1}{2}\zeta.\text{tang.}\frac{1}{2}\eta=m$, seu retentis valoribus $CP=z$, et $CA=CB=b$, vt fiat $(v-b+z)(u-b-z)=myy$. Cum autem sit

$$\text{vel } yy=vv-(b-z)^2=(v-b+z)(v+b-z)$$
$$\text{vel } yy=uu-(b+z)^2=(u-b-z)(u+b+z)$$

erit, vtroque seorsim adhibito,

vel $u-b-z=m(v+b-z)$, vel $v-b+z=m(u+b+z)$ quibus additis prodit $u+v-2b=m(u+v+2b)$ ita vt sit $u+v=\frac{2(1+m)b}{1-m}=2c$, seu $m=\frac{c-b}{c+b}$, denotante $2c$ axem transuersum. Cum igitur sit $uu-vv=4bz$, erit hac per illam diuisa $u-v=\frac{2bz}{c}$, ideoque $v=c-\frac{bz}{c}$ et $u=c+\frac{bz}{c}$, hincque $yy=\frac{cc-bb}{cc}(cc-zz)$.

30. Cum nunc sit $l\text{tang.}\frac{1}{2}\zeta+l\text{tang.}\frac{1}{2}\eta=lm$, erit differentiando $\frac{d\zeta}{\text{sin.}\zeta}+\frac{d\eta}{\text{sin.}\eta}=0$, hincque $\frac{d\zeta\,\text{sin.}\eta}{d\eta\,\text{sin.}\zeta}=-1$. Quare

Quare ex §. 15. necesse est, vt sit $\frac{P-\sqrt{(PP-QQ)}}{Q}=-1$, ideoque $P+Q=0$, vnde esse oportet

$$(A+B)(\text{cos.}\zeta+\text{cos.}\eta)+D+D\,\text{cos.}\zeta\,\text{cos.}\eta$$
$$+E\,\text{sin.}\zeta\,\text{sin.}\eta=0$$

vbi constantes ita sunt definiendae, vt haec aequatio conueniat cum natura ellipsis $\text{tang.}\frac{1}{2}\zeta.\text{tang.}\frac{1}{2}\eta=m=\frac{c-b}{c+b}$ seu hac $\frac{(1-\text{cos.}\zeta)(1-\text{cos.}\eta)}{\text{sin.}\zeta\,\text{sin.}\eta}=m=\frac{c-b}{c+b}$. Cum ergo sit $\text{sin.}\zeta\,\text{sin.}\eta=\frac{1-\text{cos.}\zeta-\text{cos.}\eta+\text{cos.}\zeta\,\text{cos.}\eta}{m}$, hoc valore ibi substituto fit

$$\left.\begin{aligned}m(A+B)(\text{cos.}\zeta+\text{cos.}\eta)+mD+mD\,\text{cos.}\zeta\,\text{cos.}\eta\\-E(\text{cos.}\zeta+\text{cos.}\eta)+E+E\,\text{cos.}\zeta\,\text{cos.}\eta\end{aligned}\right\}=0$$

quocirca hae conditiones requiruntur, vt sit $E=m(A+B)$ et $D=-\frac{E}{m}=-A-B$, hincque $E=\frac{c-b}{c+b}(A+B)$ et $C=D+E=\frac{-2b}{c+b}(A+B)$.

31. Vt iam motus rationem in hac ellipsi definiamus ob $dy=-\frac{zdz}{\sqrt{(cc-zz)}}.\frac{\sqrt{(cc-bb)}}{c}$ erit $dz^2+dy^2=\frac{dz^2(c^4-bbzz)}{c\sqrt{(cc-zz)}}$ et $\sqrt{(dz^2+dy^2)}=\frac{dz\sqrt{(c^4-bbzz)}}{c\sqrt{(cc-zz)}}$ supra autem inuenimus esse

$$dx^2+dy^2=4g\,dt^2\left(\frac{A}{v}+\frac{B}{u}+\frac{C}{2b}\right) \text{ seu}$$
$$dx^2+dy^2=4g\,dt^2\left(\frac{Ac}{cc-bz}+\frac{Bc}{cc+bz}-\frac{A-B}{c+b}\right)$$

quae transit in hanc formam:

$$dx^2+dy^2=\frac{4bg\,dt^2(A(c+z)(cc+bz)+B(c-z)(cc-bz))}{(c+b)(c^4-bbzz)}$$

vnde colligimus

$$\frac{4bccg\,dt^2}{b+c}=\frac{dz^2(c^4-bbzz)^2}{(cc-zz)(A(c+z)(cc+bz)+B(c-z)(cc-bz))}$$

 hinc-

hincque integrando

$$2ct\sqrt{\frac{bg}{b+c}} = \int \frac{(c^4 - bbzz)\,dz}{\sqrt{(cc-zz)(A(c+bz)+B(c-z)(cc-bz))}};$$

32. Si ponamus $B = 0$, casus reducitur ad vnicum centrum virium A, cuius calculum supra expediuimus; verum haec solutio cum illa minime conuenit, vnde methodus hic vsurpata non parum suspecta redditur. Cuius singularis phaenomeni causam inuestigaturus obseruo, per superiorem aequationem (§. 30.) neambas quidem litteras D et E determinari. Ex aequatione enim $(1 - \text{cos}.\zeta)(1 - \text{cos}.\eta) = m\,\text{sin}.\zeta\,\text{sin}.\eta$ quadrata colligimus $(1 - \text{cos}.\zeta)(1 - \text{cos}.\eta) = mm(1 + \text{cos}\,\zeta)(1 + \text{cos}\,\eta)$ vnde fit

$$(1 - mm)(1 + \text{cos}.\zeta\,\text{cos}.\eta) = (1 + mm)(\text{cos}.\zeta + \text{cos}.\eta).$$

Cum nunc esse debeat

$$(E + MD)(1 + \text{cos}.\zeta\,\text{cos}.\eta) = (E - m(A+B))(\text{cos}.\zeta + \text{cos}.\eta)$$

sufficit, vt sit

$$E(1+mm) + Dm(1+mm) = E(1-mm) - (A+B)m(1-mm)$$

vnde fit $2Em + D(1+mm) + (A+B)(1-mm) = 0$, seu $E = -\frac{(A+B)(1-mm)}{2m} - \frac{D(1+mm)}{2m}$, hincque

$$C = D + E = -\frac{(A+B)(1-mm)}{2m} - \frac{D(1-m)^2}{2m}.$$

33. Hinc vitium methodi, qua hic sum vsus, eo clarius in oculos incurrit. Cum enim quantitas D maneat indeterminata, etiamsi curua a corpore M descripta sit data, celeritas corporis M in quolibet orbitae suae puncto non esset determinata, sed quasi arbitrio nostro relinqueretur. Nam pro vertice ellipsis foco A

