$$64 + 32y = 14p - 14p + p^2y = -56 + 2y,$$

which gives $y = -4$; that is, the known case again. If we chose to destroy the second terms, we should have $64 = 14p$, and $p = \frac{32}{7}$; and, consequently, dividing the other terms by y^2, we should obtain

$$32 + 8y = -14 + p^2 - 2py, \text{ or}$$
$$32 + 8y = \tfrac{338}{49} - \tfrac{64}{7}y; \text{ whence}$$
$$y = -\tfrac{71}{28}, \text{ and } x = -\tfrac{15}{28};$$

that is to say, the same values that we found before.

145. We may proceed, in the same manner, with respect to the general formula

$$a + bx + cx^2 + dx^3 + ex^4,$$

when we know one case, as $x = h$, in which it becomes a square, k^2. The constant method is to suppose $x = h + y$: from this, we obtain a formula of as many terms as the other, the first of them being k^2. If, after that, we express the root by $k + py + qy^2$; and determine p and q so, that the second and third terms may disappear; the last two, being divisible by y^3, will be reduced to a simple equation of the first degree, from which we may easily obtain the value of y, and, consequently, that of x also.

Still, however, we shall be obliged, as before, to exclude a great number of cases in the application of this method; those, for instance, in which the value found for x is no other than $x = h$, which was given, and in which, consequently, we could not advance one step. Such cases shew either that the formula is impossible in itself, or that we have yet to find some other case in which it becomes a square.

146. And this is the utmost length to which mathematicians have yet advanced, in the resolution of formulæ, that are affected by the sign of the square root. No discovery has hitherto been made for those, in which the quantities under the sign exceed the fourth degree; and when formulæ occur which contain the fifth, or a higher power of x, the artifices which we have explained are not sufficient to resolve them, even although a case be given.

That the truth of what is now said may be more evident, we shall consider the formula

$$k^2 + bx + cx^2 + dx^3 + ex^4 + fx^5,$$

the first term of which is already a square. If, as before, we suppose the root of this formula to be $k + px + qx^2$, and determine p and q, so as to make the second and third terms disappear, there will still remain three terms, which, when divided by x^3, form an equation of the second degree; and x evidently cannot be expressed, except by a new irrational quantity. But if we were to suppose the root to be $k + px + qx^2 + rx^3$, its square would rise to the sixth power; and, consequently, though we should even determine p, q, and r, so as to remove the second, third, and fourth terms, there would still remain the fourth, the fifth, and the sixth powers; and, dividing by x^4, we should again have an equation of the second degree, which we could not resolve without a radical sign. This seems to indicate that we have really exhausted the subject of transforming formulæ into squares: we may now, therefore, proceed to quantities affected by the sign of the cube root.

CHAP. X.

Of the Method *of rendering rational the irrational* Formula $\sqrt[3]{(a + bx + cx^2 + dx^3)}$.

147. It is here required to find such values of x, that the formula $a + bx + cx^2 + dx^3$ may become a cube, and that we may be able to extract its cube root. We see immediately that no such solution could be expected, if the formula exceeded the third degree; and we shall add, that if it were only of the second degree, that is to say, if the term dx^3 disappeared, the solution would not be easier. With regard to the case in which the last two terms disappear, and in which it would be required to reduce the formula $a + bx$ to a cube, it is evidently attended with no difficulty; for we have only to make $a + bx = p^3$, to find at once $x = \frac{p^3 - a}{b}$.

148. Before we proceed farther on this subject, we must again remark, that when neither the first nor the last term is a cube, we must not think of resolving the formula, unless we already know a case in which it becomes a cube, whether that case readily occurs, or whether we are obliged to find it out by trial.

So that we have three kinds of formulæ to consider. One is, when the first term is a cube; and as then the formula is expressed by $f^3 + bx + cx^2 + dx^3$, we immediately perceive the known case to be that of $x = 0$. The second class comprehends the formula $a + bx + cx^2 + g^3x^3$; that is to say, the case in which the last term is a cube. The third class is composed of the two former; and comprehends the cases in which both the first term and the last are cubes.

149. *Case* 1. Let $f^3 + bx + cx^2 + dx^3$ be the proposed formula, which is to be transformed into a cube.

Suppose its root to be $f + px$; and, consequently, that the formula itself is equal to the cube

$$f^3 + 3f^2px + 3fp^2x^2 + p^3x^3;$$

as the first terms disappear of themselves, we shall determine p, so as to make the second terms also disappear; namely, by making $b = 3f^2p$, or $p = \frac{b}{3f^2}$; then the remaining terms being divided by x^2, give $c + dx = 3fp^2 + p^3x$; or $x = \frac{c - 3fp^2}{p^3 - d}$.

If the last term, dx^3, had not been in the formula, we might have simply supposed the cube root to be f, and should have then had $f^3 = f^3 + bx + cx^2$, or $b + cx = 0$, and $x = -\frac{b}{c}$; but this value would not have served to find others.

150. *Case* 2. If, in the second place, the proposed expression has this form, $a + bx + cx^2 + g^3x^3$, we may represent its cube root by $p + gx$, the cube of which is $p^3 + 3p^2gx + 3gp^2x^2 + g^3x^3$; so that the last terms destroy each other. Let us now determine p, so that the last terms but one may likewise disappear: which will be done by supposing $c = 3g^2p$, or $p = \frac{c}{3g^2}$, and the other terms will then give $a + bx = p^3 + 3gp^2x$; whence we find

$$x = \frac{a - p^3}{3gp^2 - b}.$$

If the first term, a, had been wanting, we should have contented ourselves with expressing the cube root by gx, and should have had

$$g^3x^3 = bx + cx^2 + g^3x^3, \text{ or } b + cx = 0,$$

whence $x = -\frac{b}{c}$; but this is of no use for finding other values.

