that which we explained first; for it applies successfully to to all kinds of equations: whereas the other often requires the equation to be prepared in a certain manner, without which it cannot be employed; and of this we have seen a proof in different examples.

QUESTIONS FOR PRACTICE.

1. Given $x^3 + 2x^2 - 23x - 70 = 0$, to find x. *Ans.* $x = 5{\cdot}13450$.
2. Given $x^3 - 15x^2 + 63x - 50 = 0$, to find x. *Ans.* $x = 1{\cdot}028039$.
3. Given $x^4 - 3x^2 - 75x = 10000$, to find x. *Ans.* $x = 10{\cdot}2615$.
4. Given $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 54321$, to find x. *Ans.* $x = 8{\cdot}4144$.
5. Let $120x^3 + 3657x^2 - 38059x = 8007115$, to find x. *Ans.* $x = 34{\cdot}6532$.

END OF PART I.

ELEMENTS OF ALGEBRA.

PART II.

Containing the Analysis *of* Indeterminate Quantities.

CHAP. I.

Of the Resolution *of* Equations *of the* First Degree, *which contain more than one unknown* Quantity.

ARTICLE I.

IT has been shewn, in the First Part, how one unknown quantity is determined by a single equation, and how we may determine two unknown quantities by means of two equations, three unknown quantities by three equations, and so on; so that there must always be as many equations as there are unknown quantities to determine, at least when the question itself is determinate.

When a question, therefore, does not furnish as many equations as there are unknown quantities to be determined, some of these must remain undetermined, and depend on our will; for which reason, such questions are said to be *indeterminate*; forming the subject of a particular branch of algebra, which is called *Indeterminate Analysis*.

2. As in those cases we may assume any numbers for one, or more unknown quantities, they also admit of several solutions: but, on the other hand, as there is usually annexed the condition, that the numbers sought are to be integer and positive, or at least rational, the number of all the possible solutions of those questions is greatly limited: so that often there are very few of them possible; at other

times, there may be an infinite number, but such as are not readily obtained; and sometimes, also, none of them are possible. Hence it happens, that this part of analysis frequently requires artifices entirely appropriate to it, which are of great service in exercising the judgment of beginners, and giving them dexterity in calculation.

3. To begin with one of the easiest questions. Let it be required to find two positive, integer numbers, the sum of which shall be equal to 10.

Let us represent those members by x and y; then we have $x + y = 10$; and $x = 10 - y$, where y is so far only determined, that this letter must represent an integer and positive number. We may therefore substitute for it all integer numbers from 1 to infinity: but since x must likewise be a positive number, it follows, that y cannot be taken greater than 10, for otherwise x would become negative; and if we also reject the value of $x = 0$, we cannot make y greater than 9; so that only the following solutions can take place:

If $y = 1, 2, 3, 4, 5, 6, 7, 8, 9$,
then $x = 9, 8, 7, 6, 5, 4, 3, 2, 1$.

But, the last four of these nine solutions being the same as the first four, it is evident, that the question really admits only of five different solutions.

If three numbers were required, the sum of which might make 10, we should have only to divide one of the numbers already found into two parts, by which means we should obtain a greater number of solutions.

4. As we have found no difficulty in this question, we will proceed to others, which require different considerations.

Question 1. Let it be required to divide 25 into two parts, the one of which may be divisible by 2, and the other by 3.

Let one of the parts sought be $2x$, and the other $3y$; we shall then have $2x + 3y = 25$; consequently $2x = 25 - 3y$; and dividing by 2, we obtain

$x = \frac{25-3y}{2}$; whence we conclude, in the first place, that $3y$ must be less than 25, and, consequently, y is less than 8. Also, if, from this value of x, we take out as many integers as we possibly can, that is to say, if we divide by the denominator 2, we shall have $x = 12 - y + \frac{1-y}{2}$; whence it follows, that $1 - y$, or rather $y - 1$, must be divisible by

2. Let us, therefore, make $y - 1 = 2z$; and we shall have $y = 2z + 1$, so that

$x = 12 - 2z - 1 - z = 11 - 3z$.

And, since y cannot be greater than 8, we must not substitute any numbers for z which would render $2z + 1$ greater than 8; consequently, z must be less than 4, that is to say, z cannot be taken greater than 3, for which reasons we have the following answers:

If we make	$z = 0$	$z = 1$	$z = 2$	$z = 3$,
we have	$y = 1$	$y = 3$	$y = 5$	$y = 7$,
and	$x = 11$	$x = 8$	$x = 5$	$x = 2$.

Hence, the two parts of 25 sought, are

$22 + 3$, $16 + 9$, $10 + 15$, or $4 + 21$.

5. *Question* 2. To divide 100 into two such parts, that the one may be divisible by 7, and the other by 11.

Let $7x$ be the first part, and $11y$ the second. Then we must have $7x + 11y = 100$; and, consequently,

$$x = \frac{100-11y}{7} = \frac{98+2-7y-4y}{7}, \text{ or}$$

$$x = 14 - y + \frac{2-4y}{7};$$

wherefore $2 - 4y$, or $4y - 2$, must be divisible by 7.

Now, if we can divide $4y - 2$ by 7, we may also divide its half, $2y - 1$, by 7. Let us therefore make $2y - 1 = 7z$, or $2y = 7z + 1$, and we shall have $x = 14 - y - 2z$; but, since $2y = 7z + 1 = 6z + z + 1$, we shall have

$y = 3z + \frac{z+1}{2}$. Let us therefore make $z + 1 = 2u$, or $z = 2u - 1$; which supposition gives $y = 3z + u$; and, consequently, we may substitute for u every integer number that does not make x or y negative. Now, as y becomes $= 7u - 3$, and $x = 19 - 11u$, the first of these expressions shews that $7u$ must exceed 3; and according to the second, $11u$ must be less than 19, or u less than $\frac{19}{11}$: so that u cannot be 2; and since it is impossible for this number to be 0, we must have $u = 1$: which is the only value that this letter can have. Hence, we obtain $x = 8$, and $y = 4$; and the two parts of 100 which were required, are 56, and 44.

6. *Question* 3. To divide 100 into two such parts, that dividing the first by 5, there may remain 2; and dividing the second by 7, the remainder may be 4.

Since the first part, divided by 5, leaves the remainder 2, let us suppose it to be $5x + 2$; and, for a similar reason, we may represent the second part by $7y + 4$: we shall thus have

$$5x + 7y + 6 = 100, \text{ or } 5x = 94 - 7y = 90 + 4 - 5y - 2y;$$

whence we obtain $x = 18 - y + \frac{4-2y}{5}$. Hence it follows, that $4 - 2y$, or $2y - 4$, or the half $y - 2$, must be divisible by 5. For this reason, let us make $y - 2 = 5z$, or $y = 5z + 2$, and we shall have $x = 16 - 7z$; whence we conclude, that $7z$ must be less than 16, and z less than $\frac{16}{7}$, that is to say, z cannot exceed 2. The question proposed, therefore, admits of three answers:

1. $z = 0$ gives $x = 16$, and $y = 2$; whence the two parts are $82 + 18$.

2. $z = 1$ gives $x = 9$, and $y = 7$; and the two parts are $47 + 53$.

3. $z = 2$ gives $x = 2$, and $y = 12$; and the two parts are $12 + 88$.

7. *Question* 4. Two women have together 100 eggs: one says to the other; 'When I count my eggs by eights, there is an overplus of 7.' The second replies: 'If I count mine by tens, I find the same overplus of 7.' How many eggs had each?

As the number of eggs belonging to the first woman, divided by 8, leaves the remainder 7; and the number of eggs belonging to the second, divided by 10, gives the same remainder 7; we may express the first number by $8x + 7$, and the second by $10y + 7$; so that $8x + 10y + 14 = 100$, or $8x = 86 - 10y$, or $4x = 43 - 5y = 40 + 3 - 4y - y$. Consequently, if we make $y - 3 = 4z$, so that $y = 4z + 3$, we shall have

$$x = 10 - 4z - 3 - z = 7 - 5z;$$

whence it follows, that $5z$ must be less than 7, or z less than 2; that is to say, we shall only have the two following answers:

1. $z = 0$ gives $x = 7$, and $y = 3$; so that the first woman had 63 eggs, and the second 37.

2. $z = 1$ gives $x = 2$, and $y = 7$; therefore the first woman had 23 eggs, and the second had 77.

8. *Question* 5. A company of men and women spent 1000 sous at a tavern. The men paid each 19 sous, and each woman 13. How many men and women were there?

Let the number of men be x, and that of the women y, we shall then have the equation

$$19x + 13y = 1000, \text{ or}$$

$13y = 1000 - 19x = 988 + 12 - 13x - 6x$, and

$$y = 76 - x + \frac{12-6x}{13};$$

whence it follows, that $12 - 6x$, or $6x - 12$, or $x - 2$, the sixth part of that number must be divisible by 13. If, therefore, we make $x - 2 = 13z$, we shall have $x = 13z + 2$, and $y = 76 - 13z - 2 - 6z$, or $y = 74 - 19z$; which shews that z must be less than $\frac{74}{19}$, and, consequently, less than 4; so that the four following answers are possible:

1. $z = 0$ gives $x = 2$, and $y = 74$: in which case there were 2 men and 74 women; the former paid 38 sous, and the latter 962 sous.

2. $z = 1$ gives the number of men $x = 15$, and that of women $y = 55$; so that the former spent 285 sous, and the latter 715 sous.

3. $z = 2$ gives the number of men $x = 28$, and that of the women $y = 36$; therefore the former spent 532 sous, and the latter 468 sous.

4. $z = 3$ gives $x = 41$, and $y = 17$; so that the men spent 779 sous, and the women 221 sous.

9. *Question* 6. A farmer lays out the sum of 1770 crowns in purchasing horses and oxen; he pays 31 crowns for each horse, and 21 crowns for each ox. How many horses and oxen did he buy?

Let the number of horses be x, and that of oxen y; we shall then have $31x + 21y = 1770$, or $21y = 1770 - 31x = 1764 + 6 - 21x - 10x$; that is to say,

$y = 84 - x + \frac{6-10x}{21}$. Therefore $10x - 6$, and likewise its half $5x - 3$, must be divisible by 21. If we now suppose $5x - 3 = 21z$, we shall have $5x = 21z + 3$, and hence $y = 84 - x - 2z$. But, since

$x = \frac{21z+3}{5} = 4z + \frac{z+3}{5}$, we must also make $z + 3 = 5u$;

which supposition gives

$$z = 5u - 3,\ x = 21u - 12, \text{ and}$$

$$y = 84 - 21u + 12 - 10u + 6 = 102 - 31u;$$

hence it follows, that u must be greater than 0, and yet less than 4, which furnishes the following answers:

1. $u = 1$ gives the number of horses $x = 9$, and that of oxen $y = 71$; wherefore the former cost 279 crowns, and the latter 1491; in all 1770 crowns.

2. $u = 2$ gives $x = 30$, and $y = 40$; so that the horses cost 930 crowns, and the oxen 840 crowns, which together make 1770 crowns.

3. $u = 3$ gives the number of the horses $x = 51$, and that of the oxen $y = 9$; the former cost 1581 crowns, and the latter 189 crowns; which together make 1770 crowns.

10. The questions which we have hitherto considered lead all to an equation of the form $ax + by = c$, in which a, b, and c, represent integer and positive numbers, and in which the values of x and y must likewise be integer and positive. Now, if b is negative, and the equation has the form $ax - by = c$, we have questions of quite a different kind, admitting of an infinite number of answers, which we shall treat of before we conclude the present chapter.

The simplest questions of this sort are such as the following. Required two numbers, whose difference may be 6. If, in this case, we make the less number x, and the greater y, we must have $y - x = 6$, and $y = 6 + x$. Now, nothing prevents us from substituting, instead of x, all the integer numbers possible, and whatever number we assume, y will always be greater by 6. Let us, for example, make $x = 100$, we have $y = 106$; it is evident, therefore, that an infinite number of answers are possible.

11. Next follow questions, in which $c = 0$, that is to say, in which ax must simply be equal to by. Let there be required, for example, a number divisible both by 5 and by 7. If we write N for that number, we shall first have $N = 5x$, since N must be divisible by 5; farther, we shall have $N = 7y$, because the number must also be divisible by 7; we shall therefore have $5x = 7y$, and $x = \frac{7y}{5}$. Now, since 7 cannot be divided by 5, y must be divisible by 5: let us therefore make $y = 5z$, and we have $x = 7z$; so that the number sought $N = 35z$; and as we may take for z, any integer number whatever, it is evident that we can assign for N an infinite number of values; such as

35, 70, 105, 140, 175, 210, &c.

If, beside the above condition, it were also required that the number N be divisible by 9, we should first have $N = 35z$, as before, and should farther make $N = 9u$. In this manner, $35z = 9u$, and $u = \frac{35z}{9}$; where it is evident that z must be divisible by 9; therefore let $z = 9s$; and we shall then have $u = 35s$, and the number sought $N = 315s$.

12. We find more difficulty, when c is not $= 0$. For example, when $5x = 7y + 3$, the equation to which we are led, and which requires us to seek a number N such, that it may be divisible by 5, and if divided by 7, may leave the remainder 3: for we must then have $N = 5x$, and also $N = 7y + 3$; whence results the equation $5x = 7y + 3$; and, consequently,

$$x = \frac{7y+3}{5} = \frac{5y+2y+3}{5} = y + \frac{2y+3}{5}.$$

If we make $2y + 3 = 5z$, or $z = \frac{2y+3}{5}$, we have $x = y + z$; now, because $2y + 3 = 5z$, or $2y = 5z - 3$, we have

$$y = \frac{5z-3}{2}, \text{ or } y = 2z + \frac{z-3}{2}.$$

If, therefore, we farther suppose $z - 3 = 2u$, we have $z = 2u + 3$, and $y = 5u + 6$, and

$$x = y + z = (5u + 6) + (2u + 3) = 7u + 9.$$

Hence, the number sought $N = 35u + 45$, in which equation we may substitute for u not only all positive integer numbers, but also negative numbers; for, as it is sufficient that N be positive, we may make $u = -1$, which gives $N = 10$; the other values are obtained by continually adding 35; that is to say, the numbers sought are 10, 45, 80, 115, 150, 185, 220, &c.

13. The solution of questions of this sort depends on the relation of the two numbers by which we are to divide; that is, they become more or less tedious, according to the nature of those divisors. The following question, for example, admits of a very short solution:

Required a number which, divided by 6, leaves the remainder 2; and divided by 13, leaves the remainder 3.

Let this number be N; first $N = 6x + 2$, and then $N = 13y + 3$; consequently, $6x + 2 = 13y + 3$, and $6x = 13y + 1$; hence,

$$x = \frac{13y+1}{6} = 2y + \frac{y+1}{6},$$

and if we make $y + 1 = 6z$, we obtain $y = 6z - 1$, and $x = 2y + z = 13z - 2$; whence we have for the number sought $N = 78z - 10$; therefore, the question admits of the following values of N; viz.

$$N = 68, 146, 224, 302, 380, \&c.$$

which numbers form an arithmetical progression, whose difference is $78 = 6 \times 13$. So that if we know one of the values, we may easily find all the rest; for we have only to add 78 continually, or to subtract that number, as long as it is possible, when we seek for smaller numbers.

14. The following question furnishes an example of a longer and more tedious solution.

Question 8. To find a number N, which, when divided by 39, leaves the remainder 16; and such also, that if it be divided by 56, the remainder may be 27.

In the first place, we have $N = 39p + 16$; and in the second, $N = 56q + 27$; so that

$$39p + 16 = 56q + 27, \text{ or } 39p = 56q + 11, \text{ and}$$

$$p = \frac{56q+11}{39} = q + \frac{17q+11}{39} = q + r, \text{ by making}$$

$$r = \frac{17q+11}{39}. \text{ So that } 39r = 17q + 11, \text{ and}$$

$$q = \frac{39r-11}{17} = 2r + \frac{5r-11}{17} = 2r + s, \text{ by making}$$

$$s = \frac{5r-11}{17}, \text{ or } 17s = 5r - 11; \text{ whence we get}$$

$$r = \frac{17s+11}{5} = 3s + \frac{2s+11}{5} = 3s + t, \text{ by making}$$

$$t = \frac{2s+11}{5}, \text{ or } 5t = 2s + 11; \text{ whence we find}$$

$$s = \frac{5t-11}{2} = 2t + \frac{t-11}{2} = 2t + u, \text{ by making}$$

$$u = \frac{t-11}{2}; \text{ whence } t = 2u + 11.$$

Having now no longer any fractions, we may take u at pleasure, and then we have only to trace back the following values:

$$\begin{aligned} t &= 2u + 11, \\ s = 2t + u &= 5u + 22, \\ r = 3s + t &= 17u + 77, \\ q = 2r + s &= 39u + 176, \\ p = q + r &= 56u + 253, \end{aligned}$$

and, lastly, $N = 39 \times 56u + 9883$*. And the least possible value of N is found by making $u = -4$; for, by this supposition, we have $N = 1147$: and if we make $u = x - 4$, we find

$N = 2184x - 8736 + 9883$; or $N = 2184x + 1147$; which numbers form an arithmetical progression, whose first term is 1147, and whose common difference is 2184; the following being some of its leading terms:

$$1147, 3331, 5515, 7699, 9883, \&c.$$

15. We shall subjoin some other questions by way of practice.

Question 9. A company of men and women club together for the payment of a reckoning: each man pays 25 livres, and each woman 16 livres; and it is found that all the women together have paid 1 livre more than the men. How many men and women were there?

Let the number of women be p, and that of men q; then the women will have expended $16p$, and the men $25q$; so that $16p = 25q + 1$, and

$$p = \frac{25q+1}{16} = q + \frac{9q+1}{16} = q + r, \text{ or } 16r = 9q + 1,$$

$$q = \frac{16r-1}{9} = r + \frac{7r-1}{9} = r + s, \text{ or } 9s = 7r - 1,$$

$$r = \frac{9s+1}{7} = s + \frac{2s+1}{7} = s + t, \text{ or } 7t = 2s + 1,$$

$$s = \frac{7t-1}{2} = 3s + \frac{t-1}{2} = 3s + u, \text{ or } 2u = t - 1.$$

We shall therefore obtain, by tracing back our substitutions,

$$\begin{aligned} t &= 2u + 1, \\ s = 3t + u &= 7u + 3, \\ r = s + t &= 9u + 4, \\ q = r + s &= 16u + 7, \\ p = q + r &= 25u + 11. \end{aligned}$$

So that the number of women was $25u + 11$, and that of men was $16u + 7$; and in these formulæ we may substitute

* As the numbers 176 and 253 ought, respectively, to be divisible by 39 and 56; and as the former ought, by the question, to leave the remainder 16, and the latter 27, the sum 9883 is formed by multiplying 176 by 56, and adding the remainder 27 to the product: or by multiplying 253 by 39, and adding the remainder 16 to the product. Thus,

$(176 \times 56) + 27 = 9883$; and $(253 \times 39) + 16 = 9883$.

for u any integer numbers whatever. The least results, therefore, will be as follow:

Number of women,	11, 36, 61, 86, 111, &c.
——— of men,	7, 23, 39, 55, 71, &c.

According to the first answer, or that which contains the least numbers, the women expended 176 livres, and the men 175 livres; that is, one livre less than the women.

16. *Question* 10. A person buys some horses and oxen: he pays 31 crowns per horse, and 20 crowns for each ox; and he finds that the oxen cost him 7 crowns more than the horses. How many oxen and horses did he buy?

If we suppose p to be the number of the oxen, and q the number of the horses, we shall have the following equation:

$$p = \frac{31q+7}{20} = q + \frac{11q+7}{20} = q + r, \text{ or } 20r = 11q+7,$$

$$q = \frac{20r-7}{11} = r + \frac{9r-7}{11} = r + s, \text{ or } 11s = 9r-7,$$

$$r = \frac{11s+7}{9} = s + \frac{2s+7}{9} = s + t, \text{ or } 9t = 2s+7,$$

$$s = \frac{9t-7}{2} = 4t + \frac{t-7}{2} = 4t + u, \text{ or } 2u = t-7,$$

whence $t \ldots\ldots = 2u + 7$, and, consequently,

$s = 4t + u = 9u + 28$,
$r = s + t = 11u + 35$,
$q = r + s = 20u + 63$, number of horses,
$p = q + r = 31u + 98$, number of oxen.

Whence, the least positive values of p and q are found by making $u = -3$; those which are greater succeed in the following arithmetical progressions:

Number of oxen, $p = 5, 36, 67, 98, 129, 160, 191, 222, 253$, &c.

Number of horses, $q = 3, 23, 43, 63, 83, 103, 123, 143, 163$, &c.

17. If now we consider how the letters p and q, in this example, are determined by the succeeding letters, we shall perceive that this determination depends on the ratio of the numbers 31 and 20, and particularly on the ratio which we discover by seeking the greatest common divisor of these two numbers. In fact, if we perform this operation,

$$\begin{array}{l}
20)\ 31\ (1 \\
\quad\ \ 20 \\
\hline
11)\ 20\ (1 \\
\quad\ \ 11 \\
\hline
9)\ 11\ (1 \\
\quad\ \ 9 \\
\hline
2)\ 9\ (4 \\
\quad\ 8 \\
\hline
1)\ 2\ (2 \\
\quad\ 2 \\
\hline
\quad 0,
\end{array}$$

it is evident that the quotients are found also in the successive values of the letters p, q, r, s, &c. and that they are connected with the first letter to the right, while the last always remains alone. We see, farther, that the number 7 occurs only in the fifth and last equation, and is affected by the sign +, because the number of this equation is odd; for if that number had been even, we should have obtained -7. This will be made more evident by the following Table, in which we may observe the decomposition of the numbers 31 and 20, and then the determination of the values of the letters p, q, r, &c.