A propiore, quo est distantia $v = c - b$, et $u = c + b$, seu ob $\frac{c-b}{c+b} = m$, $v = \frac{2mb}{1-m}$ et $u = \frac{2b}{1-m}$, erit celeritatis quadratum $\frac{dx^2 + dy^2}{dt^2} = \frac{2g}{b}\left(\frac{A(1-m)}{2m} + \frac{B(1-m)}{2} - \frac{(A+B)(1-mm)}{2m} - \frac{D(1-m)^2}{2m}\right) = \frac{g}{mb}(-Am(1-m) - B(1-m) - D(1-m)^2)$ ideoque ipsa celeritas $\frac{\sqrt{(dx^2+dy^2)}}{dt} = \sqrt{\frac{(1-m)g}{mb}}(-Am\,B - D(1-m))$, quae cum indefinita esse nullo modo possit, manifestum est, methodum §. 30. adhibitam esse vitiosam, id quod adhuc clarius perspicitur, si ambo corpora A et B euanescentia statuamus, quo casu certe corpus M in linea recta esset incessurum, neque ergo ellipsim, quam hic assumsimus, describere poterit, etiamsi calculus noster aliter ostendat. Plurimum igitur intererit vitium huius methodi nosse, ne simili methodo alias vtentes in errores delabamur.

34. Quoniam in calculo nullum vitium deprehenditur, ipsum ratiocinium, quo vsus sum, fallax sit necesse est, quod isti innititur fundamento, quod aequationi differentiali

$$\frac{d\zeta\,\text{sin}.\eta}{d\eta\,\text{sin}.\zeta} = \frac{P \pm \sqrt{(PP+QQ)}}{Q} \text{ inde §. 15.}$$

satisfaciat aequatio finita $(1 - \text{cos}.\zeta)(1 - \text{cos}.\eta) = m\,\text{sin}.\zeta\,\text{sin}\,\eta$, (quae vtique est pro ellipsi) siquidem vna constantium D et E certo modo assumatur. Verum iam alia occasione obseruaui fieri posse, vt aequationi cuipiam differentiali aequatio quaedam finita satisfaciat, quae tamen in aequatione per integrationem inde deducta minime contineatur. Veluti aequationi $ds\sqrt{(1-zz)} = dz$ manifesto satisfacit valor $z = 1$, qui tamen in aequatione integrali $s = \alpha + \text{Arc.sin}.z$ vel $z = \text{sin}.(s - \alpha)$ neutiquam

neutiquam continetur, quicunque valor constanti arbitrariae α tribuatur. Nullum ergo est dubium, quin ob similem caussam methodus hic adhibita in errorem induxerit.

35. Cum in originem huius erroris accuratius inquirerem, praeter expectationem incidi in completam problematis propositi solutionem, ex qua omnia, quae hactenus in hoc argumento erant desiderata, perspicue cognoscentur, simulque origo erroris hic commissi ita dilucide intelligetur, vt haud difficulter in aliis similibus casibus huiusmodi errores euitare queamus. Atque hoc modo tractatio problematis mechanici tantas dilucidationes in ipsa Analysi suppeditauit, quas alias forte frustra quaesiuissemus, quod quidem non insolitum est censendum, cum pleraque artificia, quae adhuc in Analysi sunt inuenta, quaestionibus Mechanicis ac Physicis accepta sint referenda. In his enim saepe eiusmodi inuestigationes occurrunt, quae occasionem praebent, indolem aequationum accuratius rimandi, atque adeo non raro commissio errorum illustribus inuentis fuit compensata, quemadmodum mihi hoc problema tractanti vsu venit, cuius solutionem, nisi in praedictum errorem fuissem delapsus, certe nunquam inuenissem.

Solutio completa Problematis propositi.

36. Cum problema propositum pendeat ab integratione istius aequationis differentialis:

$$\frac{d\zeta\,\sin.\eta}{d\eta\,\sin.\zeta} = \frac{P+\sqrt{(PP-QQ)}}{Q}$$

posito

posito breuitatis gratia

$$P = A\cos.\eta + B\cos.\zeta + D\cos.\zeta\cos.\eta + E\sin.\zeta\sin.\eta$$

$$Q = A\cos.\zeta + \cos.\eta + D$$

eam redigo ad hanc formam:

$$\frac{d\zeta\sin.\eta + d\eta\sin.\zeta}{d\zeta\sin.\eta - d\eta\sin.\zeta} = \frac{P+Q+\sqrt{(PP-QQ)}}{P-Q+\sqrt{(PP-QQ)}} = \frac{\sqrt{(P+Q)}}{\sqrt{(P-Q)}}$$

Tum vero posito $\tang.\tfrac{1}{2}\zeta = p$, et $\tang.\tfrac{1}{2}\eta = q$, vnde cum sit $\frac{d\zeta}{\sin.\zeta} = \frac{dp}{p}$ et $\frac{d\eta}{\sin.\eta} = \frac{dq}{q}$, nostra aequatio resoluenda erit

$$\frac{qdp+pdq}{qdp-pdq} = \sqrt{\frac{P+Q}{P-Q}}.$$

37. At posito $\tang\tfrac{1}{2}\zeta = p$ et $\tang.\tfrac{1}{2}\eta = q$, erit $\sin.\zeta = \frac{2p}{1+pp}$; $\cos.\zeta = \frac{1-pp}{1+pp}$; $\sin.\eta = \frac{2p}{1+qq}$; $\cos.\eta = \frac{1-qq}{1+qq}$. Quare cum sit

$$P+Q = (A+B)(\cos.\zeta + \cos.\eta) + D(1+\cos.\zeta\cos.\eta) + E\sin.\zeta\sin.\eta$$

fiet

$$P+Q\ \frac{2(A+B)(1+ppqq)+2D(1+ppqq)+2Epq}{(1+pp)(1+qq)}.$$

Deinde quia

$$P-Q = (A-B)(\cos.\eta - \cos.\zeta) - D(1-\cos\zeta\cos.\eta) + E\sin.\zeta\sin.\eta$$

fiet

$$P-Q = \frac{2(A-B)(pp-qq)-2D(pp+qq)+4Epq}{(1+pp)(1+qq)}.$$

His ergo valoribus introductis, nostra aequatio resoluenda erit

$$\frac{qdp+pdq}{qdp-pdq} = \sqrt{\frac{(A+B)(1-ppqq)+D(1+ppqq)+2Epq}{(A-B)(pp-qq)-D(pp+qq)+2Epq}}$$

quam facile patet ad separationem variabilium perduci posse, cum posterioris membri numerator sit functio ipsius pq, in denominatore autem quantitates p et q vbique duas dimensiones compleant.

