DE CVRVA HYPERGEOMETRICA HAC AEQVATIONE EXPRESSA

$y = 1 . 2 . 3 . \ldots . x$.

Auctore

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I.

Denotante hic littera x abſciſſam et y applicatam, aequatio haec immediate nonniſi pro iis abſciſſis, quae numeris integris exprimuntur, applicatarum quantitatem indicat; hinc enim ſi fuerint

abſciſſae x . . 0, 1, 2, 3, 4, 5, 6 etc.
erunt
applicatae y . . 1, 1, 2, 6, 24, 120, 720 etc.

ita, vt dum abſciſſae ſecundum numeros naturales capiuntur, applicatae ſecundum progreſſionem hypergeometricam *Walliſii* progrediantur; quam ob cauſam etiam hanc curuam hypergeometricam appellari conueniet. Etſi autem per hanc aequationem innumerabilia quidem iſtius curuae puncta, ſed inter ſe diſcreta aſſignantur; vniuerſa tamen huius curuae indoles per eam aequationem definiri eſt cenſenda, ita vt cuique abſciſſae certa ac vi iſtius ipſius aequationis determinata reſpondeat applicata. Ratio enim iſtius aequationis omnino poſtulat, vt ſi abſciſſae cuicunque $x = p$ conueniat applicata $y = q$, tum abſciſſae $x = p + 1$ reſpondeat applicata $y = q(p + 1)$ abſciſſae vero $x = p - 1$ haec applicata $y = \frac{q}{p}$. Quam ob rem neutiquam arbitrio noſtro relinquitur per infinita illa puncta data curuam quandam parabolici generis ducere, cum omnia plane eius puncta ex ipſa aequatione determinentur.

II.

Praeter has autem applicatas, quae abſciſſis per numeros integros expreſſis reſpondent, imprimis notari merentur, quae inter eas ex aequo interiacent; et omnes per eam, quam abſciſſae $x = \frac{1}{2}$ reſpondere et quantitati $\frac{1}{2}\sqrt{\pi}$ aequari olim oſtendi, determinantur. Cum igitur ſit $\sqrt{\pi} = 1,77245385090548$; hae applicatae coniunctim tam pro abſciſſis poſitiuis quam negatiuis ſequenti modo ſe habebunt:

pro

pro abſciſſis poſitiuis		pro abſciſſis negatiuis	
x	eſt applicata y	x	eſt applicata y
0	1	0	$+1$
$\frac{1}{2}$	0,8862269	$-\frac{1}{2}$	$+1,7724538$
1	1	-1	$\pm\infty$
$1\frac{1}{2}$	1,3293404	$-1\frac{1}{2}$	$-3,5449077$
2	2	-2	$\mp\infty$
$2\frac{1}{2}$	3,3233509	$-2\frac{1}{2}$	$+2,3632718$
3	6	-3	$\pm\infty$
$3\frac{1}{2}$	11,6317284	$-3\frac{1}{2}$	$-0,9453087$
4	24	-4	$\mp\infty$
$4\frac{1}{2}$	52,3427777	$-4\frac{1}{2}$	$+0,2700882$
5	120	-5	$\pm\infty$
$5\frac{1}{2}$	287,8852775	$-5\frac{1}{2}$	$-0,0600196$
6	720	-6	$\mp\infty$
$6\frac{1}{2}$	1871,2543038	$-6\frac{1}{2}$	$+0,0109126$
7	5040	-7	$\pm\infty$

Hinc delineaui iſtam curuam in fig. 1. expreſſam quae ab abſciſſa negatiua $x=-1$, vbi applicata fit infinita vsque ad $x=3$, vbi fit $y=6$ porrigitur, hinc vero continuo in infinitam aſcendere eſt intelligenda; ſiniſtrorſum vero, vbi pro ſingulis abſciſſarum valoribus integris applicatae abeunt in aſymtotas, vltra $x=-1$ non expreſſi. Tab. I. Fig. 1.

III.

Conſideratio huius curuae plures ſuppeditat quaeſtiones haud parum curioſas, eius naturae accuratius cognoſcendae inſeruientes, quarum euolutio eo maio-

maiori attentione digna videtur, quod aequatio pro curua more solito explicari nequit. Huiusmodi quaestiones primo circa determinationem reliquorum curuae punctorum praeter ea, quae facile assignare licet, versantur. Deinde in singulis punctis positio tangentis insignem inuestigationem requirit, quo facilius tractus totius curuae definiri queat. Tum vero ex inspectione figurae perspicuum est inter abscissas $x=0$ et $x=1$, alicubi applicatam omnium minimam esse debere; cuius adeo tam locum quam ipsam quantitatem assignari operae erit pretium.

Praeterea vero inter binas abscissas negatiuas $-1, -2, -3, -4, -5$ etc. vbi applicatae in infinitum extenduntur, necesse est dari quoque applicatas minimas, quae quo magis sinistrorsum progrediamur, continuo minores euadunt, donec tandem prorsus euanescant. Denique etiam quaestio de radio curvaminis in singulis curuae punctis attentionem nostram meretur, isque imprimis curuae locus notatu dignus videtur, vbi curuatura est maxima, siquidem manifestum est, in elongatione ab axe curuae ramos continuo propius ad lineam rectam accedere. Has igitur quaestiones resoluere institui.

Quaestio prima.

Pro curua hypergeometrica inuenire aequationem continuam inter abscissam x *et applicatam* y, *quae aeque locum habeat, siue pro* x *capiatur numerus integer, siue fractus quicunque.*

4. Cum

4. Cum aequatio propofita $y = 1.2.3 \ldots x$ locum proprie habere nequeat, nifi x fit numerus integer; eam in aliam formam transfundi oportet, quae ab hac conditione fit liberata; quod pluribus modis per expreffiones in infinitum excurrentes fieri poteft, inter quas primum occurrit ifta:

$$y = \frac{1}{1+x}\left(\tfrac{2}{1}\right)^x \cdot \frac{2}{2+x}\left(\tfrac{3}{2}\right)^x \cdot \frac{3}{3+x}\left(\tfrac{4}{3}\right)^x \cdot \frac{4}{4+x}\left(\tfrac{5}{4}\right)^x. \text{ etc.}$$

qui factores in infinitum continuari debent. Ratio huius expreffionis inde eft manifefta, quod quo plures capiantur factores, veritas eo propius, fumtis autem infinitis, accurate obtineatur: fi enim factorum numerus fit $= n$, habetur

$$y = \frac{1}{1+x} \cdot \frac{2}{2+x} \cdot \frac{3}{3+x} \cdots \cdots \frac{n}{n+x}(n+1)^x$$

cuius numerator fi ita repraefentetur:

$$1.2.3 \ldots \ldots x(x+1)(x+2)(x+3) \ldots n$$

denominator vero ita

$$(1+x)(2+x)(3+x) \ldots n(n+1)(n+2) \ldots (n+x)$$

deletis factoribus communibus prouenit

$$y = \frac{1.2.3 \ldots \ldots \ldots \ldots \ldots x}{(n+1)(n+2)(n+3) \ldots (n+x)}(n+1)^x.$$

Quare fi n fit numerus infinitus, ob denominatoris fingulos factores $= n+1$ eorumque numerorum $= x$, totus denominator per multiplicatorem $(n+1)^x$ tollitur, proditque aequatio propofita $y = 1.2.3 \ldots x$.

5. Haec forma aliquanto generalior reddi poteft; cum enim totum negotium eo redeat, vt multi-

multiplicator $(n+1)^x$ postremo denominatori $(n+1)(n+2)(n+3) \ldots \ldots (n+x)$ aequiualeat, casu quo numerus n est infinitus, euidens est huic conditioni quoque satisfieri, si multiplicator ille in genere statuatur $(n+a)^x$ existente a numero quocunque finito; maxime vero hanc formulam ad institutum fore accommodatam, si litterae a medius quidam valor inter 1 et x veluti $a = \frac{1+x}{2}$ seu $a = \sqrt{x}$ tribuatur. Nunc vero necesse est hunc multiplicatarem $(n+a)^x$ in tot factores, quot numerus n continet vnitates, resolui, quod commode hac resolutione praestatur:

$$(n+a)^x = a^x . \left(\frac{a+1}{a}\right)^x . \left(\frac{a+2}{a+1}\right)^x . \left(\frac{a+3}{a+2}\right)^x \ldots \left(\frac{a+n}{a+n-1}\right)^x .$$

Quocirca pro abscissa quacunque x habebimus applicatam:

$$y = a^x . \frac{1}{1+x}\left(\frac{a+1}{a}\right)^x . \frac{2}{2+x}\left(\frac{a+2}{a+1}\right)^x . \frac{3}{3+x}\left(\frac{a+3}{a+2}\right)^x . \text{ etc. in infinitum}$$

quae expressio semper veritati est consentanea, quicunque numerus pro a accipiatur, promtissime autem ad veritatem perducet, si sumatur $a = \frac{1+x}{2}$, vnde fiet:

$$y = \left(\frac{1+x}{2}\right)^x . \frac{1}{1+x}\left(\frac{3+x}{1+x}\right)^x . \frac{2}{2+x}\left(\frac{5+x}{3+x}\right)^x . \frac{3}{3+x}\left(\frac{7+x}{5+x}\right)^x . \text{ etc.}$$

quae expressio ex infinitis factoribus formae $\frac{m}{m+x}\left(\frac{a+m}{a+m-1}\right)^x$ praeter primum a^x constat, et quo plures quouis casu inuicem multiplicantur, eo propius ad ad veritatem accedetur. Hinc autem naſcitur prima expreſſio, ſi ſumatur $a = 1$.

6. Eo magis autem haec expreſſio ad vſum eſt accommodata, quo promtius factores ad vnitatem conuergunt, id quod euenit ſumendo $a = \frac{1+x}{2}$, tum vero calculus eo facilius expedietur, quo minores numeri loco x ſubſtituuntur, ſemper autem ſufficit applicatas pro abſciſſis x vnitate vel adeo nihilo minoribus inueſtigaſſe, quoniam inde facili negotio applicatae per abſciſſis $x+1$, $x+2$, $x+3$, $x+4$ etc. deriuantur. Sit igitur $x = \frac{\alpha}{\beta}$ exiſtente $\alpha < \beta$, eritque

$$y = \left(\frac{\alpha+\beta}{2\beta}\right)^{\frac{\alpha}{\beta}} \cdot \frac{\beta}{\alpha+\beta}\left(\frac{3\beta+\alpha}{\beta+\alpha}\right)^{\frac{\alpha}{\beta}} \cdot \frac{2\beta}{\alpha+2\beta}\left(\frac{5\beta+\alpha}{3\beta+\alpha}\right)^{\frac{\alpha}{\beta}} \cdot \frac{3\beta}{\alpha+3\beta}\left(\frac{7\beta+\alpha}{5\beta+\alpha}\right)^{\frac{\alpha}{\beta}} \cdot \text{etc.}$$

vnde applicatae poteſtas y^β ita prodit expreſſa:

$$y^\beta = \left(\frac{\alpha+\beta}{2\beta}\right)^\alpha \cdot \frac{\beta^\beta(3\beta+\alpha)^\alpha}{(\beta+\alpha)^\beta(\beta+\alpha)^\alpha} \cdot \frac{(2\beta)^\beta(5\beta+\alpha)^\alpha}{(2\beta+\alpha)^\beta(3\beta+\alpha)^\alpha} \cdot \frac{(3\beta)^\beta(7\beta+\alpha)^\alpha}{(3\beta+\alpha)^\beta(5\beta+\alpha)^\alpha} \text{ etc.}$$

Pro abſciſſa autem $x = -\frac{\alpha}{\beta}$ applicata y hinc colligetur

$$y^\beta = \left(\frac{2\beta}{\beta-\alpha}\right)^\alpha \cdot \frac{\beta^\beta(\beta-\alpha)^\alpha}{(\beta-\alpha)^\beta(3\beta-\alpha)^\alpha} \cdot \frac{(2\beta)^\beta(3\beta-\alpha)^\alpha}{(2\beta-\alpha)^\beta(5\beta-\alpha)^\alpha} \cdot \frac{(3\beta)^\beta(5\beta-\alpha)^\alpha}{(3\beta-\alpha)^\beta(7\beta-\alpha)^\alpha} \cdot \text{etc.}$$

Sumamus exempli gratia $x = \frac{1}{2}$ et impetrabimus:

$$y^2 = \frac{3}{4} \cdot \frac{2 \cdot 2 \cdot 7}{3 \cdot 3 \cdot 3} \cdot \frac{4 \cdot 4 \cdot 11}{5 \cdot 5 \cdot 7} \cdot \frac{6 \cdot 6 \cdot 15}{7 \cdot 7 \cdot 11} \cdot \frac{8 \cdot 8 \cdot 19}{9 \cdot 9 \cdot 15} \cdot \text{etc.}$$

cuius factor in genere cum ſit $\frac{2n \cdot 2n \cdot (4n+3)}{(2n+1)(2n+1)(4n-1)} = \frac{16n^3+12nn}{16n^3+12nn-1} = 1 + \frac{1}{(2n+1)^2(4n-1)}$, hinc intelligitur, quam

 promte

promte hi factores ad vnitatem accedunt, erit igitur:

$$y^2 = \tfrac{3}{4}\left(1+\tfrac{1}{3^2 \cdot 3}\right)\left(1+\tfrac{1}{5^2 \cdot 7}\right)\left(1+\tfrac{1}{7^2 \cdot 11}\right)\left(1+\tfrac{1}{9^2 \cdot 15}\right)\left(1+\tfrac{1}{11^2 \cdot 19}\right) \text{ etc.}$$

vbi quidem nouimus esse $y^2 = \frac{\pi}{4}$. Sin autem statuamus $x = -\frac{1}{2}$, cui conuenit $y = \sqrt{\pi}$ erit ex altera expressione

$$\pi = 4 \cdot \frac{2 \cdot 2 \cdot 1}{1 \cdot 1 \cdot 5} \cdot \frac{4 \cdot 4 \cdot 5}{3 \cdot 3 \cdot 9} \cdot \frac{6 \cdot 6 \cdot 9}{5 \cdot 5 \cdot 13} \cdot \frac{8 \cdot 8 \cdot 13}{7 \cdot 7 \cdot 17} \text{ etc.}$$

seu $\pi = 4\left(1-\frac{1}{1^2 \cdot 5}\right)\left(1-\frac{1}{3^2 \cdot 9}\right)\left(1-\frac{1}{5^2 \cdot 13}\right)\left(1-\frac{1}{7^2 \cdot 17}\right)$ etc. inde vero est $\pi = 3\left(1+\frac{1}{3^2 \cdot 3}\right)\left(1+\frac{1}{5^2 \cdot 7}\right)\left(1+\frac{1}{7^2 \cdot 11}\right)\left(1+\frac{1}{9^2 \cdot 15}\right)$ etc.

ita vt altera crescendo, altera decrescendo ad veritatem appropinquet.

7. Commodius autem calculus instituetur, si expressio nostra in singulis factoribus abrumpatur, tum enim sequentes formulae prodibunt continuo propius ad veritatem accedentes:

$$y = \frac{1}{1+x}\left(\frac{3+x}{2}\right)^x$$

$$y = \frac{1}{1+x} \cdot \frac{2}{2+x}\left(\frac{5+x}{2}\right)^x$$

$$y = \frac{1}{1+x} \cdot \frac{2}{2+x} \cdot \frac{3}{3+x}\left(\frac{7+x}{2}\right)^x$$

$$y = \frac{1}{1+x} \cdot \frac{2}{2+x} \cdot \frac{3}{3+x} \cdot \frac{4}{4+x}\left(\frac{9+x}{2}\right)^x$$

$$y = \frac{1}{1+x} \cdot \frac{2}{2+x} \cdot \frac{3}{3+x} \cdot \frac{4}{4+x} \cdot \frac{5}{5+x}\left(\frac{11+x}{2}\right)^x.$$

Quia

Quia ſi loco x ſcribatur $-x$ prodit applicata $= \frac{y}{x}$ erit per ſimiles formulas:

$$y = \left(\frac{2+x}{2}\right)^{x-1}$$

$$y = \frac{2}{1+x}\left(\frac{4+x}{2}\right)^{x-1}$$

$$y = \frac{2}{1+x} \cdot \frac{3}{2+x}\left(\frac{6+x}{2}\right)^{x-1}$$

$$y = \frac{2}{1+x} \cdot \frac{3}{2+x} \cdot \frac{4}{3+x}\left(\frac{8+x}{2}\right)^{x-1}$$

$$y = \frac{2}{1+x} \cdot \frac{3}{2+x} \cdot \frac{4}{3+x} \cdot \frac{5}{4+x}\left(\frac{10+x}{2}\right)^{x-1}.$$

Quare poſito $x = \frac{1}{2}$ pro applicata $y = \frac{1}{2}\sqrt{\pi}$ duplex ſeries formularum eo conuergentium reſultat:

$$\frac{1}{2}\sqrt{\pi} = \frac{2}{3}\sqrt{\frac{7}{4}}$$

$$\frac{1}{2}\sqrt{\pi} = \frac{2 \cdot 4}{3 \cdot 5}\sqrt{\frac{11}{4}}$$

$$\frac{1}{2}\sqrt{\pi} = \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}\sqrt{\frac{15}{4}}$$

$$\frac{1}{2}\sqrt{\pi} = \frac{2 \cdot 4 \cdot 6 \cdot 8}{3 \cdot 5 \cdot 7 \cdot 9}\sqrt{\frac{19}{4}}$$

etc.

$$\frac{1}{2}\sqrt{\pi} = \sqrt{\frac{4}{5}}$$

$$\frac{1}{2}\sqrt{\pi} = \frac{4}{3}\sqrt{\frac{4}{9}}$$

$$\frac{1}{2}\sqrt{\pi} = \frac{4 \cdot 6}{3 \cdot 5}\sqrt{\frac{4}{13}}$$

$$\frac{1}{2}\sqrt{\pi} = \frac{4 \cdot 6 \cdot 8}{3 \cdot 5 \cdot 7}\sqrt{\frac{4}{17}}$$

$$\frac{1}{2}\sqrt{\pi} = \frac{4 \cdot 6 \cdot 8 \cdot 10}{3 \cdot 5 \cdot 7 \cdot 9}\sqrt{\frac{4}{21}}$$

etc.