151. *Case* 3. Lastly, let the formula be

$$f^3 + bx + cx^2 + g^3x^3,$$

in which the first and the last terms are both cubes. It is evident that we may consider this as belonging to either of the two preceding cases; and, consequently, that we may obtain two values of x.

But beside this, we may also represent the root by $f + gx$, and then make the formula equal to the cube

$$f^3 + 3f^2gx + 3fg^2x^2 + g^3x^3;$$

and likewise, as the first and last terms destroy each other, the others being divisible by x, we arrive at the equation $b + cx = 3f^2g + 3fg^2x$, which gives

$$x = \frac{b - 3f^2g}{3fg^2 - c}.$$

152. On the contrary, when the given formula belongs not to any of the above three cases, we have no other resource than to try to find such a value for x as will change it into a cube; then, having found such a value, for example, $x = h$, so that $a + bh + ch^2 + dh^3 = k^3$, we suppose $x = h + y$, and find, by substitution,

$$\begin{array}{l} a \\ bh + by \\ ch^2 + 2chy + cy^2 \\ dh^3 + 3dh^2y + 3dhy^2 + dy^3 \\ \hline k^3 + (b + 2ch + 3dh^2)y + (c + 3dh)y^2 + dy^3. \end{array}$$

This new formula belonging to the first case, we know how to determine y, and therefore shall find a new value of x, which may then be employed for finding other values.

153. Let us endeavour to illustrate this method by some examples.

Suppose it were required to transform into a cube the formula $1 + x + x^2$, which belongs to the first case. We might at once make the cube root 1, and should find $x + x^2 = 0$, that is, $x(1 + x) = 0$, and, consequently, either $x = 0$, or $x = -1$; but from this we can draw no conclusion. Let us therefore represent the cube root by $1 + px$; and as its cube is $1 + 3px + 3p^2x^2 + p^3x^3$, we shall have $3p = 1$, or $p = \frac{1}{3}$; by which means the other terms, being divided by x^2, give $3p^2 + p^3x = 1$, or $x = \frac{1-3p^2}{p^3}$. Now, $p = \frac{1}{3}$, so that $x = \frac{\frac{2}{3}}{\frac{1}{27}} = 18$, and our formula becomes $1 + 18 + 324 = 343$, and the cube root $1 + px = 7$. If now we proceed, by making $x = 18 + y$, our formula will assume the form $343 + 37y + y^2$, and by the first rule we must suppose its cube root to be $7 + py$; comparing it then with the cube

$$343 + 147py + 21p^2y^2 + p^3y^3,$$

it is evident we must make $147p = 37$, or $p = \frac{37}{147}$; the other terms give the equation $21p^2 + p^3y = 1$, whence we obtain the value of

$$y = \frac{1-21p^2}{p^3} = -\frac{147\times(147^2-21\times 37^2)}{37^3} = -\frac{1049580}{50653}, \ldots$$

which may lead, in the same manner, to new values.

154. Let it now be required to make the formula $2 + x^2$ equal to a cube. Here, as we easily get the case $x = 5$, we shall immediately make $x = 5 + y$, and shall have $27 + 10y + y^2$; supposing now its cube root to be $3 + py$, so that the formula itself may be $27 + 27py + 9p^2y^2 + p^3y^3$, we shall have to make $27p = 10$, or $p = \frac{10}{27}$; therefore $1 = 9p^2 + p^3y$, and

$$y = \frac{1-9p^2}{p^3} = -\frac{27\times(27^2-9\times 10^2)}{1000} = -\frac{4617}{1000}, \text{ and}$$

$x = \frac{383}{1000}$; therefore our formula becomes $2+x^2 = \frac{2146689}{1000000}$, the cube root of which must be $3 + py = \frac{129}{100}$.

155. Let us also see whether the formula, $1 + x^3$, can become a cube in any other cases beside the evident ones of $x = 0$, and $x = -1$. We may here remark first, that though this formula belongs to the third class, yet the root $1 + x$ is of no use to us, because its cube, $1 + 3x + 3x^2 + x^3$, being equal to the formula, gives $3x + 3x^2 = 0$, or $3x(1+x) = 0$, that is, again, $x = 0$, or $x = -1$.

If we made $x = -1 + y$, we should have to transform into a cube the formula $3y - 3y^2 + y^3$, which belongs to the second case; so that, supposing its cube root to be $p + y$, or the formula itself equal to the cube $p^3 + 3p^2y + 3py^2 + y^3$, we should have $3p = -3$, or $p = -1$, and thence the equation $3y = p^3 + 3p^2y = -1 + 3y$, which gives $y = \frac{1}{0}$, or infinity; so that we obtain nothing more from this second supposition. In fact, it is in vain to seek for other values of x; for it may be demonstrated, that the sum of two cubes, as $t^3 + x^3$, can never become

a cube*; so that, by making $t = 1$, it follows that the formula, $x^3 + 1$, can never become a cube, except in the cases already mentioned.

156. In the same manner, we shall find that the formula, $x^3 + 2$, can only become a cube in the case of $x = -1$. This formula belongs to the second case; but the rule there given cannot be applied to it, because the middle terms are wanting. It is by supposing $x = -1 + y$, which gives $1 + 3y - 3y^2 + y^3$, that the formula may be managed according to all the three cases, and that the truth of what we have advanced may be demonstrated. If, in the first case, we make the root $= 1 + y$, whose cube is $1 + 3y - 3y^2 + y^3$, we have $-3y^2 = 3y^2$, which can only be true when $y = 0$: and if, according to the second case, the root be $-1 + y$, or the formula equal to $-1 + 3y - 3y^2 + y^3$, we have $1 + 3y = -1 + 3y$, and $y = \frac{2}{0}$, or an infinite value; lastly, the third case requires us to suppose the root to be $1 + y$, which has already been done for the first case.