 38.

38. Hunc in finem statuamus $pq = r$ et $\frac{p}{q} = s$ vt sit $p = \sqrt{rs}$ et $q = \sqrt{\frac{r}{s}}$, vnde ob $pdq + qdp = dr$ et $qdp - pdq = qqds$ fiet

$$\frac{dr}{qqds} = \sqrt{\frac{(A+B)(1-rr)+D(1+rr)+2Er}{qq((A+B)(ss-1)-D(ss+1)+2Es)}} \text{ seu}$$
$$\frac{dr}{ds} = \sqrt{\frac{r((A+B)(1-rr)+D(1+rr)+2Er)}{s((A-B)(ss-1)-D(ss+1)+2Es)}}$$

ex qua forma separatio variabilium r et s manifesta est, erit enim

$$\frac{dr}{\sqrt{r(A+B+D+2Er-(A+B-D)rr)}} = \frac{ds}{\sqrt{s(-A+B-D+2Es+(A-B-D)ss)}}$$

Vel si ponamus $r = xx$ et $s = yy$, habebitur

$$\frac{dx}{\sqrt{(A+B+D+2Exx-(A+B-D)x^4)}} = \frac{dy}{\sqrt{(-A+B-D+2Eyy+(A-B-D)y^4)}}$$

Quia autem r et s valores habere possent negatiuos, haec transformatio incommodum implicare posset.

39. Verum etsi hoc modo ad aequationem separatam peruenimus, tamen vtriusque partis integratio magna laborat difficultate, cum neque per circuli quadraturam, neque per logarithmos, absolui possit; constructio autem per arcus sectionum conicarum hic parum lucis esset allatura. Atque haec difficultas non minuitur, etsi statuamus $B = 0$, quo tamen casu solutio aliunde est nota; quin etiam casus $A = 0$ et $B = 0$, quo linea a corpore M descripta certo est recta, haud minore difficultate impeditur. Necesse igitur est, vt his casibus ambae quantitates transcendentes, quae ex vtraque integratione nascuntur, eiusmodi inter se teneant relationem, vt adeo aequationem algebraicam inter r et s complectantur. Ex quo nouus aperitur campus in aequationes algebraicas, quae forte in huiusmodi aequationibus differentialibus continentur, inquirendi. Atque in hoc negotio alia adhuc methodus non constat praeter eam, quam ante aliquot annos exposui, et cuius ope infinitos arcus, tam ellipticos, quam hyperbolicos, inter se comparaui, quae mihi iam tum maximum vsum aliquando habitura videbatur.

40. Sed antequam ad hanc methodum confugiam, haud abs re erit, originem erroris supra commissi indicare, quae nunc quidem est manifesta. Cum enim ad aequationem differentialem separatam inter r et s peruenerimus, euidens est, ei satisfieri, si vel ipsi r eiusmodi valor constans α tribuatur, vt fiat

$$A + B + D + 2Er - (A + B - D)rr = 0$$

vel ipsi s eiusmodi valor constans β, vt fiat

$$-A + B - D + 2Es + (A - B - D)ss = 0$$

quod vtrumque duobus modis fieri potest. Atque inde quidem $r = \alpha$, sequitur $pq = \text{tang.}\frac{1}{2}\zeta \text{tang.}\frac{1}{2}\eta = \alpha$, quae est aequatio pro ellipsi, hinc autem $s = \beta$ prodit $\frac{p}{q} = \beta$ seu $\text{tang.}\frac{1}{2}\zeta = \beta \text{tang.}\frac{1}{2}\eta$, quae est aequatio pro hyperbola. Neque vero hae curuae problema soluunt, nisi iidem valores in aequatione integrata contineantur. Euidens ergo est, nos in similem errorem illapsuros fuisse, si corporis M motum in hyperbola fieri assumsissemus.

41. Perpendamus iam aequationem integralem, quam casu $B = 0$ supra §. 22 ex nostra aequatione differentiali eliciuimus, quae erit

$$\frac{\sin.\zeta\cos.\eta}{\sin.\eta} = \frac{A + D\cos\zeta}{G} + \sin.\zeta\sqrt{\left(\frac{2E}{G} + \frac{AA - DD}{GG} - 1\right)}$$

haec-

haecque, posito tang. $\frac{1}{2}\zeta = p$ et tang. $\frac{1}{2}\eta = q$, porroque $pq = r$ et $\frac{p}{q} = s$, abit in hanc formam:

$$\left.\begin{matrix} GG(r+s)^2 + 2G(A+D)(r-s) & -8EGrs + (A+D)^2 \\ +2G(A-D)rs(r-s) & -2(AA\,DD)rs \\ & +(A-D)^2 rrss \end{matrix}\right\} = 0$$

quae est ergo aequatio integralis completa, huic aequationi differentiali conueniens

$$\frac{dr}{\sqrt{r(A+D+2Er-(A-D)rr)}} = \frac{ds}{\sqrt{s(-A-D+2Es+(A-D)ss)}}$$

quandoquidem in illa noua constans arbitraria G continetur.

42. Vt generaliter in talem aequationem integralem inquiramus, ponamus breuitatis gratia

$$\frac{A+B}{2E} = m;\ \frac{A-B}{2E} = n, \text{ et } \frac{D}{2E} = k$$

vt aequatio hoc modo integranda, siquidem id fieri potest, sit

$$\frac{dr}{\sqrt{r(m+k+r-(m-k)rr)}} = \frac{ds}{\sqrt{s(-n-k+s+(n-k)ss)}}$$

cuius integralem in hac forma contineri fingamus:

$$\mathfrak{A} + 2\mathfrak{B}r + 2\beta s + \mathfrak{C}rr + \gamma ss + 2\mathfrak{D}rs + 2\mathfrak{E}rrs + 2\varepsilon rss + \mathfrak{F}rrss = 0$$

vnde deducimus:

$$rr = \frac{-2\mathfrak{B}r - 2\mathfrak{D}rs - 2\varepsilon rss - \mathfrak{A} - 2\beta s - \gamma ss}{\mathfrak{C} + 2\mathfrak{E}s + \mathfrak{F}ss} \text{ et}$$

$$ss = \frac{-2\beta s - 2\mathfrak{D}rs - 2\mathfrak{E}rrs - \mathfrak{A} - 2\mathfrak{B}r - \mathfrak{C}rr}{\gamma + 2\varepsilon r + \mathfrak{F}rr}.$$

Tum vero est differentiando:

$$dr(\mathfrak{B} + \mathfrak{C}r + \mathfrak{D}s + 2\mathfrak{E}rs + \varepsilon ss + \mathfrak{F}rss) + ds(\beta + \gamma s + \mathfrak{D}r + \mathfrak{E}rr + 2\varepsilon rs + \mathfrak{F}rrs) = 0.$$

43.