8. Huiusmodi autem producta commodiſſime per logarithmos euoluuntur; ac primo quidem ex forma generali numerum quemcunque a implicante nanciſcimur:

$$ly = x\,la + x\,l\frac{a+1}{a} + x\,l\frac{a+2}{a+1} + x\,l\frac{a+3}{a+2} + x\,l\frac{a+4}{a+3} \text{ etc.}$$
$$- l(1+x) - l(1+\tfrac{x}{2}) - l(1+\tfrac{x}{3}) - l(1+\tfrac{x}{4}) \text{ etc.}$$

et ſumto $a = \frac{1+x}{2}$, vt haec ſeries maxime conuergens reddatur:

$$ly = xl\frac{1+x}{2} + xl\frac{x+3}{x+1} + xl\frac{x+5}{x+3} + xl\frac{x+7}{x+5} + xl\frac{x+9}{x+7} \text{ etc.}$$
$$-l(1+x) - l(1+\tfrac{x}{2}) - l(1+\tfrac{x}{3}) - l(1+\tfrac{x}{4}) \text{ etc.}$$

Sumtis igitur his logarithmis naturalibus, cum ſit in genere:

$$xl\frac{x+2m+1}{x+2m-1} = \frac{2x}{x+2m} + \frac{2x}{3(x+2m)^3} + \frac{2x}{5(x+2m)^5} + \frac{2x}{7(x+2m)^7} + \text{etc.}$$

$$\text{et } l(1+\tfrac{x}{m}) = \frac{2x}{x+2m} + \frac{2x^3}{3(x+2m)^3} + \frac{2x^5}{5(x+4m)^5} + \frac{2x^7}{7(x+2m)^7} + \text{etc.}$$

ſequentem formam infinitis ſeriebus conſtantem adipiſcimur:

$$ly = xl\frac{1+x}{2} + \tfrac{2}{3}x(1-xx)\left(\frac{1}{(x+2)^3} + \frac{1}{(x+4)^3} + \frac{1}{(x+6)^3} + \frac{1}{(x+8)^3} + \text{etc.}\right)$$
$$+ \tfrac{2}{5}x(1-x^4)\left(\frac{1}{(x+2)^5} + \frac{1}{(x+4)^5} + \frac{1}{(x+6)^5} + \frac{1}{(x+8)^5} + \text{etc.}\right)$$
$$+ \tfrac{2}{7}x(1-x^6)\left(\frac{1}{(x+2)^7} + \frac{1}{(x+4)^7} + \frac{1}{(x+6)^7} + \frac{1}{(x+8)^7} + \text{etc.}\right)$$
$$+ \tfrac{2}{9}x(1-x^8)\left(\frac{1}{(x+2)^9} + \frac{1}{(x+4)^9} + \frac{1}{(x+6)^9} + \frac{1}{(x+8)^9} + \text{etc.}\right)$$
$$\text{etc.}$$

9. Primae ſeriei ſumamus definitum terminorum numerum qui ſit $= n$, et cum ſuperior pars ad vnicum membrum $xl(a+n)$ redigatur, erit

$$ly = xl(a+n) - l(1+x) - l(1+\tfrac{1}{2}x) - l(1+\tfrac{1}{3}x) \ldots - l(1+\tfrac{1}{n}x)$$

quae expreſſio eo propius ad veritatem accedit, quo maior capiatur numerus n. Sit igitur n numerus praemagnus ac primo quidem habebimus $l(n+a) = ln + \frac{a}{n} - \frac{aa}{2n^2} + \frac{a^3}{3n^3} -$ etc. vbi loco a ſumi $\frac{1+x}{2}$ conueniet; tum vero poſita breuitatis gratia fractione $0,5772156649015325 = \Delta$, nouimus eſſe ſummam progreſſionis harmonicae:

$1 + \frac{1}{2}$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\ldots\ldots+\frac{1}{n}=\Delta+ln+\frac{1}{2n}-\frac{1}{12nn}+\frac{1}{120n^4}-\text{etc.}$$

vnde cum ſit:

$$l(n+a)=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}\ldots\ldots+\frac{1}{n}-\Delta-\frac{1}{2n}+\frac{1}{12nn}-\frac{1}{120n^4}$$
$$+\frac{a}{n}-\frac{aa}{2nn}+\frac{a^3}{3n^3}$$

colligimus ſumto $a=\frac{a+x}{2}$

$$ly=-\Delta x+x+\frac{1}{2}x+\frac{1}{3}x\ \ldots\ldots+\frac{1}{n}x+\frac{xx}{2n}$$
$$-l(1+x)-l(1+\tfrac{1}{2}x)-l(1+\tfrac{1}{3}x)-l(1+\tfrac{1}{n}x)-\frac{x-6xx-3x^3}{24nn}.$$

Reuera ergo augendo numerum n in infinitum erit:

$$ly=-\Delta x+x+\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x+\text{etc.}$$
$$-l(1+x)-l(1+\tfrac{1}{2}x)-l(1+\tfrac{1}{3}x)-l(1+\tfrac{1}{4}x)-\text{etc.}$$

et ſingulis logarithmis per ſeries euolutis:

$$ly=-\Delta x+\frac{1}{2}xx(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc.})$$
$$-\frac{1}{3}x^3(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\text{etc.})$$
$$+\frac{1}{4}x^4(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.})$$
$$-\frac{1}{5}x^5(1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\text{etc.})$$
$$\text{etc.}$$

10. Praeter has autem formulas, quibus cuique abſciſſae x conueniens applicata y aſſignatur, methodus mea progreſſiones indefinite ſummandi ſingularem ſuppeditat expreſſionem ad eundem ſcopum accommodatam.

Cum enim ſit $ly=l1+l2+l3+l4\ldots+lx$ hanc progreſſionem indefinite ſummari oportet; introducendo autem valores numericos:

 $A=\frac{1}{6}$

$A = \frac{1}{6}$, $B = \frac{1}{90}$, $C = \frac{1}{945}$, $D = \frac{1}{9450}$, $E = \frac{1}{93555}$, $F = \frac{691}{1.\,3.\,5.\,\ldots\ldots\,15.\,315}$ etc. quorum progreſſio ita eſt comparata vt ſit

$$5B = 2AA;\quad 7C = 4AB;\quad 9D = 4AC + 2BB;\quad 11E = 4AD + 4BC \text{ etc.}$$

oſtendi alibi fore

$$ly = \tfrac{1}{2}l2\pi + (x + \tfrac{1}{2})lx - x + \frac{A}{2x} - \frac{1.\,2\,B}{2^3 x^3} + \frac{1.\,2.\,3.\,4\,C}{2^5 x^5} - \frac{1.\,2.\,3.\,4.\,5.\,6\,D}{2^7 x^7} + \text{etc.}$$

quae ſeries prae ſuperioribus hunc praeſtat vſum, vt quo maiores capiantur abſciſſae x eo promtius verum valorem applicatae y exhibeat. Cum igitur ſi abſciſſae x conueniat applicata y, abſciſſae maiori $x+n$ conueniat applicata $y(x+1)(x+2)(x+3)\ldots(x+n)$, habebimus ſemper per ſeriem valde conuergentem:

$$ly = \tfrac{1}{2}l2\pi - l(x+1) - l(x+2) - l(x+3) \ldots\ldots - l(x+n) + (x+n+\tfrac{1}{2})l(x+n) - x - n + \frac{A}{2(x+n)} - \frac{1.\,2\,B}{2^3(x+n)^3} + \frac{1.\,2.\,3.\,4\,C}{2^5(x+n)^5} - \frac{1.\,2.\,3.\,4.\,5.\,6\,D}{2^7(x+n)^7} + \text{etc.}$$

Quodſi ergo e denotet numerum, cuius logarithmus naturalis $=1$, breuitatis gratia ponatur:

$$\frac{A}{2(x+n)} - \frac{1.\,2\,B}{2^3(x+n)^3} + \frac{1.\,2.\,3.\,4\,C}{2^5(x+n)^5} - \text{etc.} = s$$

concludimus a logarithmus ad numeros regrediendo

$$y = \frac{\sqrt{2\pi(x+n)}}{(x+1)(x+2)(x+3)\ldots\ldots(x+n)}\left(\frac{x+n}{e}\right)^{x+n} e^s$$

vbi numerus integer n arbitrio noſtro relinquitur, quo

quo maior is autem accipiatur, eo facilius verum valorem ipſius s inuenire licet.

11. Denique etiam applicatam y per formulam integralem exhibere licet, poſita enim abſciſſa $x = p$, nouaque introducta variabili u, prae qua quantitas p vt conſtans tractetur, erit applicata $y = \int du (l\frac{1}{u})^p$ ſiquidem integratio a valore $u = 0$ vsque ad valorem $u = 1$ extendatur. Vel ſi forma exponentiali vti malimus, erit quoque

$$y = \int e^{-v} v^p \, dv$$

integrationem a valore $v = 0$ ad $v = \infty$ extendendo. Ex his quidem formulis, quoties abſciſſa p eſt numerus integer, integratio ſtatim praebet $y = 1 . 2 . 3 \ldots \ldots p$. at ſi p fuerit numerus fractus, hinc ſimul intelligitur ad quodnam genus quantitatum tranſcendentium valor ipſius y referri debeat. Alio autem loco oſtendi, quomodo tum integrale per quadraturas curuarum algebraicarum exprimi queat.

12. En ergo plurimas ſolutiones quaeſtionis noſtrae primae, qua pro qualibet abſciſſa x etiamſi numero non integro exprimatur, valor applicatae y reperiebatur: quarum praecipuas ſimul aſpectui expoſuiſſe iuuabit, vt inde quouis caſu ea, quae maxime ad vſum accommodata videatur, eligi queat:

I. $y = \frac{1}{1+x}\left(\frac{2}{1}\right)^x \cdot \frac{2}{2+x}\left(\frac{3}{2}\right)^x \cdot \frac{3}{3+x}\left(\frac{4}{3}\right)^x \cdot \frac{4}{4+x}\left(\frac{5}{4}\right)^x \cdot$ etc.

II. $y = \left(\frac{1+x}{2}\right)^x \cdot \frac{1}{1+x}\left(\frac{3+x}{2+x}\right)^x \cdot \frac{2}{2+x}\left(\frac{5+x}{3+x}\right)^x \cdot \frac{3}{3+x}\left(\frac{7+x}{5+x}\right)^x \cdot$ etc.

III.

$$\text{III. } ly = xl\tfrac{2}{1} + xl\tfrac{3}{2} + xl\tfrac{4}{3} + xl\tfrac{5}{4} + \text{etc.}$$
$$- l(1+x) - l(1+\tfrac{1}{2}x) - l(1+\tfrac{1}{3}x) - l(1+\tfrac{1}{4}x) - \text{etc.}$$

$$\text{IV. } ly = xl\tfrac{1+x}{2} + xl\tfrac{x+3}{x+1} + xl\tfrac{x+5}{x+3} + xl\tfrac{x+7}{x+5} + xl\tfrac{x+9}{x+7} + \text{etc.}$$
$$- l(1+x) - l(1+\tfrac{1}{2}x) - l(1+\tfrac{1}{3}x) - l(1+\tfrac{1}{4}x) - \text{etc.}$$

$$\text{V. } ly = -\Delta x + x + \tfrac{1}{2}x + \tfrac{1}{3}x + \tfrac{1}{4}x + \text{etc.}$$
$$- l(1+x) - l(1+\tfrac{1}{2}x) - l(1+\tfrac{1}{3}x) - l(1+\tfrac{1}{4}x) - \text{etc.}$$

$$\text{VI. } ly = -\Delta x + \tfrac{1}{2}xx\left(1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \text{etc.}\right)$$
$$+ \tfrac{1}{3}x^3\left(1 + \tfrac{1}{2^3} + \tfrac{1}{3^3} + \tfrac{1}{4^3} + \text{etc.}\right)$$
$$+ \tfrac{1}{4}x^4\left(1 + \tfrac{1}{2^4} + \tfrac{1}{3^4} + \tfrac{1}{4^4} + \text{etc.}\right)$$
$$+ \tfrac{1}{5}x^5\left(1 + \tfrac{1}{2^5} + \tfrac{1}{3^5} + \tfrac{1}{4^5} + \text{etc.}\right)$$
$$+ \text{etc.}$$

$$\text{VII. } ly = \tfrac{1}{2}l2\pi + (x+\tfrac{1}{2})lx - x + \frac{A}{2x} - \frac{1.2\,B}{2^3\,x^3} + \frac{1.2.3.4\,C}{2^5\,x^5} - \frac{1.2\ldots6\,D}{2^7\,x^7} + \text{etc.}$$

existente $\Delta = 0,5772156649014225$ et

$$A = \tfrac{1}{6},\ B = \tfrac{1}{90},\ C = \tfrac{1}{945},\ D = \tfrac{1}{9450},\ E = \tfrac{1}{93555} \text{ etc.}$$

Tum in tribus postremis formis logarithmos naturales accipi oportet.

Quaestio secunda.

In curua hypergeometrica ad quoduis eius punctum directionem tangentis definire.

13. Hic igitur assumimus pro quauis abscissa x valorem applicatae y iam esse inuentum; et cum directio tangentis ratione differentialium $\frac{dy}{dx}$ definiatur, quippe qua fractione tangens anguli, quo curvae

vae tangens in loco propoſito ad axem inclinatur, exprimi ſolet, tantum opus eſt, vt quandam formularum inuentarum differentiemus. In hunc autem finem formula V maxime videtur idonea, ex qua colligimus:

$$\frac{dy}{y\,dx} = -\Delta + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.}$$
$$- \frac{1}{1+x} - \frac{1}{2+x} - \frac{1}{3+x} - \frac{1}{4+x} - \text{etc.}$$

quae expreſſio in hanc concinniorem contrahitur:

$$\frac{dy}{y\,dx} = -\Delta + \frac{x}{1+x} + \frac{x}{2(2+x)} + \frac{x}{3(3+x)} + \frac{x}{4(4+x)} + \text{etc.}$$

vnde ſtatim patet, ſi x ſit numerus integer negatiuus, fieri non ſolum applicatam y, ſed etiam formulam $\frac{dy}{dx}$ infinitam, ita vt in his locis ipſae applicatae, vtpote aſymtotae fiant tangentes. Ponamus autem in genere angulum, quem tangens cum axe conſtituit $= \Phi$ vt ſit $\frac{dy}{dx} = \text{tang.}\,\Phi$.

14. Primum ergo hinc definiamus tangentes pro abſciſſis x, quae numeris poſitiuis exprimuntur, ſiquidem applicatae y ſponte dantur.

I. Sit ergo $x = 0$, et ob $y = 1$ fit

$$\frac{dy}{dx} = -\Delta = -0,5772156649 = \text{tang.}\,\Phi.$$

vnde fit angulus $\Phi = -29^\circ, 59', 39''$, vbi ſignum $-$ indicat, tangentem dextrorſum in axem incidere, cum eoque angulum tantum non 30° conſtituere.

II. Sit $x = 1$ et ob $y = 1$ fit $\frac{dy}{dx} = 1 - \Delta = 0,422784335 = \text{tang.}\,\Phi$, hincque angulus $\Phi = 22^\circ, 55'$.

III. Sit $x=2$ et ob $y=2$ fit $\frac{dy}{dx}=2(1+\frac{1}{2}-\Delta)=1,845568670$ $=\text{tang.}\,\Phi$ hincque angulus $\Phi=61^\circ, 33'$.

IV. Sit $x=3$ et ob $y=6$ fit $\frac{dy}{dx}=6(1+\frac{1}{2}+\frac{1}{3}-\Delta)=\text{tang.}\,\Phi$ seu tang. $\Phi=7,536706010$ et $\Phi=82^\circ, 26'$.

V. Sit $x=4$ et ob $y=24$ fit $\frac{dy}{dx}=24(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\Delta)$ hincque tang. $\Phi=36,146824040$ et $\Phi=88^\circ, 25'$.

In genere igitur si abscissa x aequetur numero integro cuicunque n, ob $y=1.\,2\,.\,.\,.\,.\,.\,.\,n$ erit

$\frac{dy}{dx}=\text{tang.}\,\Phi=1.\,2.\,3\,.\,.\,.\,.\,n(1+\frac{1}{2}+\frac{1}{3}+\,.\,.\,.\,.\,+\frac{1}{n}-\Delta)$.

15. Definiamus hinc etiam tangentes pro locis intermediis, ac primo quidem ad abscissas positiuas relatis:

I. Sit $x=\frac{1}{2}$, erit $y=\frac{1}{2}\sqrt{\pi}$ atque

$\frac{dy}{ydx}=-\Delta+1-\frac{2}{3}+\frac{1}{2}-\frac{2}{5}+\frac{1}{3}-\frac{2}{7}$ etc.

seu $\frac{dy}{ydx}=-\Delta+2(\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\text{etc.})=-\Delta+2(1-l2)$

hincque $\frac{dy}{dx}=\text{tang.}\,\Phi=y(2(1-l2)-\Delta)=0,0364899739\,.\,y$

II. Sit $x=\frac{3}{2}$ erit $y=\frac{1.\,3}{2.\,2}\sqrt{\pi}$ atque

$\frac{dy}{ydx}=-\Delta+2(1+\frac{1}{3}-l2)$ unde fit

$\frac{dy}{dx}=\text{tang.}\,\Phi=y(2(1+\frac{1}{3}-l2)-\Delta)=0,7031566405\,.\,y$

III. Sit $x=\frac{5}{2}$ erit $y=\frac{1.\,3.\,5}{2.\,2.\,2}\sqrt{\pi}$ et $\frac{dy}{ydx}=-\Delta+2(1+\frac{1}{3}+\frac{1}{5}-l2)$

hinc tang. $\Phi=y(2(1+\frac{1}{3}+\frac{1}{5}-l2)-\Delta)=1,1031566405\,.\,y$.

Cum

Cum nunc ſit $\frac{1}{2}\sqrt{\pi}.(2(1-l2)-\Delta)=0{,}0323383973$ erit pro his caſibus:

$$x=\tfrac{1}{2};\; y=0{,}8862269 \quad \text{tang.}\,\Phi=0{,}0323384$$
$$x=\tfrac{3}{2};\; y=1{,}3293404 \quad \text{tang.}\,\Phi=0{,}9347345$$
$$x=\tfrac{5}{2};\; y=3{,}3233509 \quad \text{tang.}\,\Phi=3{,}6661767$$
$$x=\tfrac{7}{2};\; y=11{,}6317284 \quad \text{tang.}\,\Phi=16{,}1549694$$
$$x=\tfrac{9}{2};\; y=52{,}3427777 \quad \text{tang.}\,\Phi=84{,}3290907$$

etc.

16. Antequam vlterius progrediar, obſeruo ſi fuerit pro abſciſſa quacunque

$$x=p;\quad y=q;\quad \text{tang.}\,\Phi=r$$

tum pro abſciſſa ſequente fore

$$x=p+1;\; y=q(p+1)\ \text{et}\ \text{tang.}\,\Phi=r(p+1)+q$$

pro abſciſſa autem antecedente

$$x=p-1;\; y=\frac{q}{p};\quad \text{et}\ \text{tang.}\,\Phi=\frac{r}{p}-\frac{q}{pp}$$

vnde ſuperiores valores facile retro continuare poterimus:

$$x=\tfrac{1}{2};\; y=0{,}8862269;\; \text{tang.}\,\Phi=0{,}0323384$$
$$x=-\tfrac{1}{2};\; y=1{,}7724538;\; \text{tang.}\,\Phi=-3{,}4802308$$
$$x=-\tfrac{3}{2};\; y=-3{,}5449077;\; \text{tang.}\,\Phi=-0{,}1293538$$
$$x=-\tfrac{5}{2};\; y=+2{,}3632718;\; \text{tang.}\,\Phi=+1{,}6617504$$
$$x=-\tfrac{7}{2};\; y=-0{,}9453087;\; \text{tang.}\,\Phi=-1{,}0428236$$
$$x=-\tfrac{9}{2};\; y=+0{,}2700882;\; \text{tang.}\,\Phi=+0{,}3751176$$
$$x=-\tfrac{11}{2};\; y=-0{,}0600196;\; \text{tang.}\,\Phi=-0{,}0966971$$
$$x=-\tfrac{13}{2};\; y=+0{,}0109126;\; \text{tang.}\,\Phi=+0{,}0195654$$

etc.

17. Eadem aequatio differentialis ei curuae puncto μ inueniendo inseruit, vbi applicata est minima seu tangens axi parallela. Posito igitur $\frac{dy}{dx} = 0$, abscissa respondens x ex hac aequatione quaeri debet:

$$\Delta = \frac{x}{1+x} + \frac{x}{2(2+x)} + \frac{x}{3(3+x)} + \frac{x}{4(4+x)} + \frac{x}{5(5+x)} + \text{etc.}$$

quae euoluitur in hanc:

$$\begin{aligned} \Delta = &+ x\left(1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \text{etc.}\right) \\ &- x^2\left(1 + \tfrac{1}{2^3} + \tfrac{1}{3^3} + \tfrac{1}{4^3} + \text{etc.}\right) \\ &+ x^3\left(1 + \tfrac{1}{2^4} + \tfrac{1}{3^4} + \tfrac{1}{4^4} + \text{etc.}\right) \\ &- x^4\left(1 + \tfrac{1}{2^5} + \tfrac{1}{3^5} + \tfrac{1}{5^5} + \text{etc.}\right) \\ &\text{etc.} \end{aligned}$$

Summis autem harum serierum proximis substitutis erit

$$\begin{aligned} 0 = &+ 0{,}5772156649 - 1{,}6449340668\, x \\ &+ 1{,}2020569032\, x^2 - 1{,}0823232337\, x^3 \\ &+ 1{,}0369277551\, x^4 - 1{,}0173430620\, x^5 \\ &+ 1{,}0083492774\, x^6 - 1{,}0040773562\, x^7 \\ &+ 1{,}0020083928\, x^8 - 1{,}0009945751\, x^9 \\ &+ 1{,}0004941886\, x^{10} - 1{,}0002460866\, x^{11} \\ &+ 1{,}0001227233\, x^{12} - 1{,}0000612481\, x^{13} \\ &+ 1{,}0000305882\, x^{14} - 1{,}0000152823\, x^{15} \\ &\text{etc.} \end{aligned}$$

Sin autem duae primae fractiones retineantur, sequens series multo magis conuergens emergit

0 =

$$0 = +0,5772156649 - \frac{x}{1+x} - \frac{x}{2(2+x)}$$
$$+0,0770569032\,x^2 - 0,3949340668\,x$$
$$+0,0056777551\,x^4 - 0,0198232337\,x^3$$
$$+0,00[illegible]367774\,x^6 - 0,0017180620\,x^5$$
$$+0,0000552678\,x^8 - 0,0001711062\,x^7$$
$$+0,0000059074\,x^{10} - 0,0000180126\,x^9$$
$$+0,0000006530\,x^{12} - 0,0000019460\,x^{11}$$
$$+0,0000000706\,x^{14} - 0,0000002130\,x^{13}$$
$$+0,0000000078\,x^{16} - 0,0000000235\,x^{15}.$$

Hinc proxime reperitur $x = \frac{1}{2}$, verum haec applicata minima facilius ope sequentis quaestionis definietur.