$31 = 1 \times 20 + 11$	$p = 1 \times q + r$
$20 = 1 \times 11 + 9$	$q = 1 \times r + s$
$11 = 1 \times 9 + 2$	$r = 1 \times s + t$
$9 = 4 \times 2 + 1$	$s = 4 \times t + u$
$2 = 2 \times 1 + 0$	$t = 2 \times u + 7.$

18. In the same manner, we may represent the example in Art. 14.

$56 = 1 \times 39 + 17$	$p = 1 \times q + r$
$39 = 2 \times 17 + 5$	$q = 2 \times r + s$
$17 = 3 \times 5 + 2$	$r = 3 \times s + t$
$5 = 2 \times 2 + 1$	$s = 2 \times t + u$
$2 = 2 \times 1 + 0$	$t = 2 \times u + 11.$

19. And, in the same manner, we may analyse all questions of this kind. For, let there be given the equation $bp = aq + n$, in which a, b, and n, are known numbers; then, we have only to proceed as we should do to find the greatest common divisor of the numbers a and b, and we

may immediately determine p and q by the succeeding letters, as follows:

$$\text{Let}\left\{\begin{array}{l} a = Ab + c \\ b = Bc + d \\ c = Cd + e \\ d = De + f \\ e = Ef + g \\ f = Fg + o \end{array}\right. \text{ and we shall find } \left\{\begin{array}{l} p = Aq + r \\ q = Br + s \\ r = Cs + t \\ s = Dt + u \\ t = Eu + v \\ u = Fv \pm n. \end{array}\right.$$

We have only to observe farther, that in the last equation the sign $+$ must be prefixed to n, when the number of equations is odd; and that, on the contrary, we must take $-n$, when the number is even: by these means, the questions which form the subject of the present chapter may be readily answered, of which we shall give some examples.

20. *Question* 11. Required a number, which, being divided by 11, leaves the remainder 3; but being divided by 19, leaves the remainder 5.

Call this number N; then, in the first place, we have $N = 11p + 3$, and in the second, $N = 19q + 5$; therefore, we have the equation $11p = 19q + 2$, which furnishes the following Table:

$$\begin{array}{r|l} 19 = 1 \times 11 + 8 & p = q + r \\ 11 = 1 \times 8 + 3 & q = r + s \\ 8 = 2 \times 3 + 2 & r = 2s + t \\ 3 = 1 \times 2 + 1 & s = t + u \\ 2 = 2 \times 1 + 0 & t = 2u + 2, \end{array}$$

where we may assign any value to u, and determine by it the preceding letters successively. We find,

$$\begin{array}{l} t \ldots\ldots = 2u + 2 \\ s = t + u = 3u + 2 \\ r = 2s + t = 8u + 6 \\ q = r + s = 11u + 8 \\ p = q + r = 19u + 14; \end{array}$$

whence we obtain the number sought $N = 209u + 157$; therefore 157 is the least number that can express N, or satisfy the terms of the question.

21. *Question* 12. To find a number N such, that if we divide it by 11, there remains 3, and if we divide it by 19, there remains 5; and farther, if we divide it by 29, there remains 10.

The last condition requires that $N = 29p + 10$; and as we have already performed the calculation (in the last question) for the two others, we must, in consequence of that

result, have $N = 209u + 157$, instead of which we shall write $N = 209q + 157$; so that $29p + 10 = 209q + 157$, or $29p = 209q + 147$; whence we have the following Table;

$$\begin{array}{l|l} 209 = 7 \times 29 + 6; & \\ 29 = 4 \times 6 + 5; & \text{wherefore} \left\{\begin{array}{l} p = 7q + r, \\ q = 4r + s, \\ r = s + t, \\ s = 5t - 147. \end{array}\right. \\ 6 = 1 \times 5 + 1; & \\ 5 = 5 \times 1 + 0; & \end{array}$$

And, if we now retrace these steps, we have

$$\begin{array}{l} s \ldots\ldots = 5t - 147, \\ r = s + t = 6t - 147, \\ q = 4r + s = 29t - 735, \\ p = 7q + r = 209t - 5292^*. \end{array}$$

So that $N = 6061t - 153458$: and the least number is found by making $t = 26$, which supposition gives $N = 4128$.

22. It is necessary, however, to observe, in order that an equation of the form $bp = aq + n$ may be resolvible, that the two numbers a and b must have no common divisor; for, otherwise, the question would be impossible, unless the number n had the same common divisor.

If it were required, for example, to have $9p = 15q + 2$; since 9 and 15 have a common divisor 3, and which is not a divisor of 2, it is impossible to resolve the question, because $9p - 15q$ being always divisible by 3, can never become $= 2$. But if in this example $n = 3$, or $n = 6$, &c. the question would be posssible: for it would be sufficient first to divide by 3; since we should obtain $3p = 5q + 1$, an equation easily resolvible by the rule already given. It is evident, therefore, that the numbers a, b, ought to have no common divisor, and that our rule cannot apply in any other case.

23. To prove this still more satisfactorily, we shall consider the equation $9p = 15q + 2$ according to the usual method. Here we find

$$p = \frac{15q+2}{9} = q + \frac{6q+2}{9} = q + r; \text{ so that}$$

$$9r = 6q + 2, \text{ or } 6q = 9r - 2; \text{ or}$$

$$q = \frac{9r-2}{6} = r + \frac{3r-2}{6} = r + s; \text{ so that } 3r - 2 = 6s,$$

* That is, $-5292 \times 29 = -153468$; to which if the remainder $+10$ required by the question be added, the sum is -153458.

or $3r = 6s + 2$: consequently, $r = \frac{6s+2}{3} = 2s + \frac{2}{3}$.

Now, it is evident, that this can never become an integer number, because s is necessarily an integer; which shews the impossibility of such questions *.

CHAP. II.

Of the Rule *which is called* Regula Cæci, *for determining by means of two* Equations, *three or more* Unknown Quantities.

24. In the preceding chapter, we have seen how, by means of a single equation, two unknown quantities may be determined, so far as to express them in integer and positive numbers. If, therefore, we had two equations, in order that the question may be indeterminate, those equations must contain more than two unknown quantities. Questions of this kind occur in the common books of arithmetic; and are resolved by the rule called *Regula Cæci*, *Position*, or *The Rule of False*; the foundation of which we shall now explain, beginning with the following example:

25. *Question* 1. Thirty persons, men, women, and children, spend 50 crowns in a tavern; the share of a man is 3 crowns, that of a woman 2 crowns, and that of a child is 1 crown; how many persons were there of each class?

If the number of men be p, of women q, and of children r, we shall have the two following equations;

1. $p + q + r = 30$, and
2. $3p + 2q + r = 50$,

from which it is required to find the value of the three letters p, q, and r, in integer and positive numbers. The first equation gives $r = 30 - p - q$; whence we immediately conclude that $p + q$ must be less than 30; and, substituting this value of r in the second equation, we have $2p + q + 30 = 50$; so that $q = 20 - 2p$, and $p + q =$

* See the Appendix to this chapter, at Art. 3. of the Additions by De la Grange.

$20 - p$, which evidently is also less than 30. Now, as we may, in this equation, assume all numbers for p which do not exceed 10, we shall have the following eleven answers: the number of men p, of women q, and of children r, being as follow:

$p =$	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10;
$q =$	20,	18,	16,	14,	12,	10,	8,	6,	4,	2,	0;
$r =$	10,	11,	12,	13,	14,	15,	16,	17,	18,	19,	20;

and, if we omit the first and the last, there will remain 9.

26. *Question* 2. A certain person buys hogs, goats, and sheep, to the number of 100, for 100 crowns; the hogs cost him $3\frac{1}{2}$ crowns apiece; the goats, $1\frac{1}{3}$ crown; and the sheep, $\frac{1}{2}$ a crown. How many had he of each?

Let the number of hogs be p, that of the goats q, and of the sheep r, then we shall have the two following equations:

1. $p + q + r = 100$,
2. $3\frac{1}{2}p + 1\frac{1}{3}q + \frac{1}{2}r = 100$;

the latter of which being multiplied by 6, in order to remove the fractions, becomes, $21p + 8q + 3r = 600$. Now, the first gives $r = 100 - p - q$; and if we substitute this value of r in the second, we have $18p + 5q = 300$, or $5q = 300 - 18p$; and $q = 60 - \frac{18p}{5}$: consequently, $18p$ must be divisible by 5, and therefore, as 18 is not divisible by 5, p must contain 5 as a factor. If we therefore make $p = 5s$, we obtain $q = 60 - 18s$, and $r = 13s + 40$; in which we may assume for the value of s any integer number whatever, provided it be such, that q does not become negative: but this condition limits the value of s to 3; so that if we also exclude 0, there can only be three answers to the question; which are as follow:

When	$s =$ 1,	2,	3,
We have	$p =$ 5,	10,	15,
	$q =$ 42,	24,	6,
	$r =$ 53,	66,	79.

27. In forming such examples for practice, we must take particular care that they may be possible; in order to which, we must observe the following particulars:

Let us represent the two equations, to which we were just now brought, by

1. $x + y + z = a$, and
2. $fx + gy + hz = b$,

in which f, g, and h, as well as a and b, are given numbers.

Now, if we suppose that among the numbers f, g, and h, the first, f, is the greatest, and h the least, since we have $fx + fy + fz$, or $(x + y + z)f = fa$, (because $x + y + z = a$) it is evident, that $fx + fy + fz$ is greater than $fx + gy + hz$; consequently, fa must be greater than b, or b must be less than fa. Farther, since $hx + hy + hz$, or $(x + y + z)h = ha$, and $hx + hy + hz$ is undoubtedly less than $fx + gy + hz$, ha must be less than b, or b must be greater than ha. Hence it follows, that if b be not less than fa, and also greater than ha, the question will be impossible: which condition is also expressed, by saying that b must be contained between the limits fa and ha; and care must also be taken that it may not approach either limit too nearly, as that would render it impossible to determine the other letters.

In the preceding example, in which $a = 100$, $f = 3\frac{1}{2}$, and $h = \frac{1}{2}$, the limits were 350 and 50. Now, if we suppose $b = 51$, instead of 100, the equations will become

$$x + y + z = 100, \text{ and } 3\tfrac{1}{2}x + 1\tfrac{1}{3}y + \tfrac{1}{2}z = 51;$$

or, removing the fractions, $21x + 8y + 3z = 306$; and if the first be multiplied by 3, we have $3x + 3y + 3z = 300$. Now, subtracting this equation from the other, there remains $18x + 5y = 6$; which is evidently impossible, because x and y must be integer and positive numbers*.

28. Goldsmiths and coiners make great use of this rule, when they propose to make, from three or more kinds of metal, a mixture of a given value, as the following example will shew.

Question 3. A coiner has three kinds of silver, the first of 7 ounces, the second of $5\frac{1}{2}$ ounces, the third of $4\frac{1}{2}$ ounces, fine per marc†; and he wishes to form a mixture of the weight of 30 marcs, at 6 ounces: how many marcs of each sort must he take?

If he take x marcs of the first kind, y marcs of the second, and z marcs of the third, he will have $x + y + z = 30$, which is the first equation.

Then, since a marc of the first sort contains 7 ounces of fine silver, the x marcs of this sort will contain $7x$ ounces of such silver. Also, the y marcs of the second sort will contain $5\frac{1}{2}y$ ounces, and the z marcs of the third sort will contain $4\frac{1}{2}z$ ounces, of fine silver; so that the whole mass will contain $7x + 5\frac{1}{2}y + 4\frac{1}{2}z$ ounces of fine silver. As this mixture is to weigh 30 marcs, and each of these marcs must contain 6 ounces of fine silver, it follows that the whole mass

* Vide Article 22.

† A *marc* is eight ounces.

will contain 180 ounces of fine silver; and thence results the second equation, $7x + 5\frac{1}{2}y + 4\frac{1}{2}z = 180$, or $14x + 11y + 9z = 360$. If we now subtract from this equation nine times the first, or $9x + 9y + 9z = 270$, there remains $5x + 2y = 90$, an equation which must give the values of x and y in integer numbers; and with regard to the value of z, we may derive it from the first equation $z = 30 - x - y$. Now, the former equation gives $2y = 90 - 5x$, and $y = 45 - \frac{5x}{2}$; therefore, if $x = 2u$, we shall have $y = 45 - 5u$, and $z = 3u - 15$; which shews that u must be greater than 4, and yet less than 10. Consequently, the question admits of the following solutions:

If	$u =$ 5,	6,	7,	8,	9,
Then	$x =$ 10,	12,	14,	16,	18,
	$y =$ 20,	15,	10,	5,	0,
	$z =$ 0,	3,	6,	9,	12.

29. Questions sometimes occur, containing more than three unknown quantities; but they are also resolved in the same manner, as the following example will shew.

Question 4. A person buys 100 head of cattle for 100 pounds; viz. oxen at 10 pounds each, cows at 5 pounds, calves at 2 pounds, and sheep at 10 shillings each. How many oxen, cows, calves, and sheep, did he buy?

Let the number of oxen be p, that of the cows q, of calves r, and of sheep s. Then we have the following equations:

$$1.\quad p + q + r + s = 100;$$
$$2.\quad 10p + 5q + 2r + \tfrac{1}{2}s = 100;$$

or, removing the fractions, $20p + 10q + 4r + s = 200$: then subtracting the first equation from this, there remains $19p + 9q + 3r = 100$; whence

$$3r = 100 - 19p - 9q, \text{ and}$$
$$r = 33 + \tfrac{1}{3} - 6p - \tfrac{1}{3}p - q; \text{ or}$$
$$r = 33 - 6p - 3q + \frac{1-p}{3};$$

whence $1 - p$, or $p - 1$, must be divisible by 3; therefore if we make

$$p - 1 = 3t, \text{ we have}$$

$$p = 3t + 1$$
$$q = q$$
$$r = 27 - 19t - 3q$$
$$s = 72 + 2q + 16t;$$

whence it follows, that $19t + 3q$ must be less than 27, and that, provided this condition be observed, we may give any value to q and t. We have therefore to consider the following cases:

1. If $t = 0$	2. If $t = 1$
we have $p = 1$	$p = 4$
$q = q$	$q = q$
$r = 27 - 3q$	$r = 8 - 3q$
$s = 72 + 2q.$	$s = 88 + 2q.$

We cannot make $t = 2$, because r would then become negative.

Now, in the first case, q cannot exceed 9; and, in the second, it cannot exceed 2; so that these two cases give the following solutions, the first giving the following ten answers:

	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
$p =$	1	1	1	1	1	1	1	1	1	1
$q =$	0	1	2	3	4	5	6	7	8	9
$r =$	27	24	21	18	15	12	9	6	3	0
$s =$	72	74	76	78	80	82	84	86	88	90.

And the second furnishes the three following answers;

	1.	2.	3.
$p =$	4	4	4
$q =$	0	1	2
$r =$	8	5	2
$s =$	88	90	92.

There are, therefore, in all, thirteen answers, which are reduced to ten if we exclude those that contain *zero*, or 0.

30. The method would still be the same, even if the letters in the first equation were multiplied by given numbers, as will be seen from the following example.

Question 5. To find three such integer numbers, that if the first be multiplied by 3, the second by 5, and the third by 7, the sum of the products may be 560; and if we multiply the first by 9, the second by 25, and the third by 49, the sum of the products may be 2920.

If the first number be x, the second y, and the third z, we shall have the two equations,

$$1.\ 3x + 5y + 7z = 560.$$
$$2.\ 9x + 25y + 49z = 2920.$$

And here, if we subtract three times the first, or $9x + 15y + 21z = 1680$, from the second, there remains $10y + 28z = 1240$; dividing by 2, we have $5y + 14z = 620$; whence

we obtain $y = 124 - \frac{14z}{5}$: so that z must be divisible by 5. If therefore we make $z = 5u$, we shall have $y = 124 - 14u$; which values of y and z being substituted in the first equation, we have $3x - 35u + 620 = 560$; or $3x = 35u - 60$, and $x = \frac{35u}{3} - 20$; therefore we shall make $u = 3t$, from which we obtain the following answer, $x = 35t - 20$, $y = 124 - 42t$, and $z = 15t$, in which we must substitute for t an integer number greater than 0 and less than 3: so that we are limited to the two following answers:

$$\text{If} \left.\begin{array}{l} t = 1, \\ t = 2, \end{array}\right\} \text{we have} \left\{\begin{array}{l} x = 15, y = 82, z = 15. \\ x = 50, y = 40, z = 30. \end{array}\right.$$

CHAP. III.

Of Compound Indeterminate Equations, *in which one of the* Unknown Quantities *does not exceed the* First Degree.

31. We shall now proceed to indeterminate equations, in which it is required to find two unknown quantities, one of them being multiplied by the other, or raised to a power higher than the first, whilst the other is found only in the first degree. It is evident that equations of this kind may be represented by the following general expression:

$$a + bx + cy + dx^2 + exy + fx^3 + gx^2y + hx^4 + kx^3y + , \&c. = 0.$$

As in this equation y does not exceed the first degree, that letter is easily determined; but here, as before, the values both of x and of y must be assigned in integer numbers.

We shall consider some of those cases, beginning with the easiest.

32. *Question* 1. To find two such numbers, that their product added to their sum may be 79.

Call the numbers sought x and y: then we must have $xy + x + y = 79$; so that $xy + y = 79 - x$, and

$$y = \frac{79 - x}{x + 1} = \frac{79}{x+1} + \frac{-x}{x+1} = -1 + \frac{80}{x+1},$$ from which we see that $x + 1$ must be a divisor of 80. Now, 80 having

several divisors, we shall also have several values of x, as the following Table will shew:

The divisors of 80 are	1	2	4	5	8	10	16	20	40	80
therefore $x =$	0	1	3	4	7	9	15	19	39	79
and $y =$	79	39	19	15	9	7	4	3	1	0

But as the answers in the bottom line are the same as those in the first, inverted, we have, in reality, only the five following; viz.

$$x = 0,\ 1,\ 3,\ 4,\ 7, \text{ and}$$
$$y = 79,\ 39,\ 19,\ 15,\ 9.$$

33. In the same manner, we may also resolve the general equation $xy + ax + by = c$; for we shall have $xy + by = c - ax$, and $y = \frac{c - ax}{x + b}$, or $y = \frac{ab + c}{x + b} - a$; that is to say, $x + b$ must be a divisor of the known number $ab + c$; so that each divisor of this number gives a value of x. If we therefore make $ab + c = fg$, we have $y = \frac{fg}{x+b} - a$; and supposing $x + b = f$, or $x = f - b$, it is evident, that $y = g - a$; and, consequently, that we have also two answers for every method of representing the number $ab + c$ by a product, such as fg. Of these two answers, one is $x = f - b$, and $y = g - a$, and the other is obtained by making $x + b = g$, in which case $x = g - b$, and $y = f - a$.

If, therefore, the equation $xy + 2x + 3y = 42$ were proposed, we should have $a = 2$, $b = 3$, and $c = 42$; consequently, $y = \frac{48}{x+3} - 2$. Now, the number 48 may be represented in several ways by two factors, as fg: and in each of those cases we shall always have either $x = f - 3$, and $y = g - 2$; or else $x = g - 3$, and $y = f - 2$. The analysis of this example is as follows:

Factors	1 × 48		2 × 24		3 × 16		4 × 12		6 × 8	
	x	y	x	y	x	y	x	y	x	y
Numbers	−2	46	−1	22	0	14	1	10	3	6
or	45	−1	21	0	13	1	9	2	5	4

34. The equation may be expressed still more generally, by writing $mxy = ax + by + c$; where a, b, c, and m, are

given numbers, and it is required to find integers for x and y that are not known.

If we first separate y, we shall have $y = \frac{ax + c}{mx - b}$; and removing x from the numerator, by multiplying both sides by m, we have

$$my = \frac{max + mc}{mx - b} = a + \frac{mc + ab}{mx - b}.$$

We have here a fraction whose numerator is a known number, and whose denominator must be a divisor of that number; let us therefore represent the numerator by a product of two factors, as fg (which may often be done in several ways), and see if one of these factors may be compared with $mx - b$, so that $mx - b = f$. Now, for this purpose, since $x = \frac{f+b}{m}$, $f + b$ must be divisible by m; and hence it follows, that out of the factors of $mc + ab$, we can employ only those which are of such a nature, that, by adding b to them, the sums will be divisible by m. We shall illustrate this by an example.

Let the equation be $5xy = 2x + 3y + 18$. Here, we have

$$y = \frac{2x + 18}{5x - 3}, \text{ and } 5y = \frac{10x + 90}{5x - 3} = 2 + \frac{96}{5x - 3};$$

it is therefore required to find those divisors of 96 which, added to 3, will give sums divisible by 5. Now, if we consider all the divisors of 96, which are 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96, it is evident that only these three of them, viz. 2, 12, 32, will answer this condition.

Therefore,

1. If $5x - 3 = 2$, we obtain $5y = 50$, and consequently $x = 1$, and $y = 10$.
2. If $5x - 3 = 12$, we obtain $5y = 10$, and consequently $x = 3$, and $y = 2$.
3. If $5x - 3 = 32$, we obtain $5y = 5$, and consequently $x = 7$, and $y = 1$.

35. As in this general solution we have

$$my - a = \frac{mc + ab}{mx - b},$$

it will be proper to observe, that if a number, contained in the formula $mc + ab$, have a divisor of the form $mx - b$, the quotient in that case must necessarily be contained in the formula $my - a$: we may therefore express the number $mc + ab$ by a product, such as $(mx - b) \times (my - a)$. For example, let $m = 12$, $a = 5$, $b = 7$, and $c = 15$, and we have $12y - 5 = \frac{215}{12x - 7}$. Now, the divisors of 215 are 1, 5, 43, 215; and we must select from these such as are contained in the formula $12x - 7$; or such as, by adding 7 to them, the sum may be divisible by 12: but 5 is the only divisor that satisfies this condition; so that $12x - 7 = 5$, and $12y - 5 = 43$. In the same manner, as the first of these equations gives $x = 1$, we also find y, in integer numbers, from the other, namely, $y = 4$. This property is of the greatest importance with regard to the theory of numbers, and therefore deserves particular attention.

36. Let us now consider also an equation of this kind, $xy + x^2 = 2x + 3y + 29$. First, it gives us

$$y = \frac{2x - x^2 + 29}{x - 3}, \text{ or } y = -x - 1 + \frac{26}{x-3}; \text{ and}$$

$$y + x + 1 = \frac{26}{x-3}: \text{ so that } x - 3 \text{ must be a divisor of } 26;$$

and, in this case, the divisors of 26 being 1, 2, 13, 26, we obtain the three following answers:

1. $x - 3 = 1$, or $x = 4$; so that
 $y + x + 1 = y + 5 = 26$, and $y = 21$;
2. $x - 3 = 2$, or $x = 5$; so that
 $y + x + 1 = y + 6 = 13$, and $y = 7$;
3. $x - 3 = 13$, or $x = 16$; so that
 $y + x + 1 = y + 17 = 2$, and $y = -15$.