157. Let the formula $3x^3 + 3$ be also required to be transformed into a cube. This may be done, in the first place, if $x = -1$, but from that we can conclude nothing: then also, when $x = 2$; and if, in this second case, we suppose $x = 2 + y$, we shall have the formula $27 + 36y + 18y^2 + 3y^3$; and as this belongs to the first case, we shall represent its root by $3 + py$, the cube of which is $27 + 27py + 9p^2y^2 + p^3y^3$; then, by comparison, we find $27p = 36$, or $p = \frac{4}{3}$; and thence results the equation

$$18 + 3y = 9p^2 + p^3y = 16 + \tfrac{64}{27}y,$$

which gives $y = \frac{-54}{17}$, and, consequently, $x = \frac{-20}{17}$: therefore our formula $3 + 3x^3 = -\frac{9261}{4913}$, and its cube root $3 + py = \frac{21}{17}$; which solution would furnish new values, if we chose to proceed.

158. Let us also consider the formula $4 + x^2$, which becomes a cube in two cases that may be considered as known; namely, $x = 2$, and $x = 11$. If now we first make $x = 2 + y$, the formula $8 + 4y + y^2$ will be required to become a cube, having for its root $2 + \frac{1}{3}y$, and this cubed being $8 + 4y + \frac{2}{3}y^2 + \frac{1}{27}y^3$, we find $1 = \frac{2}{3} + \frac{1}{27}y$; therefore $y = 9$, and $x = 11$; which is the second given case.

If we here suppose $x = 11 + y$, we shall have $125 + 22y + y^2$, which, being made equal to the cube of $5 + py$, or to $125 + 75py + 15p^2y^2 + p^3y^3$, gives $p = \frac{22}{75}$;

* See Article 247 of this Part.

and thence $15p^2 + p^3y = 1$, or $p^3y = 1 - 15p^2 = -\frac{109}{375}$; consequently, $y = -\frac{122625}{10648}$, and $x = -\frac{5497}{10648}$.

And since x may either be negative or positive, x^2 being found alone in the given formula, let us suppose

$x = \frac{2+2y}{1-y}$, and our formula will become $\frac{8+8y^2}{(1-y)^2}$, which must be a cube; let us therefore multiply both terms by $1-y$, in order that the denominator may become a cube; and this will give $\frac{8-8y+8y^2-8y^3}{(1-y)^3}$: then we shall only have the numerator $8 - 8y + 8y^2 - 8y^3$, or if we divide by 8, only the formula $1 - y + y^2 - y^3$, to transform into a cube; which formula belongs to all the three cases. Let us, according to the first, take for the root $1 - \frac{1}{3}y$; the cube of which is $1 - y + \frac{1}{3}y^2 - \frac{1}{27}y^3$; so that we have $1 - y = \frac{1}{3} - \frac{1}{27}y$, or $27 - 27y = 9 - y$; therefore $y = \frac{9}{13}$; also, $1 + y = \frac{22}{13}$, and $1 - y = \frac{4}{13}$; whence $x = 11$, as before.

We should have exactly the same result, if we considered the formula as coming under the second case.

Lastly, if we apply the third, and take $1 - y$ for the root, the cube of which is $1 - 3y + 3y^2 - y^3$, we shall have $-1 + y = -3 + 3y$, and $y = 1$; so that $x = \frac{4}{0}$, or infinity; and, consequently, a result which is of no use.

159. But since we already know the two cases, $x = 2$, and $x = 11$, we may also make $x = \frac{2+11y}{1+y}$; for, by these means, if $y = 0$, we have $x = 2$; and if $y = \infty$, or infinity, we have $x = 11$.

Therefore, let $x = \frac{2+11y}{1-y}$, and our formula becomes $4 + \frac{4+44y+121y^2}{1+2y+y^2}$, or $\frac{8+52y+125y^2}{(1+y)^2}$. Multiply both terms by $1+y$, in order that the denominator may become a cube, and we shall only have the numerator $8 + 60y + 177y^2 + 125y^3$, to transform into a cube. And if, for this purpose, we suppose the root to be $2 + 5y$, we shall not only have the first terms disappear, but also the last. We may, therefore, refer our formula to the second case, taking $p + 5y$ for the root, the cube of which is $p^3 + 15p^2y + 75py^2 + 125y^3$; so that we must make $75p = 177$, or $p = \frac{59}{25}$; and there will result $8 + 60y = p^3 + 15p^2y$, or $-\frac{2943}{125}y = \frac{80379}{15625}$, and $y = \frac{80379}{367875}$, whence we might obtain a value of x.

But we may also suppose $x = \frac{2+11y}{1-y}$; and, in this case, our formula becomes

$$4 + \frac{4+44y+121y^2}{1-2y+y^2} = \frac{8+36y+125y^2}{(1-y)^2};$$

so that multiplying both terms by $1-y$, we have $8 + 28y + 89y^2 - 125y^3$ to transform into a cube. If we therefore suppose, according to the first case, the root to be $2 + \frac{7}{3}y$, the cube of which is $8 + 28y + \frac{98}{3}y^2 + \frac{343}{27}y^3$, we have $89 - 125y = \frac{98}{3} + \frac{343}{27}y$, or $\frac{3718}{27}y = \frac{169}{3}$; and, consequently, $y = \frac{1521}{3718} = \frac{9}{22}$; whence we get $x = 11$; that is, one of the values already known.