Quaestio tertia.

Pro dato quouis curuae hypergeometricae puncto, indolem portionis minimae istius curuae circa id punctum sitae inuestigare.

18. Pro abscissa ergo data $x = p$ inuenta sit applicata $y = q$; et nunc quaeri oportet applicatam, quae abscissae parumper ab illa discrepanti $p + \omega$ respondeat; quae applicata statuatur $= q + \psi$. Cum igitur sit secundum formulam V

$$lq = -\Delta p + p \quad + \tfrac{1}{2}p \quad + \tfrac{1}{3}p \quad + \tfrac{1}{4}p \quad + \text{etc.}$$
$$- l(1+p) - l(1+\tfrac{1}{2}p) - l(1+\tfrac{1}{3}p) - l(1+\tfrac{1}{4}p) - \text{etc.}$$

si hic loco p scribatur $p + \omega$, loco lq prodibit valor ipsius $l(q+\psi)$, quo ipso quaestio resoluetur. At si ponamus $lq = P$, scribendo $p + \omega$ loco p notum est prodire

$$l(q+\psi) = P + \frac{\omega\, dP}{1\, dp} + \frac{\omega^2\, ddP}{1 \cdot 2\, dp^2} + \frac{\omega^3\, d^3P}{1 \cdot 2 \cdot 3\, dp^3} + \frac{\omega^4\, d^4P}{1 \cdot 2 \cdot 3 \cdot 4\, dp^5} + \text{etc.}$$

Eſt vero vt vidimus:

$$\frac{dP}{dp} = -\Delta + \frac{p}{1+p} + \frac{p}{2(2+p)} + \frac{p}{3(3+p)} + \frac{p}{4(4+p)} + \text{etc.}$$

hincque porro:

$$\frac{ddP}{1 \cdot dp^2} = \frac{1}{(1+p)^2} + \frac{1}{(2+p)^2} + \frac{1}{(3+p)^2} + \frac{1}{(4+p)^2} + \text{etc.}$$

$$\frac{d^3 P}{1 \cdot 2\, dp^3} = -\frac{1}{(1+p)^3} - \frac{1}{(2+p)^3} - \frac{1}{(3+p)^3} - \frac{1}{(4+p)^3} - \text{etc.}$$

$$\frac{d^4 P}{1 \cdot 2 \cdot 3\, dp^4} = \frac{1}{(1+p)^4} + \frac{1}{(2+p)^4} + \frac{1}{(3+p)^4} + \frac{1}{(4+p)^4} + \text{etc.}$$

etc.

vnde ob $P = lq$ colligimus:

$$l\left(1 + \frac{\psi}{q}\right) = -\Delta\omega + \omega\left(\frac{p}{1+p} + \frac{p}{2(2+p)} + \frac{p}{3(3+p)} + \text{etc.}\right)$$

$$+ \tfrac{1}{2}\omega^2\left(\frac{1}{(1+p)^2} + \frac{1}{(2+p)^2} + \frac{1}{(3+p)^2} + \text{etc.}\right)$$

$$- \tfrac{1}{3}\omega^3\left(\frac{1}{(1+p)^3} + \frac{1}{(2+p)^3} + \frac{1}{(3+p)^3} + \text{etc.}\right)$$

$$+ \tfrac{1}{4}\omega^4\left(\frac{1}{(1+p)^4} + \frac{1}{(2+p)^4} + \frac{1}{(3+p)^4} + \text{etc.}\right)$$

$$- \tfrac{1}{5}\omega^5\left(\frac{1}{(1+p)^5} + \frac{1}{(2+p)^5} + \frac{1}{(3+p)^5} + \text{etc.}\right)$$

etc.

19. Hic iam coordinatae p et q vt conſtantes ſpectari poſſunt, quoniam litterae ω et ψ nouas coordinatas a dato curuae puncto ſumtas atque illis parallelas referunt; ex quarum relatione hic definita indoles curuae circa id punctum verſantis facile inveſtigatur. Quare cum iam innumerabilia curuae puncta aſſignauerimus, hinc tractus ſingularum curvae portionum inter bina illorum punctorum interiacentium vero proxime definiri poterit. Primo ſcilicet

ſcilicet ex illa aequatione differentiata colligitur vt ante inclinatio tangentis ad axem Φ, ſitque

$$\frac{d\psi}{d\omega} = \text{tang.}\,\Phi = q\left(-\Delta + \frac{p}{1+p} + \frac{p}{2(2+p)} + \frac{p}{3(3+p)} + \text{etc.}\right).$$

Deinde ſi pro aequatione differentiali breuitatis gratia ponamus $d\psi = A\,d\omega + B\omega\,d\omega + C\omega^2 d\omega + \text{etc.}$ erit radius curuaturae in dato curuae puncto

$$= \frac{(1+AA)^{\frac{3}{2}}}{B} = \frac{1}{B\,\cos.\Phi^3} \text{ ob } A = \text{tang.}\,\Phi. \text{ Eſt vero}$$

$$B = \text{tang.}\,\Phi\left(-\Delta + \frac{p}{1+p} + \frac{p}{2(2+p)} + \frac{p}{3(3+p)} + \text{etc.}\right)$$
$$+ q\left(\frac{1}{(1+p)^2} + \frac{1}{(2+p)^2} + \frac{1}{(3+p)^2} + \frac{1}{(4+p)^2} + \text{etc.}\right)$$

Vnde ſi radius curuaturae ponatur $= r$ erit

$$\frac{1}{r} = \frac{\sin.\Phi^2 \cos.\Phi}{q} + q\left(\frac{1}{(1+p)^2} + \frac{1}{(2+p)^2} + \frac{1}{(3+p)^2} + \text{etc.}\right)$$

20. Quo autem inueſtigationem directionis et curuaturae ad curuae puncta a puncto principali coordinatis p et q definito extendere queamus, ponamus breuitatis cauſa

$$-\Delta + \frac{p}{1+p} + \frac{p}{2(2+p)} + \frac{p}{3(3+p)} + \frac{p}{4(4+p)} + \text{etc} = P$$
$$\frac{1}{(1+p)^2} + \frac{1}{(2+p)^2} + \frac{1}{(3+p)^2} + \frac{1}{(4+p)^2} + \text{etc.} = Q$$
$$\frac{1}{(1+p)^3} + \frac{1}{(2+p)^3} + \frac{1}{(3+p)^3} + \frac{1}{(4+p)^3} + \text{etc.} = R$$
$$\frac{1}{(1+p)^4} + \frac{1}{(2+p)^4} + \frac{1}{(3+p)^4} + \frac{1}{(4+p)^4} + \text{etc.} = S$$
etc.

vt ſit $l\left(1 + \frac{\psi}{q}\right) = P\omega + \frac{1}{2}Q\omega^2 - \frac{1}{3}R\omega^3 + \frac{1}{4}S\omega^4 - \frac{1}{5}T\omega^5 + \text{etc.}$

Iam

Iam hinc differentiando elicimus:

$$\frac{d\psi}{d\omega} = (q+\psi)(P + Q\omega - R\omega^2 + S\omega^3 - T\omega^4 + \text{etc.})$$

atque vlterius differentiando

$$\frac{dd\psi}{d\omega^2} = (q+\psi)(P + Q\omega - R\omega^2 + S\omega^3 - T\omega^4 + \text{etc.})^2$$
$$+ (q+\psi)(Q - 2R\omega + 3S\omega^2 - 4T\omega^3 + \text{etc.})$$

$$\frac{d^3\psi}{d\omega^3} = 3(q+\psi)(Q - 2R\omega + 3S\omega^2 - 4T\omega^3 + \text{etc.})(P + Q\omega - R\omega^2 + S\omega^3 - \text{etc.})$$
$$+ (q+\psi)(P + Q\omega - R\omega^2 + S\omega^3 - T\omega^4 + \text{etc.})^3$$
$$- (q+\psi)(2R - 6S\omega + 12T\omega^2 - \text{etc.}).$$

His expeditis pro curuae puncto, quod conuenit abscissae $x = p + \omega$ et applicatae $y = q + \psi$ directio tangentis ita se habebit vt sit

$$\text{tang.}\,\Phi = \frac{d\psi}{d\omega} = (q+\psi)(P + Q\omega - R\omega^2 + S\omega^3 - T\omega^4 + \text{etc.}).$$

Tum vero posito radio curuaturae $= r$, nouimus fore

$$r = \left(1 + \frac{d\psi^2}{d\omega^2}\right)^{\frac{3}{2}} : \frac{dd\psi}{d\omega^2} = 1 : \frac{dd\psi}{d\omega^2}\,\text{cos.}\,\Phi^3$$

seu $\frac{1}{r} = \frac{dd\psi}{d\omega^2}\,\text{cos.}\,\Phi^3$, vnde pro variabilitate curuaturae elicimus:

$$-\frac{dr}{rr\,d\omega} = \frac{d^3\psi}{d\omega^3}\,\text{cos.}\,\Phi^3 - \frac{3\,dd\psi}{d\omega^2}\cdot\frac{d\Phi}{d\omega}\,\text{sin.}\,\Phi\,\text{cos.}\,\Phi^2.$$

Est vero $\frac{d\Phi}{\text{cos.}\,\Phi^2} = \frac{dd\psi}{d\omega}$ vnde conficitur:

$$-\frac{dr}{rr\,d\omega} = \frac{d^3\psi}{d\omega^3}\,\text{cos.}\,\Phi^3 - 3\left(\frac{dd\psi}{d\omega^2}\right)^2\text{sin.}\,\Phi\,\text{cos.}\,\Phi^4.$$

Quae-

Quaestio quarta.

Naturam curuae hypergeometricae circa punctum eius infimum μ, *vbi applicata est minima, inuestigare.*

21. Quoniam hoc punctum parum distat a loco, cui respondet abscissa $=\frac{1}{2}$ et applicata $=\frac{1}{2}\sqrt{\pi}$ statuamus $p=\frac{1}{2}$ vt sit $q=\frac{1}{2}\sqrt{\pi}$, hincque primo quaeramus valores litterarum P, Q, R, S etc. qui prodibunt

$$P=-\Delta+\frac{1}{3}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 7}+\text{etc.}=2(1-l2)-\Delta=0,0364899739785 7$$

$$Q=\frac{4}{3^2}+\frac{4}{5^2}+\frac{4}{7^2}+\frac{4}{9^2}+\text{etc.}=0,93480220054468$$

$$R=\frac{8}{3^3}+\frac{8}{5^3}+\frac{8}{7^3}+\frac{8}{9^3}+\text{etc.}=0,41439832211716$$

$$S=\frac{16}{3^4}+\frac{16}{5^4}+\frac{16}{7^4}+\frac{16}{9^4}+\text{etc.}=0,23484850566707$$

$$T=\frac{32}{3^5}+\frac{32}{5^5}+\frac{32}{7^5}+\frac{32}{9^5}+\text{etc.}=0,14476040831276$$

$$V=\frac{64}{3^6}+\frac{64}{5^6}+\frac{64}{7^6}+\frac{64}{9^6}+\text{etc.}=0,09261290502029$$

$$W=\frac{128}{3^7}+\frac{128}{5^7}+\frac{128}{7^7}+\frac{128}{9^7}+\text{etc.}=0,06035822809843$$

Deinde vero est $q=\frac{1}{2}\sqrt{\pi}=0,88622692545274$.

22. Hinc iam ante omnia definiamus locum μ, vbi applicata est omnium minima, quem cum leues approximationes ostendant respondere abscissae $x=0,4616$, posito $p+\omega=\frac{1}{2}+\omega=0,4616$, colligitur proxime $\omega=-0,0383$, qui iam valor ex aequatione $\frac{d\psi}{d\omega}=0$ seu

$$P+Q\omega-R\omega^2+S\omega^3-T\omega^4+\text{etc.}=0$$

accuratius inueftigari debet. Cum igitur fit prope $\omega = -\frac{1}{26}$ ftatuatur $\omega = -\frac{1}{26} - z$, et facta fubftitutione neceffe eft fiat . . . $0{,}0364899739785 7 =$

$$
\begin{aligned}
&+0{,}0359539307901 8 + 0{,}934802200\,z \\
&+0{,}0006130152694 0 + 0{,}031876794\,z + 0{,}414398\,zz \\
&+0{,}0000133618858 5 + 0{,}001042227\,z + 0{,}027097\,zz \\
&+0{,}0000003167790 0 + 0{,}000032945\,z + 0{,}001285\,zz \\
&+0{,}0000000077947 9 + 0{,}000001013\,z + 0{,}000053\,zz \\
&+0{,}0000000001953 8 + 0{,}000000030\,z + 0{,}000002\,zz \\
&+0{,}0000000000049 6
\end{aligned}
$$

13

$$
\begin{aligned}
&\,0{,}0365806327197 0 + 0{,}967755211\,z + 0{,}442835\,zz \\
&\,0{,}0364899739785 7 \\
&0 = 0{,}0000906587411 3 + 0{,}967755211\,z + 0{,}442835\,zz
\end{aligned}
$$

vnde reperitur $z = -0{,}000093683 23$ hincque $\omega = -0{,}038367855 23$.

Quocirca minima applicata $m\mu$ refpondet, abfciffae $Om = 0{,}4616321447 7$. Pro applicata vero $m\mu = q + \psi$ euolui oportet aequationem

$$l\left(1 + \frac{\psi}{q}\right) = P\omega + \frac{1}{2}Q\omega^2 - \frac{1}{3}R\omega^3 + \frac{1}{4}S\omega^4 - \frac{1}{5}T\omega^5 + \text{etc.}$$

ex qua colligitur $l\left(1 + \frac{\psi}{q}\right) = -0{,}0007040 53$ porroque $1 + \frac{\psi}{q} = 1 - 0{,}0007038 05$, ita vt fiat applicata minima $m\mu = q + \psi = 0{,}8856031945$.

23. Definiamus iam in genere ex aequatione logarithmica valorem ipfius ψ ac calculo fubducto obtinebimus:

$\frac{\psi}{q} =$

$$\frac{\psi}{q} = +0,0364899740\,\omega + 0,468066860\,\omega^2$$
$$- 0,121069221\,\omega^3 + 0,16321479\,\omega^4$$
$$- 0,09360753\,\omega^5 \text{ etc.}$$

qui termini si quidem ω valde paruum accipiatur sufficiunt. Ponamus autem breuitatis gratia

$$\frac{\psi}{q} = \mathfrak{P}\omega + \mathfrak{Q}\omega^2 - \mathfrak{R}\omega^3 + \mathfrak{S}\omega^4 - \mathfrak{T}\omega^5 \text{ vt fit}$$
$$\mathfrak{P} = 0,0364899740; \quad \mathfrak{Q} = 0,468066860$$
$$\mathfrak{R} = 0,121069221; \quad \mathfrak{S} = 0,16321479$$
$$\mathfrak{T} = 0,09360753$$

atque hinc habebimus:

$$\frac{d\psi}{d\omega} = q(\mathfrak{P} + 2\mathfrak{Q}\omega - 3\mathfrak{R}\omega^2 + 4\mathfrak{S}\omega^3 - 5\mathfrak{T}\omega^4)$$
$$\frac{dd\psi}{d\omega^2} = q(2\mathfrak{Q} - 6\mathfrak{R}\omega + 12\mathfrak{S}\omega^2 - 20\mathfrak{T}\omega^3).$$

Quodsi iam hinc radium curuaturae in loco infimo μ vbi est $\omega = -0,0383678523$ indagare velimus, quoniam ibi est $\frac{d\psi}{d\omega} = 0$, erit is $= \frac{d\omega^2}{dd\psi}$. Ponatur in hoc loco radius curuaturae $= r$ et cum sit

$$\frac{1}{r} = 2q(\mathfrak{Q} - 3\mathfrak{R}\omega + 6\mathfrak{S}\omega^2 - 10\mathfrak{T}\omega^3) = 0,9669949$$

prodit pro puncto μ radius curuaturae $r = 1,'66893$.

24. Determinationes has puncti curuae infimi μ ideo omni studio inuestigaui, quod non sine ratione suspicari licebat quemadmodum hoc punctum singulari praerogatiua est praeditum, ita numeros eius indolem exhibentes elegantiam quandam in se esse complexuros, ac nisi satis simpliciter siue ratio-

 naliter

naliter siue irrationaliter exprimantur, ad simplicius saltem genus quoddam transcendentium quantitatum relatum iri. Praeter expectationem autem vsu venit, vt tale criterium elegantiae neque in abscissa $Om = 0,46163214477$ neque in applicata $m\mu = 0,8856031945$ neque in radio curuaturae ibidem $= 1,166893$ appareat; nulla enim affinitas neque cum numeris rationalibus neque irrationalibus duntaxat simplicioribus, neque cum quadratura circuli, nec logarithmis vel exponentialibus deprehenditur. Cum etiam si abscissa Om vt logarithmus consideretur, numerus ei conueniens aliquid promittere videri posset, hunc numerum quaesiui et inueni $= 1,5866616$, in quo autem nulla affinitas cum quantitatibus cognitis cernitur.

25. Antequam huic speculationi finem imponam, obseruasse iuuabit formulam $1.\ 2.\ 3\ldots\ldots x$ etiam per sequentem seriem indefinite exprimi posse

$$x^x - x(x-1)^x + \frac{x(x-1)}{1.\ 2}(x-2)^x - \frac{x(x-1)(x-2)}{1.\ 2.\ 3}(x-3)^x + \text{etc.}$$

quippe quae quoties x est numerus integer positiuus sponte dat illud productum $1.\ 2.\ 3\ldots\ldots x$. Hoc vero etiam praestat ista expressio latius patens:

$$a^x - x(a-1)^x + \frac{x(x-1)}{1.\ 2}(a-2)^x - \frac{x(x-1)(x-2)}{1.\ 2.\ 3}(a-3)^x + \text{etc.}$$

erit enim si loco x successiue substituantur numeri 0, 1, 2, 3 etc. vt sequitur:

$a^0 = 1$

$$a^0 = 1$$
$$a^1 - (a-1)^1 = 1$$
$$a^2 - 2(a-1)^2 + (a-2)^2 = 1.\,2$$
$$a^3 - 3(a-1)^3 + 3(a-2)^3 - (a-3)^3 = 1.\,2.\,3$$
$$a^4 - 4(a-1)^4 + 6(a-2)^4 - 4(a-3)^4 + (a-4)^4 = 1.\,2.\,3.\,4$$
$$a^5 - 5(a-1)^5 + 10(a-2)^5 - 10(a-3)^5 + 5(a-4)^5 - (a-5)^5 = 1.\,2.\,3.\,4.\,5.$$
etc.

26. Manifesta haec quidem sunt ex iis, quae de differentiis cuiusque ordinis progressionum algebraicarum sunt demonstrata, verumtamen ex ipsa harum serierum natura veritas haud facile euincitur; vnde sequens demonstratio non superflua videtur. Cum pro exponentibus minoribus x res per se sit perspicua, ratiocinium ita instruo vt concessa pro casu $x = n$ veritate, eam quoque pro casu $x = n+1$ locum habere sim ostensurus.