This last value, being negative, must be omitted; and, for the same reason, we cannot include the last case, $x - 3 = 26$.

37. It would be unnecessary to analyse any more of these formulæ, in which we find only the first power of y, and higher powers of x; for these cases occur but seldom, and, besides, they may always be resolved by the method which we have explained. But when y also is raised to the second power, or to a degree still higher, and we wish to determine its value by the above rules, we obtain radical signs, which contain the second, or higher powers of x; and it is then necessary to find such values of x, as will destroy the radical signs, or the irrationality. Now, the great art of *Indeterminate Analysis* consists in rendering those surd, or incommensurable formulæ rational: the methods of performing which will be explained in the following chapters*.

* See the Appendix to this chapter, at Art. 4, of the Additions by De la Grange.

QUESTIONS FOR PRACTICE.

1. Given $24x = 13y + 16$, to find x and y in whole numbers. *Ans.* $x = 5$, and $y = 8$.

2. Given $87x + 256y = 15410$, to find the least value of x, and the greatest of y, in whole positive numbers. *Ans.* $x = 30$, and $y = 12800$.

3. What is the number of all the possible values of x, y, and z, in whole numbers, in the equation $5x + 7y + 11z = 224$? *Ans.* 60.

4. How many old guineas at 21*s.* 6*d*; and pistoles at 17*s*, will pay 100*l.*? and in how many ways can it be done? *Ans.* Three different ways; that is, 19, 62, 105 pistoles, and 78, 44, 10 guineas.

5. A man bought 20 birds for 20 pence; consisting of geese at 4 pence, quails at $\frac{1}{2}$*d.* and larks at $\frac{1}{4}$*d.* each; how many had he of each? *Ans.* Three geese, 15 quails, and 2 larks.

6. A, B, and C, and their wives P, Q, and R, went to market to buy hogs; each man and woman bought as many hogs, as they gave shillings for each; A bought 25 hogs more than Q, and B bought 11 more than P. Also each man laid out three guineas more than his wife. Which two persons were, respectively, man and wife? *Ans.* B and Q, C and P, A and R.

7. To determine whether it be possible to pay 100*l.* in guineas and moidores only? *Ans.* It is not possible.

8. I owe my friend a shilling, and have nothing about me but guineas, and he has nothing but louis d'ors, valued at 17*s.* each; how must I acquit myself of the debt? *Ans.* I must pay him 13 guineas, and he must give me 16 louis d'ors.

9. In how many ways is it possible to pay 1000*l.* with crowns, guineas, and moidores only? *Ans.* 70734.

10. To find the least whole number, which being divided by the nine whole digits respectively, shall leave no remainders. *Ans.* 2520.

CHAP. IV.

On the Method *of rendering* Surd Quantities *of the form* $\sqrt{}(a + bx + cx^2)$ Rational.

38. It is required in the present case to determine the values which are to be adopted for x, in order that the formula $a + bx + cx^2$ may become a real square; and, consequently, that a rational root of it may be assigned. Now, the letters a, b, and c, represent given numbers; and the determination of the unknown quantity depends chiefly on the nature of these numbers; there being many cases in which the solution becomes impossible. But even when it is possible, we must content ourselves at first with being able to assign rational values for the letter x, without requiring those values also to be integer numbers; as this latter condition produces researches altogether peculiar.

39. We suppose here that the formula extends no farther than the second power of x; the higher dimensions require different methods, which will be explained in their proper places.

We shall observe first, that if the second power were not in the formula, and c were $= 0$, the problem would be attended with no difficulty; for if $\sqrt{}(a + bx)$ were the given formula, and it were required to determine x, so that $a + bx$ might be a square, we should only have to make $a + bx = y^2$, whence we should immediately obtain $x = \frac{y^2 - a}{b}$. Now, whatever number we substitute here for y, the value of x would always be such, that $a + bx$ would be a square, and consequently, $\sqrt{}(a + bx)$ would be a rational quantity.

40. We shall therefore begin with the formula $\sqrt{}(1 + x^2)$; that is to say, we are to find such values of x, that, by adding unity to their squares, the sums may likewise be squares; and as it is evident that those values of x cannot be integers, we must be satisfied with finding the fractions which express them.

41. If we supposed $1 + x^2 = y^2$, since $1 + x^2$ must be a square, we should have $x^2 = y^2 - 1$, and $x = \sqrt{}(y^2 - 1$; so that in order to find x we should have to seek numbers for y, whose squares, diminished by unity, would also leave squares; and, consequently, we should be led to a question as difficult as the former, without advancing a single step.

It is certain, however, that there are real fractions, which, being substituted for x, will make $1 + x^2$ a square; of which we may be satisfied from the following cases:

1. If $x = \frac{3}{4}$, we have $1 + x^2 = \frac{25}{16}$; and consequently $\sqrt{}(1 + x^2) = \frac{5}{4}$.

2. $1 + x^2$ becomes a square likewise, if $x = \frac{4}{3}$, which gives $\sqrt{}(1 + x^2) = \frac{5}{3}$.

3. If we make $x = \frac{5}{12}$,(we obtain $1 + x^2 = \frac{169}{144}$, the square root of which is $\frac{13}{12}$.

But it is required to shew how to find these values of x, and even all possible numbers of this kind.

42. There are two methods of doing this. The first requires us to make $\sqrt{}(1 + x^2) = x + p$; from which supposition we have $1 + x^2 = x^2 + 2px + p^2$, where the square x^2 destroys itself; so that we may express x without a radical sign. For, cancelling x^2 on both sides of the equation, we obtain $2px + p^2 = 1$; whence we find $x = \frac{1 - p^2}{2p}$; a quantity in which we may substitute for p any number whatever less than unity.

Let us therefore suppose $p = \frac{m}{n}$, and we have

$$x = \frac{1 - \frac{m^2}{n^2}}{\frac{2m}{n}},$$

and if we multiply both terms of this fraction by n^2, we shall find $x = \frac{n^2 - m^2}{2mn}$.

43. In order, therefore, that $1 + x^2$ may become a square, we may take for m and n all possible integer numbers, and in this manner find an infinite number of values for x.

Also, if we make, in general, $x = \frac{n^2 - m^2}{2mn}$, we find, by squaring, $1 + x^2 = 1 + \frac{n^4 - 2m^2n^2 + m^4}{4m^2n^2}$, or, by putting $1 = \frac{4m^2}{4m^2}$ in the numerator, $1 + x^2 = \frac{n^4 + 2m^2n^2 + m^4}{4m^2n^2}$; a fraction which is really a square, and gives

$$\sqrt{}(1 + x^2) = \frac{n^2 + m^2}{2mn}.$$

We shall exhibit, according to this solution, some of the least values of x.

If $n = 2,\ 3,\ 3,\ 4,\ 4,\ 5,\ 5,\ 5,\ 5,$
and $m = 1,\ 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 3,\ 4,$
We have $x = \frac{3}{4},\ \frac{4}{3},\ \frac{5}{12},\ \frac{15}{8},\ \frac{7}{24},\ \frac{12}{5},\ \frac{21}{20},\ \frac{8}{15},\ \frac{9}{40}.$

44. We have, therefore, in general,

$$1 + \frac{(n^2-m^2)^2}{(2mn)^2} = \frac{(n^2+m^2)^2}{(2mn)^2};$$

and, if we multiply this equation by $(2mn)^2$, we find

$$(2mn)^2 + (n^2 - m^2)^2 = (n^2 + m^2)^2;$$

so that we know, in a general manner, two squares, whose sum gives a new square. This remark will lead to the solution of the following question:

To find two square numbers, whose sum is likewise a square number.

We must have $p^2 + q^2 = r^2$; we have therefore only to make $p = 2mn$, and $q = n^2 - m^2$, then we shall have $r = n^2 + m^2$.

Farther, as $(n^2 + m^2)^2 - (2mn)^2 = (n^2 - m^2)^2$, we may also resolve the following question:

To find two squares, whose difference may also be a square number.

Here, since $p^2 - q^2 = r^2$, we have only to suppose $p = n^2 + m^2$, and $q = 2mn$, and we obtain $r = n^2 - m^2$. We might also make $p = n^2 + m^2$, and $q = n^2 - m^2$, from which we should find $r = 2mn$.

45. We spoke of two methods of giving the form of a square to the formula $1 + x^2$. The other is as follows:

If we suppose $\sqrt{(1+x^2)} = 1 + \frac{mx}{n}$, we shall have

$1 + x^2 = 1 + \frac{2mx}{n} + \frac{m^2x^2}{n^2}$; subtracting 1 from both sides,

we have $x^2 = \frac{2mx}{n} + \frac{m^2x^2}{n^2}$. This equation may be divided

by x, so that we have $x = \frac{2m}{n} + \frac{m^2x}{n^2}$, or $n^2x = 2mn + m^2x$,

whence we find $x = \frac{2mn}{n^2 - m^2}$. Having found this value of

x, we have

$1 + x^2 = 1 + \frac{4m^2n^2}{n^4-2m^2n^2+m^4} = \frac{n^4+2m^2n^2+m^4}{n^4-2m^2n^2+m^4}$, which is

the square of $\frac{n^2+m^2}{n^2-m^2}$. Now, as we obtain from that, the

equation $1 + \frac{(2mn)^2}{(n^2-m^2)^2} = \frac{(n^2+m^2)^2}{(n^2-m^2)^2}$, we shall have, as be-
fore, $(n^2-m^2)^2 + (2mn)^2 = (n^2 + m^2)^2$;
that is, the same two squares, whose sum is also a square.
46. The case which we have just analysed furnishes two
methods of transforming the general formula $a + bx + cx^2$
into a square. The first of these methods applies to all
the cases in which c is a square; and the second to those in
which a is a square. We shall consider both these sup-
positions.
First, let us suppose that c is a square, or that the given
formula is $a + bx + f^2x^2$. Since this must be a square,

we shall make $\sqrt{(a + bx + f^2x^2)} = fx + \frac{m}{n}$, and shall thus

have $a + bx + f^2x^2 = f^2x^2 + \frac{2mfx}{n} + \frac{m^2}{n^2}$, in which the

terms containing x^2 destroy each other, so that

$a + bx = \frac{2mfx}{n} + \frac{m^2}{n^2}$. If we multiply by n^2, we obtain

$n^2a + n^2bx = 2mnfx + m^2$; hence we find $x = \frac{m^2 - n^2a}{n^2b - 2mnf}$; and,

substituting this value for x, we shall have

$\sqrt{(a + bx + f^2x^2)} = \frac{m^2f - n^2af}{n^2b - 2mnf} + \frac{m}{n} = \frac{mnb - m^2f - n^2af}{n^2b - 2mnf}.$

47. As we have got a fraction for x, namely,

$\frac{m^2 - n^2a}{n^2b - 2mnf}$, let us make $x = \frac{p}{q}$, then $p = m^2 - n^2a$, and

$q = n^2b - 2mnf$; so that the formula $a + \frac{bp}{q} + \frac{f^2p^2}{q^2}$ is a

square; and as it continues a square, though it be mul-
tiplied by the square q^2, it follows, that the formula
$aq^2 + bpq + f^2p^2$ is also a square, if we suppose $p = m^2 - n^2a$,
and $q = n^2b - 2mnf$. Hence it is evident, that an infinite
number of answers, in integer numbers, may result from
this expression, because the values of the letters m and n are
arbitrary.
48. The second case which we have to consider, is that in
which a, or the first term, is a square. Let there be pro-
posed, for example, the formula $f^2 + bx + cx^2$, which it
is required to make a square. Here, let us suppose

$\sqrt{(f^2 + bx + cx^2)} = f + \frac{mx}{n}$, and we shall have

$f^2 + bx + cx^2 = f^2 + \frac{2fmx}{n} + \frac{m^2x^2}{n^2}$, in which equation the terms f^2 destroying each other, we may divide the remaining terms by x, so that we obtain

$$b + cx = \frac{2mf}{n} + \frac{m^2x}{n^2}, \text{ or}$$
$$n^2b + n^2cx = 2mnf + m^2x, \text{ or}$$
$$x(n^2c - m^2) = 2mnf - n^2b; \text{ or, lastly,}$$
$$x = \frac{2mnf - n^2b}{n^2c - m^2}.$$

If we now substitute this value instead of x, we have

$$\sqrt{(f^2 + bx + cx^2)} = f + \frac{2m^2f - mnb}{n^2c - m^2} = \frac{n^2cf + m^2f - mnb}{n^2c - m^2};$$

and making $x = \frac{p}{q}$, we may, in the same manner as before, transform the expression $f^2q^2 + bpq + cp^2$, into a square, by making $p = 2mnf - n^2b$, and $q = n^2c - m^2$.

49. Here we have chiefly to distinguish the case in which $a = 0$, that is to say, in which it is required to make a square of the formula $bx + cx^2$; for we have only to suppose $\sqrt{(bx + cx^2)} = \frac{mx}{n}$, from which we have the equation $bx + cx^2 = \frac{m^2x^2}{n^2}$; which, divided by x, and multiplied by n^2, gives $bn^2 + cn^2x = m^2x$; and, consequently,

$$x = \frac{bn^2}{m^2 - cn^2}.$$

If we seek, for example, all the triangular numbers that are at the same time squares, it will be necessary that $\frac{x^2 + x}{2}$, which is the form of triangular numbers, must be a square; and, consequently, $2x^2 + 2x$ must also be a square. Let us, therefore, suppose $\frac{m^2x^2}{n^2}$ to be that square, and we shall have $2n^2x + 2n^2 = m^2x$, and $x = \frac{2n^2}{m^2 - 2n^2}$; in which value we may substitute, instead of m and n, all possible numbers; but we shall generally find a fraction for x, though sometimes we may obtain an integer number. For example, if $m = 3$, and $n = 2$, we find $x = 8$, the triangular number of which, or 36, is also a square.

We may also make $m = 7$, and $n = 5$; in this case, $x = -50$, the triangle of which, 1225, is at the same time the triangle of $+49$, and the square of 35. We should have obtained the same result by making $n = 7$ and $m = 10$; for, in that case, we should also have found $x = 49$.

In the same manner, if $m = 17$ and $n = 12$, we obtain $x = 288$, its triangular number is

$$\frac{x(x+1)}{2} = \frac{288 \times 289}{2} = 144 \times 289,$$

which is a square, whose root is $12 \times 17 = 204$.

50. We may remark, with regard to this last case, that we have been able to transform the formula $bx + cx^2$ into a square from its having a known factor, x; this observation leads to other cases, in which the formula $a + bx + cx^2$ may likewise become a square, even when neither a nor c are squares.

These cases occur when $a + bx + cx^2$ may be resolved into two factors; and this happens when $b^2 - 4ac$ is a square: to prove which, we may remark, that the factors depend always on the roots of an equation; and that, therefore, we must suppose $a + bx + cx^2 = 0$. This being laid down, we have $cx^2 = -bx - a$, or

$$x^2 = -\frac{bx}{c} - \frac{a}{c}, \text{ whence we find}$$

$$x = -\frac{b}{2c} \pm \sqrt{\left(\frac{b^2}{4c^2} - \frac{a}{c}\right)}, \text{ or } x = -\frac{b}{2c} \pm \frac{\sqrt{(b^2 - 4ac)}}{2c},$$

and, it is evident, that if $b^2 - 4ac$ be a square, this quantity becomes rational.

Therefore let $b^2 - 4ac = d^2$; then the roots will be $\frac{-b \pm d}{2c}$, that is to say, $x = \frac{-b \pm d}{2c}$; and, consequently, the divisors of the formula $a + bx + cx^2$ are $x + \frac{b - d}{2c}$, and $x + \frac{b + d}{2c}$. If we multiply these factors together, we are brought to the same formula again, except that it is divided by c; for the product is $x^2 + \frac{bx}{c} + \frac{b^2}{4c^2} - \frac{d^2}{4c^2}$; and since $d^2 = b^2 - 4ac$, we have

$x^2 + \frac{bx}{c} + \frac{b^2}{4c^2} - \frac{b^2}{4c^2} + \frac{4ac}{4c^2} = x^2 + \frac{bx}{c} + \frac{a}{c}$; which being multiplied by c, gives $cx^2 + bx + a$. We have, therefore, only to multiply one of the factors by c, and we obtain the formula in question expressed by the product,

$$(cx + \frac{b}{2} - \frac{d}{2}) \times (x + \frac{b}{2c} + \frac{d}{2c});$$

and it is evident that this solution must be applicable whenever $b^2 - 4ac$ is a square.

51. From this results the third case, in which the formula $a + bx + cx^2$ may be transformed into a square; which we shall add to the other two.

52. This case, as we have already observed, takes place, when the formula may be represented by a product, such as $(f + gx) \times (h + kx)$. Now, in order to make a square of this quantity, let us suppose its root, or

$\sqrt{(f + gx) \times (h + kx)} = \frac{m(f + gx)}{n}$; and we shall then have $(f + gx) \times (h + kx) = \frac{m^2(f + gx)^2}{n^2}$; and, dividing this equation by $f + gx$, we have $h + kx = \frac{m^2(f + gx)}{n^2}$; or

$$hn^2 + kn^2x = fm^2 + gm^2x;$$

and, consequently, $x = \frac{fm^2 - hn^2}{kn^2 - gm^2}$.

To illustrate this, let the following questions be proposed.

Question 1. To find all the numbers, x, such, that if 2 be subtracted from twice their square, the remainder may be a square.

Since $2x^2 - 2$ is the quantity which is to be a square, we must observe, that this quantity is expressed by the factors, $2 \times (x + 1) \times (x - 1)$. If, therefore, we suppose its root $= \frac{m(x + 1)}{n}$, we have $2(x + 1) \times (x - 1) = \frac{m^2(x + 1)^2}{n^2}$; dividing by $x + 1$, and multiplying by n^2, we obtain

$$2n^2x - 2n^2 = m^2x + m^2, \text{ and } x = \frac{m^2 + 2n^2}{2n^2 - m^2}.$$

If, therefore, we make $m = 1$, and $n = 1$, we find $x = 3$, and $2x^2 - 2 = 16 = 4^2$.

If $m = 3$ and $n = 2$, we have $x = -17$. Now, as x is

only found in the second power, it is indifferent whether we take $x = -17$, or $x = +17$; either supposition equally gives $2x^2 - 2 = 576 = 24^2$.

53. *Question* 2. Let the formula $6 + 13x + 6x^2$ be proposed to be transformed into a square. Here, we have $a = 6$, $b = 13$, and $c = 6$, in which neither a nor c is a square. If, therefore, we try whether $b^2 - 4ac$ becomes a square, we obtain 25; so that we are sure the formula may be represented by two factors; and those factors are

$$(2 + 3x) \times (3 + 2x).$$

If $\frac{m(2+3x)}{n}$ is their root, we have

$$(2 + 3x) \times (3 + 2x) = \frac{m^2(2+3x)^2}{n^2},$$

which becomes $3n^2 + 2n^2x = 2m^2 + 3m^2x$, whence we find $x = \frac{2m^2 - 3n^2}{2n^2 - 3m^2} = \frac{3n^2 - 2m^2}{3m^2 - 2n^2}$. Now, in order that the numerator of this fraction may become positive, $3n^2$ must be greater than $2m^2$; and, consequently, $2m^2$ less than $3n^2$: that is to say, $\frac{m^2}{n^2}$ must be less than $\frac{3}{2}$. With regard to the denominator, if it must be positive, it is evident that $3m^2$ must exceed $2n^2$; and, consequently, $\frac{m^2}{n^2}$ must be greater than $\frac{2}{3}$. If, therefore, we would have the positive values of x, we must assume such numbers for m and n, that $\frac{m^2}{n^2}$ may be less than $\frac{3}{2}$, and yet greater than $\frac{2}{3}$.

For example, let $m = 6$, and $n = 5$; we shall then have $\frac{m^2}{n^2} = \frac{36}{25}$, which is less than $\frac{3}{2}$, and evidently greater than $\frac{2}{3}$, whence $x = \frac{3}{58}$.

54. This third case leads us to consider also a fourth, which occurs whenever the formula $a + bx + cx^2$ may be resolved into two such parts, that the first is a square, and the second the product of two factors: that is to say, in this case, the formula must be represented by a quantity of the form $p^2 + qr$, in which the letters p, q, and r express quantities of the form $f + gx$. It is evident that the rule for this

case will be to make $\sqrt{(p^2+qr)} = p + \frac{mq}{n}$; for we shall thus obtain $p^2+qr = p^2 + \frac{2mpq}{n} + \frac{m^2q^2}{n^2}$, in which the terms p^2 vanish; after which we may divide by q, so that we find $r = \frac{2mp}{n} + \frac{m^2q}{n^2}$, or $n^2r = 2nmp + m^2q$, an equation from which x is easily determined. This, therefore, is the fourth case in which our formula may be transformed into a square; the application of which is easy, and we shall illustrate it by a few examples.

55. *Question* 3. Required a number, x, such, that double its square, shall exceed some other square by unity; that is, if we subtract unity from this double square, the remainder may be a square.

For instance, the case applies to the number 5, whose square 25, taken twice, gives the number 50, which is greater by 1 than the square 49.