But let us rather consider our formula with reference to the third case, and suppose its root to be $2 - 5y$; the cube of this binomial being $8 - 60y + 150y^2 - 125y^3$, we shall have $28 + 89y = -60 + 150y$; therefore $y = \frac{88}{61}$, whence we get $x = -\frac{1090}{27}$; so that our formula becomes $\frac{1191016}{729}$, or the cube of $\frac{106}{9}$.

160. The foregoing are the methods which we at present know, for reducing such formulæ as we have considered, either to squares, or to cubes, provided the highest power of the unknown quantity does not exceed the fourth power in the former case, nor the third in the latter.

We might also add the problem for transforming a given formula into a biquadrate, in the case of the unknown quantity not exceeding the second degree. But it will be perceived, that, if such a formula as $a + bx + cx^2$ were proposed to be transformed into a biquadrate, it must in the first place be a square; after which it will only remain to transform the root of that square into a new square, by the rules which we have given.

If $x^2 + 7$, for example, is to be made a biquadrate, we first make it a square, by supposing

$$x = \frac{7p^2-q^2}{2pq}, \text{ or } x = \frac{q^2-7p^2}{2pq};$$

the formula then becomes equal to the square

$$\frac{q^4-14q^2p^2+49p^4}{4p^2q^2} + 7 = \frac{q^4+14q^2p^2+49p^4}{4p^2q^2},$$

the root of which $\frac{7p^2+q^2}{2pq}$ must likewise be transformed into a square; for this purpose, let us multiply the two terms by $2pq$, in order that the denominator becoming a square, we may have only to consider the numerator $2pq(7p^2 + q)$. Now, we cannot make a square of this formula, without having previously found a satisfactory case; so that supposing $q = pz$, we must have the formula

$$2p^2z(7p^2 + p^2z^2) = 2p^4z(7 + z^2),$$

and, consequently, if we divide by p^4, the formula $2z(7+z^2)$ must become a square. The known case is here $z = 1$, for which reason we shall make $z = 1 + y$, and we shall thus have

$$(2 + 2y) \times (8 + 2y + y^2) = 16 + 20y + 6y^2 + 2y^3,$$

the root of which we shall suppose to be $4 + \frac{5}{2}y$; then its square will be $16 + 20y + \frac{25}{4}y^2$, which, being made equal to the formula, gives $6 + 2y = \frac{25}{4}$; therefore $y = \frac{1}{8}$, and $z = \frac{9}{8}$: also, $z = \frac{q}{p}$; so that $q = 9$, and $p = 8$, which makes $x = \frac{367}{144}$, and the formula $7 + x^2 = \frac{279841}{20736}$. If we now extract the square root of this fraction, we find $\frac{529}{144}$; and taking the square root of this also, we find $\frac{23}{12}$; consequently, the given formula is the biquadrate of $\frac{23}{12}$.

161. Before we conclude this chapter, we must observe, that there are some formulæ, which may be transformed into cubes in a general manner; for example, if cx^2 must be a cube, we have only to make its root $= px$, and we find $cx^2 = p^3x^3$, or $c = p^3x$, that is, $x = \frac{c}{p^3}$, or $x = cq^3$, if we write $\frac{1}{q}$ instead of p.

The reason of this evidently is, that the formula contains a square, on which account, all such formulæ, as $a(b+cx)^2$, or $ab^2 + 2abcx + ac^2x^2$, may very easily be transformed into cubes. In fact, if we suppose its cube root to be $\frac{b+cx}{q}$, we shall have the equation $a(b + cx)^2 = \frac{(b + cx)^3}{q^3}$, which, divided by $(b + cx)^2$, gives $a = \frac{b + cx}{q^3}$, whence we get $x = \frac{aq^3 - b}{c}$, a value in which q is arbitrary.

This shews how useful it is to resolve the given formulæ into their factors, whenever it is possible: on this subject, therefore, we think it will be proper to dwell at some length in the following chapter.

CHAP. XI.

Of the Resolution *of the* Formula $ax^2 + bxy + cy^2$ *into its* Factors.

162. The letters x and y shall, in the present formula, represent only integer numbers; for it has been sufficiently seen, from what has been already said, that, even when we were confined to fractional results, the question may always be reduced to integer numbers. For example, if the number sought, x, be a fraction, we have only to make $x = \frac{t}{u}$, and may always assign t and u in integer numbers; and as this fraction may be reduced to its lowest terms, we shall consider the numbers t and u as having no common divisor.

Let us suppose, therefore, in the present formula, that x and y are only integer numbers, and endeavour to determine what values must be given to these letters, in order that the formula may have two or more factors. This preliminary inquiry is very necessary, before we can shew how to transform this formula into a square, a cube, or any higher power.

163. There are three cases to be considered here. The first, when the formula is really decomposed into two rational factors; which happens, as we have already seen, when $b^2 - 4ac$ becomes a square.

The second case is that in which those two factors are equal; and in which, consequently, the formula is a square.

The third case is, when the formula has only irrational factors, whether they be simply irrational, or at the same time imaginary. They will be simply irrational, when $b^2 - 4ac$ is a positive number without being a square; and they will be imaginary, if $b^2 - 4ac$ be negative.

164. If, in order to begin with the first case, we suppose that the formula is resolvible into two rational factors, we may give it this form, $(fx + gy) \times (hx + ky)$, which already contains two factors. If we then wish it to contain, in a general manner, a greater number of factors, we have only to make $fx + gy = pq$, and $hx + ky = rs$; our formula will then become equal to the product $pqrs$; and will thus necessarily contain four factors, and we may increase this number

at pleasure. Now, from these two equations we obtain a double value for x, namely, $x = \frac{pq - gy}{f}$, and $x = \frac{rs - ky}{h}$, which gives $hpq - hgy = frs - fky$; consequently,

$y = \frac{frs - hpq}{fk - hg}$, and $x = \frac{kpq - grs}{fk - hg}$: but if we choose to have x and y expressed in integer numbers, we must give such values to the letters p, q, r, and s, that the numerator may be really divisible by the denominator; which happens either when p and r, or q and s, are divisible by that denominator.