Sit ergo

I. $a^n - n(a-1)^n + \frac{n(n-1)}{1.\,2}(a-2)^n - \text{etc.} = N = 1.\,2.\,3\,..\,n$

et quia summa N non ab a pendet erit etiam

II. $(a-1)^n - n(a-2)^n + \frac{n(n-1)}{1.\,2}(a-3)^n - \text{etc.} = N$

quae ab illa subtracta relinquit

III. $a^n - \frac{(n+1)}{1}(a-1)^n + \frac{(n+1)n}{1.\,2}(a-2)^n - \frac{(n+1)n(n-1)}{1.\,2.\,3}(a-3)^n + \text{etc.} = 0$

haec multiplicetur per a vt prodeat

IV. $a^{n+1} - \frac{(n+1)}{1}a(a-1)^n + \frac{(n+1)n}{1.\,2}a(a-2)^n - \frac{(n+1)n(n-1)}{1.\,2.\,3}a(a-3)^n + \text{etc.} = 0$

huic addatur aequatio II in $n+1$ ducta, nempe:

V. $+(n+1)1(a-1)^n - \frac{(n+1)n}{1.\ 2}2(a-2)^n + \frac{(n+1)n(n-1)}{1.\ 2.\ 3}3(a-3)^n - \text{etc} = (n+1)N$

atque aggregatum IV + V dabit

VI. $a^{n+1} - \frac{(n+1)}{1}(a-1)^{n+1} + \frac{(n+1)n}{1.\ 2}(a-2)^{n+1} - \frac{(n+1)n(n-1)}{1.\ 2.\ 3}(a-3)^{n+1} + \text{etc.} = (n+1)N$

vbi ob $N = 1.\ 2.\ 3 \ldots n$ erit $(n+1)N = 1.\ 2.\ 3 \ldots (n+1)$.

Euictum ergo eſt, quod ſi propoſitio noſtra

$$a^x - x(a-1)^x + \frac{x(x-1)}{1.\ 2}(a-2)^x - \frac{x(x-1)(x-2)}{1.\ 2.\ 3}(a-3)^x + \text{etc.} = 1.\ 2.\ 3 \ldots x$$

vera fuerit caſu $x = n$, eam quoque veram eſſe caſu $x = n+1$. Quoniam igitur ea manifeſto vera eſt caſu $x = 1$, hinc ſequitur eam quoque veram eſſe pro omnibus numeris integris poſitiuis loco x aſſumtis.

27. Quanquam autem haec expreſſio ſatis eſt elegans et omni attentione digna, tamen ad noſtrum inſtitutum, cui curua hypergeometrica eſt propoſita, minus eſt accommodata quoniam pro caſibus quibus x eſt numerus fractus, haec ſeries non ſolum in infinitum excurrit, ſed etiam ſi denominator eſt numerus par, terminos imaginarios complectitur, ita vt eius valorem ne appropinquando quidem colligere liceat. Ita poſito $x = \frac{1}{2}$ prodit haec ſeries infinita:

$$\sqrt{a} - \tfrac{1}{2}\sqrt{(a-1)} - \tfrac{1}{2}.\tfrac{1}{4}\sqrt{(a-2)} - \tfrac{1}{2}.\tfrac{1}{4}.\tfrac{3}{6}\sqrt{(a-3)} - \tfrac{1}{2}.\tfrac{1}{4}.\tfrac{3}{6}.\tfrac{5}{8}\sqrt{(a-4)} - \text{etc.}$$

cuius valorem eſſe $= \frac{1}{2}\sqrt{\pi}$, vix quisquam oſtendere poterit. Pari modo ſumendo $x = -\frac{1}{2}$ ex ſuperioribus quidem iam nouimus eſſe,

$\sqrt{\pi} =$

$$\sqrt{\pi} = \frac{1}{\sqrt{a}} + \frac{1}{2\sqrt{(a-1)}} + \frac{1 \cdot 3}{2 \cdot 4 \sqrt{(a-2)}} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \sqrt{(a-3)}} + \text{etc.}$$

Nihilo vero minus vberior huius feriei inueftigatio geometris merito commendatur, imprimis fi ei amplior extenfio inducatur, atque hac forma repraefentetur:

$$s = x^n - m(x-1)^n + \frac{m(m-1)}{1 \cdot 2}(x-2)^n - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}(x-3)^n + \text{etc.}$$

feu enim ftudio adhibito, mox admodum infignes proprietates deprehenduntur, quarum euolutio omnem attentionem noftram mereri videtur. Equidem quae mihi circa eam obferuare contigit phaenomena prorfus fingularia hic in medium afferam.

Obferuationes circa hanc feriem.

$$s = x^n - m(x-1)^n + \frac{m(m-1)}{1 \cdot 2}(x-2)^n - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}(x-3)^n + \text{etc.}$$

I. In praecedentibus igitur iam demonftraui, fi fuerit exponens $n = m$, fore huius feriei fummam

$$s = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot m$$

ita vt ea hoc cafu non a numero x pendeat. Hinc autem primo colligo fi fuerit $n = m-1$, tum fore $s = 0$. Cum enim fumto $n = m$ fit

$$♄ \ldots 1 \cdot 2 \cdot 3 \ldots m = x^m - m(x-1)^m + \frac{m(m-1)}{1 \cdot 2}(x-2)^m - \text{etc.}$$

erit fcribendo $x-1$ loco x et $m-1$ loco m fimili modo:

$$♃ \ldots 1 \cdot 2 \cdot 3 \ldots (m-1) = (x-1)^{m-1} - (m-1)(x-2)^{m-1} + \frac{(m-1)(m-2)}{1 \cdot 2}(x-2)^{m-1} - \text{etc.}$$

Iam

Iam illa aequatio hoc modo referatur:

$$♂ \ldots 1.2.3\ldots m = x.x^{m-1} - mx(x-1)^{m-1} + \frac{m(m-1)}{1.2}x(x-2)^{m-1} - \text{etc.}$$
$$+ m(x-1)^{m-1} - \frac{m(m-1)}{1}(x-2)^{m-1} + \text{etc.}$$

aequatio autem ♃ per m multiplicata dat:

$$☉\ 1.2.3\ldots m = m(x-1)^{m-1} - \frac{m(m-1)}{1}(x-2)^{m-1} + \frac{m(m-1)(m-2)}{1.2}(x-2)^{m-1} - \text{etc.}$$

quae ab illa ♂ subtracta et diuisione per x facta praebet

$$♀\ 0 = x^{m-1} - \frac{m}{1}(x-1)^{m-1} + \frac{m(m-1)}{1.2}(x-2)^{m-1} - \text{etc.}$$

quae est ipsa aequatio proposita pro casu $n = m-1$, cuius idcirco valor est $= 0$.

II. Eodem modo ostenditur seriei propositae summam s quoque euanescere casu $n = m-2$. Series enim illa ♀ hoc modo repraesentetur:

$$☿ = 0 = x.x^{m-2} - \frac{m}{1}x(x-1)^{m-2} + \frac{m(m-1)}{1.2}x(x-2)^{m-2} - \text{etc.}$$
$$+ m(x-1)^{m-2} - \frac{m(m-1)}{1}(x-2)^{m-2} + \text{etc.}$$

et si in eadem serie ♀ scribatur $x-1$ loco x et $m-1$ loco m, tota vero series per m multiplicetur, fit

$$☽ \ldots\ 0 = m(x-1)^{m-2} - \frac{m(m-1)}{1}(x-2)^{m-2} + \text{etc.}$$

Hac ab illa subtracta residuum per x diuidatur, prodibitque:

$$0 = x^{m-2} - \frac{m}{1}(x-1)^{m-2} + \frac{m(m-1)}{1.2}(x-2)^{m-2} - \text{etc.}$$

Sicque

Sicque seriei propositae summa s etiam euanescit casu $n = m - 2$, parique modo ostendi potest eam quoque euanescere casibus $n = m - 3$, $n = m - 4$, et in genere $n = m - i$, existente i numero quocunque integro positiuo. Teneatur ergo seriei propositae summam esse $s = 1.2.3 \ldots m$ casu $n = m$, casibus autem quibus exponens n minor est numero m summam in nihilum abire, siquidem numeri m et n sint integri, seu saltem $n - m$ numerus integer positiuus.

III. Quo igitur indolem reliquorum casuum perscrutemur singulos terminos nostrae seriei euoluamus et secundum potestates ipsius x disponamus, quo pacto consequemur:

$$
\begin{aligned}
s = {} & x^n\left(1 - m + \frac{m(m-1)}{1.\,2} - \frac{m(m-1)(m-2)}{1.\,2.\,3} + \text{etc.}\right) \\
& + n x^{n-1}\left(m - \frac{2m(m-1)}{1.\,2} + \frac{3m(m-1)(m-2)}{1.\,2.\,3} - \text{etc.}\right) \\
& - \frac{n(n-1)}{1.\,2} x^{n-2}\left(m - \frac{4m(m-1)}{1.\,2} + \frac{9m(m-1)(m-2)}{1.\,2.\,3} - \text{etc.}\right) \\
& + \frac{n(n-1)(n-2)}{1.\,2.\,3} x^{n-3}\left(m - \frac{8m(m-1)}{1.\,2} + \frac{27m(m-1)(m-2)}{1.\,2.\,3} - \text{etc.}\right) \\
& \text{etc.}
\end{aligned}
$$

quarum singularum serierum summas sequenti modo inueniemus; prima aliquanto generalius exhibeatur, et cum eius summa sit cognita:

$$1 - mu + \frac{m(m-1)}{1.\,2}u^2 - \frac{m(m-1)(m-2)}{1.\,2.\,3}u^3 + \text{etc.} = (1-u)^m$$

continuo eam differentiemus, et perpetuo loco du restituamus u, fietque signis mutatis:

$$mu - \frac{2m(m-1)}{1.\;2}u^2 + \frac{3m(m-1)(m-2)}{1.\;2.\;3}u^3 - \text{etc.} = mu(1-u)^{m-1}$$

$$mu - \frac{2^2 m(m-1)}{1.\;2}u^2 + \text{etc.} = mu(1-u)^{m-1} - m(m-1)u^2(1-u)^{m-2}$$

$$mu - \frac{2^3 m(m-1)}{1.\;2}u^2 + \text{etc.} = mu(1-u)^{m-1} - 3m(m-1)u^2(1-u)^{m-2}$$
$$+ m(m-1)(m-2)u^3(1-u)^{m-3}$$

$$mu - \frac{2^4 m(m-1)}{1.\;2}u^2 + \text{etc.} = mu(1-u)^{m-1} - 7m(m-1)uu(1-u)^{m-2} + 6m(m-1)$$
$$(m-2)u^3(1-u)^{m-3}$$
$$- m(m-1)(m-2)(m-3)u^4(1-u)^{m-4}$$

etc.

Hic ergo iam ſcribi oportet $u = 1$, quo facto omnes termini in quouis ordine euaneſcunt praeter eos, vbi exponens ipſius $1-u$ fit $= 0$.

IV. Tribuantur nunc ſucceſſiue ipſi m valores $1, 2, 3, 4, 5$ etc. et loco coefficientis in genere $\frac{n(n-1)(n-2)\ldots(n-i)}{1.\;2.\;3\ldots(i+1)}$ ſcribatur breuitatis gratia $\left(\frac{n-i}{i+1}\right)$, quo facto nanciſcimur valores ſequentes:

ſi	erit
$m=1$	$\frac{s}{1} = \left(\frac{n}{1}\right)x^{n-1} - \left(\frac{n-1}{2}\right)x^{n-2} + \left(\frac{n-2}{3}\right)x^{n-3} - \left(\frac{n-3}{4}\right)x^{n-4} + \left(\frac{n-4}{5}\right)x^{n-5} - \text{etc.}$
$m=2$	$\frac{s}{1.2} = \left(\frac{n-1}{2}\right)x^{n-2} - 3\left(\frac{n-2}{3}\right)x^{n-3} + 7\left(\frac{n-3}{4}\right)x^{n-4} - 15\left(\frac{n-4}{5}\right)x^{n-5} + 31\left(\frac{n-5}{6}\right)x^{n-6} - \text{etc.}$
$m=3$	$\frac{s}{1.2.3} = \left(\frac{n-2}{3}\right)x^{n-3} - 6\left(\frac{n-3}{4}\right)x^{n-4} + 25\left(\frac{n-4}{5}\right)x^{n-5} - 90\left(\frac{n-5}{6}\right)x^{n-6} + 301\left(\frac{n-6}{7}\right)x^{n-7} + \text{etc.}$
$m=4$	$\frac{s}{1.2..4} = \left(\frac{n-3}{4}\right)x^{n-4} - 10\left(\frac{n-4}{5}\right)x^{n-5} + 65\left(\frac{n-5}{6}\right)x^{n-6} - 350\left(\frac{n-6}{7}\right)x^{n-7} + 1701\left(\frac{n-7}{8}\right)x^{n-8} + \text{etc.}$
$m=5$	$\frac{s}{1.2..5} = \left(\frac{n-4}{5}\right)x^{n-5} - 15\left(\frac{n-5}{6}\right)x^{n-6} + 140\left(\frac{n-6}{7}\right)x^{n-7} - 1050\left(\frac{n-7}{8}\right)x^{n-8} + 6951\left(\frac{n-8}{9}\right)x^{n-9} + \text{etc.}$
$m=6$	$\frac{s}{1.2..6} = \left(\frac{n-5}{6}\right)x^{n-6} - 21\left(\frac{n-6}{7}\right)x^{n-7} + 266\left(\frac{n-7}{8}\right)x^{n-8} - 2646\left(\frac{n-8}{9}\right)x^{n-9} + 22827\left(\frac{n-9}{10}\right)x^{n-10} + \text{etc.}$

vbi

vbi formatio coefficientium numericorum ex antecedentibus est manifesta, est nempe pro postrema sexta serie:

$21 = 6 . 1 + 15$; $266 = 6 . 21 + 140$; $2646 = 6 . 266 + 1050$ etc.

Atque hinc statim perspicitur, si fuerit $n < m$ valorem ipsius s euanescere, in postrema enim serie si $n < 6$ ideoque vel 5 vel 4 vel 3 etc. erit $(\frac{n-5}{6}) = 0$, $(\frac{n-6}{7}) = 0$ etc.

Tum vero etiam si sit $n = m$, euidens est fore $\frac{s}{1 . 2 . . . . m} = 1$, est enim in infima serie:

$(\frac{6-5}{6}) = 1$, $(\frac{6-6}{7}) = 0$, $(\frac{6-7}{8}) = 0$, $(\frac{6-8}{9}) = 0$ etc.

Euolutio casuum $n = m + 1$.

V. Hinc primo casus euoluamus, quibus est $n = m + 1$, et forma postrema praebet

si	has summas
$m = 1, n = 2$	$\frac{s}{1} = 2x - 1$
$m = 2, n = 3$	$\frac{s}{1 . 2} = 3x - 3$
$m = 3, n = 4$	$\frac{s}{1 . 2 . 3} = 4x - 6$
$m = 4, n = 5$	$\frac{s}{1 . 2 . 3 . 4} = 5x - 10$
$m = 5, n = 6$	$\frac{s}{1 . 2 . . . . 5} = 6x - 15$
	etc

vbi priores coefficientes ipsius x ipsi n, numeri absoluti autem trigonalibus ipsius n aequentur, habebimus in genere

si fit	hanc aequationem
$n = m + 1$	$\frac{s}{1 \cdot 2 \ldots m} = (m+1)x - \frac{m(m+1)}{1 \cdot 2} = (m+1)(x - \frac{m}{2})$

ita vt sit

$$x^{m+1} - m(x-1)^{m+1} + \frac{m(m-1)}{1 \cdot 2}(x-2)^{m+1} - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}(x-3)^{m+1} + \text{etc.}$$
$$= 1 \cdot 2 \cdot 3 \ldots \ldots (m+1)(x - \tfrac{m}{2}).$$

Euolutio casuum $n = m + 2$.

VI. Pro his ergo casibus habebimus

si fuerit	has aequationes
$m = 1, n = 3$	$\frac{s}{1} = 3x^2 - 3 \cdot 1 x + 1 \cdot 1 = 3(xx - x + \frac{2}{6})$
$m = 2, n = 4$	$\frac{s}{1 \cdot 2} = 6x^2 - 4 \cdot 3 x + 1 \cdot 7 = 6(xx - 2x + \frac{7}{6})$
$m = 3, n = 5$	$\frac{s}{1 \cdot 2 \cdot 3} = 10x^2 - 5 \cdot 6 x + 1 \cdot 25 = 10(xx - 3x + \frac{15}{6})$
$m = 4, n = 6$	$\frac{s}{1 \cdot 2 \cdot 3 \cdot 4} = 15x^2 - 6 \cdot 10 x + 1 \cdot 65 = 15(xx - 4x + \frac{26}{6})$
$m = 5, n = 7$	$\frac{s}{1 \cdot 2 \ldots 5} = 21x^2 - 7 \cdot 15 x + 1 \cdot 140 = 21(xx - 5x + \frac{40}{6})$
$m = 6, n = 8$	$\frac{s}{1 \cdot 2 \ldots 6} = 28x^2 - 8 \cdot 21 x + 1 \cdot 266 = 28(xx - 6x + \frac{57}{6})$

etc.

quae formae ita repraesentari possunt

si fuerit	erit
$m = 1; n = 3$	$\frac{s}{1} = \frac{2 \cdot 3}{1 \cdot 2}(xx - x + \frac{1 \cdot 4}{12})$
$m = 2; n = 4$	$\frac{s}{1 \cdot 2} = \frac{3 \cdot 4}{1 \cdot 2}(xx - 2x + \frac{2 \cdot 7}{12})$
$m = 3; n = 5$	$\frac{s}{1 \cdot 2 \cdot 3} = \frac{4 \cdot 5}{1 \cdot 2}(xx - 3x + \frac{3 \cdot 10}{12})$
$m = 4; n = 6$	$\frac{s}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{5 \cdot 6}{1 \cdot 2}(xx - 4x + \frac{4 \cdot 13}{12})$
$m = 5; n = 7$	$\frac{s}{1 \cdot 2 \cdot 3 \ldots 5} = \frac{6 \cdot 7}{1 \cdot 2}(xx - 5x + \frac{5 \cdot 16}{12})$
$m = 6; n = 8$	$\frac{s}{1 \cdot 2 \ldots 6} = \frac{7 \cdot 8}{1 \cdot 2}(xx - 6x + \frac{6 \cdot 19}{12})$

vnde

vnde manifesto sequitur, si in genere sit $n = m + 2$

fore $\frac{s}{1 \cdot 2 \cdot \ldots \cdot m} = \frac{m+1}{1} \cdot \frac{m+2}{2} \left(xx - mx + \frac{m(3m+1)}{12}\right)$

seu $\frac{s}{1 \cdot 2 \cdot \ldots \cdot m} = \frac{m+1}{1} \cdot \frac{m+2}{2} \left(\left(x - \frac{m}{2}\right)^2 + \frac{m}{12}\right)$.

Ergo hinc obtinetur ista summatio

$$x^{m+2} - m(x-1)^{m+2} + \frac{m(m-1)}{1 \cdot 2}(x-2)^{m+2} - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}(x-3)^{m+2} + \text{etc.}$$
$$= 1 \cdot 2 \cdot 3 \ldots \ldots (m+2)\left(\tfrac{1}{2}\left(x - \tfrac{m}{2}\right)^2 + \tfrac{m}{24}\right).$$

Euolutio casuum $n = m + 3$.