According to this enunciation, $2x^2 - 1$ must be a square; and as we have, by the formula, $a = -1$, $b = 0$, and $c = 2$, it is evident that neither a nor c is a square; and farther, that the given quantity cannot be resolved into two factors, since $b^2 - 4ac = 8$ which is not a square; so that none of the first three cases will apply. But, according to the fourth, this formula may be represented by

$$x^2 + (x^2 - 1) = x^2 + (x-1) \times (x+1).$$

If, therefore, we suppose its root $= x + \frac{m(x+1)}{n}$, we shall have

$$x^2 + (x+1) \times (x-1) = x^2 + \frac{2mx(x+1)}{n} + \frac{m^2(x+1)^2}{n^2}.$$

This equation, after having expunged the terms x^2, and divided the other terms by $x + 1$, gives

$$n^2x - n^2 = 2mnx + m^2x + m^2;$$ whence we find

$$x = \frac{m^2+n^2}{n^2-2mn-m^2};$$ and, since in our formula, $2x^2 - 1$, the square x^2 alone is found, it is indifferent whether we take positive or negative values for x. We may at first even write $-m$, instead of $+m$, in order to have

$$x = \frac{m^2+n^2}{n^2+2mn-m^2}.$$

If we make $m = 1$, and $n = 1$, we find $x = 1$, and $2x^2 - 1 = 1$; or if we make $m = 1$, and $n = 2$, we find $x = \frac{5}{7}$, and $2x^2 - 1 = \frac{1}{49}$; lastly, if we suppose $m = 1$, and $n = -2$, we find $x = -5$, or $x = +5$, and $2x^2 - 1 = 49$.

56. *Question* 4. To find numbers whose squares doubled and increased by 2, may likewise be squares.

Such a number, for instance, is 7, since the double of its square is 98, and if we add 2 to it, we have the square 100.

We must, therefore, have $2x^2 + 2$ a square; and as $a = 2$, $b = 0$, and $c = 2$, so that neither a nor c, nor $b^2 - 4ac$, the last being $= -16$, are squares, we must, therefore, have recourse to the fourth rule.

Let us suppose the first part to be 4, then the second will be $2x^2 - 2 = 2(x+1) \times (x-1)$, which presents the quantity proposed in the form

$$4 + (x+1) \times (x-1).$$

Now, let $2 + \frac{m(x+1)}{n}$ be its root, and we shall have the equation

$$4 + 2(x+1) \times (x-1) = 4 + \frac{4m(x+1)}{n} + \frac{m^2(x+1)^2}{n^2},$$

in which the squares 4, are destroyed; so that after having divided the other terms by $x + 1$, we have $2n^2x - 2n^2 = 4mn + m^2x + m^2$; and consequently,

$$x = \frac{4mn+m^2+2n^2}{2n^2-m^2}.$$

If, in this value, we make $m = 1$, and $n = 1$, we find $x = 7$, and $2x^2 + 2 = 100$. But if $m = 0$, and $n = 1$, we have $x = 1$, and $2x^2 + 2 = 4$.

57. It frequently happens, also, when none of the first three rules applies, that we are still able to resolve the formula into such parts as the fourth rule requires, though not so readily as in the foregoing examples.

Thus, if the question comprises the formula $7 + 15x + 13x^2$, the resolution we speak of is possible, but the method of performing it does not readily occur to the mind. It requires us to suppose the first part to be $(1 - x)^2$ or $1 - 2x + x^2$, so that the other may be $6 + 17x + 12x^2$: and we perceive that this part has two factors, because $17^2 - (4 \times 6 \times 12)$, $= 1$, is a square. The two factors therefore are $(2 + 3x) \times (3 + 4x)$; so that the formula becomes $(1 - x)^2 + (2 + 3x) \times (3 + 4x)$, which we may now resolve by the fourth rule.

But, as we have observed, it cannot be said that this analysis is easily found; and, on this account, we shall explain a general method for discovering, beforehand, whether the resolution of any such formula be possible or not; for there is an infinite number of them which cannot be resolved at all: such, for instance, as the formula $3x^2 + 2$, which can in no case whatever become a square. On the other hand, it is sufficient to know a single case, in which a formula is possible, to enable us to find all its answers; and this we shall explain at some length.

58. From what has been said, it may be observed, that all the advantage that can be expected on these occasions, is to determine, or suppose, any case in which such a formula as $a + bx + cx^2$, may be transformed into a square; and the method which naturally occurs for this, is to substitute small numbers successively for x, until we meet with a case which gives a square.

Now, as x may be a fraction, let us begin with substituting for x the general fraction $\frac{t}{u}$; and, if the formula

$$a + \frac{bt}{u} + \frac{ct^2}{u^2}$$ which results from it, be a square, it will be so also after having been multiplied by u^2; so that it only remains to try to find such integer values for t and u, as will make the formula $au^2 + btu + ct^2$ a square; and it is evident, that after this, the supposition of $x = \frac{t}{u}$ cannot fail to give the formula $a + bx + cx^2$ equal to a square.

But if, whatever we do, we cannot arrive at any satisfactory case, we have every reason to suppose that it is altogether impossible to transform the formula into a square; which, as we have already said, very frequently happens.

59. We shall now shew, on the other hand, that when one satisfactory case is determined, it will be easy to find all the other cases which likewise give a square; and it will be perceived, at the same time, that the number of those solutions is always infinitely great.

Let us first consider the formula $2 + 7x^2$, in which $a = 2$, $b = 0$, and $c = 7$. This evidently becomes a square, if we suppose $x = 1$; let us therefore make $x = 1 + y$, and, by substitution, we shall have $x^2 = 1 + 2y + y^2$, and our

formula becomes $9 + 14y + 7y^2$, in which the first term is a square; so that we shall suppose, conformably to the second rule, the square root of the new formula to be $3 + \frac{my}{n}$, and we shall thus obtain the equation

$$9 + 14y + 7y^2 = 9 + \frac{6my}{n} + \frac{m^2y^2}{n^2},$$ in which we may expunge 9 from both sides, and divide by y: which being done, we shall have $14n^2 + 7n^2y = 6mn + m^2y$; whence

$$y = \frac{6mn - 14n^2}{7n^2 - m^2};$$ and, consequently,

$$x = \frac{6mn - 7n^2 - m^2}{7n^2 - m^2},$$ in which we may substitute any values we please for m and n.

If we make $m = 1$, and $n = 1$, we have $x = -\frac{1}{3}$; or, since the second power of x stands alone, $x = +\frac{1}{3}$, wherefore $2 + 7x^2 = \frac{25}{9}$.

If $m = 3$, and $n = 1$, we have $x = -1$, or $x = +1$.

But if $m = 3$, and $n = -1$, we have $x = 17$; which gives $2 + 7x^2 = 2025$, the square of 45.

If $m = 8$, and $n = 3$, we shall then have, in the same manner, $x = -17$, or $x = +17$.

But, by making $m = 8$, and $n = -3$, we find $x = 271$; so that $2 + 7x^2 = 514089 = 717^2$.

60. Let us now examine the formula $5x^2 + 3x + 7$, which becomes a square by the supposition of $x = -1$. Here, if we make $x = y - 1$, our formula will be changed into this:

$$\begin{array}{r} 5y^2 - 10y + 5 \\ + \ 3y - 3 \\ + 7 \\ \hline 5y^2 - \ 7y + 9, \end{array}$$

the square root of which we shall suppose to be $3 - \frac{my}{n}$; by which means we shall have

$$5y^2 - 7y + 9 = 9 - \frac{6my}{n} + \frac{m^2y^2}{n^2},$$ or

$$5n^2y - 7n^2 = -6mn + m^2y;$$ whence we deduce

$$y = \frac{7n^2 - 6mn}{5n^2 - m^2};$$ and, lastly, $$x = \frac{2n^2 - 6mn + m^2}{5n^2 - m^2}.$$

If $m = 2$, and $n = 1$, we have $x = -6$, and consequently $5x^2 + 3x + 7 = 169 = 13^2$.

But if $m = -2$ and $n = 1$, we find $x = 18$, and $5x^2 + 3x + 7 = 1681 = 41^2$.

61. Let us now consider the formula, $7x^2 + 15x + 13$, in which we must begin with the supposition of $x = \frac{t}{u}$. Having substituted and multiplied u^2, we obtain $7t^2 + 15tu + 13u^2$, which must be a square. Let us therefore try to adopt some small numbers as the values of t and u.

$$\left.\begin{array}{l} \text{If } t = 1, \text{ and } u = 1, \\ t = 2, \text{ and } u = 1, \\ t = 2, \text{ and } u = -1, \\ t = 3, \text{ and } u = 1, \end{array}\right\} \text{the formula will become} \left\{\begin{array}{l} = 35 \\ = 71 \\ = 11 \\ = 121. \end{array}\right.$$

Now, 121 being a square, it is proof that the value of $x = 3$ answers the required condition; let us therefore suppose $x = y + 3$, and we shall have, by substituting this value in the formula,

$$7y^2 + 42y + 63 + 15y + 45 + 13, \text{ or}$$
$$7y^2 + 57y + 121.$$

Therefore let the root be represented by $11 + \frac{my}{n}$, and we shall have $7y^2 + 57y + 121 = 121 + \frac{22my}{n} + \frac{m^2y^2}{n^2}$, or $7n^2y + 57n^2 = 22mn + m^2y$; whence

$$y = \frac{57n^2 - 22mn}{m^2 - 7n^2}, \text{ and } x = \frac{36n^2 - 22mn + 3m^2}{m^2 - 7n^2}.$$

Suppose, for example, $m = 3$, and $n = 1$; we shall then find $x = -\frac{3}{2}$, and the formula becomes

$$7x^2 + 15x + 13 = \tfrac{25}{4} = (\tfrac{5}{2})^2.$$

If $m = 1$, and $n = 1$, we find $x = -\frac{17}{6}$; if $m = 3$, and $n = -1$, we have $x = \frac{129}{2}$, and the formula

$$7x^2 + 15x + 13 = \tfrac{120409}{4} = (\tfrac{347}{2})^2.$$

62. But frequently it is only lost labor to endeavour to find a case, in which the proposed formula may become a square. We have already said that $3x^2 + 2$ is one of those unmanageable formulæ; and, by giving it, according to this rule, the form $3t^2 + 2u^2$, we shall perceive that, whatever values we give to t and u, this quantity never becomes a square number. As the formulæ of this kind are very numerous, it will be worth while to fix on some characters, by which their impossibility may be perceived, in order that we may be often saved the trouble of useless trials; which shall form the subject of the following chapter*.

CHAP. V.

Of the Cases *in which the* Formula $a + bx + cx^2$ *can never become a* Square.

63. As our general formula is composed of three terms, we shall observe, in the first place, that it may always be transformed into another, in which the middle term is wanting. This is done by supposing $x = \frac{y - b}{2c}$; which substitution changes the formula into

$$a + \frac{by - b^2}{2c} + \frac{y^2 - 2by + b^2}{4c}; \text{ or } \frac{4ac - b^2 + y^2}{4c};$$

and since this must be a square, let us make it equal to $\frac{z^2}{4}$, we shall then have $4ac - b^2 + y^2 = \frac{4cz^2}{4}, = cz^2$, and, consequently, $y^2 = cz^2 + b^2 - 4ac$. Whenever, therefore, our formula is a square, this last $cz^2 + b^2 - 4ac$ will be so likewise; and reciprocally, if this be a square, the proposed formula will be a square also. If therefore we write t, instead of $b^2 - 4ac$, the whole will be reduced to determining whether a quantity of the form $cz^2 + t$ can become a square or not. And as this formula consists only of two terms, it is certainly much easier to judge from that whether it be possible or not; but in any further inquiry we must be guided by the nature of the given numbers c and t.

64. It is evident that if $t = 0$, the formula cz^2 can become a square only when c is a square; for the quotient arising from the division of a square by another square being likewise a square, the quantity cz^2 cannot be a square, unless

* See the Appendix to this chapter, at Article 5. of the Additions by De la Grange.

$\frac{cz^2}{z^2}$, that is to say, c, be one. So that when c is not a square, the formula cz^2 can by no means become a square; and on the contrary, if c be itself a square, cz^2 will also be a square, whatever number be assumed for z.

65. If we wish to consider other cases, we must have recourse to what has been already said on the subject of different kinds of numbers, considered with relation to their division by other numbers.

We have seen, for example, that the divisor 3 produces three different kinds of numbers. The first comprehends the numbers which are divisible by 3, and may be expressed by the formula $3n$.

The second kind comprehends the numbers which, being divided by 3, leave the remainder 1, and are contained in the formula $3n + 1$.

To the third class belong numbers which, being divided by 3, leave 2 for the remainder, and which may be represented by the general expression $3n + 2$.

Now, since all numbers are comprehended in these three formulæ, let us therefore consider their squares. First, if the question relate to a number included in the formula $3n$, we see that the square of this quantity being $9n^2$, it is divisible not only by 3, but also by 9.

If the given number be included in the formula $3n + 1$, we have the square $9n^2 + 6n + 1$, which, divided by 3, gives $3n^2 + 2n$, with the remainder 1; and which, consequently, belongs to the second class, $3n + 1$. Lastly, if the number in question be included in the formula $3n + 2$, we have to consider the square $9n^2 + 12n + 4$; and if we divide it by 3, we obtain $3n^2 + 4n + 1$, and the remainder 1; so that this square belongs, as well as the former, to the class $3n + 1$.

Hence it is obvious, that square numbers are only of two kinds with relation to the number 3; for they are either divisible by 3, and in this case are necessarily divisible also by 9; or they are not divisible by 3, in which case the remainder is always 1, and never 2; for which reason, no number contained in the formula $3n + 2$ can be a square.

66. It is easy, from what has just been said, to shew, that the formula $3x^2 + 2$ can never become a square, whatever integer, or fractional number, we choose to substitute for x. For, if x be an integer number, and we divide the formula $3x^2 + 2$ by 3, there remains 2; therefore it cannot be a square. Next, if x be a fraction, let us express it by $\frac{t}{u}$, supposing it already reduced to its lowest terms, and that t and u have no common divisor. In order, therefore, that $\frac{3t^2}{u^2} + 2$ may be a square, we must obtain, after multiplying by u^2, $3t^2 + 2u^2$ also a square. Now, this is impossible; for the number u is either divisible by 3, or it is not: if it be, t will not be so, for t and u have no common divisor, since the fraction $\frac{t}{u}$ is in its lowest terms. Therefore, if we make $u = 3f$, as the formula becomes $3t^2 + 18f^2$, it is evident that it can be divided by 3 only once, and not twice, as it must necessarily be if it were a square; in fact, if we divide by 3, we obtain $t^2 + 6f^2$. Now, though one part, $6f^2$, is divisible by 3, yet the other, t^2, being divided by 3, leaves 1 for a remainder.

Let us now suppose that u is not divisible by 3, and see what results from that supposition. Since the first term is divisible by 3, we have only to learn what remainder the second term, $2u^2$, gives. Now, u^2 being divided by 3, leaves the remainder 1, that is to say, it is a number of the class $3n + 1$; so that $2u^2$ is a number of the class $6n + 2$; and dividing it by 3, the remainder is 2; consequently, the formula $3t^2 + 2u^2$, if divided by 3, leaves the remainder 2, and is certainly not a square number.

67. We may, in the same manner, demonstrate, that the formula $3t^2 + 5u^2$, likewise can never become a square, nor any one of the following:

$$3t^2 + 8u^2,\ 3t^2 + 11u^2,\ 3t^2 + 14u^2, \&c.$$

in which the numbers 5, 8, 11, 14, &c. divided by 3, leave 2 for a remainder. For, if we suppose that u is divisible by 3, and, consequently, that t is not so, and if we make $u = 3n$, we shall always be brought to formulæ divisible by 3, but not divisible by 9: and if u were not divisible by 3, and consequently u^2 a number of the kind $3n + 1$, we should have the first term, $3t^2$, divisible by 3, while the second terms, $5u^2$, $8u^2$, $11u^2$, &c. would have the forms $15n + 5$, $24n + 8$, $33n + 11$, &c. and, when divided by 3, would constantly leave the remainder 2.

68. It is evident that this remark extends also to the general formula, $3t^2 + (3n + 2) \times u^2$, which can never become a square, even by taking negative numbers for n. If, for example, we should make $n = -1$, I say, it is im-

possible for the formula $3t^2 - u^2$ to become a square. This is evident, if u be divisible by 3: and if it be not, then u^2 is a number of the kind $3n + 1$, and our formula becomes $3t^2 - 3n - 1$, which, being divided by 3, gives the remainder -1, or $+2$; and in general, if n be $= -m$, we obtain the formula $3t^2 - (3m - 2)u^2$, which can never become a square.

69. So far, therefore, are we led by considering the divisor 3; if we now consider 4 also as a divisor, we see that every number may be comprised in one of the four following formulæ;

$$4n,\ 4n + 1,\ 4n + 2,\ 4n + 3.$$

The square of the first of these classes of numbers is $16n^2$; and, consequently, it is divisible by 16.

That of the second class, $4n + 1$, is $16n^2 + 8n + 1$; which if divided by 8, the remainder is 1; so that it belongs to the formula $8n + 1$.

The square of the third class, $4n + 2$, is $16n^2 + 16n + 4$; which if we divide by 16, there remains 4; therefore this square is included in the formula $16n + 4$.

Lastly, the square of the fourth class, $4n + 3$, being $16n^2 + 24n + 9$, it is evident that dividing by 8 there remains 1.

70. This teaches us, in the first place, that all the even square numbers are either of the form $16n$, or $16n + 4$; and, consequently, that all the other even formulæ, namely,

$$16n+2,\ 16n+6,\ 16n+8,\ 16n+10,\ 16n+12,\ 16n+14,$$

can never become square numbers.

Secondly, that all the odd squares are contained in the formula $8n + 1$; that is to say, if we divide them by 8, they leave a remainder of 1. And hence it follows, that all the other odd numbers, which have the form either of $8n + 3$, or of $8n + 5$, or of $8n + 7$, can never be squares.

71. These principles furnish a new proof, that the formula $3t^2 + 2u^2$ cannot be a square. For, either the two numbers t and u are both odd, or the one is even and the other odd. They cannot be both even, because in that case they would, at least, have the common divisor 2. In the first case, therefore, in which both t^2 and u^2 are contained in the formula $8n + 1$, the first term $3t^2$, being divided by 8, would leave the remainder 3, and the other term $2u^2$ would leave the remainder 2; so that the whole remainder would be 5: consequently, the formula in question cannot be a square. But, if the second case be supposed, and t be even, and u odd, the first term $3t^2$ will be divisible by 4, and the second term $2u^2$, if divided by 4, will leave the remainder 2; so that the two terms together, when divided by 4, leave a remainder of 2, and therefore cannot form a square. Lastly, if we were to suppose u an even number, as $2s$, and t odd, so that t^2 is of the form $8n + 1$, our formula would be changed into this, $24n + 3 + 8s^2$; which, divided by 8, leaves 3, and therefore cannot be a square.

This demonstration extends to the formula $3t^2 + (8n+2)u^2$; also to this, $(8m + 3)t^2 + 2u^2$, and even to this, $(8m + 3)t^2 + (8n + 2)u^2$; in which we may substitute for m and n all integer numbers, whether positive or negative.

72. But let us proceed farther, and consider the divisor 5, with respect to which all numbers may be ranged under the five following classes:

$$5n,\ 5n + 1,\ 5n + 2,\ 5n + 3,\ 5n + 4.$$

We remark, in the first place, that if a number be of the first class, its square will have the form $25n^2$; and will consequently be divisible not only by 5, but also by 25.

Every number of the second class will have a square of the form $25n^2 + 10n + 1$; and as dividing by 5 gives the remainder 1, this square will be contained in the formula $5n + 1$.

The numbers of the third class will have for their square $25n^2 + 20n + 4$; which, divided by 5, gives 4 for the remainder.

The square of a number of the fourth class is $25n^2 + 30n + 9$; and if it be divided by 5, there remains 4.

Lastly, the square of a number of the fifth class is $25n^2 + 40n + 16$; and if we divide this square by 5, there will remain 1.

When a square number therefore cannot be divided by 5, the remainder after division will always be 1, or 4, and never 2, or 3: hence it follows, that no square number can be contained in the formula $5n + 2$, or $5n + 3$.

73. From this it may be proved, that neither the formula $5t^2 + 2u^2$, nor $5t^2 + 3u^2$, can be a square. For, either u is divisible by 5, or it is not: in the first case, these formulæ will be divisible by 5, but not by 25; therefore they cannot be squares. On the other hand, if u be not divisible by 5, u^2 will either be of the form $5n + 1$, or $5n + 4$. In the first of these cases, the formula $5t^2 + 2u^2$ becomes $5t^2 + 10n + 2$; which, divided by 5, leaves a remainder of 2; and the formula $5t^2 + 3u^2$ becomes $5t^2 + 15n + 3$; which, being divided by 5, gives a remainder of 3; so that neither the one nor the other can be a square. With regard to the case of $u^2 = 5n + 4$, the first formula becomes $5t^2 + 10n + 8$;

which, divided by 5, leaves 3; and the other becomes $5t^2 + 15n + 12$, which, divided by 5, leaves 2; so that in this case also, neither of the two formulæ can be a square.

For a similar reason, we may remark, that neither the formula $5t^2 + (5n + 2)u^2$, nor $5t^2 + (5n + 3)u^2$, can become a square, since they leave the same remainders that we have just found. We might even in the first term write $5mt^2$, instead of $5t^2$, provided m be not divisible by 5.

74. Since all the even squares are contained in the formula $4n$, and all the odd squares in the formula $4n + 1$; and, consequently, since neither $4n + 2$, nor $4n + 3$, can become a square, it follows that the general formula $(4m + 3)t^2 + (4n + 3)u^2$ can never be a square. For if t be even, t^2 will be divisible by 4, and the other term, being divided by 4, will give 3 for a remainder; and, if we suppose the two numbers t and u odd, the remainders of t^2 and of u^2 will be 1; consequently, the remainder of the whole formula will be 2: now, there is no square number, which, when divided by 4, leaves a remainder of 2.

We shall remark, also, that both m and n may be taken negatively, or $= 0$, and still the formulæ $3t^2 + 3u^2$, and $3t^2 - u^2$, cannot be transformed into squares.

75. In the same manner as we have found for a few divisors, that some kinds of numbers can never become squares, we might determine similar kinds of numbers for all other divisors.

If we take the divisor 7, we shall have to distinguish seven different kinds of numbers, the squares of which we shall also examine.

Kinds of numbers.		Their squares are of the kind,	
1.	$7n$	$49n^2$	$7n$
2.	$7n + 1$	$49n^2 + 14n + 1$	$7n + 1$
3.	$7n + 2$	$49n^2 + 28n + 4$	$7n + 4$
4.	$7n + 3$	$49n^2 + 42n + 9$	$7n + 2$
5.	$7n + 4$	$49n^2 + 56n + 16$	$7n + 2$
6.	$7n + 5$	$49n^2 + 70n + 25$	$7n + 4$
7.	$7n + 6$	$49n^2 + 84n + 36$	$7n + 1$.