43. Ex illis autem aequationibus radicem extrahendo obtinemus

$$r(\mathfrak{C} + 2\mathfrak{E}s + \mathfrak{F}ss) + \mathfrak{B} + \mathfrak{D}s + \varepsilon ss = S = \sqrt{((\mathfrak{B} + \mathfrak{D}s + \varepsilon ss)^2 - (\mathfrak{A} + 2\beta s + \gamma ss)(\mathfrak{C} + 2\mathfrak{E}s + \mathfrak{F}ss))}$$

atque

$$s(\gamma + 2\varepsilon r + \mathfrak{F}rr) + \beta + \mathfrak{D}r + \mathfrak{E}rr = -R = -\sqrt{((\beta + \mathfrak{D}r + \mathfrak{E}rr)^2 - (\mathfrak{A} + 2\mathfrak{B}r + \mathfrak{C}rr)(\gamma + 2\varepsilon r + \mathfrak{F}rr))}$$

qui valores in differentiali adhibito praebent

$$S\,dr - R\,ds = 0 \text{ seu } \frac{dr}{R} = \frac{ds}{S}.$$

Superest ergo tantum, vt formulae irrationales R et S iis, quas nostra aequatio resoluenda continet, aequales efficiantur, seu vt fiat

$$R = \sqrt{((m+k)r + rr - (m-k)r^3)} \text{ et}$$

$$S = \sqrt{(-(n+k)s + ss + (n-k)s^3)}.$$

44. Cum igitur hic vtrinque tam termini primi constantes, quam vltimi r^4 et s^4 continentes, sint nulli, fieri debet

$$\mathfrak{B}\mathfrak{B} - \mathfrak{A}\mathfrak{C} = 0;\ \varepsilon\varepsilon - \gamma\mathfrak{F} = 0;\ \beta\beta - \mathfrak{A}\gamma = 0;\ \mathfrak{E}\mathfrak{E} - \mathfrak{C}\mathfrak{F} = 0.$$

Erit ergo

$$\mathfrak{C} = \frac{\mathfrak{B}\mathfrak{B}}{\mathfrak{A}} = \frac{\mathfrak{E}\mathfrak{E}}{\mathfrak{F}} \text{ et } \gamma = \frac{\varepsilon\varepsilon}{\mathfrak{F}} = \frac{\beta\beta}{\mathfrak{A}}$$

hincque $\frac{\mathfrak{A}}{\mathfrak{F}} = \frac{\mathfrak{B}\mathfrak{B}}{\mathfrak{E}\mathfrak{E}} = \frac{\beta\beta}{\varepsilon\varepsilon}$.

Porro ob terminos rr et ss fieri oportet

$$\mathfrak{D}\mathfrak{D} + 2\mathfrak{B}\varepsilon - \mathfrak{A}\mathfrak{F} - 4\beta\mathfrak{E} - \gamma\mathfrak{C} = x$$

$$\mathfrak{D}\mathfrak{D} + 2\beta\mathfrak{E} - \mathfrak{A}\mathfrak{F} - 4\mathfrak{B}\varepsilon - \gamma\mathfrak{C} = x$$

 vnde

vnde fit $6\mathfrak{B}\varepsilon = 6\beta\mathfrak{E}$, seu $\frac{\mathfrak{B}}{\mathfrak{E}} = \frac{\beta}{\varepsilon}$, tum vero

$$\mathfrak{D}\mathfrak{D} - 2\beta\mathfrak{E} - \mathfrak{A}\mathfrak{F} - \gamma\mathfrak{C} = 1, \text{ seu}$$

$$\mathfrak{D}\mathfrak{D} - 2\beta\mathfrak{E} - \mathfrak{A}\mathfrak{F} - \frac{\beta\mathfrak{E}\mathfrak{E}}{\mathfrak{A}\mathfrak{F}} = 1, \text{ hincque}$$

$$\mathfrak{A}\mathfrak{F}(\mathfrak{D}\mathfrak{D} - 1) = (\beta\mathfrak{E} + \mathfrak{A}\mathfrak{F})^2$$

vel $\beta\mathfrak{E} = -\mathfrak{A}\mathfrak{F} + \sqrt{\mathfrak{A}\mathfrak{F}(\mathfrak{D}\mathfrak{D} - 1)} = \mathfrak{B}\varepsilon$

siue, cum sit $\mathfrak{F} = \frac{\mathfrak{A}\mathfrak{E}\mathfrak{E}}{\mathfrak{B}\mathfrak{B}}$, erit

$$\mathfrak{D}\mathfrak{D} = 2\beta\mathfrak{E} + \frac{\mathfrak{A}\mathfrak{A}\mathfrak{E}\mathfrak{E}}{\mathfrak{B}\mathfrak{B}} + \frac{\mathfrak{B}\mathfrak{B}\mathfrak{B}\mathfrak{B}}{\mathfrak{A}\mathfrak{A}} + 1 = \left(\frac{\mathfrak{A}\mathfrak{E}}{\mathfrak{B}} + \frac{\mathfrak{B}\mathfrak{B}}{\mathfrak{A}}\right)^2 + 1.$$

45. Reliqui termini dant

$$2\beta\mathfrak{D} - 2\mathfrak{A}\varepsilon - 2\mathfrak{B}\gamma = m + k$$

$$2\mathfrak{D}\mathfrak{E} - 2\mathfrak{C}\varepsilon - 2\mathfrak{B}\mathfrak{F} = -m + k$$

$$2\mathfrak{B}\mathfrak{D} - 2\mathfrak{A}\mathfrak{E} - 2\beta\mathfrak{C} = -n - k$$

$$2\mathfrak{D}\varepsilon - 2\gamma\mathfrak{E} - 2\beta\mathfrak{F} = -n - k$$

quarum summa praebet hanc aequalitatem:

$$\mathfrak{D}(\beta + \mathfrak{B} + \varepsilon + \mathfrak{E}) - \mathfrak{A}(\varepsilon + \mathfrak{E}) - \mathfrak{F}(\beta + \mathfrak{B}) - \mathfrak{C}\varepsilon - \gamma\mathfrak{E} - \mathfrak{B}\gamma - \beta\mathfrak{C} = 0.$$