VII. Pro his casibus habebimus

si fuerit	has aequationes
$m = 1; n = 4$	$\frac{s}{1} = 4x^3 - 6 \cdot 1x^2 + 4 \cdot 1x - 1 \cdot 1$
$m = 2; n = 5$	$\frac{s}{1 \cdot 2} = 10x^3 - 10 \cdot 3x^2 + 5 \cdot 7x - 1 \cdot 15$
$m = 3; n = 6$	$\frac{s}{1 \cdot 2 \cdot 3} = 20x^3 - 15 \cdot 6x^2 + 6 \cdot 25x - 1 \cdot 90$
$m = 4; n = 7$	$\frac{s}{1 \cdot 2 \cdot \cdot 4} = 35x^3 - 21 \cdot 10x^2 + 7 \cdot 65x - 1 \cdot 350$
$m = 5; n = 8$	$\frac{s}{1 \cdot 2 \cdot \cdot \cdot 5} = 56x^3 - 28 \cdot 15x^2 + 8 \cdot 140x - 1 \cdot 1050$

quae hoc modo repraesententur:

$$\frac{s}{1} = \frac{2 \cdot 3 \cdot 4}{1 \cdot 2 \cdot 3}\left(x^3 - \frac{3}{2}x^2 + \frac{1 \cdot 4}{4}x - \frac{1 \cdot 1 \cdot 2}{8}\right)$$
$$\frac{s}{1 \cdot 2} = \frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3}\left(x^3 - \frac{6}{2}x^2 + \frac{2 \cdot 7}{4}x - \frac{2 \cdot 2 \cdot 3}{8}\right)$$
$$\frac{s}{1 \cdot 2 \cdot 3} = \frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3}\left(x^3 - \frac{9}{2}x^2 + \frac{3 \cdot 10}{4}x - \frac{3 \cdot 3 \cdot 4}{8}\right)$$
$$\frac{s}{1 \cdot 2 \cdot \cdot 4} = \frac{5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3}\left(x^3 - \frac{12}{2}x^2 + \frac{4 \cdot 13}{4}x - \frac{4 \cdot 4 \cdot 5}{8}\right)$$
$$\frac{s}{1 \cdot 2 \cdot \cdot 5} = \frac{6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 3}\left(x^3 - \frac{15}{2}x^2 + \frac{5 \cdot 16}{4}x - \frac{5 \cdot 5 \cdot 6}{8}\right)$$

etc.

vnde in genere concluditur pro casu $n = m + 3$

$$\frac{s}{1 \cdot 2 \ldots m} = \frac{m+1}{1} \cdot \frac{m+2}{2} \cdot \frac{m+3}{3} \left(x^3 - \frac{3m}{2} x^2 + \frac{m(3m+1)}{4} x - \frac{mm(m+1)}{8}\right)$$

$$= \frac{m+1}{1} \cdot \frac{m+2}{2} \cdot \frac{m+3}{3} \left(\left(x - \frac{m}{2}\right)^3 + \frac{m}{4}\left(x - \frac{m}{2}\right)\right)$$

ita vt iam consequamur:

$$x^{m+3} - m(x-1)^{m+3} + \frac{m(m-1)}{1 \cdot 2}(x-2)^{m+3} - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}(x-3)^{m+3} \text{ etc.}$$

$$= 1 \cdot 2 \cdot 3 \ldots (m+3)\left(\frac{1}{6}\left(x - \frac{m}{2}\right)^3 + \frac{m}{24}\left(x - \frac{m}{2}\right)\right).$$

Praeparatio ad casus sequentes.

VIII. Cum § 4. formulas tantum ad casum $m = 6$ produxerimus, conemur pro iis formam generalem eruere. In hunc finem statuamus $n = m + \lambda$, et ad abbreuiandum loco talis expressionis $\frac{k(k-1)(k-2)(k-3) \ldots (k-i+1)}{1 \cdot 2 \cdot 3 \cdot 4 \ldots i}$ scribamus $\left(\frac{k}{i}\right)$ ita vt k denotet primum factorem numeratoris, i vero vltimum denominatoris. Ponamus igitur esse pro casu

$$m-1 \;\Big|\; \frac{s}{1 \cdot 2 \ldots (m-1)} = \left(\frac{m+\lambda}{m-1}\right)x^{\lambda+1} - A\left(\frac{m+\lambda}{m}\right)x^{\lambda} + B\left(\frac{m+\lambda}{m+1}\right)x^{\lambda-1} - C\left(\frac{m+\lambda}{m+2}\right)x^{\lambda-2} \text{ etc.}$$

$$m \;\Big|\; \frac{s}{1 \cdot 2 \cdot 3 \ldots m} = \left(\frac{m+\lambda}{m}\right)x^{\lambda} - A^{1}\left(\frac{m+\lambda}{m+1}\right)x^{\lambda-1} + B^{1}\left(\frac{m+\lambda}{m+2}\right)x^{\lambda-2} - C^{1}\left(\frac{m+\lambda}{m+3}\right)x^{\lambda-3} \text{ etc.}$$

ita vt A^1, B^1, C^1, D^1 etc. sint ii coefficientes, quos inuestigari oportet. Ex lege autem istarum formularum vidimus esse; $A^1 = m \cdot 1 + A$; $B^1 = mA^1 + B$; $C^1 = mB^1 + C$; $D^1 = mC^1 + D$ etc. vbi euidens est esse $A = \frac{m(m-1)}{1 \cdot 2}$ et $A^1 = \frac{(m+1)m}{1 \cdot 2}$ seu nostro signando modo $A = \left(\frac{m}{2}\right)$ et $A^1 = \left(\frac{m+1}{2}\right)$. Iam pro sequentibus operationibus obseruo esse:

$$\left(\frac{m+\mu+1}{\nu}\right) - \left(\frac{m+\mu}{\nu}\right) = \left(\frac{m+\mu}{\nu-1}\right)$$

quod

quod facile inde patet, quod fit euoluendo:

$$\left(\frac{m+\mu+1}{\nu}\right) = \frac{(m+\mu+1)(m+\mu)(m+\mu-1)\ldots\ldots(m+\mu+2-\nu)}{1.\quad 2.\quad 3\quad .\quad .\quad \nu}$$

$$\left(\frac{m+\mu}{\nu}\right) = \frac{(m+\mu)(m+\mu-1)\ldots\ldots(m+\mu+2-\nu)(m+\mu+1-\nu)}{1.\quad 2\quad .\quad .\quad .\quad .\quad .\quad (\nu-1).\quad \nu}$$

vnde perſpicitur eſſe $\left(\frac{m+1}{2}\right) - \left(\frac{m}{2}\right) = \left(\frac{m}{1}\right) = m.$

IX. Iam vt fiat

$$B' - B = mA' = \left(\frac{m+1}{2}\right)m = 3\left(\frac{m+1}{3}\right) + \left(\frac{m+1}{2}\right)$$

ſtatuamus $B = \alpha\left(\frac{m+1}{4}\right) + \beta\left(\frac{m+1}{3}\right)$

hincque $B' = \alpha\left(\frac{m+2}{4}\right) + \beta\left(\frac{m+2}{3}\right)$

prodibitque

$$B' - B = \alpha\left(\frac{m+1}{3}\right) + \beta\left(\frac{m+1}{2}\right)$$

vnde fit $\alpha = 3$ et $\beta = 1$ ita vt fit

$$B' = 3\left(\frac{m+2}{4}\right) + \left(\frac{m+2}{3}\right).$$

Pro ſequentibus operationibus autem notetur eſſe in genere:

$$\left(\frac{m+\mu}{\nu}\right)m = (\nu+1)\left(\frac{m+\mu}{\nu+1}\right) + (\nu-\mu)\left(\frac{m+\mu}{\nu}\right)$$

quippe quae forma prodit, ſi valor ipſius $\left(\frac{m+\mu}{\nu}\right)$ ſupra euolutus multiplicetur per

$$m = m+\mu-\nu+\nu-\mu = (\nu+1).\ \frac{m+\mu-\nu}{\nu+1} + (\nu-\mu).$$

X. His obſeruatis cum eſſe debeat $C'-C = mB'$, ob

$$\left(\frac{m+2}{4}\right)m = 5\left(\frac{m+2}{5}\right) + 2\left(\frac{m+2}{4}\right)$$

et

et $\left(\frac{m+2}{3}\right) = 4\left(\frac{m+2}{4}\right) + 1\left(\frac{m+2}{3}\right)$ erit

$$m\,B^{\mathrm{I}} = 15\left(\frac{m+2}{6}\right) + 10\left(\frac{m+2}{4}\right) + 1\left(\frac{m+2}{3}\right)$$

ſtatuatur ergo

$$C = 15\left(\frac{m+2}{6}\right) + 10\left(\frac{m+2}{5}\right) + 1\left(\frac{m+2}{4}\right)$$

hinc $C^{\mathrm{I}} = 15\left(\frac{m+3}{6}\right) + 10\left(\frac{m+3}{5}\right) + 1\left(\frac{m+3}{4}\right)$.

XI. Simili modo cum eſſe debeat $D^{\mathrm{I}} - D = mC^{\mathrm{I}}$ quia eſt

$$m\left(\frac{m+3}{6}\right) = 7\left(\frac{m+3}{7}\right) + 3\left(\frac{m+3}{6}\right)$$

$$m\left(\frac{m+3}{5}\right) = 6\left(\frac{m+3}{6}\right) + 2\left(\frac{m+3}{5}\right)$$

$$m\left(\frac{m+3}{4}\right) = 5\left(\frac{m+3}{5}\right) + 1\left(\frac{m+3}{4}\right)$$

erit

$$mC^{\mathrm{I}} = 105\left(\frac{m+3}{7}\right) + 105\left(\frac{m+3}{6}\right) + 25\left(\frac{m+3}{5}\right) + \left(\frac{m+3}{4}\right)$$

vnde colligimus

$$D^{\mathrm{I}} = 105\left(\frac{m+4}{8}\right) + 105\left(\frac{m+4}{7}\right) + 25\left(\frac{m+4}{6}\right) + 1\left(\frac{m+4}{5}\right)$$

XII. Porro ob $E^{\mathrm{I}} - E = mD^{\mathrm{I}}$ quia eſt

$$m\left(\frac{m+4}{8}\right) = 9\left(\frac{m+4}{9}\right) + 4\left(\frac{m+4}{8}\right)$$

$$m\left(\frac{m+4}{7}\right) = 8\left(\frac{m+4}{8}\right) + 3\left(\frac{m+4}{7}\right)$$

$$m\left(\frac{m+4}{6}\right) = 7\left(\frac{m+4}{7}\right) + 2\left(\frac{m+4}{6}\right)$$

$$m\left(\frac{m+4}{5}\right) = 6\left(\frac{m+4}{6}\right) + 1\left(\frac{m+4}{5}\right)$$

colli-

colligimus

$$mD^{I} = 945\left(\frac{m+4}{9}\right) + 1260\left(\frac{m+4}{8}\right) + 490\left(\frac{m+4}{7}\right) + 56\left(\frac{m+4}{6}\right) + 1\left(\frac{m+4}{5}\right)$$

hincque

$$E^{I} = 945\left(\frac{m+5}{10}\right) + 1260\left(\frac{m+5}{9}\right) + 490\left(\frac{m+5}{8}\right) + 56\left(\frac{m+5}{7}\right) + 1\left(\frac{m+5}{6}\right)$$

et vlterius progrediendo

$$F^{I} = 10395\left(\frac{m+6}{12}\right) + 17325\left(\frac{m+6}{11}\right) + 9450\left(\frac{m+6}{10}\right) + 1918\left(\frac{m+6}{9}\right) + 119\left(\frac{m+6}{8}\right) + \left(\frac{m+6}{7}\right).$$

Euolutio casus $n = m + \lambda$.

XIII. Pro serie ergo nostra casu quo $n = m + \lambda$

$$s = x^{m+\lambda} - \frac{m}{1}(x-1)^{m+\lambda} + \frac{m(m-1)}{1.\ 2.}(x-2)^{m+\lambda} - \frac{m(m-1)(m-2)}{1.\ 2.\ 3}(x-3)^{m+\lambda} + \text{etc.}$$

si aequationem generalem supra §. VIII. exhibitam diuidamus per $\left(\frac{m+\lambda}{m}\right) = \frac{(m+\lambda)(m+\lambda-1)(m+\lambda-2)\ldots\ldots(\lambda+1)}{1.\ 2.\ 3\ \ldots\ldots\ m}$ perueniemus ad hanc expressionem

$$\frac{s}{(\lambda+1)(\lambda+2)\ldots(\lambda+m)} = x^{\lambda} - \frac{\lambda}{m+1}A^{I}x^{\lambda-1} + \frac{\lambda(\lambda-1)}{(m+1)(m+2)}B^{I}x^{\lambda-2} - \frac{\lambda(\lambda-1)(\lambda-2)}{(m+1)(m+2)(m+3)}C^{I}x^{\lambda-3} + \text{etc.}$$

vbi loco litterarum A^{I}, B^{I}, C^{I}, D^{I} etc. sequentes valores substitui oportet:

$$A^{I} = \left(\frac{m+1}{2}\right) = \frac{(m+1)m}{1.\ 2}$$

$$B^{I} = 3\left(\frac{m+2}{4}\right) + \left(\frac{m+2}{3}\right)$$

$$C^{I} = 15\left(\frac{m+3}{6}\right) + 10\left(\frac{m+3}{5}\right) + \left(\frac{m+3}{4}\right)$$

$$D^{I} = 105\left(\frac{m+4}{8}\right) + 105\left(\frac{m+4}{7}\right) + 25\left(\frac{m+4}{6}\right) + \left(\frac{m+4}{5}\right)$$

$$E^{I} = 945\left(\frac{m+5}{10}\right) + 1260\left(\frac{m+5}{9}\right) + 490\left(\frac{m+5}{8}\right) + 56\left(\frac{m+5}{7}\right) + \left(\frac{m+5}{6}\right)$$

$$F^{I} = 10395\left(\frac{m+6}{12}\right) + 17325\left(\frac{m+6}{11}\right) + 9450\left(\frac{m+6}{10}\right) + 1918\left(\frac{m+6}{9}\right) + 119\left(\frac{m+6}{8}\right) + \left(\frac{m+6}{7}\right)$$

 vbi

vbi est $10395 = 11.945$; $17325 = 10.1260 + 5.945$

$$9450 = 9.490 + 4.1260$$
$$1918 = 8.56 + 3.490$$
$$119 = 7.1 + 2.56$$
$$1 = 6.0 + 1.1$$

hinc si pro valore sequente ponatur:

$$G' = \alpha\left(\frac{m+7}{14}\right) + \beta\left(\frac{m+7}{13}\right) + \gamma\left(\frac{m+7}{12}\right) + \delta\left(\frac{m+7}{11}\right) + \varepsilon\left(\frac{m+7}{10}\right) + \zeta\left(\frac{m+7}{9}\right) + \eta\left(\frac{m+7}{8}\right)$$

hi coefficientes ita determinabuntur:

$\alpha = 13.10395$	$\varepsilon = 9.119 + 3.1918$
$\beta = 12.17325 + 6.10395$	$\zeta = 8.1 + 2.119$
$\gamma = 11.9450 + 5.17325$	$\eta = 7.0 + 1.1$
$\delta = 10.1918 + 4.9450$	

XIV. Iidem autem valores commodius ita exprimentur:

$$A' = \left(\frac{m+1}{2}\right).1$$

$$B' = \left(\frac{m+2}{3}\right)\left(1 + 3.\frac{m-1}{4}\right)$$

$$C' = \left(\frac{m+3}{4}\right)\left(1 + 10.\frac{m-1}{5} + 15.\frac{m-1}{5}.\frac{m-2}{6}\right)$$

$$D' = \left(\frac{m+4}{5}\right)\left(1 + 25.\frac{m-1}{6} + 105.\frac{m-1}{6}.\frac{m-2}{7} + 105.\frac{m-1}{6}.\frac{m-2}{7}.\frac{m-3}{8}\right)$$

$$E' = \left(\frac{m+5}{6}\right)\left(1 + 56.\frac{m-1}{7} + 490.\frac{m-1}{7}.\frac{m-2}{8} + 1260.\frac{m-1}{7}.\frac{m-2}{8}.\frac{m-3}{9} + 945.\frac{m-1}{7}.\frac{m-2}{8}.\frac{m-3}{9}.\frac{m-4}{10}\right)$$

$$F' = \left(\frac{m+6}{7}\right)\left(1 + 119.\frac{m-1}{8} + 1918.\frac{m-1}{8}.\frac{m-2}{9} + 9450.\frac{m-1}{8}.\frac{m-2}{9}.\frac{m-3}{10} + 17325.\frac{m-1}{8}.\frac{m-2}{9}.\frac{m-3}{10}.\frac{m-4}{11} + 10395.\frac{m-1}{8}.\frac{m-2}{9}.\frac{m-3}{10}.\frac{m-4}{11}.\frac{m-5}{12}\right)$$

cuius

cuius progressionis lex quo facilius perspiciatur, ponamus in genere

$$M^{I} = \left(\frac{m+\mu-1}{\mu}\right)\left(1 + \alpha . \frac{m-1}{\mu+1} + \beta . \frac{m-1}{\mu+1} . \frac{m-2}{\mu+2} + \gamma . \frac{m-1}{\mu+1} . \frac{m-2}{\mu+2} . \frac{m-3}{\mu+3} + \text{etc.}\right.$$

et sequentem

$$N^{I} = \left(\frac{m+\mu}{\mu+1}\right)\left(1 + \alpha^{I} . \frac{m-1}{\mu+2} + \beta^{I} . \frac{m-1}{\mu+2} . \frac{m-2}{\mu+3} + \gamma^{I} . \frac{m-1}{\mu+2} . \frac{m-2}{\mu+3} . \frac{m-3}{\mu+4} + \text{etc.}\right.$$

atque hi coefficientes hoc modo per praecedentes determinantur

$$\alpha^{I} = 2\alpha + \mu + 1;$$
$$\beta^{I} = 3\beta + (\mu+2)\alpha$$
$$\gamma^{I} = 4\gamma + (\mu+3)\beta$$
$$\delta^{I} = 5\delta + (\mu+4)\gamma$$
$$\varepsilon^{I} = 6\varepsilon + (\mu+5)\delta$$

vnde has formulas facile quousque libuerit continuare licet.