Therefore, since the squares which are not divisible by 7, are all contained in the three formulæ $7n + 1$, $7n + 2$, $7n + 4$, it is evident, that the three other formulæ, $7n + 3$, $7n + 5$, and $7n + 6$, do not agree with the nature of squares.

76. To make this conclusion still more apparent, we shall remark, that the last kind, $7n + 6$, may be also expressed

by $7n - 1$; that, in the same manner, the formula $7n + 5$ is the same as $7n - 2$, and $7n + 4$ the same as $7n - 3$. This being the case, it is evident, that the squares of the two classes of numbers, $7n + 1$, and $7n - 1$, if divided by 7, will give the same remainder 1; and that the squares of the two classes, $7n + 2$, and $7n - 2$, ought to resemble each other in the same respect, each leaving the remainder 4.

77. In general, therefore, let the divisor be any number whatever, which we shall represent by the letter d, the different classes of numbers which result from it will be

$$dn;$$
$$dn + 1,\ dn + 2,\ dn + 3,\ \&c.$$
$$dn - 1,\ dn - 2,\ dn - 3,\ \&c.$$

in which the squares of $dn + 1$, and $dn - 1$, have this in common, that, when divided by d, they leave the remainder 1, so that they belong to the same formula, $dn + 1$; in the same manner, the squares of the two classes $dn + 2$, and $dn - 2$, belong to the same formula, $dn + 4$. So that we may conclude, generally, that the squares of the two kinds, $dn + a$, and $dn - a$, when divided by d, give a common remainder a^2, or that which remains in dividing a^2 by d.

78. These observations are sufficient to point out an infinite number of formulæ, such as $at^2 + bu^2$, which cannot by any means become squares. Thus, by considering the divisor 7, it is easy to perceive, that none of these three formulæ, $7t^2 + 3u^2$, $7t^2 + 5u^2$, $7t^2 + 6u^2$, can ever become a square; because the division of u^2 by 7 only gives the remainders 1, 2, or 4; and, in the first of these formulæ, there remains either 3, or 6, or 5; in the second, 5, 3, or 6; and in the third, 6, 5, or 3; which cannot take place in square numbers. Whenever, therefore, we meet with such formulæ, we are certain that it is useless to attempt discovering any case, in which they can become squares: and, for this reason, the considerations, into which we have just entered, are of some importance.

If, on the other hand, the formula proposed is not of this nature, we have seen in the last chapter, that it is sufficient to find a single case, in which it becomes a square, to enable us to deduce from it an infinite number of similar cases.

The given formula, Art. 63, was properly $ax^2 + b$; and, as we usually obtain fractions for x, we supposed $x = \frac{t}{u}$, so that the problem, in reality, is to transform $at^2 + bu^2$ into a square.

But there is frequently an infinite number of cases, in which x may be assigned even in integer numbers; and the determination of those cases shall form the subject of the following chapter.

CHAP. VI.

Of the Cases *in* Integer Numbers, *in which the* Formula $ax^2 + b$ *becomes a* Square.

79. We have already shewn, Art. 63, how such formulæ as $a + bx + cx^2$, are to be transformed, in order that the second term may be destroyed; we shall therefore confine our present inquiries to the formula $ax^2 + b$, in which it is required to find for x only integer numbers, which may transform that formula into a square. Now, first of all, such a formula must be possible; for, if it be not, we shall not even obtain fractional values of x, far less integer ones.

80. Let us suppose then $ax^2 + b = y^2$; a and b being integer numbers, as well as x and y.

Now, here it is absolutely necessary for us to know, or to have already found a case in integer numbers; otherwise it would be lost labor to seek for other similar cases, as the formula might happen to be impossible.

We shall, therefore, suppose that this formula becomes a square, by making $x = f$, and we shall represent that square by g^2, so that $af^2 + b = g^2$, where f and g are known numbers. Then we have only to deduce from this case other similar cases; and this inquiry is so much the more important, as it is subject to considerable difficulties; which, however, we shall be able to surmount by particular artifices.

81. Since we have already found $af^2 + b = g^2$, and likewise, by hypothesis, $ax^2 + b = y^2$, let us subtract the first equation from the second, and we shall obtain a new one, $ax^2 - af^2 = y^2 - g^2$, which may be represented by factors in the following manner; $a(x + f) \times (x - f) = (y + g) \times (y - g)$, and which, by multiplying both sides by pq, becomes $apq(x + f) \times (x - f) = pq(y + g) \times (y - g)$. If we now decompound this equation, by making $ap(x + f) = q(y + g)$, and $q(x - f) = p(y - g)$, we may derive, from these two equations, values of the two letters x and y. The first, divided by q, gives $y + g = \frac{apx + apf}{q}$; and the second, divided by p, gives $y - g = \frac{qx - qf}{p}$. Subtracting this latter equation from the former, we have

$$2g = \frac{(ap^2 - q^2)x + (ap^2 + q^2)f}{pq}, \text{ or}$$

$$2gpq = (ap^2 - q^2)x + (ap^2 + q^2)f; \text{ therefore}$$

$$x = \frac{2gpq}{ap^2 - q^2} - \frac{(ap^2 + q^2)f}{ap^2 - q^2}, \text{ from which we obtain}$$

$$y = g + \frac{2gq^2}{ap^2 - q^2} - \frac{(ap^2 + q^2)fq}{(ap^2 - q^2)p} - \frac{qf}{p}.$$ And as, in this latter value, the first two terms, both containing the letter g, may be put into the form $\frac{g(ap^2 + q^2)}{ap^2 - q^2}$, and as the other two, containing the letter f, may be expressed by $\frac{2afpq}{ap^2 - q^2}$, all the terms will be reduced to the same denomination, and we shall have $y = \frac{g(ap^2 + q^2) - 2afpq}{ap^2 - q^2}$.

82. This operation seems not, at first, to answer our purpose; since having to find integer values of x and y, we are brought to fractional results; and it would be required to solve this new question,—What numbers are we to substitute for p and q, in order that the fraction may disappear? A question apparently still more difficult than our original one: but here we may employ a particular artifice, that will readily bring us to our object, which is as follows:

As every thing must be expressed in integer numbers, let us make $\frac{ap^2 + q^2}{ap^2 - q^2} = m$, and $\frac{2pq}{ap^2 - q^2} = n$, in order that we may have $x = ng - mf$, and $y = mg - naf$.

Now, we cannot here assume m and n at pleasure, since these letters must be such as will answer to what has been already determined: therefore, for this purpose, let us consider their squares, and we shall find

$$m^2 = \frac{a^2p^4 + 2ap^2q^2 + q^4}{a^2p^4 - 2ap^2q^2 + q^4}, \text{ and}$$

$$n^2 = \frac{4p^2q^2}{a^2p^4 - 2ap^2q^2 + q^4}; \text{ and hence}$$

$$m^2 - an^2 = \frac{a^2p^4 + 2ap^2q^2 + q^4 - 4ap^2q^2}{a^2p^4 - 2ap^2q^2 + q^4} = \frac{a^2p^4 - 2ap^2q^2 + q^4}{a^2p^4 - 2ap^2q^2 + q^4} = 1.$$

83. We see, therefore, that the two numbers m and n must be such, that $m^2 = an^2 + 1$. So that, as a is a known number, we must begin by considering the means of determining such an integer number for n, as will make $an^2 + 1$ a square; for then m will be the root of that square; and when we have likewise determined the number f so, that $af^2 + b$ may become a square, namely g^2, we shall obtain for x and y the following values in integer numbers; $x = ng - mf$, $y = mg - naf$; and thence, lastly, $ax^2 + b = y^2$.

84. It is evident, that having once determined m and n, we may write instead of them $-m$ and $-n$, because the square n^2 still remains the same.

But we have already shewn that, in order to find x and y in integer numbers, so that $ax^2 + b = y^2$, we must first know a case, such that $af^2 + b$ may be equal to g^2; when we have therefore found such a case, we must also endeavour to know, beside the number a, the values of m and n, which will give $an^2 + 1 = m^2$: the method for which shall be described in the sequel, and when this is done, we shall have a new case, namely, $x = ng + mf$, and $y = mg + naf$, also $ax^2 + b = y^2$.

Putting this new case instead of the preceding one, which was considered as known; that is to say, writing $ng + mf$ for f, and $mg + naf$ for g, we shall have new values of x and y, from which, if they be again substituted for x and y, we may find as many other new values as we please: so that, by means of a single case known at first, we may afterwards determine an infinite number of others.

85. The manner in which we have arrived at this solution has been very embarrassed, and seemed at first to lead us from our object, since it brought us to complicated fractions, which an accidental circumstance only enabled us to reduce: it will be proper, therefore, to explain a shorter method, which leads to the same solution.

86. Since we must have $ax^2 + b = y^2$, and have already found $af^2 + b = g^2$, the first equation gives us $b = y^2 - ax^2$, and the second gives $b = g^2 - af^2$; consequently, also, $y^2 - ax^2 = g^2 - af^2$, and the whole is reduced to determining the unknown quantities x and y, by means of the known quantities f and g. It is evident, that for this purpose we need only make $x = f$ and $y = g$; but it is also evident, that this supposition would not furnish a new case in addition to that already known. We shall, therefore, suppose that we have already found such a number for n, that $an^2 + 1$ is a square, or that $an^2 + 1 = m^2$; which being laid down, we have $m^2 - an^2 = 1$; and multiplying by this equation the one we had last, we find also $y^2 - ax^2 = (g^2 - af^2) \times (m^2 - an^2) = g^2m^2 - af^2m^2 - ag^2n^2 + a^2f^2n^2$. Let us now suppose $y = gm + afn$, and we shall have

$$g^2m^2 + 2afgmn + a^2f^2n^2 - ax^2 = g^2m^2 - af^2m^2 - ag^2n^2 + a^2f^2n^2,$$

in which the terms g^2m^2 and $a^2f^2m^2$ are destroyed; so that there remains $ax^2 = af^2m^2 + ag^2n^2 + 2afgmn$, or $x^2 = f^2m^2 + 2fgmn + g^2n^2$. Now, this formula is evidently a square, and gives $x = fm + gn$. Hence we have obtained the same formulæ for x and y as before.

87. It will be necessary to render this solution more evident, by applying it to some examples.

Question 1. To find all the integer values of x, that will make $2x^2 - 1$, a square, or give $2x^2 - 1 = y^2$.

Here, we have $a = 2$ and $b = -1$; and a satisfactory case immediately presents itself, namely, that in which $x = 1$ and $y = 1$: which gives us $f = 1$ and $g = 1$. Now, it is farther required to determine such a value of n, as will give $2n^2 + 1 = m^2$; and we see immediately, that this obtains when $n = 2$, and consequently $m = 3$; so that every case, which is known for f and g, giving us these new cases $x = 3f + 2g$, and $y = 3g + 4f$, we derive from the first solution, $f = 1$ and $g = 1$, the following new solutions:

$x = f = 1$	5	29	169	
$y = g = 1$	7	41	239, &c.	

88. *Question* 2. To find all the triangular numbers, that are at the same time squares.

Let z be the triangular root; then $\frac{z^2 + z}{2}$ is the triangle, which is to be also a square; and if we call x the root of this square, we have $\frac{z^2 + z}{2} = x^2$: multiplying by 8, we have $4z^2 + 4z = 8x^2$; and also adding 1 to each side, we have

$$4z^2 + 4z + 1 = (2z + 1)^2 = 8x^2 + 1.$$

Hence the question is to make $8x^2 + 1$ become a square;

for, if we find $8x^2 + 1 = y^2$, we shall have $y = 2z + 1$, and, consequently, the triangular root required will be

$$z = \frac{y-1}{2}.$$

Now, we have $a = 8$, and $b = 1$, and a satisfactory case immediately occurs, namely, $f = 0$ and $g = 1$. It is farther evident, that $8n^2 + 1 = m^2$, if we make $n = 1$, and $m = 3$; therefore $x = 3f + g$, and $y = 3g + 8f$; and since $z = \frac{y-1}{2}$, we shall have the following solutions:

$x = f = 0$	1	6	35	204	1189
$y = g = 1$	3	17	99	577	3363
$z = \frac{y-1}{2} = 0$	1	8	49	288	1681, &c.

89. *Question* 3. To find all the pentagonal numbers, which are at the same time squares.

If the root be z, the pentagon will be $= \frac{3z^2 - z}{2}$, which we shall make equal to x^2, so that $3z^2 - z = 2x^2$; then multiplying by 12, and adding unity, we have $36z^2 - 12z + 1 = (6z - 1)^2 = 24x^2 + 1$; also, making $24x^2 + 1 = y^2$, we have $y = 6z - 1$, and $z = \frac{y+1}{6}$.

Since $a = 24$, and $b = 1$, we know the case $f = 0$, and $g = 1$; and as we must have $24n^2 + 1 = m^2$, we shall make $n = 1$, which gives $m = 5$; so that we shall have $x = 5f + g$ and $y = 5g + 24f$; and not only $z = \frac{y+1}{6}$, but also $z = \frac{1-y}{6}$, because we may write $y = 1 - 6z$: whence we find the following results:

$x = f = 0$	1	10	99	980
$y = g = 1$	5	49	485	4801
$z = \frac{y+1}{6} = \frac{1}{3}$	1	$\frac{25}{3}$	81	$\frac{2401}{3}$
or $z = \frac{1-y}{6} = 0$	$-\frac{2}{3}$	-8	$-\frac{242}{3}$	-800, &c.

90. *Question* 4. To find all the integer square numbers, which, if multiplied by 7 and increased by 2, become squares.

It is here required to have $7x^2 + 2 = y^2$, or $a = 7$, and $b = 2$; and the known case immediately occurs, that is to say, $x = 1$; so that $x = f = 1$, and $y = g = 3$. If we next consider the equation $7n^2 + 1 = m^2$, we easily find also that $n = 3$ and $m = 8$; whence $x = 8f + 3g$, and $y = 8g + 21f$. We shall therefore have the following results:

$x = f = 1$	17	271
$y = g = 3$	45	717, &c.

91. *Question* 5. To find all the triangular numbers, that are at the same time pentagons.

Let the root of the triangle be p, and that of the pentagon q: then we must have $\frac{p^2+p}{2} = \frac{3q^2-q}{2}$, or $3q^2 - q = p^2 + p$; and, in endeavouring to find q, we shall first have

$$q^2 = \tfrac{1}{3}q + \frac{p^2+p}{3}, \text{ and}$$

$$q = \tfrac{1}{6} \pm \sqrt{(\tfrac{1}{36} + \frac{p^2+p}{3})}, \text{ or } q = \frac{1 \pm \sqrt{(12p^2+12p+1)}}{6}.$$

Consequently, it is required to make $12p^2 + 12p + 1$ become a square, and that in integer numbers. Now, as there is here a middle term $12p$, we shall begin with making $p = \frac{x-1}{2}$, by which means we shall have $12p^2 = 3x^2 - 6x + 3$, and $12p = 6x - 6$; consequently, $12p^2 + 12p + 1 = 3x^2 - 2$; and it is this last quantity, which at present we are required to transform into a square.

If, therefore, we make $3x^2 - 2 = y^2$, we shall have $p = \frac{x-1}{2}$, and $q = \frac{1+y}{6}$; so that all depends on the formula $3x^2 - 2 = y^2$; and here we have $a = 3$, and $b = -2$. Farther, we have a known case, $x = f = 1$, and $y = g = 1$; lastly, in the equation $m^2 = 3n^2 + 1$, we have $n = 1$, and $m = 2$; therefore we find the following values both for x and y, and for p and q:

First, $x = 2f + g$, and $y = 2g + 3f$; then,

$x = f = 1$	3	11	41
$y = g = 1$	5	19	71
$p = 0$	1	5	20
$q = \frac{1}{3}$	1	$\frac{10}{3}$	12
or $q = 0$	$-\frac{2}{3}$	-3	$-\frac{35}{3}$

because we have also $q = \frac{1-y}{6}$.

92. Hitherto, when the given formula contained a second term, we were obliged to expunge it, but the method we have now given may be applied, without taking away that second term, in the following manner.

Let $ax^2 + bx + c$ be the given formula, which must be a square, y^2, and let us suppose that we already know the case $af^2 + bf + c = g^2$.

Now, if we subtract this equation from the first, we shall have $a(x^2 - f^2) + b(x - f) = y^2 - g^2$, which may be expressed by factors in this manner:

$$(x - f) \times (ax + af + b) = (y - g) \times (y + g);$$

and if we multiply both sides by pq, we shall have

$$pq(x - f)(ax + af + b) = pq(y - g) \times (y + g),$$

which equation may be resolved into these two,

1. $p(x - f) = q(y - g)$,
2. $q(ax + af + b) = p(y + g)$.

Now, multiplying the first by p, and the second by q, and subtracting the first product from the second, we obtain

$$(aq^2 - p^2)x + (aq^2 + p^2)f + bq^2 = 2gpq,$$

which gives $x = \frac{2gpq}{aq^2 - p^2} - \frac{(aq^2+p^2)f}{aq^2-p^2} - \frac{bq^2}{aq^2-p^2}$.

But the first equation is $q(y - g) = p(x - f) = \ldots\ldots$

$$p\left(\frac{2gpq}{aq^2-p^2} - \frac{2afq^2}{aq^2-p^2} - \frac{bq^2}{aq^2-p^2}\right); \text{ so that } y - g =$$

$$\frac{2gp^2}{aq^2-p^2} - \frac{2afpq}{aq^2-p^2} - \frac{bpq}{aq^2-p^2}; \text{ and, consequently,}$$

$$y = g\left(\frac{aq^2+p^2}{aq^2-p^2} - \frac{2afpq}{aq^2-p^2} - \frac{bpq}{aq^2-p^2}\right).$$

Now, in order to remove the fractions, let us make, as before, $\frac{aq^2+p^2}{aq^2-p^2} = m$, and $\frac{2pq}{aq^2-p^2} = n$; and we shall have $m + 1 = \frac{2aq^2}{aq^2-p^2}$, and $\frac{q^2}{aq^2-p^2} = \frac{m+1}{2a}$; therefore

$$x = ng - mf - \frac{b(m+1)}{2a}, \text{ and } y = mg - naf - \tfrac{1}{2}bn;$$

in which the letters m and n must be such, that, as before, $m^2 = an^2 + 1$.

93. The formulæ which we have obtained for x and y, are still mixed with fractions, since some of their terms contain the letter b; for which reason they do not answer our

purpose. But if from those values we pass to the succeeding ones, we constantly obtain integer numbers; which, indeed, we should have obtained much more easily by means of the numbers p and q that were introduced at the beginning. In fact, if we take p and q, so that $p^2 = aq^2 + 1$, we shall have $aq^2 - p^2 = -1$, and the fractions will disappear. For then $x = -2gpq + f(aq^2 + p^2) + bq^2$, and $y = -g(aq^2 + p^2) + 2afpq + bpq$; but as in the known case, $af^2 + bf + c = g^2$, we find only the second power of g, it is of no consequence what sign we give that letter; if, therefore, we write $-g$ instead of $+g$, we shall have the formulæ

$$x = 2gpq + f(aq^2 + p^2) + bq^2, \text{ and}$$
$$y = g(aq^2 + p^2) + 2afpq + bpq,$$

and we shall thus be certain, at the same time, that $ax^2 + bx + c = y^2$.

Let it be required, as an example, to find the hexagonal numbers that are also squares.

We must have $2x^2 - x = y^2$, or $a = 2$, $b = -1$, and $c = 0$, and the known case will evidently be $x = f = 1$, and $y = g = 1$.

Farther, in order that we may have $p^2 = 2q^2 + 1$, we must have $q = 2$, and $p = 3$; so that we shall have $x = 12g + 17f - 4$, and $y = 17g + 24f - 6$; whence result the following values:

$x = f =$	1	25	841
$y = g =$	1	35	1189, &c.

94. Let us also consider our first formula, in which the second term was wanting, and examine the cases which make the formula $ax^2 + b$ a square in integer numbers.

Let $ax^2 + b = y^2$, and it will be required to fulfil two conditions:

1. We must know a case in which this equation exists; and we shall suppose that case to be expressed by the equation $af^2 + b = g^2$.

2. We must know such values of m and n, that $m^2 = an^2 + 1$; the method of finding which will be taught in the next chapter.

From that results a new case, namely, $x = ng + mf$, and $y = mg + anf$; this, also, will lead us to other similar cases, which we shall represent in the following manner:

$x = f$	A	B	C	D	E
$y = g$	P	Q	R	S	T, &c.

in which	$A = ng + mf$	$B = nP + mA$	$C = nQ + mB$	$D = nR + mC$
and	$P = mg + anf$	$Q = mP + anA$	$R = mQ + anB$	$S = mR + anC$, &c.

and these two series of numbers may be easily continued to any length.

95. It will be observed, however, that here we cannot continue the upper series for x, without having the under one in view; but it is easy to remove this inconvenience, and to give a rule, not only for finding the upper series, without knowing the other, but also for determining the latter without the former.

The numbers which may be substituted for x succeed each other in a certain progression, such that each term (as, for example, E), may be determined by the two preceding terms C and D, without having recourse to the terms of the second series R and S. In fact, since $E = nS + mD =$

$$n(mR + anC) + m(nR + mC) =$$

$$2mnR + an^2C + m^2C, \text{ and } nR = D - mC,$$

we therefore find

$$E = 2mD - m^2C + an^2C, \text{ or}$$
$$E = 2mD - (m^2 - an^2)C; \text{ or lastly,}$$
$$E = 2mD - C, \text{ because } m^2 = an^2 + 1,$$

and $m^2 - an^2 = 1$; from which it is evident, how each term is determined by the two which precede it.

It is the same with respect to the second series; for, since $T = mS + anD$, and $D = nR + mC$, we have $T = mS + an^2R + amnC$. Farther, $S = mR + anC$, so that $anC, = S - mR$; and if we substitute this value of anC, we have $T = 2mS - R$, which proves that the second progression follows the same law, or the same rule, as the first.

Let it be required, as an example, to find all the integer numbers, x, such, that $2x^2 - 1 = y^2$.