Cum nunc inuenerimus $\frac{\mathfrak{B}}{\mathfrak{E}} = \frac{\beta}{\varepsilon}$, ponamus $\mathfrak{B} = \lambda\beta$, et $\mathfrak{E} = \lambda\varepsilon$, erit $\mathfrak{C} = \frac{\lambda\lambda\beta\beta}{\mathfrak{A}}$; $\gamma = \frac{\beta\beta}{\mathfrak{A}}$; et $\mathfrak{F} = \frac{\varepsilon\varepsilon}{\beta\beta}\mathfrak{A}$; vnde ponatur porro $\mathfrak{A} = \mu\beta\beta$ et $\mathfrak{F} = \mu\varepsilon\varepsilon$; vt fit $\mathfrak{C} = \frac{\lambda\lambda}{\mu}$, et $\gamma = \frac{1}{\mu}$, hincque $\mathfrak{D}\mathfrak{D} = 1 + (\mu\beta\varepsilon + \frac{\lambda}{\mu})^2$; quibus valoribus, praeter hunc vltimum, ibi substitutis obtinebimus

$$\mathfrak{D}(\lambda + 1)(\beta + \varepsilon) - \mu\beta\varepsilon(\lambda + 1)(\beta + \varepsilon) - \frac{\lambda(\lambda + 1)(\beta + \varepsilon)}{\mu} = 0$$

seu $(\lambda + 1)(\beta + \varepsilon)(\mathfrak{D} - \mu\beta\varepsilon - \frac{\lambda}{\mu}) = 0$

cuius aequationis tres factores totidem praebent solutiones.

46.

46. *Resolutio I.* Sit $\lambda = -1$, erit $\mathfrak{B} = -\beta$; $\mathfrak{E} = -\varepsilon$; $\mathfrak{C} = \frac{1}{\mu}$; $\gamma = \frac{1}{\mu}$; $\mathfrak{A} = \mu\beta\beta$; $\mathfrak{F} = \mu\varepsilon\varepsilon$; hincque $\mathfrak{D}\mathfrak{D} = (\mu\beta\varepsilon - \frac{1}{\mu})^2 + 1$, vnde conditiones adimplendae erunt:

$$k = \mathfrak{D}(\beta - \varepsilon) + \frac{\beta - \varepsilon}{\mu} - \mu\beta\varepsilon(\beta - \varepsilon) = (\beta - \varepsilon)(\mathfrak{D} + \frac{1}{\mu} - \mu\beta\varepsilon)$$

$$m = \mathfrak{D}(\beta + \varepsilon) + \frac{\beta + \varepsilon}{\mu} - \mu\beta\varepsilon(\beta + \varepsilon) = (\beta + \varepsilon)(\mathfrak{D} + \frac{1}{\mu} - \mu\beta\varepsilon)$$

$$n = \mathfrak{D}(\beta + \varepsilon) + \frac{\beta + \varepsilon}{\mu} - \mu\beta\varepsilon(\beta + \varepsilon) = (\beta + \varepsilon)(\mathfrak{D} + \frac{1}{\mu} - \mu\beta\varepsilon).$$

Hinc ergo foret $m = n$, et $B = 0$, ita vt haec resolutio tantum ad casum $B = 0$ accommodari possit. Hoc igitur casu cum sit $\frac{m}{k} = \frac{\beta + \varepsilon}{\beta - \varepsilon}$, ponatur $\beta + \varepsilon = m$ et $\beta - \varepsilon = k$, vt sit $\beta = \frac{k + m}{2}$ et $\varepsilon = \frac{m - k}{2}$; oportetque esse $\mathfrak{D} + \frac{1}{\mu} - \mu\beta\varepsilon = 1$, vnde oritur

$$1 + 2(\mu\beta\varepsilon - \frac{1}{\mu}) + (\mu\beta\varepsilon - \frac{1}{\mu})^2 = 1 + (\mu\beta\varepsilon - \frac{1}{\mu})^2$$

ideoque $\mu\mu = \frac{1}{\beta\varepsilon} = \frac{4}{mm - kk}$, et $\mu = \frac{2}{\sqrt{(mm - kk)}}$. Vnde colligimus $\mathfrak{D} = 1$, eritque pro casu $m = n$ aequatio integralis

$$\mathfrak{A} + 2\mathfrak{B}r + 2\beta s + \mathfrak{C}rr + \gamma ss + 2\mathfrak{D}rs + 2\mathfrak{E}rrs + 2\varepsilon rss + \mathfrak{F}rrss = 0.$$

47. At haec aequatio integralis, quia nulla noua inest constans, non est completa; cuius ratio est, quod quantitates $\beta - \varepsilon$ et $\beta + \varepsilon$ ipsis numeris k et m non aequales, sed tantum proportionales statui debent. Sit ergo

$$\beta - \varepsilon = \frac{k}{\nu};\ \beta + \varepsilon = \frac{m}{\nu};\ \text{erit } \beta = \frac{m + k}{2\nu};\ \varepsilon = \frac{m - k}{2\nu} \text{ et}$$

$$\mathfrak{D} = \nu + \mu\beta\varepsilon - \frac{1}{\mu} = \sqrt{(1 + (\mu\beta\varepsilon - \frac{1}{\mu})^2)}, \text{ vnde fit}$$

$$\mu\beta\varepsilon - \frac{1}{\mu} = \frac{1 - \nu\nu}{2\nu} \text{ et } \mathfrak{D} = \frac{1 + \nu\nu}{2\nu}$$

 vbi

vbi ν eſt quantitas arbitraria, per quam numerus μ definiri debet. Quoniam ergo β et ε per m et k cum ν dantur, aequatio integralis erit

$$0 = \mu\beta\beta + 2\beta(s-r) + \frac{1}{\mu}(rr+ss) + \frac{1+\nu\nu}{\nu}rs + 2\varepsilon rs(s-r) + m\varepsilon\varepsilon rrss$$

quae pro β et ε ſubſtitutis valoribus, per μ multiplicando abit in hanc formam:

$$0 = \frac{\mu\mu(m+k)^2}{4\nu\nu} + \frac{\mu}{\nu}(m+k)(s-r) + rr + ss + \frac{\mu\mu}{4\nu\nu}(m-k)^2 rrss + \frac{\mu}{\nu}(m-k)rs(s-r) + \frac{\mu(1+\nu\nu)}{\nu}rs.$$