XV. Substituamus iam hos valores, ac pro summa s seriei propositae quando $n = m + \lambda$ obtinebimus sequentem expressionem:

$$\frac{s}{(\lambda+1)(\lambda+2)\ldots(\lambda+m)} =$$

$$x^{\lambda} - \frac{\lambda m}{1 . 2} x^{\lambda-1} + \frac{\lambda(\lambda-1)m}{1 . 2 . 3} x^{\lambda-2}\left(1 + \frac{3(m-1)}{4}\right)$$

$$- \frac{\lambda(\lambda-1)(\lambda-2)m}{1 . 2 . 3 . 4} x^{\lambda-3}\left(1 + 10 . \frac{m-1}{5} + 15 . \frac{m-1}{5} . \frac{m-2}{6}\right)$$

$$+ \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)m}{1 . 2 . 3 . 4 . 5} x^{\lambda-4}\left(1 + 25 . \frac{m-1}{6} + 105 . \frac{m-1}{6} . \frac{m-2}{7} + 105 . \frac{m-1}{6} . \frac{m-2}{7} . \frac{m-3}{8}\right)$$

$$- \frac{\lambda \ldots (\lambda-4)m}{1 \ldots 5 . 6} x^{\lambda-5}\left(1 + 56 . \frac{m-1}{7} + 490 . \frac{m-1}{7} . \frac{m-2}{8} + 1260 . \frac{m-1}{7} \ldots \frac{m-3}{9} \right.$$
$$\left. + 945 \frac{m-1}{7} \ldots \frac{m-4}{10}\right)$$

$$+ \frac{\lambda \ldots (\lambda-5)m}{1 \ldots 6 . 7} x^{\lambda-6}\left(1 + 119 . \frac{m-1}{8} + 1918 . \frac{m-1}{8} . \frac{m-2}{9} + 9450 . \frac{m-1}{8} \ldots \frac{m-3}{10} \right.$$
$$\left. + 17325 . \frac{m-1}{8} \ldots \frac{m-4}{11} + 10395 . \frac{m-1}{8} \ldots \frac{m-5}{12}\right\}$$

etc.

subtra-

ſubtrahatur hinc primo poteſtas

$$\left(x-\tfrac{m}{2}\right)^{\lambda}=x^{\lambda}-\frac{\lambda m}{2}x^{\lambda-1}+\frac{\lambda(\lambda-1)m^{2}}{1.\,2.\,4}x^{\lambda-2}-\frac{\lambda(\lambda-1)(\lambda-2)m^{3}}{1.\,2.\,3.\,8}x^{\lambda-3}$$
$$+\frac{\lambda\ldots(\lambda-3)m^{4}}{1\ldots 4.\,16}x^{\lambda-4}-\frac{\lambda\ldots(\lambda-4)m^{5}}{1\ldots 5.\,32}x^{\lambda-5}$$
$$+\frac{\lambda\ldots(\lambda-5)m^{6}}{1\ldots 6.\,64}x^{\lambda-6}-\text{etc.}$$

Commode autem hic euenit vt ſit

$$\frac{15}{5.\,6}=\tfrac{4}{8};\quad \frac{105}{6.\,7.\,8}=\tfrac{5}{16};\quad \frac{945}{7.\,8.\,9.\,10}=\tfrac{6}{32};\quad \frac{10395}{8.\,9.\,10.\,11.\,12}=\tfrac{7}{64}$$

cuius quidem rei ratio per ſe eſt perſpicua; quamobrem expreſſio ſuperior euoluta ſequentem induit formam

$$\left(x-\tfrac{m}{2}\right)^{\lambda}+\frac{\lambda(\lambda-1)m}{1.\,2.\,3}x^{\lambda-2}.\tfrac{1}{4}-\frac{\lambda(\lambda-1)(\lambda-2)m}{1.\,2.\,3.\,4}x^{\lambda-3}.\tfrac{m}{2}+\frac{\lambda\ldots(\lambda-3)m}{1\ldots 4.\,5}x^{\lambda-4}\left(\tfrac{5}{8}m^{2}+\tfrac{5}{48}m-\tfrac{1}{24}\right)$$
$$-\frac{\lambda\ldots(\lambda-4)m}{1\ldots 5.\,6}x^{\lambda-5}\left(\tfrac{5}{8}m^{3}+\tfrac{5}{16}m^{2}-\tfrac{1}{8}m\right)$$
$$+\frac{\lambda\ldots(\lambda-5)m}{1\ldots 6.\,7}x^{\lambda-6}\left(\tfrac{35}{64}m^{4}+\tfrac{35}{64}m^{3}-\tfrac{91}{576}m^{2}-\tfrac{7}{96}m+\tfrac{1}{36}\right).\ \text{etc.}$$

XVI. In hac expreſſione denuo poteſtas ipſius $x-\frac{m}{2}$ ſcilicet $\frac{\lambda(\lambda-1)m}{2.\,3.\,4}\left(x-\frac{m}{2}\right)^{\lambda-2}$ contineri deprehenditur qua inde ſeparata expreſſio noſtra erit:

$$\left(x-\tfrac{m}{2}\right)^{\lambda}+\frac{\lambda(\lambda-1)m}{2.\,3.\,4}\left(x-\tfrac{m}{2}\right)^{\lambda-2}+\frac{\lambda\ldots(\lambda-3)m}{1\ldots 4.\,5}x^{\lambda-4}\left(\tfrac{5}{48}m-\tfrac{1}{24}\right)$$
$$-\frac{\lambda\ldots(\lambda-4)m}{1\ldots 5.\,6}x^{\lambda-5}\left(\tfrac{5}{16}m^{2}-\tfrac{1}{8}m\right)$$
$$+\frac{\lambda\ldots(\lambda-5)m}{1\ldots 6.\,7}x^{\lambda-6}\left(\tfrac{35}{64}m^{3}-\tfrac{91}{576}m^{2}-\tfrac{7}{96}m+\tfrac{1}{36}\right)\text{etc.}$$

in qua adhuc continetur $\frac{\lambda\ldots(\lambda-3)m}{1\ldots 4.\,5}\left(\frac{5}{48}m-\frac{1}{24}\right)\left(x-\frac{m}{2}\right)^{\lambda-4}$ ac praeterea ſupereſt

$$\frac{\lambda\ldots(\lambda-5)m}{1\ldots 6.\,7}x^{\lambda-6}\left(\tfrac{35}{576}m^{2}-\tfrac{7}{96}m+\tfrac{1}{36}\right)$$

vnde

vnde ſine dubio inſuper haec poteſtas accedit:

$$+\frac{\lambda \ldots\ldots (\lambda-5)m}{1 \ldots\ldots 6 \cdot 7}\cdot\frac{35m^2-42m+16}{576}\left(x-\frac{m}{2}\right)^{\lambda-6}.$$

Quocirca aequatio noſtra ita erit comparata:

$$\frac{s}{(\lambda+1)(\lambda+2)\ldots(\lambda+m)}=\left(x-\frac{m}{2}\right)^{\lambda}+\frac{\lambda(\lambda-1)}{1\cdot 2}\cdot\frac{m}{12}\left(x-\frac{m}{2}\right)^{\lambda-2}$$
$$+\frac{\lambda(\lambda-1)\ldots(\lambda-3)}{1\cdot 2\ldots 4}\cdot\frac{m(5m-2)}{240}\left(x-\frac{m}{2}\right)^{\lambda-4}$$
$$+\frac{\lambda(\lambda-1)\ldots(\lambda-5)}{1\cdot 2\ldots 6}\cdot\frac{m(35m^2-42m+16)}{4032}\left(x-\frac{m}{2}\right)^{\lambda-6}\text{ etc.}$$

XVII. En ergo ſeriei noſtrae propoſitae generalis:

$$s=x^{m+\lambda}-\frac{m}{1}(x-1)^{m+\lambda}+\frac{m(m-1)}{1\cdot 2}(x-2)^{m+\lambda}$$
$$-\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}(x-3)^{m+\lambda}+\text{etc.}$$

eximiam transformationem, quae cum per plures ambages ſit eruta, ac pluribus operationibus admodum intricatis innixa, tantopere abſtruſa videtur, vt eius inueſtigatio directa ingentia ſubſidia in Analyſin ſit allatura. Quo autem facilius hanc transformationem perſcrutari liceat eam hoc modo repraeſentabo, vt ſit

$$\frac{s}{(\lambda+1)(\lambda+2)\ldots(\lambda+m)}=\left(x-\frac{m}{2}\right)^{\lambda}+\frac{\lambda(\lambda-1)}{1\cdot 2}\,P\left(x-\frac{m}{2}\right)^{\lambda-2}$$
$$+\frac{\lambda(\lambda-1)\ldots(\lambda-3)}{1\cdot 2\ldots 4}\,Q\left(x-\frac{m}{2}\right)^{\lambda-4}$$
$$+\frac{\lambda(\lambda-1)\ldots(\lambda-5)}{1\cdot 2\ldots 6}\,R\left(x-\frac{m}{2}\right)^{\lambda-6}$$
$$+\frac{\lambda(\lambda-1)\ldots(\lambda-7)}{1\cdot 2\ldots 8}\,S\left(x-\frac{m}{2}\right)^{\lambda-8}$$
$$+\frac{\lambda(\lambda-1)\ldots(\lambda-9)}{1\cdot 2\ldots 10}\,T\left(x-\frac{m}{2}\right)^{\lambda-10}$$
$$\text{etc.}$$

pro qua expressione hactenus quidem inueni

$$P = \frac{m}{3.\,4}$$

$$Q = \frac{m(5m - 2)}{5.\,6.\,8}$$

$$R = \frac{m(35mm - 42m + 16)}{6.\,7.\,96}$$

$$S = \frac{m(175m^3 - 420m^2 + 404m - 144)}{34560}$$

sed methodus desideratur harum litterarum valores expedite inueniendi.

XVIII. Imprimis autem hic notasse iuuabit, seriem nostram in aliam esse transmutatam, quae secundum potestates formulae $x - \frac{m}{2}$ ita progrediatur, vt earum exponentes sint λ, $\lambda - 2$, $\lambda - 4$ etc. binario continuo decrescentes; tum vero litteras P, Q, R, etc. a solo numero m pendere, ita vt neque exponens λ neque quantitas x in eas ingrediatur; praeterea vero coefficientes praefixos solum numerum λ implicare, ac legem progressionis ex euolutione binomii ortae seruare. Hac forma probe obseruata manifestum est valores litterarum P, Q, R, S etc. seorsim ex ipsa serie proposita vel ex eius transformata §. XV. cuius lex progressionis itidem est cognita elici posse, siquidem ponatur $x = \frac{m}{2}$ si enim tum capiatur $\lambda = 2$ fit $P = \frac{s}{(\lambda+1)(\lambda+2)\ldots(\lambda+m)}$ posito autem $\lambda = 4$ fit $Q = \frac{s}{(\lambda+1)(\lambda+2)\ldots(\lambda+m)}$ at posito $\lambda = 6$ fit $R = \frac{s}{(\lambda+1)(\lambda+2)\ldots(\lambda+m)}$,

etc.

XIX. Quodsi ergo hic loco $\frac{1}{(\lambda+1)(\lambda+2)\ldots(\lambda+m)}$ series supra §. XV. inuenta substituatur, atque in hunc finem breuitatis gratia ponatur:

$$\mathfrak{A} = 1$$

$$\mathfrak{B} = 1 + 3.\frac{m-1}{4}$$

$$\mathfrak{C} = 1 + 10.\frac{m-1}{5} + 15.\frac{m-1}{5}.\frac{m-2}{6}$$

$$\mathfrak{D} = 1 + 25.\frac{m-1}{6} + 105.\frac{m-1}{6}.\frac{m-2}{7} + 105.\frac{m-1}{6}.\frac{m-2}{7}.\frac{m-3}{8}$$

$$\mathfrak{E} = 1 + 56.\frac{m-1}{7} + 490.\frac{m-1}{7}.\frac{m-2}{8} + 1260.\frac{m-1}{7}\ldots\frac{m-3}{9} + 945.\frac{m-1}{7}\ldots\frac{m-4}{10}$$

$$\mathfrak{F} = 1 + 119.\frac{m-1}{8} + 1918.\frac{m-1}{8}.\frac{m-2}{9} + 9450.\frac{m-1}{8}\ldots\frac{m-3}{10} + 17325.\frac{m-1}{8}\ldots\frac{m-4}{11} + 10395.\frac{m-1}{8}\ldots\frac{m-5}{12}$$

etc.

adipiscimur sequentes valores

$$P = \frac{m^2}{2^2} - 2\mathfrak{A}\frac{m}{2}.\frac{m}{2} + \mathfrak{B}\frac{m}{3}$$

$$Q = \frac{m^4}{2^4} - 4\mathfrak{A}.\frac{m}{2}.\frac{m^3}{2^3} + 6\mathfrak{B}.\frac{m}{3}.\frac{m^2}{2^2} - 4\mathfrak{C}.\frac{m}{4}.\frac{m}{2} + \mathfrak{D}\frac{m}{5}$$

$$R = \frac{m^6}{2^6} - 6\mathfrak{A}.\frac{m}{2}.\frac{m^5}{2^5} + 15\mathfrak{B}.\frac{m}{3}.\frac{m^4}{2^4} - 20\mathfrak{C}.\frac{m}{4}.\frac{m^3}{2^3} + 15\mathfrak{D}.\frac{m}{5}.\frac{m^2}{2^2} - 6\mathfrak{E}\frac{m}{6}.\frac{m}{2} + \mathfrak{F}.\frac{m}{7}$$

quem in finem valores illos litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. euolui conueniet, vnde prodit

$$\mathfrak{A} = 1$$

$$\mathfrak{B} = \frac{3}{4}m + \frac{1}{4} = \frac{3}{4}(m + \frac{1}{3})$$

$$\mathfrak{C} = \frac{1}{2}m^2 + \frac{1}{2}m = \frac{1}{2}(mm + m)$$

$\mathfrak{D} =$

$$\mathfrak{D} = \tfrac{5}{16}m^3 + \tfrac{5}{8}m^2 + \tfrac{5}{48}m - \tfrac{1}{24} = \tfrac{5}{16}(m^3 + 2m^2 + \tfrac{1}{3}m - \tfrac{2}{15})$$

$$\mathfrak{E} = \tfrac{5}{16}m^4 + \tfrac{5}{8}m^3 + \tfrac{5}{16}m^2 - \tfrac{1}{8}m = \tfrac{6}{32}(m^4 + \tfrac{10}{3}m^3 + \tfrac{5}{3}m^2 - \tfrac{2}{3}m)$$

$$\mathfrak{F} = \tfrac{7}{64}m^5 + \tfrac{35}{64}m^4 + \tfrac{35}{64}m^3 - \tfrac{91}{576}m^2 - \tfrac{7}{96}m + \tfrac{1}{3}$$

seu . . . $\mathfrak{F} = \tfrac{7}{64}(m^5 + 5m^4 + 5m^3 - \tfrac{13}{9}m^2 - \tfrac{2}{3}m + \tfrac{16}{63})$

hic autem praeterquam in primis terminis nullus ordo perſpicitur

XX. Quod vero ſeries transformata ſecundum poteſtates quantitatis $x = \frac{m}{2}$ progrediatur, id quidem per ſolam inductionem agnouimus, verumtamen hoc neceſſario euenire ita oſtendi poteſt. Quoniam progreſſio propoſita ſimili modo deſinit, quo incipit, ita vt vltimi bini termini futuri ſint $\pm m(x-m+1)^{m+\lambda} \mp (x-m)^{m+\lambda}$, vbi ſigna ſuperiora valent ſi m ſit numerus impar, inferiora vero ſi par; ſumamus m eſſe numerum parem, (eadem enim concluſio producitur ſi fuerit impar) et ponamus $x - \frac{m}{2} = y$, eritque

$$2s = +(y+\tfrac{1}{2}m)^{m+\lambda} - \tfrac{m}{1}(y+\tfrac{1}{2}m-1)^{m+\lambda} + \tfrac{m(m-1)}{1.\,2}(y+\tfrac{1}{2}m-2)^{m+\lambda}$$
$$+(y-\tfrac{1}{2}m)^{m+\lambda} - \tfrac{m}{1}(y-\tfrac{1}{2}m+1)^{m+\lambda} + \tfrac{m(m-1)}{1.\,2}(y-\tfrac{1}{2}m+2)^{m+\lambda}$$

et facta euolutione ſecundum poteſtates ipſius $y = x - \frac{m}{2}$ reperitur;

$$s = y^{m+\lambda}\left(1 - \tfrac{m}{1} + \tfrac{m(m-1)}{1.\,2} - \text{etc.}\right)$$
$$+ \left(\tfrac{m+\lambda}{2}\right) y^{m+\lambda-2}\left(\left(\tfrac{m}{2}\right)^2 - \tfrac{m}{1}\left(\tfrac{m}{2}-1\right)^2 + \tfrac{m(m-1)}{1.\,2}\left(\tfrac{m}{2}-2\right)^2 - \text{etc.}\right)$$
$$+ \left(\tfrac{m+\lambda}{4}\right) y^{m+\lambda-4}\left(\left(\tfrac{m}{2}\right)^4 - \tfrac{m}{1}\left(\tfrac{m}{2}-1\right)^4 + \tfrac{m(m-1)}{1.\,2}\left(\tfrac{m}{2}-2\right)^4 - \text{etc.}\right)$$
$$\text{etc.}$$

Hae

Hae ſeries autem omnes euaneſcunt, donec perueniatur ad eam in qua exponentes ſunt m, eiusque ſummam nouimus eſſe $= 1 . 2 . 3 \ldots m$, omiſſis ergo prioribus, quarum ſumma ad nihilum reducitur, obtinebimus:

$$s = \left(\frac{m+\lambda}{m}\right) y^{\lambda} \left(\left(\tfrac{m}{2}\right)^m - \tfrac{m}{1}\left(\tfrac{m}{2}-1\right)^m + \tfrac{m(m-1)}{1 . 2}\left(\tfrac{m}{2}-2\right)^m - \text{etc.} \right)$$

$$+ \left(\frac{m+\lambda}{m+2}\right) y^{\lambda-2} \left(\left(\tfrac{m}{2}\right)^{m+2} - \tfrac{m}{1}\left(\tfrac{m}{2}-1\right)^{m+2} + \tfrac{m(m-1)}{1 . 2}\left(\tfrac{m}{2}-2\right)^{m+2} - \text{etc.} \right)$$

$$\text{etc.}$$

ſicque manifeſtum eſt, quod demonſtrare ſuſcepi, ſcilicet hanc ſeriem ſecundum poteſtates $y^{\lambda}, y^{\lambda-2}, y^{\lambda-4}$ etc. deſcendere.

XXI. Tribuamus huic ſeriei ſimilem formam ei quam §. XVII. habuimus, fietque

$$\frac{s}{(\lambda+1)(\lambda+2)\ldots(\lambda+m)} = \frac{y^{\lambda}}{1 . 2 \ldots m} \left(\left(\tfrac{m}{2}\right)^m - \tfrac{m}{1}\left(\tfrac{m}{2}-1\right)^m + \text{etc} \right)$$

$$+ \frac{1 . 2\, y^{\lambda-2}}{1 . 2 \ldots (m+2)} . \frac{\lambda(\lambda-1)}{1 . 2} \left(\left(\tfrac{m}{2}\right)^{m+2} - \tfrac{m}{1}\left(\tfrac{m}{2}-1\right)^{m+2} + \text{etc.} \right)$$

$$+ \frac{1 . 2 . 3 . 4\, y^{\lambda-4}}{1 . 2 \ldots (m+4)} . \frac{\lambda(\lambda-1)\ldots(\lambda-3)}{1 . 2 \ldots 4} \left(\left(\tfrac{m}{2}\right)^{m+4} - \text{etc.} \right)$$

$$\text{etc.}$$

vnde valores litterarum P, Q, R etc. nouo modo ita determinare licet

$$P = \frac{1}{3 . 4 \ldots (m+2)} \left(\left(\tfrac{m}{2}\right)^{m+2} - \tfrac{m}{1}\left(\tfrac{m}{2}-1\right)^{m+2} + \text{etc.} \right)$$

$$Q = \frac{1}{5 . 6 \ldots (m+4)} \left(\left(\tfrac{m}{2}\right)^{m+4} - \tfrac{m}{1}\left(\tfrac{m}{2}-1\right)^{m+4} + \text{etc.} \right)$$

$$R = \frac{1}{7 . 8 \ldots (m+6)} \left(\left(\tfrac{m}{2}\right)^{m+6} - \tfrac{m}{1}\left(\tfrac{m}{2}-1\right)^{m+6} + \text{etc.} \right)$$

$$S = \frac{1}{9 . 10 \ldots (m+8)} \left(\left(\tfrac{m}{2}\right)^{m+8} - \tfrac{m}{1}\left(\tfrac{m}{2}-1\right)^{m+8} + \text{etc.} \right)$$

$$\text{etc.}$$

Hic quidem ſimilium ſerierum ſummatione opus eſt quoniam vero iſtae ſeries ſolum numerum m inuoluunt, noſtra inueſtigatio ad caſum ſimpliciorem perducta eſt cenſenda. Ceterum nunc demum certo agnoſcimus has litteras tantum a numero m pendere.

XXII. Quodſi autem hic litterae m ſucceſſiue tribuamus valores definitos 1. 2. 3. 4. 5. 6 etc. totidem inde valores pro litteris P, Q, R, S etc. conſequemur, quibus cognitis, facile earum formas generales colligere licebit.

Ita pro littera P inuenienda reperiemus

ſi $m =$	0,	1,	2,	3,	4	etc.
$3 . 2^2 P =$	0,	1,	2,	3,	4	etc.
diff.		1,	1,	1,	1,	

ita vt hinc fit $3 . 2^2 P = m$ et $P = \frac{m}{2^2 . 3}$ vt ante. Porro pro littera Q

ſi $m =$	0,	1,	2,	3,	4,	5,	6
$2^4 . 3 . 5 Q =$	0,	3,	16,	39,	72,	115,	168
Diff. I.		3,	13,	23,	33,	43,	53
Diff. II.			10,	10,	10,	10,	10

erit ergo $2^4 . 3 . 5 Q = 3m + 10 \frac{m(m-1)}{1 . 2} = m(5m-2)$, hincque $Q = \frac{m(5m-2)}{2^4 . 3 . 5}$.

Eodem

Eodem modo pro littera R

si $m =$	0,	1,	2,	3	4,	5	6
$2^6 . 3 . 7 R =$	0,	3,	48,	205,	544,	1135,	2048
Diff. I.		3,	45,	157,	339,	591,	913
Diff. II.			42,	112,	182,	252,	322
Diff. III.				70,	70,	70	

vnde concluditur $2^6 . 3 . 7 R = 3 m + 21 m (m-1)$
$+ \frac{35}{3} m (m-1)(m-2)$

atque $R = \frac{m(35 m^2 - 42 m + 16)}{2^6 . 3^2 . 7}$

quos eosdem valores iam supra sumus nacti, hinc igitur eandem operationem ad litteras sequentes accommodemus.