We shall first have $f = 1$, and $g = 1$. Then $m^2 = 2n^2 + 1$, if $n = 2$, and $m = 3$; therefore, since $A = ng + mf = 5$, the first two terms will be 1 and 5; and all the succeeding ones will be found by the formula $E = 6D - C$: that is to say, each term taken six times and diminished by the preceding term, gives the next. So that the numbers x which we require, will form the following series:

1, 5, 29, 169, 985, 5741, &c.

This progression we may continue to any length; and if we choose to admit fractional terms also, we might find an infinite number of them by the method which has been already explained *.

* See the appendix to this chapter at Art. 7, of the additions by De la Grange.

CHAP. VII.

Of a particular Method, *by which the* Formula $an^2 + 1$ *becomes a* Square *in* Integers.

96. That which has been taught in the last chapter, cannot be completely performed, unless we are able to assign for any number a, a number n, such, that $an^2 + 1$ may become a square; or that we may have $m^2 = an^2 + 1$.

This equation would be easy to resolve, if we were satisfied with fractional numbers, since we should have only to make $m = 1 + \frac{np}{q}$; for, by this supposition, we have $m^2 = 1 + \frac{2np}{q} + \frac{n^2p^2}{q^2} = an^2 + 1$; in which equation, we may expunge 1 from both sides, and divide the other terms by n: then multiplying by q^2, we obtain $2pq + np^2 = anq^2$; and this equation, giving $n = \frac{2pq}{aq^2 - p^2}$, would furnish an infinite number of values for n: but as n must be an integer number, this method will be of no use, and therefore very different means must be employed in order to accomplish our object.

97. We must begin with observing, that if we wished to have $an^2 + 1$ a square, in integer numbers, (whatever be the value of a), the thing required would not be possible.

For, in the first place, it is necessary to exclude all the cases, in which a would be negative; next, we must exclude those also, in which a would be itself a square; because then an^2 would be a square, and no square can become a square, in integer numbers, by being increased by unity. We are obliged, therefore, to restrict our formula to the condition, that a be neither negative, nor a square; but whenever a is a positive number, without being a square, it is possible to assign such an integer value of n, that $an^2 + 1$ may become a square: and when one such value has been found, it will be easy to deduce from it an infinite number of others, as was taught in the last chapter: but for our purpose it is sufficient to know a single one, even the least;

and this, Pell, an English writer, has taught us to find by an ingenious method, which we shall here explain.

98. This method is not such as may be employed generally, for any number a whatever; it is applicable only to each particular case.

We shall therefore begin with the easiest cases, and shall first seek such a value of n, that $2n^2 + 1$ may be a square, or that $\sqrt{(2n^2 + 1)}$ may become rational.

We immediately see that this square root becomes greater than n, and less than $2n$. If, therefore, we express this root by $n + p$, it is obvious that p must be less than n; and we shall have $\sqrt{(2n^2 + 1)} = n + p$; then, by squaring, $2n^2 + 1 = n^2 + 2np + p^2$; therefore

$$n^2 = 2np + p^2 - 1, \text{ and } n = p + \sqrt{(2p^2 - 1)}.$$

The whole is reduced, therefore, to the condition of $2p^2 - 1$ being a square; now, this is the case if $p = 1$, which gives $n = 2$, and $\sqrt{(2n^2 + 1)} = 3$.

If this case had not been immediately obvious, we should have gone farther; and since $\sqrt{(2p^2 - 1)} > p$ *, and, consequently, $n > 2p$, we should have made $n = 2p + q$; and should thus have had

$$2p + q = p + \sqrt{(2p^2 - 1)}, \text{ or } p + q = \sqrt{(2p^2 - 1)},$$

and, squaring, $p^2 + 2pq + q^2 = 2p^2 - 1$, whence

$$p^2 = 2pq + q^2 + 1,$$

which would have given $p = q + \sqrt{(2q^2 + 1)}$; so that it would have been necessary to have $2q^2 + 1$ a square; and as this is the case, if we make $q = 0$, we shall have $p = 1$, and $n = 2$, as before. This example is sufficient to give an idea of the method; but it will be rendered more clear and distinct from what follows.

99. Let $a = 3$, that is to say, let it be required to transform the formula $3n^2 + 1$ into a square. Here we shall make $\sqrt{(3n^2 + 1)} = n + p$, which gives

$$3n^2 + 1 = n^2 + 2np + p^2, \text{ and } 2n^2 = 2np + p^2 - 1;$$

whence we obtain $n = \dfrac{p + \sqrt{(3p^2 - 2)}}{2}$. Now, since $\sqrt{(3p^2 - 2)}$ exceeds p, and, consequently, n is greater

* This sign, $>$, placed between two quantities, signifies that the former is greater than the latter; and when the angular point is turned the contrary way, as $<$, it signifies that the former is less than the latter.

than $\frac{2p}{2}$, or than p, let us suppose $n = p + q$, and we shall have

$$2p + 2q = p + \sqrt{(3p^2 - 2)}, \text{ or}$$
$$p + 2q = \sqrt{(3p^2 - 2)};$$

then, by squaring $p^2 + 4pq + 4q^2 = 3p^2 - 2$; so that

$$2p^2 = 4pq + 4q^2 + 2, \text{ or } p^2 = 2pq + 2q^2 + 1, \text{ and}$$
$$p = q + \sqrt{(3q^2 + 1)}.$$

Now, this formula being similar to the one proposed, we may make $q = 0$, and shall thus obtain $p = 1$, and $n = 1$; whence $\sqrt{(3n^2 + 1)} = 2$.

100. Let $a = 5$, that we may have to make a square of the formula $5n^2 + 1$, the root of which is greater than $2n$. We shall therefore suppose

$$\sqrt{(5n^2 + 1)} = 2n + p, \text{ or } 5n^2 + 1 = 4n^2 + 4np + p^2;$$

whence we obtain

$$n^2 = 4np + p^2 - 1, \text{ and } n = 2p + \sqrt{(5p^2 - 1)}.$$

Now, $\sqrt{(5p^2 - 1)} > 2p$; whence it follows that $n > 4p$; for which reason, we shall make $n = 4p + q$, which gives $2p + q = \sqrt{(5p^2 - 1)}$, or $4p^2 + 4pq + q^2 = 5p^2 - 1$, and $p^2 = 4pq + q^2 + 1$; so that $p = 2q + \sqrt{(5q^2 + 1)}$; and as $q = 0$ satisfies the terms of this equation, we shall have $p = 1$, and $n = 4$; therefore $\sqrt{(5n^2 + 1)} = 9$.

101. Let us now suppose $a = 6$, that we may have to consider the formula $6n^2 + 1$, whose root is likewise contained between $2n$ and $3n$. We shall, therefore, make $\sqrt{(6n^2 + 1)} = 2n + p$, and shall have

$$6n^2 + 1 = 4n^2 + 4np + p^2, \text{ or } 2n^2 = 4np + p^2 - 1;$$

and, thence, $n = p + \dfrac{\sqrt{(6p^2 - 2)}}{2}$, or $n = \dfrac{2p + \sqrt{(6p^2 - 2)}}{2}$;

so that $n > 2p$.

If, therefore, we make $n = 2p + q$, we shall have

$$4p + 2q = 2p + \sqrt{(6p^2 - 2)}, \text{ or}$$
$$2p + 2q = \sqrt{(6p^2 - 2)};$$

the squares of which are $4p^2 + 8pq + 4q^2 = 6p^2 - 2$; so that $2p^2 = 8pq + 4q^2 + 2$, and $p^2 = 4pq + 2q^2 + 1$. Lastly, $p = 2q + \sqrt{(6q^2 + 1)}$. Now, this formula resembling the first, we have $q = 0$; wherefore $p = 1$, $n = 2$, and $\sqrt{(6n^2 + 1)} = 5$.

102. Let us proceed farther, and take $a = 7$, and $7n^2 + 1 = m^2$; here we see that $m > 2n$; let us therefore make $m = 2n + p$, and we shall have

$7n^2 + 1 = 4n^2 + 4np + p^2$, or $3n^2 = 4np + p^2 - 1$;

which gives $n = \frac{2p + \sqrt{(7p^2 - 3)}}{3}$. At present, since $n > \frac{4}{3}p$, and, consequently, greater than p, let us make $n = p + q$, and we shall have $p + 3q = \sqrt{(7p^2 - 3)}$; then, squaring both sides, $p^2 + 6pq + 9q^2 = 7p^2 - 3$, so that $6p^2 = 6pq + 9q^2 + 3$, or $2p^2 = 2pq + 3q^2 + 1$; whence we get $p = \frac{q + \sqrt{(7q^2 + 2)}}{2}$. Now, we have here $p > \frac{3q}{2}$; and, consequently, $p > q$; so that making $p = q + r$, we shall have $q + 2r = \sqrt{(7q^2 + 2)}$; the squares of which are $q^2 + 4qr + 4r^2 = 7q^2 + 2$; then $6q^2 = 4qr + 4r^2 - 2$, or $3q^2 = 2qr + 2r^2 - 1$; and, lastly, $q = \frac{r + \sqrt{(7r^2 - 3)}}{3}$.

Since now $q > r$, let us suppose $q = r + s$, and we shall have

$$2r + 3s = \sqrt{(7r^2 - 3)};\ \text{then}$$
$$4r^2 + 12rs + 9s^2 = 7r^2 - 3,\ \text{or}$$
$$3r^2 = 12rs + 9s^2 + 3,\ \text{or}$$
$$r^2 = 4rs + 3s^2 + 1,\ \text{and}$$
$$r = 2s + \sqrt{(7s^2 + 1)}.$$

Now, this formula is like the first; so that making $s = 0$, we shall obtain $r = 1$, $q = 1$, $p = 2$, and $n = 3$, or $m = 8$.

But this calculation may be considerably abridged in the following manner, which may be adopted also in other cases.

Since $7n^2 + 1 = m^2$, it follows that $m < 3n$.

If, therefore, we suppose $m = 3n - p$, we shall have
$7n^2 + 1 = 9n^2 - 6np + p^2$, or $2n^2 = 6np - p^2 + 1$;

whence we obtain $n = \frac{3p + \sqrt{(7p^2 + 2)}}{2}$; so that $n < 3p$; for this reason we shall write $n = 3p - 2q$; and, squaring, we shall have $9p^2 - 12pq + 4q^2 = 7p^2 + 2$; or

$$2p^2 = 12pq - 4q^2 + 2,\ \text{and}\ p^2 = 6pq - 2q^2 + 1,$$

whence results $p = 3q + \sqrt{(7q^2 + 1)}$. Here, we can at once make $q = 0$, which gives $p = 1$, $n = 3$, and $m = 8$, as before.

103. Let $a = 8$, so that $8n^2 + 1 = m^2$, and $m < 3n$. Here, we must make $m = 3n - p$, and shall have

$$8n^2 + 1 = 9n^2 - 6np + p^2,\ \text{or}\ n^2 = 6np - p^2 + 1;$$

whence $n = 3p + \sqrt{(8p^2 + 1)}$, and this formula being already similar to the one proposed, we may make $p = 0$, which gives $n = 1$, and $m = 3$.

104. We may proceed, in the same manner, for every other number, a, provided it be positive and not a square, and we shall always be led, at last, to a radical quantity, such as $\sqrt{(at^2 + 1)}$, similar to the first, or given formula, and then we have only to suppose $t = 0$; for the irrationality will disappear, and by tracing back the steps, we shall necessarily find such a value of n, as will make $an^2 + 1$ a square.

Sometimes we quickly obtain our end; but, frequently also, we are obliged to go through a great number of operations. This depends on the nature of the number a; but we have no principles, by which we can foresee the number of operations that it will be necessary to perform. The process is not very long for numbers below 13, but when $a = 13$, the calculation becomes much more prolix; and, for this reason, it will be proper here to resolve that case.

105. Let therefore $a = 13$, and let it be required to find $13n^2 + 1 = m^2$. Here, as $m^2 > 9n^2$, and, consequently, $m > 3n$, let us suppose $m = 3n + p$; we shall then have $13n^2 + 1 = 9n^2 + 6np + p^2$, or $4n^2 = 6np + p^2 - 1$, and $n = \frac{3p + \sqrt{(13p^2 - 4)}}{4}$, which shews that $n > \frac{6}{4}p$, and therefore much greater than p. If, therefore, we make $n = p + q$, we shall have $p + 4q = \sqrt{(13p^2 - 4)}$; and, taking the squares,

$$13p^2 - 4 = p^2 + 8pq + 16q^2;$$

so that $12p^2 = 8pq + 16q^2 + 4$, or $3p^2 = 2pq + 4q^2 + 1$, and $p = \frac{q + \sqrt{(13q^2 + 3)}}{3}$. Here, $p > \frac{q + 3q}{3}$, or $p > q$; we shall proceed, therefore, by making $p = q + r$, and shall thus obtain $2q + 3r = \sqrt{(13q^2 + 3)}$; then

$$13q^2 + 3 = 4q^2 + 12qr + 9r^2,\ \text{or}$$
$$9q^2 = 12qr + 9r^2 - 3,\ \text{or}$$
$$3q^2 = 4qr + 3r^2 - 1;$$

which gives $q = \frac{2r + \sqrt{(13r^2 - 3)}}{3}$.

Again, since $q > \frac{2r + 3r}{3}$, or $q > r$, we shall make $q = r + s$, and we shall thus have $r + 3s = \sqrt{(13r^2 - 3)}$; or $13r^2 - 3 = r^2 + 6rs + 9s^2$, or $12r^2 = 6rs + 9s^2 + 3$, or $4r^2 = 2rs + 3s^2 + 1$; whence we obtain

$r = \frac{s + \sqrt{(13s^2 + 4)}}{4}$. But here $r > \frac{s + 3s}{4}$, or $r > s$; wherefore let $r = s + t$, and we shall have $3s + 4t = \sqrt{(13s^2 + 4)}$,

$$\text{and } 13s^2 + 4 = 9s^2 + 24st + 16t^2;$$

so that $4s^2 = 24st + 16t^2 - 4$, and $s^2 = 6ts + 4t^2 - 1$; therefore $s = 3t + \sqrt{(13t^2 - 1)}$. Here we have

$$s > 3t + 3t, \text{ or } s > 6t;$$

we must therefore make $s = 6t + u$; whence $3t + u = \sqrt{(13t^2 - 1)}$, and $13t^2 - 1 = 9t^2 + 6tu + u^2$; then $4t^2 = 6tu + u^2 + 1$; and, lastly,

$$t = \frac{3u + \sqrt{(13u^2 + 4)}}{4}, \text{ or } t > \frac{6u}{4}, \text{ and } > u.$$

If, therefore, we make $t = u + v$, we shall have $u + 4v = \sqrt{(13u^2 + 4)}$, and $13u^2 + 4 = u^2 + 8uv + 16v^2$; therefore $12u^2 = 8uv + 16v^2 - 4$, or $3u^2 = 2uv + 4v^2 - 1$; lastly, $u = \frac{v + \sqrt{(13v^2 - 3)}}{3}$, or $u > \frac{4v}{3}$, or $u > v$.

Let us, therefore, make $u = v + x$, and we shall have

$$2v + 3x = \sqrt{(13v^2 - 3)}, \text{ and}$$
$$13v^2 - 3 = 4v^2 + 12vx + 9x^2; \text{ or}$$

$9v^2 = 12vx + 9x^2 + 3$, or $3v^2 = 4vx + 3x^2 + 1$, and

$$v = \frac{2x + \sqrt{(13x^2 + 3)}}{3}; \text{ so that } v > \tfrac{5}{3}x, \text{ and } > x.$$

Let us now suppose $v = x + y$, and we shall have

$$x + 3y = \sqrt{(13x^2 + 3)}, \text{ and}$$
$$13x^2 + 3 = x^2 + 6xy + 9y^2, \text{ or}$$
$$12x^2 = 6xy + 9y^2 - 3, \text{ and}$$
$$4x^2 = 2xy + 3y^2 - 1; \text{ whence}$$
$$x = \frac{y + \sqrt{(13y^2 - 4)}}{4},$$

and, consequently, $x > y$. We shall, therefore, make $x = y + z$, which gives

$$3y + 4z = \sqrt{(13y^2 - 4)}, \text{ and}$$
$$13y^2 - 4 = 9y^2 + 24zy + 16z^2, \text{ or}$$
$$4y^2 = 24zy + 16z^2 + 4; \text{ therefore}$$
$$y^2 = 6yz + 4z^2 + 1, \text{ and}$$
$$y = 3z + \sqrt{(13z^2 + 1)}.$$

This formula being at length similar to the first, we may take $z = 0$, and go back as follows:

$z = 0,$	$u = v + x = 3,$	$q = r + s = 71,$
$y = 1,$	$t = u + v = 5,$	$p = q + r = 109,$
$x = y + z = 1,$	$s = 6t + u = 33,$	$n = p + q = 180,$
$v = x + y = 2,$	$r = s + t = 38,$	$m = 3n + p = 649.$

So that 180 is the least number, after 0, which we can substitute for n, in order that $13n^2 + 1$ may become a square.

106. This example sufficiently shews how prolix these calculations may be in particular cases; and when the numbers in question are greater, we are often obliged to go through ten times as many operations as we had to perform for the number 13.

As we cannot foresee the numbers that will require such tedious calculations, we may with propriety avail ourselves of the trouble which others have taken; and, for this purpose, a Table is subjoined to the present chapter, in which the values of m and n are calculated for all numbers, a, between 2 and 100; so that in the cases which present themselves, we may take from it the values of m and n, which answer to the given number a.

107. It is proper, however, to remark, that, for certain numbers, the letters m and n may be determined generally; this is the case when a is greater, or less than a square, by 1 or 2; it will be proper, therefore, to enter into a particular analysis of these cases.

108. In order to this, let $a = e^2 - 2$; and since we must have $(e^2 - 2)n^2 + 1 = m^2$, it is clear that $m < en$; therefore we shall make $m = en - p$, from which we have

$$(e^2 - 2)n^2 + 1 = e^2n^2 - 2enp + p^2, \text{ or}$$
$$2n^2 = 2enp - p^2 + 1; \text{ therefore}$$

$n = \frac{ep + \sqrt{(e^2p^2 - 2p^2 + 2)}}{2}$; and it is evident that if we make $p = 1$, this quantity becomes rational, and we have $n = e$, and $m = e^2 - 1$.

For example, let $a = 23$, so that $e = 5$; we shall then have $23n^2 + 1 = m^2$, if $n = 5$, and $m = 24$. The reason of which is evident from another consideration; for if, in the case of $a = e^2 - 2$, we make $n = e$, we shall have $an^2 + 1 = e^4 - 2e^2 + 1$; which is the square of $e^2 - 1$.

109. Let $a = e^2 - 1$, or less than a square by unity. First, we must have $(e^2 - 1)n^2 + 1 = m^2$; then, because, as before, $m < en$, we shall make $m = en - p$; and this being done, we have

$$(e^2 - 1)n^2 + 1 = e^2n^2 - 2enp + p^2, \text{ or } n^2 = 2enp - p^2 + 1;$$

wherefore $n = ep + \sqrt{(e^2p^2 - p^2 + 1)}$. Now, the irrationality disappeared by supposing $p = 1$; so that $n = 2e$, and $m = 2e^2 - 1$. This also is evident; for, since $a = e^2 - 1$, and $n = 2e$, we find

$$an^2 + 1 = 4e^4 - 4e^2 + 1,$$

or equal to the square of $2e^2 - 1$. For example, let $a = 24$, or $e = 5$, we shall have $n = 10$, and

$$24n^2 + 1 = 2401 = (49)^2 \text{*}.$$

110. Let us now suppose $a = e^2 + 1$, or a greater than a square by unity. Here we must have

$$(e^2 + 1)n^2 + 1 = m^2,$$

and m will evidently be greater than en. Let us, therefore, write $m = en + p$, and we shall have

$$(e^2 + 1)n^2 + 1 = e^2n^2 + 2enp + p^2, \text{ or } n^2 = 2enp + p^2 - 1;$$

whence $n = ep + \sqrt{(e^2p^2 + p^2 - 1)}$. Now, we may make $p = 1$, and shall then have $n = 2e$; therefore $m^2 = 2e^2 + 1$; which is what ought to be the result from the consideration, that $a = e^2 + 1$, and $n = 2e$, which gives $an^2 + 1 = 4e^4 + 4e^2 + 1$, the square of $2e^2 + 1$. For example, let $a = 17$, so that $e = 4$, and we shall have $17n^2 + 1 = m^2$; by making $n = 8$, and $m = 33$.

111. Lastly, let $a = e^2 + 2$, or greater than a square by 2. Here, we have $(e^2 + 2)n^2 + 1 = m^2$, and, as before, $m > en$; therefore we shall suppose $m = en + p$, and shall thus have

$$e^2n^2 + 2n^2 + 1 = e^2n^2 + 2enp + p^2, \text{ or}$$
$$2n^2 = 2epn + p^2 - 1, \text{ which gives}$$

$$n = \frac{ep + \sqrt{(e^2p^2 + 2p^2 - 2)}}{2}.$$

Let $p = 1$, we shall find $n = e$, and $m = e^2 + 1$; and, in fact, since $a = e^2 + 2$, and $n = e$, we have $an^2 + 1 = e^4 + 2e^2 + 1$, which is the square of $e^2 + 1$.

For example, let $a = 11$, so that $e = 3$; we shall find $11n^2 + 1 = m^2$, by making $n = 3$, and $m = 10$. If we supposed $a = 83$, we should have $e = 9$, and

$$83n^2 + 1 = m^2, \text{ where } n = 9, \text{ and } m = 82 \text{*}.$$

* In this case, likewise, the radical sign vanishes, if we make $p = 0$: and this supposition incontestably gives the least possible numbers for m and n, namely, $n = 1$, and $m = e$; that is to say, if $e = 5$, the formula $24n^2 + 1$ becomes a square by making $n = 1$; and the root of this square will be $m = e = 5$. F. T.