165. To render all this more clear, let there be given the formula $x^2 - y^2$, which is composed of the factors $(x + y) \times (x - y)$. Now, if this formula must be resolved into a greater number of factors, we may make $x + y = pq$, and $x - y = rs$; we shall then have $x = \frac{pq + rs}{2}$, and $y = \frac{pq - rs}{2}$; but, in order that these values may become integer numbers, the two products, pq and rs, must be either both even, or both odd.

For example, let $p = 7$, $q = 5$, $r = 3$, and $s = 1$, we shall have $pq = 35$, and $rs = 3$; therefore, $x = 19$, and $y = 16$; and thence $x^2 - y^2 = 105$, which is composed of the factors $7 \times 5 \times 3 \times 1$; so that this case is attended with no difficulty.

166. The second is attended with still less; namely, that in which the formula, containing two equal factors, may be represented thus: $(fx + gy)^2$, that is, by a square, which can have no other factors than those which arise from the root $fx + gy$; for if we make $fx + gy = pqr$, the formula becomes $p^2q^2r^2$, and may consequently have as many factors as we choose. We must farther remark, that one only of the two numbers x and y is determined, and the other may be taken at pleasure; for $x = \frac{pqr - gy}{f}$; and it is easy to give y such a value as will remove the fraction.

The easiest formula to manage of this kind, is x^2; if we make $x = pqr$, the square x^2 will contain three square factors, namely p^2, q^2, and r^2.

167. Several difficulties occur in considering the third case, which is that in which our formula cannot be resolved into two rational factors; and here particular artifices are

necessary, in order to find such values for x and y, that the formula may contain two, or more factors.

We shall, however, render this inquiry less difficult by observing, that our formula may be easily transformed into another, in which the middle term is wanting; for, in fact, we have only to suppose $x = \frac{z - by}{2a}$, in order to have the following formula:

$$\frac{z^2 - 2byz + b^2y^2}{4a} + \frac{byz - b^2y^2}{2a} + cy^2 = \frac{z^2 + (4ac - b^2)y^2}{4a}\text{: so that,}$$

neglecting the middle term, we shall consider the formula $ax^2 + cy^2$, and shall seek what values we must give to x and y, in order that this formula may be resolved into factors. Here it will be easily perceived, that this depends on the nature of the numbers a and c; so that we shall begin with some determinate formulæ of this kind.

168. Let us, therefore, first propose the formula $x^2 + y^2$, which comprehends all the numbers that are the sum of two squares, the least of which we shall set down; namely, those between 1 and 50:

1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32, 34, 36, 37, 40, 41, 45, 49, 50.

Among these numbers there are evidently some prime numbers which have no divisors, namely, the following: 2, 5, 13, 17, 29, 37, 41: but the rest have divisors, and illustrate this question, namely, 'What values are we to adopt for x and y, in order that the formula $x^2 + y^2$ may have divisors, or factors, and that it may have any number of factors?' We shall observe, farther, that we may neglect the cases in which x and y have a common divisor, because then $x^2 + y^2$ would be divisible by the same divisor, and even by its square. For example, if $x = 7p$ and $y = 7q$, the sum of the squares, or

$$49p^2 + 49q^2 = 49(p^2 + q^2),$$

will be divisible not only by 7, but also by 49: for which reason, we shall extend the question no farther than the formulæ, in which x and y are prime to each other.

We now easily see where the difficulty lies: for though it is evident, when the two numbers x and y are odd, that the formula $x^2 + y^2$ becomes an even number, and, consequently, divisible by 2; yet it is often difficult to discover whether the formula have divisors or not, when one of the numbers is even and the other odd, because the formula itself in that case is also odd. We do not mention the case in which x and y

are both even, because we have already said, that these numbers must not have a common divisor.

169. The two numbers x and y must therefore be prime to each other, and yet the formula $x^2 + y^2$ must contain two or more factors. The preceding method does not apply here, because the formula is not resolvible into two rational factors; but the irrational factors, which compose the formula, and which may be represented by the product

$$(x + y\sqrt{-1}) \times (x - y\sqrt{-1}),$$

will answer the same purpose. In fact, we are certain, if the formula $x^2 + y^2$ have real factors, that these irrational factors must be composed of other factors; because, if they had not divisors, their product could not have any. Now, as these factors are not only irrational, but imaginary; and farther, as the numbers x and y have no common divisor, and therefore cannot contain rational factors; the factors of these quantities must also be irrational, and even imaginary.

170. If, therefore, we wish the formula $x^2 + y^2$ to have two rational factors, we must resolve each of the two irrational factors into two other factors; for which reason, let us first suppose

$$x + y\sqrt{-1} = (p + q\sqrt{-1}) \times)r + s\sqrt{-1});$$

and since $\sqrt{-1}$ may be taken *minus*, as well as *plus*, we shall also have

$$x - y\sqrt{-1} = (p - q\sqrt{-1}) \times (r - s\sqrt{-1}).$$

Let us now take the product of these two quantities, and we shall find our formula $x^2 + y^2 = (p^2 + q^2) \times (r^2 + s^2)$; that is, it contains the two rational factors $p^2 + q^2$, and $r^2 + s^2$.