XXIII. Pro littera igitur S habebimus:

si fuerit $m =$	0,	1,	2,	3,	4,	5,	6
$2^8 . 5 . 9 S =$	0,	5,	256,	2013,	7936,	22085,	49920
Diff. I.		5,	251,	1757,	5923,	14149,	27835
Diff. II.			246,	1506,	4166,	8226,	13686
III.				1260,	2660,	4060,	5460
IV.					1400,	1400,	1400

vnde fit $2^8 . 5 . 9 S = 5 m + 123 m (m-1) + 210 m (m-1)(m-2)$
$+ \frac{175}{3} m (m-1)(m-2)(m-3)$

et $S = \frac{m(175 m^3 - 420 m^2 + 404 m - 144)}{2^8 . 3 . 5 . 9}$.

Nunc porro pro littera T habebimus:

si fuerit $m = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6$

$2^{10}.3.11\,T = 0,\ 3,\ 512,\ 7665,\ 46080,\ 174255,\ 499968$

Diff. I. $3,\ 509,\ 7153,\ 38415,\ 128175,\ 325713$

II. $506,\ 6644,\ 31262,\ 89760,\ 197538$

III. $6138,\ 24618,\ 58498,\ 107778$

IV. $18480,\ 33880,\ 49280$

V. $15400,\ 15400$

vnde fit
$$2^{10}.3.11\,T = 3m + 253\,m(m-1) + 1023\,m(m-1)(m-2) + 770\,m(m-1)(m-2)(m-3) + \tfrac{385}{3}\,m(m-1)(m-2)(m-3)(m-4)$$

et
$$T = \frac{m(385m^4 - 1540m^3 + 2684m^2 - 2288m + 768)}{2^{10}.9.11}.$$

XXIV. Hos nunc valores ita repraesentemus, quo facilius lex progressionis explorari possit:

$$P = \frac{1\,m}{12}$$

$$Q = \frac{1.3\,m}{12^2}\left(m - \tfrac{2}{5}\right)$$

$$R = \frac{1.3.5\,m}{12^3}\left(m^2 - \tfrac{6}{5}m + \tfrac{16}{35}\right)$$

$$S = \frac{1.3.5.7\,m}{12^4}\left(m^3 - \tfrac{12}{5}m^2 + \tfrac{404}{175}m - \tfrac{144}{175}\right)$$

$$T = \frac{1.3.5.7.9\,m}{12^5}\left(m^4 - \tfrac{20}{5}m^3 + \tfrac{244}{35}m^2 - \tfrac{208}{35}m + \tfrac{768}{385}\right)$$

atque hic in primis et secundis terminis lex progressionis ita est manifesta, vt iidem pro omnibus sequentibus litteris tuto assignari possint, in reliquis autem terminis nullam plane legem etiamnum obseruare licet.

XXV. Pro valore ergo litterae V inueniendo ſtatuamus

$$V = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11\, m}{12^6} \left(m^5 - \tfrac{30}{5} m^4 + \alpha m^3 - \beta m^2 + \gamma m - \delta\right).$$

Ex forma autem generali

$$V = \frac{1}{13 \cdot 14 \cdots (m+12)} \left(\left(\tfrac{m}{2}\right)^{m+12} - m\left(\tfrac{m}{2} - 1\right)^{m+12} + \tfrac{m(m-1)}{1 \cdot 2}\left(\tfrac{m}{2} - 2\right)^{m+12} - \text{etc.}\right.$$

colligimus

ſi ſit fore

$$m = 1;\; V = \frac{1}{2^{12} \cdot 13} = \frac{5 \cdot 7 \cdot 11}{2^{12} \cdot 3^3}(-5 + \alpha - \beta + \gamma - \delta)$$

$$m = 2;\; V = \frac{1}{7 \cdot 13} = \frac{5 \cdot 7 \cdot 11}{2^{11} \cdot 3^3}(-64 + 8\alpha - 4\beta + 2\gamma - \delta)$$

$$m = 3;\; V = \frac{597871}{2^{12} \cdot 5 \cdot 7 \cdot 13} = \frac{5 \cdot 7 \cdot 11}{2^{12} \cdot 3^2}(-243 + 27\alpha - 9\beta + 3\gamma - \delta)$$

$$m = 4;\; V = \frac{5461}{2^2 \cdot 5 \cdot 7 \cdot 13} = \frac{5 \cdot 7 \cdot 11}{2^{10} \cdot 3^3}(-512 + 64\alpha - 16\beta + 4\gamma - \delta)$$

$$m = 5;\; V = \frac{5838647}{2^{12} \cdot 7 \cdot 13} = \frac{25 \cdot 7 \cdot 11}{2^{12} \cdot 3^3}(-625 + 125\alpha - 25\beta + 5\gamma - \delta)$$

$$m = 6;\; V = \frac{6\,047}{2^2 \cdot 3 \cdot 13} = \frac{5 \cdot 7 \cdot 11}{2^{11} \cdot 3^2}(-0 + 216\alpha - 36\beta + 6\gamma - \delta).$$

Hinc ergo ſequentes formemus aequationes:

$$\alpha - \beta + \gamma - \delta = \frac{27}{5 \cdot 7 \cdot 11 \cdot 13} + 5$$

$$8\alpha - 4\beta + 2\gamma - \delta = \frac{27 \cdot 2048}{5 \cdot 7^2 \cdot 11 \cdot 13} + 64$$

$$27\alpha - 9\beta + 3\gamma - \delta = \frac{9 \cdot 597871}{5^2 \cdot 7^2 \cdot 11 \cdot 13} + 243$$

$$64\alpha - 16\beta + 4\gamma - \delta = \frac{27 \cdot 256 \cdot 5461}{5^2 \cdot 7^2 \cdot 11 \cdot 13} + 512$$

$$125\alpha - 25\beta + 5\gamma - \delta = \frac{27 \cdot 5838647}{5^2 \cdot 7^2 \cdot 11 \cdot 13} + 625$$

$$216\alpha - 36\beta + 6\gamma - \delta = \frac{3 \cdot 512 \cdot 63047}{5 \cdot 7^2 \cdot 11 \cdot 13} + 0.$$

Differentiae nunc primae ita ſe habebunt

$$7\alpha - 3\beta + \gamma = \frac{27 \cdot 157}{5 \cdot 7^2 \cdot 11} + 59$$
$$19\alpha - 5\beta + \gamma = \frac{9 \cdot 43627}{5^2 \cdot 7^2 \cdot 11} + 179$$
$$37\alpha - 7\beta + \gamma = \frac{9 \cdot 276629}{5^2 \cdot 7^2 \cdot 11} + 269$$
$$61\alpha - 9\beta + \gamma = \frac{27 \cdot 341587}{5^2 \cdot 7^2 \cdot 11} + 113$$
$$91\alpha - 11\beta + \gamma = \frac{3 \cdot 8373269}{5^2 \cdot 7^2 \cdot 11} - 625$$

ſecundae vero per 2 diuiſae dant

$$6\alpha - \beta = \frac{9 \cdot 268}{5^2 \cdot 7} + 60$$
$$9\alpha - \beta = \frac{9 \cdot 1513}{5^2 \cdot 7} + 45$$
$$12\alpha - \beta = \frac{9 \cdot 4858}{5^2 \cdot 7} - 78$$
$$15\alpha - \beta = \frac{3 \cdot 34409}{5^2 \cdot 7} - 369.$$

Tertiae tandem differentiae per 3 diuiſae praebent

$$\alpha = \frac{3 \cdot 249}{5 \cdot 7} - 5 = \frac{3 \cdot 669}{5 \cdot 7} - 41 = \frac{3967}{5 \cdot 7} - 97$$

quae tres aequationes praebent eundem valorem

$$\alpha = \frac{572}{5 \cdot 7} = \frac{4 \cdot 11 \cdot 13}{5 \cdot 7}.$$

ex quo valore iam reliqui definiuntur ſequenti modo

$$\beta = \frac{6 \cdot 572}{5 \cdot 7} - \frac{9 \cdot 268}{5^2 \cdot 7} - 60 = \frac{12 \cdot 1229}{5^2 \cdot 7} - 60 = \frac{4248}{175} = \frac{8 \cdot 9 \cdot 59}{175}$$
$$\gamma = 3\beta - 7\alpha + \frac{27 \cdot 157}{5 \cdot 7^2 \cdot 11} + 59 = \frac{255968}{5^2 \cdot 7^2 \cdot 11}$$
$$\delta = \alpha - \beta + \gamma - \frac{27}{5 \cdot 7 \cdot 11 \cdot 13} - 5 = \frac{1061376}{5^2 \cdot 7^2 \cdot 11 \cdot 13}.$$

XXVI.

XXVI. Conſpectui ergo ſimul exponamus litterarum P, Q, R, etc. valores hactenus inuentos

$P = \frac{1\,m}{12}$

$Q = \frac{1\cdot 3\,m}{12^2}(m - \frac{2}{5})$

$R = \frac{1\cdot 3\cdot 5\,m}{12^3}(m^2 - \frac{6}{5}m + \frac{15}{35})$

$S = \frac{1\cdot 3\cdot 5\cdot 7\,m}{12^4}(m^3 - \frac{12}{5}m^2 + \frac{404}{175}m - \frac{144}{175})$

$T = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\,m}{12^5}(m^4 - \frac{20}{5}m^3 + \frac{244}{35}m^2 - \frac{208}{35}m + \frac{768}{385})$

$V = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\,m}{12^6}(m^5 - \frac{30}{5}m^4 + \frac{572}{35}m^3 - \frac{4248}{175}m^2 + \frac{255968}{13475}m - \frac{1061376}{175175})$

Ex prioribus terminis concludo poteſtates hic occurrere, quibus ſeorſim poſitis ordo facilius perſpici poſſe videtur:

$P = \frac{1\,m}{12}$

$Q = \frac{1\cdot 3\,m}{12^2}(m - \frac{2}{5})^1$

$R = \frac{1\cdot 3\cdot 5\,m}{12^3}((m - \frac{3}{5})^2 + \frac{17}{175})$

$S = \frac{1\cdot 3\cdot 5\cdot 7\,m}{12^4}((m - \frac{4}{5})^3 + \frac{4\cdot 17}{175}m - \frac{16\cdot 17}{5\cdot 175})$

$T = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\,m}{12^5}((m - \frac{5}{5})^4 + \frac{2\cdot 17}{35}m^2 - \frac{4\cdot 17}{35}m + \frac{383}{385})$

$V = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\,m}{12^6}((m - \frac{6}{5})^5 + \frac{4\cdot 17}{35}m^3 - \frac{72\cdot 17}{175}m^2 + \frac{581296}{5^3\cdot 7^2\cdot 11}m - \frac{7118556}{5^5\cdot 7^2\cdot 11\cdot 13}$

quin etiam proxime ſequentes termini hoc modo contrahi poſſint vt prodeat

$P = \frac{1\,m}{12}\cdot 1$

$Q = \frac{1\cdot 3\,m}{12^2}(m - \frac{2}{5})$

$R = \frac{1\cdot 3\cdot 5\,m}{12^3}((m - \frac{3}{5})^2 + \frac{17}{175})$

S =

$$S = \frac{1.3.5.7\,m}{12^4}\left(\left(m-\tfrac{4}{5}\right)^3 + \tfrac{4.17}{175}\left(m-\tfrac{4}{5}\right)\right)$$

$$T = \frac{1.3.5.7.9\,m}{12^5}\left(\left(m-\tfrac{5}{5}\right)^4 + \tfrac{10.17}{175}\left(m-\tfrac{5}{5}\right)^2 + \tfrac{9}{385}\right)$$

$$V = \frac{1.3.5.7.9.11\,m}{12^6}\left(\left(m-\tfrac{6}{5}\right)^5 + \tfrac{20.17}{175}\left(m-\tfrac{6}{5}\right)^3 + \tfrac{15808}{5^3.7^2.11}\,m - \tfrac{467128}{5^5.7^2.11.13}\right).$$

Nisi vltimum valorem euoluissemus, videretur omnes has expressiones ad huiusmodi potestates reduci, quod autem nunc secus euenire agnoscimus. Quocirca ex alio fonte in legem harum litterarum inquiri oportebit.

XXVIII. Singulos igitur terminos harum formarum potius euolutos repraesentemus:

$$P = \frac{m}{4.3}$$

$$Q = \frac{mm}{16.3} - \frac{m}{8.3.5}$$

$$R = \frac{5\,m^3}{64.9} - \frac{mm}{32.3} + \frac{m}{4.9.7}$$

$$S = \frac{5.7\,m^4}{256.27} - \frac{7\,m^3}{64.9} + \frac{101\,m^2}{64.27.5} - \frac{m}{16.3.5}$$

$$T = \frac{5.7\,m^5}{1024.9} - \frac{5.7\,m^4}{256.9} + \frac{61\,m^3}{256.9} - \frac{13\,m^2}{64.9} + \frac{m}{4.3.11}$$

$$V = \frac{5.7.11\,m^6}{4096.27} - \frac{5.7.11\,m^5}{2048.9} + \frac{1573\,m^4}{1024.27} - \frac{649\,m^3}{512.3.5} + \frac{7909\,m^2}{128.3.5.7} - \frac{691\,m}{8.9.5.7.13}$$

vbi quidem inter terminos primos et secundos iam ordinem obseruauimus postremi autem omni ordine destituti videbantur, quoad valore etiam litterae V euoluto numerus 691 criterium nobis suppeditauerit, in his postremis terminis numeros *Bernoullianos* implicari.

Desi-

Defignemus ergo numeros *Bernoullianos* litteris α, β, γ, δ etc. vt fit

$$\alpha = \tfrac{1}{2};\ \beta = \tfrac{1}{6};\ \gamma = \tfrac{1}{6};\ \delta = \tfrac{3}{10};\ \varepsilon = \tfrac{5}{6};\ \zeta = \tfrac{691}{210} \text{ etc.}$$

et inter eos notemus hanc legem progreffionis:

$$\alpha = \tfrac{1}{2^1}$$

$$\beta = \frac{5 \cdot 4\, \alpha}{2^2 \cdot 1 \cdot 2 \cdot 3} - \tfrac{2}{2^3}$$

$$\gamma = \frac{7 \cdot 6\, \beta}{2^2 \cdot 1 \cdot 2 \cdot 3} - \frac{7 \cdot 6 \cdot 5 \cdot 4\, \alpha}{2^4 \cdot 1 \cdot 2 \ldots 5} + \tfrac{3}{2^5}$$

$$\delta = \frac{9 \cdot 8\, \gamma}{2^2 \cdot 1 \cdot 2 \cdot 3} - \frac{9 \cdot 8 \cdot 7 \cdot 6\, \beta}{2^4 \cdot 1 \cdot 2 \ldots 5} + \frac{9 \ldots 4\, \alpha}{2^6 \cdot 1 \ldots 7} - \tfrac{4}{2^7}$$

$$\varepsilon = \frac{11 \cdot 10\, \delta}{2^2 \cdot 1 \cdot 2 \cdot 3} - \frac{11 \ldots 8\, \gamma}{2^4 \cdot 1 \ldots 5} + \frac{11 \ldots 6\, \beta}{2^6 \cdot 1 \ldots 7} - \frac{11 \ldots 4\, \alpha}{2^8 \cdot 1 \ldots 9} - \tfrac{5}{2^9}$$

$$\zeta = \frac{13 \cdot 12\, \varepsilon}{2^2 \cdot 1 \cdot 2 \cdot 3} - \frac{13 \ldots 10\, \delta}{2^4 \cdot 1 \ldots 5} + \frac{13 \ldots 8\, \gamma}{2^6 \cdot 1 \ldots 7} - \frac{13 \ldots 6\, \beta}{2^8 \cdot 1 \ldots 9} + \frac{13 \ldots 4\, \alpha}{2^{10} \cdot 1 \ldots 11} - \tfrac{6}{2^{11}}$$

etc.

Ac poftremi litterarum P, Q, R, S etc. termini ita concinne referri poterunt

$$\frac{\alpha m}{2 \cdot 3};\ \frac{\beta m}{4 \cdot 5};\ \frac{\gamma m}{6 \cdot 7};\ \frac{\delta m}{8 \cdot 9};\ \frac{\varepsilon m}{10 \cdot 11};\ \frac{\zeta m}{12 \cdot 13}.$$

XXIX. Quo autem nunc etiam inueftigemus quomodo ipfae litterae P, Q, R, S etc. progrediantur, a qualibet eiusmodi multiplum praecedentis auferamus, vt primi termini tollantur, et quia has litteras praecedit littera $O = 1$, habebimus:

$$P - \tfrac{m}{12} O = 0$$

$$Q - \tfrac{3m}{12} P = -\tfrac{\beta m}{4 \cdot 5}$$

$$R - \tfrac{5m}{12} Q = -\tfrac{mm}{16 \cdot 9} + \tfrac{\gamma m}{6 \cdot 7} = -\tfrac{m}{12} P + \tfrac{\gamma m}{6 \cdot 7}$$

$$S - \tfrac{7m}{12} R = -\tfrac{7m^3}{128 \cdot 9} + \tfrac{3m^2}{16 \cdot 5} - \tfrac{\delta m}{8 \cdot 9}$$

$$T - \frac{9\,m}{12} S = - \frac{7\,m^4}{128.\,9} + \frac{17\,m^3}{64.\,3.\,5} - \frac{7\,m^2}{8.\,9.\,5} + \frac{\varepsilon\,m}{10.\,11}$$

$$V - \frac{11\,m}{12} T = - \frac{5.7.11\,m^5}{2048.\,27} + \frac{451\,m^4}{512.\,27} - \frac{121\,m^3}{2048.27.5} + \frac{7159\,m^2}{128.\,27.\,5.7} - \frac{\zeta\,m}{12.\,13};$$

Quodsi iam has formas penitius perpendamus, ac breuitatis gratia ponamus:

$$\frac{\alpha\,m}{2.\,3} = \alpha^1;\ \frac{\beta\,m}{4.\,5} = \beta^1;\ \frac{\gamma\,m}{6.\,7} = \gamma^1;\ \frac{\delta\,m}{8.\,9} = \delta^1;\ \text{etc.}$$

sequentem legem satis simplicem in nostris litteris P, Q, R etc. deprehendemus:

$$P - \alpha^1 = 0$$

$$Q - \tfrac{3}{1}\alpha^1 P + \frac{3.\,2.\,1}{1.\,2.\,3}\beta^1 = 0$$

$$R - \tfrac{5}{1}\alpha^1 Q + \frac{5.\,4.\,3}{1.\,2.\,3}\beta^1 P - \frac{5.\,4.\,3.\,2.\,1}{1.\,2.\,3.\,4.\,5}\gamma^1 = 0$$

$$S - \tfrac{7}{1}\alpha^1 R + \frac{7.\,6.\,5}{1.\,2.\,3}\beta^1 Q - \frac{7.\,6 \ldots 3}{1.\,2 \ldots 5}\gamma^1 P + \frac{7.\,6 \ldots 1}{1.\,2 \ldots 7}\delta^1 = 0$$

$$T - \tfrac{9}{1}\alpha^1 S + \frac{9 \ldots 7}{1 \ldots 3}\beta^1 R - \frac{9 \ldots 5}{1 \ldots 5}\gamma^1 Q + \frac{9 \ldots 3}{1 \ldots 7}\delta^1 P - \frac{9 \ldots 1}{1 \ldots 9}\varepsilon^1 = 0$$

$$V - \tfrac{11}{1}\alpha^1 T + \frac{11 \ldots 9}{1 \ldots 3}\beta^1 S - \frac{11 \ldots 7}{1 \ldots 5}\gamma^1 R + \frac{11 \ldots 5}{1 \ldots 7}\delta^1 Q - \frac{11 \ldots 3}{1 \ldots 9}\varepsilon^1 P + \frac{11 \ldots 1}{1 \ldots 11}\zeta^1 = 0$$

etc.

hae autem nouae litterae α^1, β^1, γ^1, δ^1 etc. ex praecedentibus hanc sequentur legem:

$$\alpha^1 - \frac{m}{2^2.\,3} = 0$$

$$\beta^1 - \frac{3.\,2}{2^2.1.2.3}\alpha^1 + \frac{m}{2^4.\,5} = 0$$

$$\gamma^1 - \frac{5.\,4}{2^2.1.2.3}\beta^1 + \frac{5.\,4.\,3.\,2}{2^4.\,1 \ldots 5}\alpha^1 - \frac{m}{2^6.\,7} = 0$$

$$\delta^1 - \frac{7.\,6}{2^2.1 \ldots 3}\gamma^1 + \frac{7 \ldots 4}{2^4.1 \ldots 5}\beta^1 - \frac{7 \ldots 2}{2^6.\,1 \ldots 7}\alpha^1 + \frac{m}{2^8.\,9} = 0$$

$$\varepsilon^1 - \frac{9.\,8}{2^2.1 \ldots 3}\delta^1 + \frac{9 \ldots 6}{2^4.1 \ldots 5}\gamma^1 - \frac{9 \ldots 4}{2^6.1 \ldots 7}\beta^1 + \frac{9 \ldots 2}{2^8.\,1 \ldots 9}\alpha^1 - \frac{m}{2^{10}.11} = 0$$

etc.