* Our author might have added here another very obvious case, which is when a is of the form $e^2 \pm \frac{2}{c}e$; for then by making $n = c$, our formula $an^2 + 1$, becomes $e^2c^2 \pm 2ce + 1 = (ec \pm 1)^2$. I was led to the consideration of the above form, from having observed that the square roots of all numbers included in this formula are readily obtained by the method of continued fractions, the quotient figures, from which the fractions are derived, following a certain determined law, of two terms, readily observed, and that whenever this is the case, the method which is given above is also applied with great facility. And as a great many numbers are included in the above form, I have been induced to place it here, as a means of abridging the operations in those particular cases.

The reader is indebted to Mr. P. Barlow of the Royal Academy, Woolwich, for the above note; and also for a few more in this Second Part, which are distinguished by the signature, B.

TABLE, shewing for each value of a the least numbers m and n, that will give $m^2 = an^2 + 1$*; or that will render $an^2 + 1$ a square.

a	n	m	a	n	m
2	2	3	53	9100	66249
3	1	2	54	66	485
5	4	9	55	12	89
6	2	5	56	2	15
7	3	8	57	20	151
8	1	3	58	2574	19603
10	6	19	59	69	530
11	3	10	60	4	31
12	2	7	61	226153980	1766319049
13	180	649	62	8	63
14	4	15	63	1	8
15	1	4	65	16	129
17	8	33	66	8	65
18	4	17	67	5967	48842
19	39	170	68	4	33
20	2	9	69	936	7775
21	12	55	70	30	251
22	42	197	71	413	3480
23	5	24	72	2	17
24	1	5	73	267000	2281249
26	10	51	74	430	3699
27	5	26	75	3	26
28	24	127	76	6630	57799
29	1820	9801	77	40	351
30	2	11	78	6	53
31	273	1520	79	9	80
32	3	17	80	1	9
33	4	23	82	18	163
34	6	35	83	9	82
35	1	6	84	6	55
37	12	73	85	30996	285769
38	6	37	86	1122	10405
39	4	25	87	3	28
40	3	19	88	21	197
41	320	2049	89	53000	500001
42	2	13	90	2	19
43	531	3482	91	165	1574
44	30	199	92	120	1151
45	24	161	93	1260	12151
46	3588	24335	94	221064	2143295
47	7	48	95	4	39
48	1	7	96	5	49
50	14	99	97	6377352	62809633
51	7	50	98	10	99
52	90	649	99	1	10

* See Article 8 of the additions by De la Grange.

CHAP. VIII.

Of the Method *of rendering the* Irrational Formula, $\sqrt{(a + bx + cx^2 + dx^3)}$ Rational.

112. We shall now proceed to a formula, in which x rises to the third power; after which we shall consider also the fourth power of x, although these two cases are treated in the same manner.

Let it be required, therefore, to transform into a square the formula $a + bx + cx^2 + dx^3$, and to find proper values of x for this purpose, expressed in rational numbers. As this investigation is attended with much greater difficulties than any of the preceding cases, more artifice is requisite to find even fractional values of x; and with such we must be satisfied, without pretending to find values in integer numbers.

It must here be previously remarked also, that a general solution cannot be given, as in the preceding cases; and that, instead of the number here employed leading to an infinite number of solutions, each operation will exhibit but one value of x.

113. As in considering the formula $a + bx + cx^2$, we observed an infinite number of cases, in which the solution becomes altogether impossible, we may readily imagine that this will be much oftener the case with respect to the present formula, which, besides, constantly requires that we already know, or have found, a solution. So that here we can only give rules for those cases, in which we set out from one known solution, in order to find a new one; by means of which, we may then find a third, and proceed, successively in the same manner, to others.

It does not, however, always happen, that, by means of a known solution, we can find another; on the contrary, there are many cases, in which only one solution can take place; and this circumstance is the more remarkable, as in the analyses which we have before made, a single solution led to an infinite number of other new ones.

114. We just now observed, that in order to render the transformation of the formula, $a + bx + cx^2 + dx^3$, into a square, a case must be presupposed, in which that solution is possible. Now, such a case is clearly perceived, when the

first term is itself a square already, and the formula may be expressed thus, $f^2 + bx + cx^2 + dx^3$; for it evidently becomes a square, if $x = 0$.

We shall therefore enter upon the subject, by considering this formula; and shall endeavour to see how, by setting out from the known case $x = 0$, we may arrive at some other value of x. For this purpose, we shall employ two different methods, which will be separately explained: in order to which, it will be proper to begin with particular cases.

115. Let, therefore, the formula $1 + 2x - x^2 + x^3$ be proposed, which ought to become a square. Here, as the first term is a square, we shall adopt for the root required such a quantity as will make the first two terms vanish. For which purpose, let $1 + x$ be the root, whose square is to be equal to our formula; and this will give $1 + 2x - x^2 + x^3 = 1 + 2x + x^2$, of which equation the first two terms destroy each other; so that we have $x^2 = -x^2 + x^3$, or $x^3 = 2x^2$, which, being divided by x^2, gives $x = 2$; so that the formula becomes $1 + 4 - 4 + 8 = 9$.

Likewise, in order to make a square of the formula, $4 + 6x - 5x^2 + 3x^3$, we shall first suppose its root to be $2 + nx$, and seek such a value of n as will make the first two terms disappear; hence,

$$4 + 6x - 5x^2 + 3x^3 = 4 + 4nx + n^2x^2;$$

therefore we must have $4n = 6$, and $n = \frac{3}{2}$; whence results the equation $-5x^2 + 3x^3 = n^2x^2 = \frac{9}{4}x^2$, or $3x^3 = \frac{29}{4}x^2$, which gives $x = \frac{29}{12}$; and this is the value which will make a square of the proposed formula, whose root will be

$$2 + \tfrac{3}{2}x = \tfrac{45}{8}.$$

116. The second method consists in giving the root three terms, as $f + gx + hx^2$, such, that the first three terms in the equation may vanish.

Let there be proposed, for example, the formula $1 - 4x + 6x^2 - 5x^3$, the root of which we shall suppose to be $1 - 2x + hx^2$, and we shall thus have

$1 - 4x + 6x^2 - 5x^3 = 1 - 4x + 4x^2 - 4hx^3 + h^2x^4 + 2hx^2$.

The first two terms, as we see, are immediately destroyed on both sides; and, in order to remove the third, we must make $2h + 4 = 6$; consequently, $h = 1$; by these means, and transposing $2hx^2 = 2x^2$, we obtain $-5x^3 = -4x^3 + x^4$, or $-5 = -4 + x$, so that $x = -1$.

117. These two methods, therefore, may be employed, when the first term a is a square. The first is founded on expressing the root by two terms, as $f + px$, in which f is

the square root of the first term, and p is taken such, that the second term must likewise disappear; so that there remains only to compare p^2x^2 with the third and fourth term of the formula, namely $cx^2 + dx^3$; for then that equation, being divisible by x^2, gives a new value of x, which is

$$x = \frac{p^2 - c}{d}.$$

In the second method, three terms are given to the root; that is to say, if the first term $a = f^2$, we express the root by $f + px + qx^2$; after which, p and q are determined such, that the first three terms of the formula may vanish, which is done in the following manner: since

$$f^2 + bx + cx^2 + dx^3 = f^2 + 2pfx + 2fqx^2 + p^2x^2 + 2pqx^3 + q^2x^4,$$

we must have $b = 2fp$; and, consequently, $p = \frac{b}{2f}$; farther,

$c = 2fq + p^2$; or, $q = \frac{c - p^2}{2f}$; after this, there remains the equation $dx^3 = 2pqx^3 + q^2x^4$; and, as it is divisible by x^3, we obtain from it $x = \frac{d - 2pq}{q^2}$.

118. It may frequently happen, however, even when $a = f^2$, that neither of these methods will give a new value of x; as will appear, by considering the formula $f^2 + dx^3$, in which the second and third terms are wanting.

For if, according to the first method, we suppose the root to be $f + px$, that is,

$$f^2 + dx^3 = f^2 + 2fpx + p^2x^2,$$

we shall have $2fp = 0$, and $p = 0$; so that $dx^3 = 0$; and therefore $x = 0$, which is not a new value of x.

If, according to the second method, we were to make the root $f + px + qx^2$, or

$f^2 + dx^3 = f^2 + 2fpx + p^2x^2 + 2fqx^2 + 2pqx^3 + q^2x^4$,

we should find $2fp = 0$, $p^2 + 2fq = 0$, and $q^2 = 0$; whence $dx^3 = 0$, and also $x = 0$.

119. In this case, we have no other expedient, than to endeavour to find such a value of x, as will make the formula a square; if we succeed, this value will then enable us to find new values, by means of our two methods: and this will apply even to the cases in which the first term is not a square.

If, for example, the formula $3 + x^3$ must become a square; as this takes place when $x = 1$, let $x = 1 + y$, and we shall thus have $4 + 3y + 3y^2 + y^3$, the first term of which is a square. If, therefore, we suppose, according to the first method, the root to be $2 + py$, we shall have

$$4 + 3y + 3y^2 + y^3 = 4 + 4py + p^2y^2.$$

In order that the second term may disappear, we must make $4p = 3$; and, consequently, $p = \frac{3}{4}$; whence $3 + y = p^2$, and $y = p^2 - 3 = \frac{9}{16} - \frac{48}{16} = \frac{-39}{16}$; therefore $x = \frac{-23}{16}$, which is a new value of x.

If, again, according to the second method, we represent the root by $2 + py + qy^2$, we shall have

$$4 + 3y + 3y^2 + y^3 = 4 + 4py + 4qy^2 + p^2y^2 + 2pqy^3 + q^2y^4,$$

from which the second term will be removed, by making $4p = 3$, or $p = \frac{3}{4}$; and the fourth, by making $4q + p^2 = 3$, or $q = \frac{3 - p^2}{4} = \frac{39}{64}$; so that $1 = 2pq + q^2y$; whence we obtain $y = \frac{1 - 2pq}{q^2}$, or $y = \frac{352}{1521}$; and, consequently,

$$x = \tfrac{1873}{1521}.$$

120. In general, if we have the formula

$$a + bx + cx^2 + dx^3,$$

and know also that it becomes a square when $x = f$, or that $a + bf + cf^2 + df^3 = g^2$, we may make $x = f + y$, and shall hence obtain the following new formula:

$$\begin{array}{l} a \\ + bf + by \\ + cf^2 + 2cfy + cy^2 \\ + df^3 + 3df^2y + 3dfy^2 + dy^3 \\ \hline g^2 + (b + 2cf + 3df^2)y + (c + 3df)y^2 + dy^3. \end{array}$$

In this formula, the first term is a square; so that the two methods above given may be applied with success, as they will furnish new values of y, and consequently of x also, since $x = f + y$.

121. But often, also, it is of no avail even to have found a value of x. This is the case with the formula $1 + x^3$, which becomes a square when $x = 2$. For if, in consequence of this, we make $x = 2 + y$, we shall get the formula $9 + 12y + 6y^2 + y^3$, which ought also to become a square.

Now, by the first rule, let the root be $3 + py$, and we shall have $9 + 12y + 6y^2 + y^3 = 9 + 6py + p^2y^2$, in which we must have $6p = 12$, and $p = 2$; therefore $6 + y = p^2 = 4$, and $y = -2$, which, since we made $x = 2 + y$, this gives $x = 0$; that is to say, a value from which we can derive nothing more.

Let us also try the second method, and represent the root by $3 + py + qy^2$; this gives

$$9 + 12y + 6y^2 + y^3 = 9 + 6py + 6qy^2 + p^2y^2 + 2pqy^3 + q^2y^4,$$

in which we must first have $6p = 12$, and $p = 2$; then $6q + p^2 = 6q + 4 = 6$, and $q = \frac{1}{3}$; farther,

$$1 = 2pq + q^2y = \tfrac{4}{3} + \tfrac{1}{9}y;$$

hence $y = -3$, and, consequently, $x = -1$, and $1 + x^3 = 0$; from which we can draw no further conclusion, because, if we wished to make $x = -1 + z$, we should find the formula, $3z - 3z^2 + z^3$, the first term of which vanishes; so that we cannot make use of either method.

We have therefore sufficient grounds to suppose, after what has been attempted, that the formula $1 + x^3$ can never become a square, except in these three cases; namely, when

1. $x = 0$, 2. $x = -1$, and 3. $x = 2$.

But of this we may satisfy ourselves from other reasons.

122. Let us consider, for the sake of practice, the formula $1 + 3x^3$, which becomes a square in the following cases; when

1. $x = 0$, 2. $x = -1$, 3. $x = 2$,

and let us see whether we shall arrive at other similar values.

Since $x = 1$ is one of the satisfactory values, let us suppose $x = 1 + y$, and we shall thus have

$$1 + 3x^3 = 4 + 9y + 9y^2 + 3y^3.$$

Now, let the root of this new formula be $2 + py$, so that $4 + 9y + 9y^2 + 3y^3 = 4 + 4py + p^2y^2$. We must have $9 = 4p$, and $p = \frac{9}{4}$, and the other terms will give $9 + 3y = p^2 = \frac{81}{16}$, and $y = -\frac{21}{16}$; consequently, $x = -\frac{5}{16}$, and $1 + 3x^3$ becomes a square, namely, $-\frac{3721}{4096}$, the root of which is $-\frac{61}{64}$, or $+\frac{61}{64}$: and, if we chose to proceed, by making $x = -\frac{5}{16} + z$, we should not fail to find new values.

Let us also apply the second method to the same formula, and suppose the root to be $2 + py + qy^2$; which supposition gives

$$4 + 9y + 9y^2 + 3y^3 = \left\{ \begin{array}{c} 4 + 4py + 4qy^2 + 2pqy^3 + q^2y^4; \\ + p^2y^2 \end{array} \right\}$$

therefore, we must have $4p = 9$, or $p = \frac{9}{4}$, and $4q + p^2 = 9 = 4q + \frac{81}{16}$, or $q = \frac{63}{64}$: and the other terms will give $3 = 2pq + q^2y = \frac{567}{128} + q^2y$, or $567 + 128q^2y = 384$, or $128q^2y = -183$; that is to say,

$$128 \times (\tfrac{63}{64})^2y = -183, \text{ or } \frac{63^2}{32}y = -183.$$

So that $y = -\frac{1952}{1323}$, and $x = -\frac{629}{1323}$; and these values will furnish new ones, by following the methods which have been pointed out.

123. It must be remarked, however, that if we gave ourselves the trouble of deducing new values from the two, which the known case of $x = 1$ has furnished, we should arrive at fractions extremely prolix; and we have reason to be surprised that the case, $x = 1$, has not rather led us to the other, $x = 2$, which is no less evident. This, indeed, is an imperfection of the present method, which is the only mode of proceeding hitherto known.

We may, in the same manner, set out from the case $x = 2$, in order to find other values. Let us, for this purpose, make $x = 2 + y$, and it will be required to make a square of the formula, $25 + 36y + 18y^2 + 3y^3$. Here, if we suppose its root, according to the first method, to be $5 + py$, we shall have

$$25 + 36y + 18y^2 + 3y^3 = 25 + 10py + p^2y^2;$$

and, consequently, $10p = 36$, or $p = \frac{18}{5}$: then expunging the terms which destroy each other, and dividing the others by y^2, there results $18 + 3y = p^2 = \frac{324}{25}$; consequently, $y = -\frac{42}{25}$, and $x = \frac{8}{25}$; whence it follows, that $1 + 3x^3$ is a square, whose root is $5 + py = -\frac{131}{125}$, or $+\frac{131}{125}$.

In the second method, it would be necessary to suppose the root $= 5 + py + qy^2$, and we should then have

$$25 + 36y + 18y^2 + 3y^3 = \left\{ \begin{array}{r} 25 + 10py + 10qy^2 + 2pqy^3 \\ + p^2y^2 + q^3yy^3; \end{array} \right\}$$

the second and third terms would disappear by making $10p = 36$, or $p = \frac{18}{5}$, and $10q + p^2 = 18$, or $10q = 18 - \frac{324}{25} = \frac{126}{25}$, or $q = \frac{63}{125}$; and then the other terms, divided by y^3, would give $2pq + q^2y = 3$, or $q^2y = 3 - 2pq = -\frac{393}{625}$; that is, $y = -\frac{3275}{1323}$, and $x = -\frac{629}{1323}$.

124. This calculation does not become less tedious and difficult, even in the cases where, setting out differently, we can give a general solution; as, for example, when the formula proposed is $1 - x - x^2 + x^3$, in which we may make, generally, $x = n^2 - 1$, by giving any value whatever to n: for, let $n = 2$; we have then $x = 3$, and the formula becomes $1 - 3 - 9 + 27 = 16$. Let $n = 3$, we have then $x = 8$, and the formula becomes $1 - 8 - 64 + 512 = 441$, and so on.

But it should be observed, that it is to a very peculiar circumstance we owe a solution so easy, and this circumstance is readily perceived by analysing our formula into factors; for we immediately see, that it is divisible by

$1 - x$, that the quotient will be $1 - x^2$, that this quotient is composed of the factors $(1 + x) \times (1 - x)$; and, lastly, that our formula,

$$1 - x - x^2 + x^3 = (1-x) \times (1+x) \times (1-x) = (1-x)^2 \times (1+x).$$

Now, as it must be a □ [*square*], and as a □, when divisible by a □, gives a □ for the quotient*, we must also have $1 + x = □$; and, conversely, if $1 + x$ be a □, it is certain that $(1 - x)^2 \times (1 + x)$ will be a square; we have therefore only to make $1 + x = n^2$, and we immediately obtain $x = n^2 - 1$.

If this circumstance had escaped us, it would have been difficult even to have determined only five or six values of x by the preceding methods.

125. Hence we conclude, that it is proper to resolve every formula proposed into factors, when it can be done; and we have already shewn how this is to be done, by making the given formula equal to 0, and then seeking the root of this equation; for each root, as $x = f$, will give a factor $f - x$; and this inquiry is so much the easier, as here we seek only rational roots, which are always divisors of the known term, or the term which does not contain x.

126. This circumstance takes place also in our general formula, $a + bx + cx^2 + dx^3$, when the first two terms disappear, and it is consequently the quantity $cx^2 + dx^3$ that must be a square; for it is evident, in this case, that by dividing by the square x^2, we must also have $c + dx$ a square; and we have therefore only to make $c + dx = n^2$, in order to have $x = \frac{n^2 - c}{d}$, a value which contains an infinite number of answers, and even all the possible answers.

127. In the application of the first of the two preceding methods, if we do not choose to determine the letter p, for the sake of removing the second term, we shall arrive at another irrational formula, which it will be required to make rational.

For example, let $f^2 + bx + cx^2 + dx^3$ be the formula proposed, and let its root $= f + px$. Here we shall have $f^2 + bx + cx^2 + dx^3 = f^2 + 2fpx + p^2x^2$, from which the first terms vanish; dividing, therefore by x, we obtain

* The mathematical student, who may wish to acquire an extensive knowledge of the many curious properties of numbers, is referred, once for all, to the second edition of Legendre's celebrated Essai sur la Theorie des Nombres; or to Mr. Barlow's Elementary Investigation of the same subject.

$b + cx + dx^2 = 2fp + p^2x^2$, an equation of the second degree, which gives

$$x = \frac{p^2 - c + \sqrt{(p^4 - 2cp^2 + 8dfp + c^2 - 4bd)}}{2d}.$$

So that the question is now reduced to finding such values of p, as will make the formula $p^4 - 2cp^2 + 8dfp + c^2 - 4bd$ become a square. But as it is the fourth power of the required number p which occurs here, this case belongs to the following chapter.

CHAP. IX.

Of the Method *of rendering* Rational *the incommensurable* Formula $\sqrt{(a + bx + cx^2 + dx^3 + ex^4)}$.

128. We are now come to formulæ, in which the indeterminate number, x, rises to the fourth power; and this must be the limit of our researches on quantities affected by the sign of the square root; since the subject has not yet been prosecuted far enough to enable us to transform into squares any formulæ, in which higher powers of x are found.

Our new formula furnishes three cases: the first, when the first term, a, is a square; the second, when the last term, ex^4, is a square; and the third, when both the first term and the last are squares. We shall consider each of these cases separately.

129. 1st. Resolution of the formula

$$\sqrt{(f^2 + bx + cx^2 + dx^3 + ex^4)}.$$

As the first term of this is a square, we might, by the first method, suppose the root to be $f + px$, and determine p in such a manner, that the first two terms would disappear, and the others be divisible by x^2; but we should not fail still to find x^2 in the equation, and the determination of x would depend on a new radical sign. We shall therefore have recourse to the second method; and represent the root by $f + px + qx^2$; and then determine p and q, so as to remove the first three terms, and then dividing by x^3, we shall arrive at a simple equation of the first degree, which will give x without any radical signs.

130. If, therefore, the root be $f + px + qx^2$, and for that reason

$$f^2 + bx + cx^2 + dx^3 + ex^4 =$$
$$f^2 + 2fpx + p^2x^2 + 2fqx^2 + 2pqx^3 + q^2x^4,$$

the first terms disappear of themselves; with regard to the second, we shall remove them by making $b = 2fp$, or $p = \frac{b}{2f}$; and, for the third, we must make $c = 2fq + p^2$, or $q = \frac{c - p^2}{2f}$. This being done, the other terms will be divisible by x^3, and will give the equation $d + ex = 2pq + q^2x$, from which we find

$$x = \frac{d - 2pq}{q^2 - e}, \text{ or } x = \frac{2pq - d}{e - q^2}.$$

131. Now, it is easy to see that this method leads to nothing, when the second and third terms are wanting in our formula; that is to say, when $b = 0$, and $c = 0$; for then $p = 0$, and $q = 0$; consequently, $x = -\frac{d}{e}$, from which we can commonly draw no conclusion, because this case evidently gives $dx^3 + ex^4 = 0$; and, therefore, our formula becomes equal to the square f^2. But it is chiefly with respect to such formulæ as $f^2 + ex^4$, that this method is of no advantage, since in this case we have $d = 0$, which gives $x = 0$, and this leads no farther. It is the same, when $b = 0$, and $d = 0$; that is to say, the second and fourth terms are wanting, in which case the formula is

$f^2 + cx^2 + ex^4$; for, then $p = 0$, and $q = \frac{c}{2f}$, whence $x = 0$, as we may immediately perceive, from which no further advantage can result.