Sit $\frac{\mu}{\nu} = 2f$, vt f ſit conſtans arbitraria, ſumtoque $f(mm-kk) = \frac{1}{f} + 1 - \nu\nu$, erit aequatio integralis completa

$$0 = ff(m+k)^2 + 2f(m+k)(s-r) + rr + ss + ff(m-k)^2 rrss + 2f(m-k)rs(s-r) + 2f(1+\nu\nu)rs.$$

48. Cum vero ſit $\nu\nu = 1 + \frac{1}{f} - f(mm-kk)$, erit $2f(1+\nu\nu) = 4f + 2 - 2ff(mm-kk)$; hincque aequatio integralis completa ita ſe euoluta habebit:

$$0 = ff(m+k)^2 + 2f(m+k)(s-r) + (r+s)^2 + ff(m-k)^2 rrss + 2f(m-k)rs(s-r) + 4frs - 2ff(mm-kk)rs$$

quae extracta radice induit hanc formam:

$$s - r + f(m+k) + f(m-k)rs = 2\sqrt{rs(ff(mm-kk)-f-1)}$$

et facta reſtitutione $s = \frac{p}{q}$; $r = pq$ fit

$$\frac{p(1-qq)}{q} + f(m+k) + f(m-k)pp = 2p\sqrt{(ff(mm-kk)-f-1}$$

quae cum integrali completo ſupra exhibito conuenit. Verum probe notandum, hoc integrale tantum ad caſum $B = 0$ pertinere.

49.

49. *Reſolutio II.* Ponamus nunc $\varepsilon = -\beta$, habebimusque primo $\mathfrak{B} = \lambda\beta$; $\mathfrak{E} = -\lambda\beta$; $\mathfrak{C} = \frac{\lambda}{\mu}$; $\gamma = \frac{1}{\mu}$; $\mathfrak{A} = \mu\beta\beta$; $\mathfrak{F} = \mu\beta\beta$ et $\mathfrak{D}\mathfrak{D} = 1 + (\frac{\lambda}{\mu} - \mu\beta\beta)^2$. Tum vero hinc concludimus:

$$k = \beta(1-\lambda)(\mathfrak{D} - \frac{\lambda}{\mu} + \mu\beta\beta)$$

$$m = \beta(1+\lambda)(\mathfrak{D} - \frac{\lambda}{\mu} + \mu\beta\beta) = -n$$

ita vt haec reſolutio locum non inueniat, niſi ſit $m + n = 0$, ideoque $A = 0$. Pro hoc autem caſu erit porro $\frac{k}{m} = \frac{1-\lambda}{1+\lambda}$, hincque $\lambda = \frac{m-k}{m+k}$, vnde ſequitur $k = \frac{2\beta k}{m+k}(\mathfrak{D} - \frac{\lambda}{\mu} + \mu\beta\beta)$ ſeu $\mathfrak{D} = \frac{\lambda}{\mu} - \mu\beta\beta + \frac{m+k}{2\beta}$; ergo

$$1 = \frac{m+k}{\beta}(\frac{\lambda}{\mu} - \mu\beta\beta) + \frac{(m+k)^2}{4\beta\beta} \quad \text{et}$$

$$\frac{\lambda}{\mu} - \mu\beta\beta = \frac{\beta}{m+k} - \frac{(m+k)}{4\beta}, \text{ ideoque } \mathfrak{D} = \frac{\beta}{m+k} + \frac{m+k}{4\beta}.$$

Littera β manet indefinita, et μ definitur per hanc aequationem: $\frac{m-k}{\mu(m+k)} - \mu\beta\beta = \frac{\beta}{m+k} - \frac{(m+k)}{4\beta}$.

50. His valoribus ſubſtitutis reſultat aequatio integralis completa pro caſu $A = 0$:

$$0 = \mu\beta\beta + 2\lambda\beta r + 2\beta s + \frac{\lambda\lambda}{\mu}rr + \frac{1}{\mu}ss + \frac{2\beta rs}{m+k} + \frac{(m+k)}{2\beta}rs - 2\lambda\beta rrs - 2\beta rss + \mu\beta\beta rrss$$

Statuamus $\mu\beta = f$, erit $\frac{m-k}{m+k} - ff = \frac{f}{m+k} - \frac{\mu\mu(m+k)}{4f}$ et illa aequatio per μ multiplicata erit

$$0 = ff + 2\lambda fr + 2fs + \lambda\lambda rr + ss\frac{2frs}{m+k} + \frac{\mu\mu(m+k)}{2f}rs - 2\lambda frrs - 2frss + ffrrss$$

quae ob $\frac{\mu\mu(m-k)}{2f} = \frac{2f}{m+k} + 2ff - 2\lambda$ induit hanc formam:

$$0 = ff(1+rs)^2 + (\lambda r - s)^2 + \frac{4frs}{m+k} + 2f(\lambda r + s) - 2frs(\lambda r + s)$$

ſeu

seu hanc:

$$0 = ff(1-rs)^2 + (\lambda r + s)^2 + 2f(1-rs)(\lambda r + s) + \frac{4frs}{m+k} + 4ffrs - 4\lambda rs$$

quae extracta radice praebet

$$f(1-rs) + \lambda r + s = 2\sqrt{rs}\left(\frac{m-k}{m+k} - ff - \frac{f}{m+k}\right)$$

Estque haec solutio similis omnino praecedenti, dum illa ad casum $B=0$, haec vero ad casum $A=0$, adstringitur.

51. Tertius factor $\mathfrak{D} - \mu\beta\varepsilon - \frac{\lambda}{\mu}$ nihil monstrat, quia eius annihilatio cum aequatione $\mathfrak{D}\mathfrak{D} = 1 + (\mu\beta\varepsilon + \frac{\lambda}{\mu})^2$ consistere nequit, sicque duos tantum casus habemus, resolutionem algebraicam admittentes, scilicet si vel $B=0$, vel $A=0$. Praeterea vero etiam tertius casus supra euolutus, quo erat $A=B$, et $D=0$ $E=0$, hic sponte se offert, tum enim aequatio §. 38. abit in hanc: $\frac{dr}{\sqrt{(A+B)r(1-rr)}} = \frac{ds}{\sqrt{s.0}}$, quae subsistere nequit, nisi sit $ds=0$, ideoque $s = \frac{p}{q} = \frac{\text{tang.}\frac{1}{2}\zeta}{\text{tang.}\frac{1}{2}\eta} = \text{Const.}$ qua aequatione hyperbola definitur. Reliquis casibus constructio aequationis

$$\frac{dr}{\sqrt{r(A+B+D+2Er-(A+B-D)rr)}} = \frac{ds}{\sqrt{s(B-A-D+2Es-(B-A+D)ss)}}$$

in subsidium est vocanda. Quod enim haec aequatio in genere integrale algebraicum non admittat, vel ex casu $D=A+B$ patet, quo prius membrum a quadratura sectionum conicarum pendet, posterius vero altiores quadraturas postulat.