Quo-

Quocirca nunc quidem quaeſtionem circa ſeriem illam ſingularem, quam hactenus ſum contemplatus, perfecte ſolutam dediſſe ſum cenſendus, vnde ſolutionem hic ſuccincte ſum propoſiturus.

Problema.

Propoſita hac progreſſione indefinita:

$$s = x^{m+\lambda} - \frac{m}{1}(x-1)^{m+\lambda} + \frac{m(m-1)}{1.\ 2}(x-2)^{m+\lambda} - \frac{m(m-1)(m-2)}{1.\ 2.\ 3}(x-3)^{m+\lambda} + \text{etc.}$$

eius ſummam aſſignare; ſiquidem λ fuerit numerus quicunque integer poſitiuus.

Solutio.

Denotent litterae $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. numeros *Bernoullianos*, ita vt ſit

$$\mathfrak{A} = \tfrac{1}{2};\ \mathfrak{B} = \tfrac{1}{6};\ \mathfrak{C} = \tfrac{1}{6};\ \mathfrak{D} = \tfrac{3}{10};\ \mathfrak{E} = \tfrac{5}{6};$$

$$\mathfrak{F} = \tfrac{691}{210};\ \mathfrak{G} = \tfrac{35}{2};\ \mathfrak{H} = \tfrac{3617}{30};\ \mathfrak{I} = \tfrac{43867}{42};$$

$$\mathfrak{K} = \frac{1222277}{110};\ \mathfrak{L} = \frac{854513}{6}$$

$$\mathfrak{M} = \frac{1181820455}{546};\ \mathfrak{N} = \frac{76977927}{2}$$

$$\mathfrak{O} = \frac{23749461029}{30};\ \mathfrak{P} = \frac{8615841276005}{462}$$

$$\mathfrak{Q} = \frac{84802531453387}{170};\ \mathfrak{R} = \frac{90219075042845}{6}$$

etc.

 Quos

Quos numeros ita progredi obſeruaui vt ſit

$$\mathfrak{A} = \tfrac{1}{2}$$

$$\mathfrak{B} = \tfrac{4}{2} \cdot \tfrac{\mathfrak{A}^2}{3}$$

$$\mathfrak{C} = \tfrac{6}{2} \cdot \tfrac{2\mathfrak{A}\mathfrak{B}}{3}$$

$$\mathfrak{D} = \tfrac{8}{2} \cdot \tfrac{2\mathfrak{A}\mathfrak{C}}{3} + \tfrac{8 \cdot 7 \cdot 6}{2 \cdot 3 \cdot 4} \cdot \tfrac{\mathfrak{B}^2}{5}$$

$$\mathfrak{E} = \tfrac{10}{2} \cdot \tfrac{2\mathfrak{A}\mathfrak{D}}{3} + \tfrac{10 \cdot 9 \cdot 8}{2 \cdot 3 \cdot 4} \cdot \tfrac{2\mathfrak{B}\mathfrak{C}}{5}$$

$$\mathfrak{F} = \tfrac{12}{2} \cdot \tfrac{2\mathfrak{A}\mathfrak{E}}{3} + \tfrac{12 \cdot 11 \cdot 10}{2 \cdot 3 \cdot 4} \cdot \tfrac{2\mathfrak{B}\mathfrak{D}}{5} + \tfrac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot \tfrac{\mathfrak{C}\mathfrak{C}}{7}$$

$$\mathfrak{G} = \tfrac{14}{2} \cdot \tfrac{2\mathfrak{A}\mathfrak{F}}{3} + \tfrac{14 \cdot 13 \cdot 12}{2 \cdot 3 \cdot 4} \cdot \tfrac{2\mathfrak{B}\mathfrak{E}}{5} + \tfrac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot \tfrac{2\mathfrak{C}\mathfrak{D}}{7}$$

etc.

Hinc iam quaerantur numeri P, Q, R, S etc. vt ſit

$$P = \tfrac{1\mathfrak{A}m}{1 \cdot 2 \cdot 3}$$

$$Q = \tfrac{3\mathfrak{A}m}{1 \cdot 2 \cdot 3} P - \tfrac{3 \cdot 2 \cdot 1 \mathfrak{B}m}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

$$R = \tfrac{5\mathfrak{A}m}{1 \cdot 2 \cdot 3} Q - \tfrac{5 \cdot 4 \cdot 3 \cdot \mathfrak{B}m}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} P + \tfrac{5 \cdot 4 \cdots 1 \mathfrak{C}m}{1 \cdot 2 \cdots 7}$$

$$S = \tfrac{7\mathfrak{A}m}{1 \cdot 2 \cdot 3} R - \tfrac{7 \cdot 6 \cdot 5 \mathfrak{B}m}{1 \cdot 2 \cdots 5} Q + \tfrac{7 \cdots 3 \mathfrak{C}m}{1 \cdot 2 \cdots 7} P - \tfrac{7 \cdots 1 \mathfrak{D}m}{1 \cdot 2 \cdots 9}$$

vbi lex progreſſionis etiam eſt perſpicua.

Hac ſerie inuenta ſumma quaeſita s ita exprimetur:

$$\frac{s}{(\lambda+1)(\lambda+2)\ldots(\lambda+m)} = \left(x-\tfrac{m}{2}\right)^\lambda + \tfrac{\lambda(\lambda-1)}{1 \cdot 2} P \left(x-\tfrac{m}{2}\right)^{\lambda-2}$$

$$+ \tfrac{\lambda \ldots (\lambda-3)}{1 \cdots 4} Q \left(x-\tfrac{m}{2}\right)^{\lambda-4}$$

$$+ \tfrac{\lambda \ldots (\lambda-5)}{1 \cdots 6} R \left(x-\tfrac{m}{2}\right)^{\lambda-6}$$

$$+ \tfrac{\lambda \ldots (\lambda-7)}{1 \cdots 8} S \left(x-\tfrac{m}{2}\right)^{\lambda-8}$$

etc.

vbi

vbi notetur si forte numerus m non sit integer valorem producti $(\lambda+1)(\lambda+2)\ldots(\lambda+m)$ per artificia alibi exposita definiri posse.

Corollarium I.

Si loco numerorum *Bernoullianorum* numeros iis cognatos, quibus ad potestatum reciprocarum summas sum vsus, introducere velimus, hosque numeros litteris A, B, C, D etc. designemus, vt sit $A = \frac{1}{6}$, $B = \frac{1}{90}$, $C = \frac{1}{945}$, $D = \frac{1}{9450}$; $E = \frac{1}{93555}$ quoniam hi numeri a prioribus ita pendent vt sit

$$\mathfrak{A} = \frac{1.2.3}{2^1}A;\ \mathfrak{B} = \frac{1\ldots\ldots5}{2^3}B;\ \mathfrak{C} = \frac{1\ldots\ldots7}{2^5}C;\ \text{etc.}$$

inter se autem ita connectuntur vt sit:

$$5B = 2A^2;\ 7C = 4AB;\ 9D = 4AC + 2BB;$$

$$11E = 4AD + 4BC;\ 13F = 4AE + 4BD + 2CC\ \text{etc.}$$

Tum ex his numeris litterae P, Q, R, S etc. ita determinabuntur:

$$P = \frac{1Am}{2}$$

$$Q = \frac{3Am}{2}P - \frac{3.2.1\,Bm}{2^3}$$

$$R = \frac{5Am}{2}Q - \frac{5.4.3\,Bm}{2^3}P + \frac{5.4\ldots1\,Cm}{2^5}$$

$$S = \frac{7Am}{2}R - \frac{7.6.5\,Bm}{2^3}Q + \frac{7.6\ldots3\,Dm}{2^5}P - \frac{7.6\ldots1\,Em}{2^7}$$

etc.

Corollarium 2.

Si pro variis valoribus numeri λ summam progressionis proposito hoc signandi modo $\int(\lambda)$ indicemus, ac iam loco λ successiue scribamus numeros 0, 1, 2, 3, 4 etc. pro his casibus summae $\int(0)$, $\int(1)$, $\int(2)$, $\int(3)$ etc. sequenti modo exprimentur ponendo breuitatis gratia $x - \frac{m}{2} = y$:

$$\frac{\int(0)}{1.2 \ldots m} = 1$$

$$\frac{\int(1)}{2.3 \ldots (m+1)} = y$$

$$\frac{\int(2)}{3.4 \ldots (m+2)} = y^2 + P$$

$$\frac{\int(3)}{4.5 \ldots (m+3)} = y^3 + 3Py$$

$$\frac{\int(4)}{5.6 \ldots (m+4)} = y^4 + 6Py^2 + Q$$

$$\frac{\int(5)}{6.7 \ldots (m+5)} = y^5 + 10Py^3 + 5Qy$$

$$\frac{\int(6)}{7.8 \ldots (m+6)} = y^6 + 15Py^4 + 15Qy^2 + R$$

etc.

Corollarium 3.

Hinc ergo istae summae sequenti modo singulae ex antecedentibus definiri possunt

$$\int(1) = \frac{m+1}{1} y \int(0)$$

$$\int(2) = \frac{m+2}{2} y \int(1) + \frac{(m+2)(m+1)}{2.2} mA \int(0)$$

$$\int(3) = \frac{m+3}{3} y \int(2) + \frac{(m+3)(m+2)}{2.3} mA \int(1)$$

$\int(4)$

$$\int(4)=\frac{m+4}{4}y\int(3)+\frac{(m+4)(m+3)}{2.\ 4}mA\int(2)-\frac{(m+4)\ldots(m+1)}{2^3.\ 4}mB\int(0)$$

$$\int(5)=\frac{m+5}{5}y\int(4)+\frac{(m+5)(m+4)}{2.\ 5}mA\int(3)-\frac{(m+5)\ldots(m+2)}{2^3.\ 5}mB\int(1)$$

$$\int(6)=\frac{m+6}{6}y\int(5)+\frac{(m+6)(m+5)}{2.\ 6}mA\int(4)-\frac{(m+6)\ldots(m+3)}{2^3.\ 6}mB\int(2)+\frac{(m+6)\ldots(m+1)}{2^5.\ 6}mC\int(0)$$

$$\int(7)=\frac{m+7}{7}y\int(6)+\frac{(m+7)(m+6)}{2.\ 7}mA\int(5)-\frac{(m+7)\ldots(m+4)}{2^3.\ 7}mB\int(3)+\frac{(m+7)\ldots(m+2)}{2^5.\ 7}mC\int(1)$$

$$\int(8)=\frac{m+8}{8}y\int(7)+\frac{(m+8)(m+7)}{2.\ 8}mA\int(6)-\frac{(m+8)\ldots(m+5)}{2^3.\ 8}mB\int(4)+\frac{(m+8)\ldots(m+2)}{2^5.\ 8}mC\int(2)$$

$$-\frac{(m+8)\ldots(m+1)}{2^7.\ 8}mD\int(0)$$

quae lex progreſſionis inſpicienti mox fit manifeſta.

Concluſio.

Nunc haud multo difficilius erit hoc negotium longe generalius expedire, ita vt, ſi $\phi:x$ denotet functionem quamcunque ipſius x ſummam huius ſeriei

$$s=\phi:x-m\phi.(x-1)+\frac{m(m-1)}{1.\ 2}\phi:(x-2)-\frac{m(m-1)(m-2)}{1.\ 2.\ 3}\phi:(x-3)$$

aſſignare queamus. Perſpicuum enim eſt hanc formam, differentiam ordinis m exhibere iſtius progreſſionis

$$\phi:x;\ \phi:(x-1);\ \phi:(x-2);\ \phi:(x-3)\ \text{etc.}$$

Ex iis enim quae in Inſtitutionibus Calculi Differentialis pag. 343. in medium attuli, ſi ponamus $\phi:x=y$, colligitur differentias ſingulorum ordinum eſſe:

Δy

$$\Delta y = \frac{dy}{dx} - \frac{ddy}{2dx^2} + \frac{d^3y}{2.3.dx^3} - \frac{d^4y}{2.3.4dx^4} + \frac{d^5y}{2\dots5dx^5} - \text{etc.}$$

$$\Delta^2 y = \frac{d^2y}{dx^2} - \frac{3d^3y}{3dx^3} + \frac{7d^4y}{3.4dx^4} - \frac{15d^5y}{3.4.5dx^5} + \frac{31d^6y}{2\dots6dx^6} - \text{etc.}$$

$$\Delta^3 y = \frac{d^3y}{dx^3} - \frac{6d^4y}{4dx^4} + \frac{25d^5y}{4.5.dx^5} - \frac{90d^6y}{4.5.6dx^6} + \frac{301d^7y}{4\dots7dx^7} - \text{etc.}$$

$$\Delta^4 y = \frac{d^4y}{dx^4} - \frac{10d^5y}{5dx^5} + \frac{65d^6y}{5.6dx^6} - \frac{350d^7y}{5.6.7dx^7} + \frac{1701d^8y}{5\dots6dx^8} - \text{etc.}$$

etc.

qui coefficientes cum ſint illi ipſi, quos ſupra §. IV habuimus, eodem modo intelligemus differentiam ordinis m ſeu $\Delta^m y$, hoc eſt ipſam ſummam ſeriei propoſitae fore

$$s = \frac{d^m y}{dx^m} - \frac{A^1 d^{m+1} y}{(m+1)dx^{m+1}} + \frac{B^1 d^{m+2} y}{(m+1)(m+2)dx^{m+2}}$$
$$- \frac{C^1 d^{m+3} y}{(m+1)\dots(m+3)dx^{m+3}} + \text{etc.}$$

quos coefficientes A^1, B^1, C^1 etc. ſupra §. 13. determinaui. Quocirca erit

$$\frac{A^1}{m+1} = \frac{m}{2}$$

$$\frac{B^1}{(m+1)(m+2)} = \frac{m}{1.2.3} + \frac{3m(m-1)}{1.2.3.4}$$

$$\frac{C^1}{(m+1)\dots(m+3)} = \frac{m}{1.2.3.4} + \frac{10m(m-1)}{1.2\dots5} + \frac{15m(m-1)(m-2)}{1.2\dots6}$$

$$\frac{D^1}{(m+1)\dots(m+4)} = \frac{m}{1.2\dots5} + \frac{25m(m-1)}{1.2\dots6} + \frac{105m(m-1)(m-2)}{1.2\dots7}$$
$$+ \frac{105m(m-1)(m-2)(m-3)}{1.2\dots8}$$

etc.

Quodſi iam nunc ponamus $\phi:(x-\frac{m}{2}) = v$, ita vt v oriatur ex y, ſi loco x ſcribatur $x-\frac{m}{2}$, erit vtique

$$\frac{d^m v}{dx^m} = \frac{d^m y}{dx^m} - \frac{m d^{m+1} y}{2dx^{m+1}} + \frac{m^2 d^{m+2} y}{2.4dx^{m+2}} - \text{etc.}$$

quae

quae aequatio si inde subtrahatur, calculus idem prorsus erit instituendus, quem supra expediuimus. Vnde introducendo easdem litteras P, Q, R, S etc. quas supra definiuimus, obtinebimus sequentem summae s valorem:

$$s = \frac{d^m v}{dx^m} + \frac{P d^{m+2} v}{1.2\, dx^{m+2}} + \frac{Q d^{m+4} v}{1.2.4\, dx^{m+4}} + \frac{R d^{m+6} v}{1.2\ldots 6\, dx^{m+6}} + \frac{S d^{m+8} v}{1.2\ldots 8\, dx^{m+8}} + \text{etc.}$$

atque hinc si sumatur $y = \Phi : x = x^{m+\lambda}$ et $v = (x - \frac{m}{2})^{m+\lambda}$ manifesto eadem summatio sequitur quam ante eruimus, sicque totum negotium redit ad litteras P, Q, R, S etc. quarum indolem ex numeris *Bernoullianis* supra deriuaui.

Hinc statim liquet, quod ante minus apparebat, si in functione y vel v numerus dimensionum minor fuerit quam exponens m, quem quidem numerum integrum positiuum esse oportet, tum omnia differentialia ordinis m et superiorum in nihilum abire, foreque summam $s = 0$.

Deinde hinc etiam planior patet via ad valores litterarum P, Q, R, S etc. inueniendos. Cum enim posito

$$s = \frac{d^m y}{dx^m} - \frac{\alpha d^{m+1} y}{dx^{m+1}} + \frac{\beta d^{m+2} y}{dx^{m+2}} - \frac{\gamma d^{m+3} y}{dx^{m+3}} + \text{etc.}$$

ſit $\alpha = \frac{m}{1.2}$

$$\beta = \frac{m}{1.2.3} + \frac{3m(m-1)}{1.2.3.4}$$

$$\gamma = \frac{m}{1 \ldots 4} + \frac{10m(m-1)}{1 \ldots 5} + \frac{15m \ldots (m-2)}{1 \ldots 6}$$

$$\delta = \frac{m}{1 \ldots 5} + \frac{25m(m-1)}{1 \ldots 6} + \frac{105m \ldots (m-2)}{1 \ldots 7} + \frac{105m \ldots (m-3)}{1 \ldots 8}$$

etc.

functio autem y ex functione $v = \phi : (x - \frac{m}{2})$ naſcatur ſi in hac loco x ſcribatur $x + \frac{m}{2}$ erit in genere

$$\frac{d^n y}{dx^n} = \frac{d^n v}{dx^n} + \frac{m}{2} \cdot \frac{d^{n+1} v}{dx^{n+1}} + \frac{m^2}{2.4} \cdot \frac{d^{n+2} v}{dx^{n+2}} + \frac{m^3}{2.4.6} \cdot \frac{d^{n+3} v}{dx^{n+3}} + \text{etc.}$$

vnde ſi loco differentialium ipſius y, haec differentialia ipſius v ſubſtituantur, fiet

$$s = \frac{d^n v}{dx^n} + \left(\frac{m}{2} - \alpha\right) \frac{d^{n+1} v}{dx^{n+1}} + \left(\frac{m^2}{2.4} - \frac{m}{2}\alpha + \beta\right) \frac{d^{n+2} v}{dx^{n+2}}$$

$$+ \left(\frac{m^3}{2.4.6} - \frac{m^2}{2.4}\alpha + \frac{m}{2}\beta - \gamma\right) \frac{d^{n+3} v}{dx^{n+3}} + \text{etc.}$$

ſicque habebimus:

$$\frac{m}{2} - \alpha = 0$$

$$\frac{m^2}{2.4} - \frac{m}{2}\alpha + \beta = \frac{P}{1.2}$$

$$\frac{m^3}{2.4.6} - \frac{m^2}{2.4}\alpha + \frac{m}{2}\beta - \gamma = 0$$

$$\frac{m^4}{2.4.6.8} - \frac{m^3}{2.4.6}\alpha + \frac{m^2}{2.4}\beta - \frac{m}{2}\gamma + \delta = \frac{Q}{1.2.3.4}$$

$$\frac{m^5}{2.4.6.8.10} - \frac{m^4}{2.4.6.8}\alpha + \frac{m^3}{2.4.6}\beta - \frac{m^2}{2.4}\gamma + \frac{m}{2}\delta - \varepsilon = 0$$

etc.

facile enim perſpicitur has expreſſiones alternatim euaneſcere debere.

QVO-