It remains, therefore, to determine the values of x and y, which must likewise be rational. Now, the supposition we have made, gives

$$x + y\sqrt{-1} = pr - qs + ps\sqrt{-1} + qr\sqrt{-1}, \text{ and}$$
$$x - y\sqrt{-1} = pr - qs - ps\sqrt{-1} - qr\sqrt{-1}.$$

If we add these formulæ together, we shall have $x = pr - qs$; if we subtract them from each other, we find

$$2y\sqrt{-1} = 2ps\sqrt{-1} + 2qr\sqrt{-1}, \text{ or } y = ps + qr.$$

Hence it follows, if we make $x = pr - qs$, and $y = ps + qr$, that our formula $x^2 + y^2$ must have two factors, since we find $x^2 + y^2 = (p^2 + q^2) \times (r^2 + s^2)$. If, after this, a greater number of factors be required, we have only to assign, in the same manner, such values to p and q, that $p^2 + q^2$ may have two factors; we shall then have three

factors in all, and the number might be augmented by this method to any length.

171. As in this solution we have found only the second powers of p, q, r, and s, we may also take these letters *minus*. If q, for example, be negative, we shall have $x = pr + qs$, and $y = ps - qr$; but the sum of the squares will be the same as before; which shews, that when a number is equal to a product, such as $(p^2 + q^2) \times (r^2 + s^2)$, we may resolve it into two squares in two ways; for we have first found $x = pr - qs$, and $y = ps - qr$, and then also

$$x = pr + qs, \text{ and } y = ps - qr.$$

For example, let $p = 3$, $q = 2$, $r = 2$, and $s = 1$: then we shall have the product $13 \times 5 = 65 = x^2 + y^2$; in which $x = 4$, and $y = 7$; or $x = 8$, and $y = 1$; since in both cases $x^2 + y^2 = 65$. If we multiply several numbers of this class, we shall also have a product, which may be the sum of two squares in a greater number of ways. For example, if we multiply together $2^2 + 1^2 = 5$, $3^2 + 2^2 = 13$, and $4^2 + 1^2 = 17$, we shall find 1105, which may be resolved into two squares in four ways, as follows:

$$1.\ 33^2 + 4^2, \quad 2.\ 32^2 + 9^2,$$
$$3.\ 31^2 + 12^2, \quad 4.\ 24^2 + 23^2.$$

172. So that among the numbers that are contained in the formula $x^2 + y^2$, are found, in the first place, those which are, by multiplication, the product of two or more numbers, prime to each other; and, secondly, those of a different class. We shall call the latter *simple factors* of the formula $x^2 + y^2$, and the former *compound factors*; then the simple factors will be such numbers as the following:

1, 2, 5, 9, 13, 17, 29, 37, 41, 49, &c.

and in this series we shall distinguish two kinds of numbers; one are prime numbers, as 2, 5, 13, 17, 29, 37, 41, which have no divisor, and are all (except the number 2), such, that if we subtract 1 from them, the remainder will be divisible by 4; so that all these numbers are contained in the expression $4n + 1$. The second kind comprehends the square numbers 9, 49, &c. and it may be observed, that the roots of these squares, namely, 3, 7, &c. are not found in the series, and that their roots are contained in the formulæ $4n - 1$. It is also evident, that no number of the form $4n - 1$ can be the sum of two squares; for since all numbers of this form are odd, one of the two squares must be even, and the other odd. Now, we have already seen, that all even squares are divisible by 4, and that the odd squares are contained in the formula $4n + 1$: if we therefore add together an even and an odd square, the sum will always have the form of $4n + 1$, and never of $4n - 1$. Farther, every prime number, which belongs to the formula $4n + 1$, is the sum of two squares; this is undoubtedly true, but it is not easy to demonstrate it*.

173. Let us proceed farther, and consider the formula $x^2 + 2y^2$, that we may see what values we must give to x and y, in order that it may have factors. As this formula is expressed by the imaginary factors $(x + y\sqrt{-2}) \times (x - y\sqrt{-2})$, it is evident, as before, that, if it have divisors, these imaginary factors must likewise have divisors. Suppose, therefore,

$$x + y\sqrt{-2} = (p + q\sqrt{-2}) \times (r + s\sqrt{-2}),$$

whence it immediately follows, that

$$x - y\sqrt{-2} = (p - q\sqrt{-2}) \times (r - s\sqrt{-2}),$$

and we shall have

$$x^2 + 2y^2 = (p^2 + 2q^2) \times (r^2 + 2s^2);$$

so that this formula has two factors, both of which have the same form. But it remains to determine the values of x and y, which produce this transformation. For this purpose, we shall consider that, since

$$x + y\sqrt{-2} = pr - 2qs + qr\sqrt{-2} + ps\sqrt{-2}, \text{ and}$$
$$x - y\sqrt{-2} = pr - 2qs - qr\sqrt{-2} - ps\sqrt{-2},$$

we have the sum $2x = 2pr - 4qs$; and, consequently, $x = pr - 2qs$: also the difference

$$2y\sqrt{-2} = 2qr\sqrt{-2} + 2ps\sqrt{-2};$$

so that $y = qr + ps$. When, therefore, our formula $x^2 + 2y^2$ has factors, they will always be numbers of the same kind as the formula; that is to say, one will have the form $p^2 + 2q^2$, and the other the form $r^2 + 2s^2$; and, in order that this may be the case, x and y may also be determined in two different ways, because q may be either positive or negative; for we shall first have $x = pr - 2qs$, and $y = ps + qr$; and, in the second place, $x = pr + 2qs$, and $y = ps - qr$.

174. This formula $x^2 + 2y^2$ comprehends therefore all the numbers which result from adding together a square and twice another square. The following is an enumeration of these numbers as far as 50;

1, 2, 3, 4, 6, 8, 9, 11, 12, 16, 17, 18, 19, 22, 24, 25, 27, 32, 33, 34, 36, 38, 41, 43, 44, 49, 50.

* The curious reader may see it demonstrated by Gauss, in his "Disquisitiones Arithmeticæ;" and by De la Grange, in the Memoirs of Berlin, 1768.