52. Inuenta autem relatione inter r et s, vnde simul ratio angulorum ζ et η innotescit, cognitio motus per tempus hauritur. Cum enim sit

$$vvuud\zeta d\eta = 2gadt^2(A\cos.\zeta + B\cos.\eta + D)$$

ob

ob $v = \frac{a\sin.\eta}{\sin.(\zeta+\eta)}$ et $u = \frac{a\sin.\zeta}{\sin.(\zeta+\eta)}$, atque tang. $\frac{1}{2}\zeta = p$ et tang. $\frac{1}{2}\eta = q$, erit $d\zeta = \frac{dp\sin.\zeta}{p}$, et $d\eta = \frac{dq\sin.\eta}{q}$, seu

$$d\zeta = \frac{2dp}{1+pp} \text{ et } d\eta = \frac{2dq}{1+qq}, \text{ hincque porro}$$

$$v = \frac{aq(1+pp)}{(p+q)(1-pq)} \text{ et } u = \frac{ap(1+qq)}{(p+q)(1-pq)}, \text{ vnde fit}$$

$$\frac{4a^3ppq(1+pp)(1+qq)dpdq}{(p+q)^4(1-pq)^4} = 2gdt^2\left(\frac{A(1-pp)}{1+pp} + \frac{B(1-qq)}{1+qq} + D\right).$$

Fiat iam porro $pq = r$, $\frac{p}{q} = s$, seu $pp = rs$ et $qq = \frac{r}{s}$ erit $2pdp = rds + sdr$ et $2qdq = \frac{sdr - rds}{ss}$, ergo $4pqdpdq = \frac{ssdr^2 - rrds^2}{ss}$, hincque tandem:

$$\frac{a^3(r+s)(1+rs)(ssdr^2 - rrds^2)}{rs(1+s)^4(1-r)^4} = 2gdt^2\left(\frac{A(1-rs)}{1+rs} + \frac{B(s-r)}{r+s} + D\right).$$

53. Ponamus nunc breuitatis gratia:

$$RR = r(A+B+D+2Er-(A+B-D)rr)$$

$$SS = s(B-A-D+2Es-(B-A+D)ss)$$

vt sit $\frac{dr}{R} = \frac{ds}{S}$, statuamusque

$$\frac{dr}{R} = \frac{ds}{S} = dV, \text{ vt sit } dr = RdV \text{ et } ds = SdV, \text{ fietque}$$

$$2gdt^2 = \frac{a^3(r+s)^2(1+rs)^2(RRss - SSrr)dV^2}{rs(1+s)^4(1-r)^4(A(r+s)(1-rs)+B(s-r)(1+rs)+D(s+r)(1+rs))}.$$

Est vero

$$RRss - SSrr = rs(A(r+s)(1-rs) + B(s-r)(1+rs) + D(r+s)(1+rs))$$

quo valore substituto prodit

$$2gdt^2 = \frac{a^3(r+s)^2(1+rs)^2dV^2}{(1+s)^4(1-r)^4}, \text{ ideoque}$$

$$dt\sqrt{\frac{2g}{a}} = \frac{a(r+s)(1+rs)dV}{(1+s)^2(1-r)^2} = adV\left(\frac{r}{(1-r)^2} + \frac{r}{(1+s)^2}\right)$$

ita vt sit

$$\frac{dt\sqrt{2g}}{a\sqrt{a}} = \frac{dr\sqrt{r}}{(1-r)^2\sqrt{(A+B+D+2Er-(A+B-D)rr)}} + \frac{ds\sqrt{s}}{(1+s)^2\sqrt{(B-A-D+2Es-(B-A+D)s^2)}};$$

sicque

ſicque etiam determinatio temporis ad integrationem formularum ſimplicium eſt perducta.

54. Cum igitur hoc problema, quod primo aſpectu vix facilius quam id, quo omnia tria corpora mobilia aſſumuntur, viſum erat, perfecte reſoluere licuerit, maiorem ſpem concipimus, fore aliquando, vt et iſtud problema, cui tanquam fundamento vniuerſa Aſtronomia inniti eſt cenſenda, reſoluatur. Equidem fateor, me hinc nullam adhuc viam ad hunc ſcopum perueniendi perſpicere, ſed etiam ad hoc plurimos ac fortaſſe operoſiſſimos conatus requiri agnoſco. Caeterum circa hoc ipſum problema, quod hic tractaui, obſeruationem Geometris forte haud ingratam adiicio, ſcilicet praeter caſus hic euolutos, innumerabiles alios dari, quibus curua, a corpore M deſcripta, futura ſit algebraica, quarum inueſtigatio Analyſi haud contemnenda incrementa allatura videtur.

55. Quanquam autem ſolutionem huius problematis ad quadraturas curuarum perduximus, tamen moleſtum foret, curuam a corpore M deſcriptam definire multoque magis ad datum tempus locum corporis aſſignare. Sin autem huiusmodi caſus in mundo exiſteret, operae pretium eſſet, hanc ſolutionem accuratius euoluere, quod hoc modo commodiſſime fieri poſſe videtur. Pro tali ſcilicet caſu, poſtquam per plura tentamina conſtantes A, B, D, E proxime ſaltem innotuerint deinceps corrigendae, tabula condi debebit pro ſingulis valoribus ipſius r valores conuenientes litterae s referens, cui deinceps tabula tempora t exhibens adiungi deberet, ex qua porro viciſſim pro dato tempore t valores litterarum r et s, hincque angulos ζ et η concludere liceret. Quae determinatio ſi cum obſeruationibus minus conueniret, indicio id eſſet, conſtantes non recte eſſe aſſumtas, ſicque tandem pluribus huiusmodi tabulis conſtructis, veritas inde non difficulter erueretur.