132. 2d. Resolution of the formula

$$\sqrt{(a + bx + cx^2 + dx^3 + g^2x^4)}.$$

We might reduce this formula to the preceding case, by supposing $x = \frac{1}{y}$; for, as the formula

$$a + \frac{b}{y} + \frac{c}{y^2} + \frac{d}{y^3} + \frac{g^2}{y^4}$$

must then be a square, and remain a square if multiplied by the square y^4, we have only to perform this multiplication, in order to obtain the formula

$$ay^4 + by^3 + cy^2 + dy + g^2,$$

which is quite similar to the former, only inverted.

But it is not necessary to go through this process; we have only to suppose the root to be $gx^2 + px + q$, or, inversely, $q + px + gx^2$, and we shall thus have

$$a + bx + cx^2 + dx^3 + g^2x^4 =$$
$$q^2 + 2pqx + 2gqx^2 + p^2x^2 + 2gpx^3 + g^2x^4.$$

Now, the fifth and sixth terms destroying each other, we shall first determine p so, that the fourth terms may also destroy each other; which happens when $d = 2gp$, or $p = \frac{d}{2g}$; we shall then likewise determine q, in order to remove the third terms, making for this purpose

$$c = 2gq + p^2, \text{ or } q = \frac{c - p^2}{2g};$$

which done, the first two terms will furnish the equation $a + bx = q^2 + 2pqx$; whence we obtain

$$x = \frac{a - q^2}{2pq - b}, \text{ or } x = \frac{q^2 - a}{b - 2pq}.$$

133. Here, again, we find the same imperfection that was before remarked, in the case where the second and fourth terms are wanting; that is to say, $b = 0$, and $d = 0$; because we then find $p = 0$, and $q = \frac{c}{2g}$; therefore $x = \frac{a - q^2}{0}$: now, this value being infinite, leads no farther than the value, $x = 0$, in the first case; whence it follows, that this method cannot be at all employed with respect to expressions of the form $a + cx^2 + g^2x^4$.

134. 3d. Resolution of the formula

$$\sqrt{(f^2 + bx + cx^2 + dx^3 + g^2x^4)}.$$

It is evident that we may employ for this formula both the methods that have been made use of; for, in the first place, since the first term is a square, we may assume $f + px + qx^2$ for the root, and make the first three terms vanish; then, as the last term is likewise a square, we may also make the root $q + px + gx^2$, and remove the last three terms; by which means we shall find even two values of x.

But this formula may be resolved also by two other methods, which are peculiarly adapted to it.

In the first, we suppose the root to be $f + px + gx^2$, and p is determined such, that the second terms destroy each other; that is to say,

$$f^2 + bx + cx^2 + dx^3 + g^2x^4 =$$
$$f^2 + 2fpx + 2fgx^2 + p^2x^2 + 2gpx^3 + g^2x^4.$$

Then, making $b = 2fp$, or $p = \frac{b}{2f}$; and since by these means both the second terms, and the first and last, are destroyed, we may divide the others by x^2, and shall have the equation $c + dx = 2fg + p^2 + 2gpx$, from which we obtain $x = \frac{c - 2fg - p^2}{2gp - d}$, or $x = \frac{p^2 + 2fg - c}{d - 2gp}$. Here, it ought to be particularly observed, that as g is found in the formula only in the second power, the root of this square, or g, may be taken negatively as well as positively; and, for this reason, we may obtain also another value of x; namely,

$$x = \frac{c + 2fg - p^2}{-2gp - d}, \text{ or } x = \frac{p^2 - 2fg - c}{2gp + d}.$$

135. There is, as we observed, another method of resolving this formula; which consists in first supposing the root, as before, to be $f + px + gx^2$, and then determining p in such a manner, that the fourth terms may destroy each other; which is done by supposing in the fundamental equation, $d = 2gp$, or $p = \frac{d}{2g}$; for, since the first and the last terms disappear likewise, we may divide the other by x, and there will result the equation $b + cx = 2fp + 2fgx + p^2x$, which gives $x = \frac{b - 2fp}{2fg + p^2 - c}$. We may farther remark, that as the square f^2 is found alone in the formula, we may suppose its root to be $-f$, from which we shall have $x = \frac{b + 2fp}{p^2 - 2fg - c}$. So that this method also furnishes two new values of x; and, consequently, the methods we have employed give, in all, six new values.

136. But here again the inconvenient circumstance occurs, that, when the second and the fourth terms are wanting, or when $b = 0$, and $d = 0$, we cannot find any value of x which answers our purpose; so that we are unable to resolve the formula $f^2 + cx^2 + gx^4$. For, if $b = 0$, and

$d=0$, we have, by both methods, $p=0$; the former giving $x=\frac{c-2fg}{0}$, and the other giving $x=0$; neither of which are proper for furnishing any further conclusions.

137. These then are the three formulæ, to which the methods hitherto explained may be applied; and, if in the formula proposed neither term be a square, no success can be expected, until we have found one such value of x as will make the formula a square.

Let us suppose, therefore, that our formula becomes a square in the case of $x=h$, or that

$$a+bh+ch^2+dh^3+eh^4=k^2;$$

if we make $x=h+y$, we shall have a new formula, the first term of which will be k^2; that is to say, a square, which will, consequently, fall under the first case: and we may also use this transformation, after having determined by the preceding methods one value of x, for instance, $x=h$; for we have then only to make $x=h+y$, in order to obtain a new equation, with which we may proceed in the same manner. And the values of x, that may be found in this manner, will furnish new ones; which will also lead to others, and so on.

138. But it is to be particularly remarked, that we can in no way hope to resolve those formulæ in which the second and fourth terms are wanting, until we have found one solution; and, with regard to the process that must be followed after that, we shall explain it by applying it to the formula $a+ex^4$, which is one of those that most frequently occur.

Suppose, therefore, we have found such a value of $x=h$, that $a+eh^4=k^2$; then if we would find, from this, other values of x, we must make $x=h+y$, and the following formula, $a+eh^4+4eh^3y+6eh^2y^2+4ehy^3+ey^4$, must be a square. Now, this formula being reducible to $k^2+4eh^3y+6eh^2y^2+4ehy^3+ey^4$, it therefore belongs to the first of our three cases; so that we shall represent its square root by $k+py+qy^2$; and, consequently, the formula itself will be equal to the square

$$k^2+2kpy+p^2y^2+2kqy^2+2pqy^3+q^2y^4;$$

from which we must first remove the second term by determining p, and consequently q; that is to say, by making $4eh^3=2kp$, or $p=\frac{2eh^3}{k}$; and $6eh^2=2kq+p^2$, or

$$q=\frac{6eh^2-p^2}{2k}=\frac{3eh^2k^2-2e^2h^6}{k^3}*=\frac{eh^2(3k^2-2eh^4)}{k^3};$$

or, lastly, $q=\frac{eh^2(k^2+2a)}{k^3}$†, because $eh^4=k^2-a$; after which, the remaining terms, $4ehy^3+ey^4$, being divided by y^3, will give $4eh+ey=2pq+q^2y$, whence we find $y=\frac{4eh-2pq}{q^2-e}$; and the numerator of this fraction may be thrown into the form $\frac{4ehk^4-4e^2h^5(k^2+2a)}{k^4}$‡,

or, because $eh^4=k^2-a$, into this,

$$\frac{4ehk^4-4eh(k^2-a)\times(k^2+2a)}{k^4}=\frac{4eh(-ak^2+2a^2)}{k^4}=\frac{4aeh(2a-k^2)}{k^4}.$$

With regard to the denominator q^2-e, since $q=\frac{eh^2(k^2+2a)}{k^3}$, and $eh^4=k^2-a$, it becomes

$$\frac{e(k^2-a)\times(k^2+2a)^2-ek^6}{k^6}=\frac{e(3ak^4-4a^3)}{k^6}=\frac{ea(3k^4-4a^2)}{k^6},$$

so that the value sought will be

$$y=\frac{4aeh(2a-k^2)}{k^4}\times\frac{k^6}{ae(3k^4-4a^2)}, \text{ or,}$$

$$y=\frac{4hk^2(2a-k^2)}{3k^4-4a^2}; \text{ and, consequently,}$$

$$x=y+h=\frac{h(8ak^2-k^4-4a^2)}{3k^4-4a^2}, \text{ or}$$

$$x=\frac{h(k^4-8ak^2+4a^2)}{4a^2-3k^4},$$

* By multiplying $6eh^2-p^2$ by k^2, and substituting for k^2p^2 its equal, $2eh^4$.

† For since $k^2=a+eh^4$, therefore $3k^2-2eh^4=3a+eh^4=k^2+2a$.

‡ Here $4eh=\frac{4ehk^4}{k^4}$, also $q=\frac{eh^2(k^2+2a)}{k^3}$, and $p=\frac{2eh^3}{k}$; therefore $2pq=\frac{4e^2h^5(k^2+2a)}{k^4}$; and, consequently,

$$4eh-2pq=\frac{4ehk^4-4e^2h^5(k^2+2a)}{k^4}. \quad \text{B.}$$

If, therefore, we substitute this value of x in the formula $a + ex^4$, it becomes a square; and its root, which we have supposed to be $k + py + qy^2$, will have this form,

$$k + \frac{8k(k^2-a)\times(2a-k^2)}{3k^4-4a^2} + \frac{16k(k^2-a)\times(k^2+2a)\times(2a-k^2)^2}{(3k^4-4a^2)^2};$$

because, as we have seen, $p = \frac{2eh^3}{k}$, $q = \frac{eh^2(k^2+2a)}{k^3}$,

$y = \frac{4hk^2(2a-k^2)}{3k^4-4a^2}$, and $eh^4 = k^2 - a$ *.

139. Let us continue the investigation of the formula $a + ex^4$; and, since the case $a + eh^4 = k^2$ is known, let us consider it as furnishing two different cases; because $x = +h$, and $x = -h$; for which reason we may transform our formula into another of the third class, in which the first term and the last are squares. This transformation is made by an artifice, which is often of great utility, and which consists in making $x = \frac{h(1+y)}{1-y}$: by which means the formula becomes

$$\frac{a(1-y)^4 + eh^4(1+y)^4}{(1-y)^4}, \text{ or rather}$$

$$\frac{k^2 + 4(k^2-2a)y + 6k^2y^2 + 4(k^2-2a)y^3 + k^2y^4}{(1-y)^4}.$$

Now, let us suppose the root of this formula, according to the third case, to be $\frac{k+py-ky^2}{(1-y)^2}$; so that the numerator of our formula must be equal to the square

$$k^2 + 2kpy + p^2y^2 - 2k^2y^2 - 2kpy^3 + k^2y^4;$$

and, removing the second terms, by making

$$4k^2 - 8a = 2kp; \text{ or } p = \frac{2k^2-4a}{k}; \text{ and dividing the}$$

* Thus,

$$py = \frac{2eh^3}{k} \times \frac{4hk^2(2a-k^2)}{3k^4-4a^2} = \frac{8eh^4k(2a-k^2)}{3k^4-4a^2} = \frac{8k(k^2-a)\times(2a-k^2)}{3k^4-4a^2};$$

$$\text{also, } qy^2 = \frac{eh^2(k^2+2a)}{k^3} \times \frac{16h^2k^4(2a-k^2)}{(3k^4-4a^2)^2} = \frac{16eh^4k(k^2+2a)\times(2a-k^2)^2}{(3k^4-4a^2)^2}$$

$$= \frac{16k(k^2-a)\times(k^2+2a)\times(2a-k^2)^2}{(3k^4-4a^2)^2}, \text{ by substituting } eh^4 = k^2 - a.$$

B.

other terms by y^2, we shall have

$$6k^2 + 4y(k^2 - 2a) = -2k^2 + p^2 - 2kpy, \text{ or}$$

$$y(4k^2 - 8a + 2kp) = p^2 - 8k^2; \text{ or}$$

$$p = \frac{2k^2-4a}{k}, \text{ and } pk = 2k^2 - 4a; \text{ so that}$$

$$y(8k^2 - 16a) = \frac{-4k^4 - 16ak^2 + 16a^2}{k^2}, \text{ and}$$

$$y = \frac{-k^4 - 4ak^2 + 4a^2}{k^2(2k^2-4a)}.$$

If we now wish to find x, we have, first,

$$1 + y = \frac{k^4 - 8ak^2 + 4a^2}{k^2(2k^2-4a)};$$

and, in the second place,

$$1 - y = \frac{3k^4 - 4a^2}{k^2(2k^2-4a)}; \text{ so that}$$

$$\frac{1+y}{1-y} = \frac{k^4 - 8ak^2 + 4a^2}{3k^4 - 4a^2}; \text{ and, consequently,}$$

$$x = \frac{h(k^4 - 8ak^2 + 4a^2)}{3k^4 - 4a^2};$$

but this is just the same value that we found before, with regard to the even powers of x.

140. In order to apply this result to an example, let it be required to make the formula $2x^4 - 1$ a square. Here, we have $a = -1$, and $e = 2$; and the known case when the formula becomes a square, is that in which $x = 1$; so that $h = 1$, and $k^2 = 1$; that is, $k = 1$; therefore, we shall have the new value, $x = \frac{1+8+4}{3-4} = -13$; and since the fourth power of x is found alone, we may also write $x = +13$, whence $2x^4 - 1 = 57121 = (239)^2$.

If we now consider this as the known case, we have $h = 13$ and $k = 239$; and shall obtain a new value of x, namely,

$$\frac{13 \times (239^4 + 8 \times 239^2 + 4)}{3 \times 239^4 - 4} = \frac{42422452969}{9788425919}.$$

141. We shall consider, in the same manner, a formula rather more general, $a + cx^2 + ex^4$, and shall take for the known case, in which it becomes a square, $x = h$; so that $a + ch^2 + eh^4 = k^2$.

And, in order to find other values from this, let us suppose $x = h + y$, and our formula will assume the following form:

$$\begin{array}{l} a \\ ch^2 + 2chy + cy^2 \\ eh^4 + 4eh^3y + 6eh^2y^2 + 4ehy^3 + ey^4 \\ \hline k^2 + (2ch + 4eh^3)y + (c + 6eh^2)y^2 + 4ehy^3 + ey^4. \end{array}$$

The first term being a square, we shall suppose the root of this formula to be $k + py + qy^2$; and the formula itself will necessarily be equal to the square

$$k^2 + 2kpy + p^2y^2 + 2kqy^2 + 2pqy^3 + q^2y^4;$$

then determining p and q, in order to expunge the second and third terms, we shall have for this purpose

$$2ch + 4eh^3 = 2kp;\ \text{or}\ p = \frac{ch + 2eh^3}{k};\ \text{and}$$

$$c + 6eh^2 = 2kq + p^2;\ \text{or}\ q = \frac{c + 6eh^2 - p^2}{2k}.$$

Now, the last two terms of the general equation being divisible by y^3, they are reduced to

$$4eh + ey = 2pq + q^2y;$$

which gives $y = \dfrac{4eh - 2pq}{q^2 - e}$, and, consequently, the value also of $x = h + y$. If we now consider this new case as the given one, we shall find another new case, and may proceed, in the same manner, as far as we please.

142. Let us illustrate the preceding article, by applying it to the formula $1 - x^2 + x^4$, in which $a = 1$, $c = -1$, and $e = 1$. The known case is evidently $x = 1$; and, therefore, $h = 1$, and $k = 1$. If we make $x = 1 + y$, and the square root of our formula $1 + py + qy^2$, we must first have $p = \dfrac{ch + 2eh^3}{k} = 1$, and then $q = \dfrac{c + 6eh^2 - p^2}{2k} = \frac{4}{2} = 2$. These values give $y = 0$, and $x = 1$. Now, this is the known case, and we have not arrived at a new one; but it is because we may prove, from other considerations, that the proposed formula can never become a square, except in the cases of $x = 0$, and $x = \pm 1$.

143. Let there be given, also, for an example, the formula $2 - 3x^2 + 2x^4$; in which $a = 2$, $c = -3$, and $e = 2$. The known case is readily found; that is, $x = 1$; so that $h = 1$, and $k = 1$: if, therefore, we make $x = 1 + y$, and the root $= 1 + py + qy^2$, we shall have $p = 1$, and $q = 4$; whence $y = 0$, and $x = 1$; which, as before, leads to nothing new.

144. Again, let the formula be $1 + 8x^2 + x^4$; in which $a = 1$, $c = 8$, and $e = 1$. Here a slight consideration is sufficient to point out the satisfactory case, namely, $x = 2$; for, by supposing $h = 2$, we find $k = 7$; so that making $x = 2 + y$, and representing the root by $7 + py + qy^2$, we shall have $p = \frac{32}{7}$, and $q = \frac{272}{343}$; whence

$$y = -\tfrac{5880}{2911},\ \text{and}\ x = -\tfrac{58}{2911};$$

and we may omit the sign *minus* in these values. But we may observe, farther, in this example, that, since the last term is already a square, and must therefore remain a square also in the new formula, we may here apply the method which has been already taught for cases of the third class. Therefore, as before, let $x = 2 + y$, and we shall have

$$\begin{array}{l} 1 \\ 32 + 32y + 8y^2 \\ 16 + 32y + 24y^2 + 8y^3 + y^4 \\ \hline 49 + 64y + 32y^2 + 8y^3 + y^4, \end{array}$$

an expression which we may now transform into a square in several ways. For, in the first place, we may suppose the root to be $7 + py + y^2$; and, consequently, the formula equal to the square

$$49 + 14py + p^2y^2 + 14y^2 + 2py^3 + y^4;$$

but then, after destroying $8y^3$, and $2py^3$, by supposing $2p = 8$, or $p = 4$, dividing the other terms by y, and deriving from the equation

$$64 + 32y = 14p + 14y + p^2y = 56 + 30y,$$

the value of $y = -4$, and of $x = -2$, or $x = +2$, we come only to the case that is already known.

Farther, if we seek to determine such a value for p, that the second terms may vanish, we shall have $14p = 64$, and $p = \frac{32}{7}$; and the other terms, when divided by y^2, form the equation $14 + p^2 + 2py = 32 + 8y$, or $\frac{1710}{49} + \frac{64}{7}y = 32 + 8y$, whence we find $y = -\frac{71}{28}$; and, consequently, $x = -\frac{15}{28}$, or $x = +\frac{15}{28}$; and this value transforms our formula into a square, whose root is $\frac{1441}{784}$. Farther, as $-y^2$ is no less the root of the last term than $+y^2$, we may suppose the root of the formula to be $7 + py - y^2$, or the formula itself equal to $49 + 14py + p^2y^2 - 14y^2 - 2py^3 + y^4$. And here we shall destroy the last terms but one, by making $-2p = 8$, or $p = -4$; then, dividing the other terms by y, we shall have

$$64 + 32y = 14p - 14p + p^2y = -56 + 2y,$$

which gives $y = -4$; that is, the known case again. If we chose to destroy the second terms, we should have $64 = 14p$, and $p = \frac{32}{7}$; and, consequently, dividing the other terms by y^2, we should obtain

$$32 + 8y = -14 + p^2 - 2py, \text{ or}$$
$$32 + 8y = \tfrac{338}{49} - \tfrac{64}{7}y; \text{ whence}$$
$$y = -\tfrac{71}{28}, \text{ and } x = -\tfrac{15}{28};$$

that is to say, the same values that we found before.

145. We may proceed, in the same manner, with respect to the general formula

$$a + bx + cx^2 + dx^3 + ex^4,$$

when we know one case, as $x = h$, in which it becomes a square, k^2. The constant method is to suppose $x = h + y$: from this, we obtain a formula of as many terms as the other, the first of them being k^2. If, after that, we express the root by $k + py + qy^2$; and determine p and q so, that the second and third terms may disappear; the last two, being divisible by y^3, will be reduced to a simple equation of the first degree, from which we may easily obtain the value of y, and, consequently, that of x also.

Still, however, we shall be obliged, as before, to exclude a great number of cases in the application of this method; those, for instance, in which the value found for x is no other than $x = h$, which was given, and in which, consequently, we could not advance one step. Such cases shew either that the formula is impossible in itself, or that we have yet to find some other case in which it becomes a square.

146. And this is the utmost length to which mathematicians have yet advanced, in the resolution of formulæ, that are affected by the sign of the square root. No discovery has hitherto been made for those, in which the quantities under the sign exceed the fourth degree; and when formulæ occur which contain the fifth, or a higher power of x, the artifices which we have explained are not sufficient to resolve them, even although a case be given.

That the truth of what is now said may be more evident, we shall consider the formula

$$k^2 + bx + cx^2 + dx^3 + ex^4 + fx^5,$$

the first term of which is already a square. If, as before, we suppose the root of this formula to be $k + px + qx^2$, and determine p and q, so as to make the second and third terms disappear, there will still remain three terms, which, when divided by x^3, form an equation of the second degree; and x evidently cannot be expressed, except by a new irrational quantity. But if we were to suppose the root to be $k + px + qx^2 + rx^3$, its square would rise to the sixth power; and, consequently, though we should even determine p, q, and r, so as to remove the second, third, and fourth terms, there would still remain the fourth, the fifth, and the sixth powers; and, dividing by x^4, we should again have an equation of the second degree, which we could not resolve without a radical sign. This seems to indicate that we have really exhausted the subject of transforming formulæ into squares: we may now, therefore, proceed to quantities affected by the sign of the cube root.

CHAP. X.

Of the Method *of rendering rational the irrational* Formula $\sqrt[3]{(a + bx + cx^2 + dx^3)}$.

147. It is here required to find such values of x, that the formula $a + bx + cx^2 + dx^3$ may become a cube, and that we may be able to extract its cube root. We see immediately that no such solution could be expected, if the formula exceeded the third degree; and we shall add, that if it were only of the second degree, that is to say, if the term dx^3 disappeared, the solution would not be easier. With regard to the case in which the last two terms disappear, and in which it would be required to reduce the formula $a + bx$ to a cube, it is evidently attended with no difficulty; for we have only to make $a + bx = p^3$, to find at once $x = \frac{p^3 - a}{b}$.

148. Before we proceed farther on this subject, we must again remark, that when neither the first nor the last term is a cube, we must not think of resolving the formula, unless we already know a case in which it becomes a cube, whether that case readily occurs, or whether we are obliged to find it out by trial.

So that we have three kinds of formulæ to consider. One is, when the first term is a cube; and as then the formula is expressed by $f^3 + bx + cx^2 + dx^3$, we imme-