We shall divide these numbers, as before, into simple and compound; the simple, or those which are not compounded of the preceding numbers, are these: 1, 2, 3, 11, 17, 19, 25, 41, 43, 49, all which, except the squares 25 and 49, are prime numbers; and we may remark, in general, that, if a number is prime, and is not found in this series, we are sure to find its square in it. It may be observed, also, that all prime numbers contained in our formula, either belong to the expression $8n + 1$, or $8n + 3$; while all the other prime numbers, namely, those which are contained in the expressions $8n + 5$, and $8n + 7$, can never form the sum of a square and twice a square: it is farther certain, that all the prime numbers which are contained in one of the other formulæ, $8n + 1$, and $8n + 3$, are always resolvible into a square added to twice a square.

175. Let us proceed to the examination of the general formula $x^2 + cy^2$, and consider by what values of x and y we may transform it into a product of factors.

We shall proceed as before; that is, we shall represent the formula by the product

$$(x + y\sqrt{-c}) \times (x - y\sqrt{-c}),$$

and shall likewise express each of these factors by two factors of the same kind; that is, we shall make

$$x + y\sqrt{-c} = (p + q\sqrt{-c}) \times (r + s\sqrt{-c}), \text{ and}$$
$$x - y\sqrt{-c} = (p - q\sqrt{-c}) \times (r - s\sqrt{-c}); \text{ whence}$$
$$x^2 + cy^2 = (p^2 + cq^2) \times (r^2 + cs^2).$$

We see, therefore, that the factors are again of the same kind with the formula. With regard to the values of x and y, we shall readily find $x = pr + cqs$, and $y = qr - ps$; or $x = pr - cqs$, and $y = ps + qr$; and it is easy to perceive how the formula may be resolved into a greater number of factors.

176. It will not now be difficult to obtain factors for the formula $x^2 - cy^2$; for, in the first place, we have only to write $-c$, instead of $+c$; but, farther, we may find them immediately in the following manner. As our formula is equal to the product

$$(x + y\sqrt{c}) \times (x - y\sqrt{c}),$$

let us make $x + y\sqrt{c} = (p + q\sqrt{c}) \times (rs + \sqrt{c})$, and $x - y\sqrt{c} = (p - q\sqrt{c}) \times (r - s\sqrt{c})$, and we shall immediately have $x^2 - cy^2 = (p^2 - cq^2) \times (r^2 - cs^2)$; so that this formula, as well as the preceding, is equal to a product whose factors resemble it in form. With regard to the values of x and y, they will likewise be found to be double; that is to say, we shall have

$x = pr + cqs$, and $y = qr + ps$; we shall also have $x = pr - cqs$, and $y = ps - qr$. If we chose to make trial, and see whether we obtain from these values the product already found, we should have, by trying the first,

$$x^2 = p^2r^2 + 2cpqrs + c^2q^2s^2, \text{ and}$$
$$y^2 = p^2s^2 + 2pqrs + q^2r^2, \text{ or}$$
$$cy^2 = cp^2s^2 + 2cpqrs + cq^2r^2; \text{ so that}$$
$$x^2 - cy^2 = p^2r^2 - cp^2s^2 + c^2q^2s^2 - cq^2r^2,$$ which is just the product already found, $(p^2 - cq^2) \times (r^2 - cs^2)$.

177. Hitherto we have considered the first term as without a coefficient; but we shall now suppose that term to be multiplied also by another letter, and shall seek what factors the formula $ax^2 + cy^2$ may contain.

Here it is evident that our formula is equal to the product $(x\sqrt{a} + y\sqrt{-c}) \times (x\sqrt{a} - y\sqrt{-c})$, and, consequently, that it is required to give factors also to these two factors. Now, in this a difficulty occurs; for if, according to the second method, we make

$x\sqrt{a} + y\sqrt{-c} = (p\sqrt{a} + q\sqrt{-c}) \times (r\sqrt{a} + s\sqrt{-c}) = apr - cqs + ps\sqrt{-ac} + qr\sqrt{-ac}$, and
$x\sqrt{a} - y\sqrt{-c} = (p\sqrt{a} - q\sqrt{-c}) \times (r\sqrt{a} - s\sqrt{-c}) = apr - cqs - ps\sqrt{-ac} - qr\sqrt{-ac}$, we shall have
$2x\sqrt{a} = 2apr - 2cqs$, and
$2y\sqrt{-c} = 2ps\sqrt{-ac} + 2qr\sqrt{-ac}$; that is to say, we have found both for x and for y irrational values, which cannot here be admitted.

178. But this difficulty may be removed thus: let us make

$x\sqrt{a} + y\sqrt{-c} = (p\sqrt{a} + q\sqrt{-c}) \times (r + s\sqrt{-ac}) = pr\sqrt{a} - cqs\sqrt{a} + qr\sqrt{-c} + aps\sqrt{-c}$, and
$x\sqrt{a} - y\sqrt{-c} = (p\sqrt{a} - q\sqrt{-c}) \times (r - s\sqrt{-ac}) = pr\sqrt{a} - cqs\sqrt{a} - qr\sqrt{-c} - aps\sqrt{-c}$. This supposition will give the following values for x and y; namely, $x = pr - cqs$, and $y = qr + aps$; and our formula, $ax^2 + cy^2$, will have the factors $(ap^2 + cq^2) \times (r^2 + acs^2)$, one of which only is of the same form with the formula, the other being different.

179. There is still, however, a great affinity between these two formulæ, or factors; since all the numbers contained in the first, if multiplied by a number contained in the second, revert again to the first. We have already seen, that two numbers of the second form $x^2 + acy^2$, which

returns to the formula $x^2 + cy^2$, and which we have already considered, if multiplied together, will produce a number of the same form.