56. Cum autem inprimis et corporis M loca noſſe conueniat, vbi eius diſtantia ab alterutro punctorum fixorum A et B eſt maxima vel minima, quemadmodum hoc definiri oporteat videamus. Quia diſtantia AM eſt $v = \frac{a \sin. \eta}{\sin. (\zeta + \eta)}$, eius differentiale

$$\frac{dv}{a} = \frac{d\eta \cos. \eta \sin. (\zeta + \eta) - (d\zeta + d\eta) \sin. \eta \cos. (\zeta + \eta)}{\sin. (\zeta + \eta)^2} = \frac{d\eta \sin. \zeta - d\zeta \sin. \eta \cos. (\zeta + \eta)}{\sin. (\zeta + \eta)^2}$$

nihilo aequale poſitum, ſeu $\frac{d\eta}{\sin. \eta} = \frac{d\zeta}{\sin. \zeta} \cos. (\zeta + \eta)$ indicabit loca, quibus diſtantia AM eſt vel maxima, vel minima. Poſito ergo tang. $\frac{1}{2}\zeta = p$ et tang $\frac{1}{2}\eta = q$, ob cof. $(\zeta + \eta) \frac{(1-pp)(1-qq) - 4pq}{(1+pp)(1+qq)}$, habebimus $\frac{dq}{q} = \frac{dp}{p} \cdot \frac{(1-pq)^2 - (p+q)^2}{(1+pp)(1+qq)}$. Factoque porro $pq = r$, $\frac{p}{q} = s$, ſeu $p = \sqrt{rs}$, $q = \sqrt{\frac{r}{s}}$, fiet $(\frac{dr}{r} - \frac{ds}{s})(s(1+rr) + r(1+ss)) = (\frac{dr}{r} + \frac{ds}{s})(s(1-r)^2 - r(1+s)^2)$ ſeu $dr(1+s)^2 = ds(1-r)^2$. Vbi ergo eſt $\frac{dr}{(1-r)^2} = \frac{ds}{(1+s)^2}$ ibi diſtantia $AM = v$ eſt vel maxima vel minima.

57. Cum igitur ſupra inuenerimus $\frac{dr}{R} = \frac{ds}{S}$, pro his locis habemus $\frac{R}{(1-r)^2} = \frac{S}{(1+s)^2}$, vnde relatio inter quantitates finitas r et s eruitur, quae eſt:

$$r(1+s)^4 (A + B + D + 2Er - (A + B - D) rr) =$$
$$s(1-r)^4 (B - A - D + 2Es - (B - A + D) ss)$$

 vnde

vnde pofito breuitatis gratia

$$\frac{A+B+D}{2E}=m, \quad \frac{B-A-D}{2E}=n$$

$$\frac{A+B-D}{2E}=\mu, \quad \frac{B-A+D}{2E}=\nu$$

vt fit $\mu+\nu=m+n$, enafcitur haec aequatio

$$\left.\begin{array}{lllll} +mr+rr & -\mu r^3 & +4(n-\mu)r^4 s-nr^4 s & -r^4 ss & \\ -ns+4(m+n)rs & +(4-6n)rrs+4(m-\nu)rs^3 & +(4-6\mu)r^3 ss-4(\mu+\nu)r^3 s^3 & +\nu r^4 s^3 & \\ -ss & +(4+6m)rss & +(4+6r)rrs^3+rrs^4 & +\mu r^3 s^4 & \\ & +\nu s^2 & +mrs^4 & & \end{array}\right\}=0$$

quae aequatio in genere nullos factores habere videtur. At aequatio inter p et q erit

$$(p+q)^4(m+pq-\mu ppqq)=(1-pq)^4(nqq+pq-\nu pp).$$

58. Reftitutis autem ipfis angulis ζ et η, inter eos aequatio pro hoc cafu, quo diftantia v fit vel maxima vel minima, ita fe habebit:

$$\text{fin.}\left(\tfrac{\zeta+\eta}{2}\right)^4(D(1+\text{cof.}\zeta\,\text{cof.}\eta)+(A+B)(\text{cof.}\zeta+\text{cof.}\eta) + E\,\text{fin.}\zeta\,\text{fin.}\eta)=$$

$$\text{cof.}\left(\tfrac{\zeta+\eta}{2}\right)^4(D(\text{cof.}\zeta\,\text{cof.}\eta-1)+(B-A)(\text{cof.}\zeta-\text{cof.}\eta) + E\,\text{fin.}\zeta\,\text{fin.}\eta)$$

vbi eft $\text{fin}\left(\frac{\zeta+\eta}{2}\right)^4=\frac{1}{4}(1-\text{cof}(\zeta+\eta))^2$ et $\text{cof.}\left(\frac{\zeta+\eta}{2}\right)^4 =\frac{1}{4}(1+\text{cof.}(\zeta+\eta))^2$ vnde colligimus:

$$(1+\text{cof.}(\zeta+\eta)^2)(A\,\text{cof.}\zeta+B\,\text{cof.}\eta+D)=2\,\text{cof.}(\zeta+\eta)$$
$$(A\,\text{cof.}\eta+B\,\text{cof.}\zeta+D\,\text{cof.}\zeta\,\text{cof.}\eta+E\,\text{fin.}\zeta\,\text{fin.}\eta)$$

quae autem aequatio aeque parum refolutionem admittit. Caeterum quia permutatis angulis ζ, η et maffis A et B aequatio non mutatur, eadem loca indicat, vbi diftantia $BM=u$ fit vel maxima vel minima. Verum radice quadrata extracta bini cafus a fe inuicem feparantur.

DE

DE MOTV VIBRATORIO TYMPANORVM.

Auctore

L. EVLERO.

1.

Quae adhuc a Geometris de motu vibratorio funt inueftigata, ad corpora tantum vna dimenfione praedita, vel quae potius tanquam talia confiderare liceat, funt reftricta, cuius modi funt cordae tenfae, et laminae elafticae, quarum vnica tantum dimenfio fecundum longitudinem extenfa in calculum ingreditur, caeteris neglectis. Hinc quomodo fuperficies, cuiusmodi eft linteum vel membrana extenfa, ad motum vibratorium fit comparata, a nemine adhuc, quantum mihi quidem conftat, eft definitum. Pertinet huc imprimis doctrina fonorum, quos tympanum tenfum ac pulfum edere folet, cuius doctrinae hic quidem prima quafi fundamenta iacere conftitui, quae eo magis omni attentione digna videtur, quod nouum fere calculi genus requirit; cum enim in cordarum vibrationibus infinita varietas locum habeat, in vibrationibus membranarum infinities maiorem varietatem admitti oportere euidens eft, quam idcirco calculus exhibere debet.

2.