It only remains, therefore, to examine to what formula we are to refer the product of two numbers of the first kind, or of the form $ax^2 + cy^2$.

For this purpose, let us multiply the two formulæ $(ap^2 + cq^2) \times (ar^2 + cs^2)$, which are of the first kind. It is easy to see that this product may be represented in the following manner: $(apr + cqs)^2 + ac(ps - qr)^2$. If, therefore, we suppose

$$apr + cqs = x, \text{ and } ps - qr = y,$$

we shall have the formula $x^2 + acy^2$, which is of the last kind. Whence it follows, that if two numbers of the first kind, $ax^2 + cy^2$, be multiplied together, the product will be a number of the second kind. If we represent the numbers of the first kind by I, and those of the second by II, we may represent the conclusion to which we have been led, abridged as follows:

I × I gives II; I × II gives I; II × II gives II.

And this shews much better what the result ought to be, if we multiply together more than two of these numbers; namely, that I × I × I gives I; that I × I × II gives II; that I × II × II gives I; and lastly, that II × II × II gives II.

180. In order to illustrate the preceding Article, let $a = 2$, and $c = 3$; there will result two kinds of numbers, one contained in the formula $2x^2 + 3y^2$, the other contained in the formula $x^2 + 6y^2$. Now, the numbers of the first kind, as far as 50, are

1st, 2, 3, 5, 8, 11, 12, 14, 18, 20, 21, 27, 29, 30, 32, 35, 44, 45, 48, 50;

and the numbers of the second kind, as far as 50, are

2d, 1, 4, 6, 7, 9, 10, 15, 16, 22, 24, 25, 28, 31, 33, 36, 40, 42, 49.

If, therefore, we multiply a number of the first kind, for example, 35, by a number of the second, suppose 31, the product 1085 will undoubtedly be contained in the formula $2x^2 + 3y^2$; that is, we may find such a number for y, that $1085 - 3y^2$ may be the double of a square, or $= 2x^2$: now, this happens, first, when $y = 3$, in which case $x = 23$; in the second place, when $y = 11$, so that $x = 19$; in the third place, when $y = 13$, which gives $x = 17$; and, in the fourth place, when $y = 19$, whence $x = 1$.

We may divide these two kinds of numbers, like the others, into *simple* and *compound* numbers: we shall apply this latter term to such as are composed of two or more of the smallest numbers of either kind; so that the simple numbers of the first kind will be 2, 3, 5, 11, 29; and the compound numbers of the same class will be 8, 12, 14, 18, 20, 27, 30, 32, 35, 40, 45, 48, 50, &c.

The simple numbers of the second class will be 1, 7, 31: and all the rest of this class will be compound numbers namely, 4, 6, 9, 10, 15, 16, 22, 24, 25, 28, 33, 36, 40, 42, 49.

CHAP. XII.

Of the Transformation *of the* Formula $ax^2 + cy^2$ *into* Squares, *and higher* Powers.

181. We have seen that it is frequently impossible to reduce numbers of the form $ax^2 + cy^2$ to squares; but whenever it is possible, we may transform this formula into another, in which $a = 1$.

For example, the formula $2p^2 - q^2$ may become a square; for, as it may be represented by

$$(2p + q)^2 - 2(p + q)^2,$$

we have only to make $2p + q = x$, and $p + q = y$, and we shall get the formula $x^2 - 2y^2$, in which $a = 1$, and $c = 2$. A similar transformation always takes place, whenever such formulæ can be made squares. Thus, when it is required to transform the formula $ax^2 + cy^2$ into a square, or into a higher power, (provided it be even) we may, without hesitation, suppose $a = 1$, and consider the other cases as impossible.

182. Let, therefore, the formula $x^2 + cy^2$ be proposed, and let it be required to make it a square. As it is composed of the factors $(x + y\sqrt{-c}) \times (x - y\sqrt{-c})$, these factors must either be squares, or squares multiplied by the same number. For, if the product of two numbers, for example, pq, must be a square, we must have $p = r^2$, and $q = s^2$; that is to say, each factor is of itself a square; or $p = mr^2$, and $q = ms^2$; and therefore these factors are squares multiplied both by the same number. For which reason, let us make $x + y\sqrt{-c} = m(p + q\sqrt{-c})^2$; it will follow that $x - y\sqrt{-c} = m(p - q\sqrt{-c})^2$, and we shall have $x^2 + cy^2 = m^2(p^2 + cq^2)^2$, which is a square.

Farther, in order to determine x and y, we have the equations $x + y\sqrt{-c} = mp^2 + 2mpq\sqrt{-c} - mcq^2$, and $x - y\sqrt{-c} = mp^2 - 2mpq\sqrt{-c} - mcq^2$; in which x is necessarily equal to the rational part, and $y\sqrt{-c}$ to the irrational part; so that $x = mp^2 - mcq^2$, and $y\sqrt{-c} = 2mpq\sqrt{-c}$, or $y = 2mpq$; and these are the values of x and y that will transform the expression $x^2 + cy^2$ into a square, $m^2(p^2 + cq^2)^2$, the root of which is $mp^2 + mcq^2$.

183. If the numbers x and y have not a common divisor, we must make $m = 1$. Then, in order that $x^2 + cy^2$ may become a square, it will be sufficient to make $x = p^2 - cq^2$, and $y = 2pq$, which will render the formula equal to the square $(p^2 + cq^2)^2$.

Or, instead of making $x = p^2 - cq^2$, we may also suppose $x = cq^2 - p^2$, since the square x^2 is still left the same.

Besides, the same formulæ having been already found by methods altogether different, there can be no doubt with regard to the accuracy of the method which we have now employed. In fact, if we wish to make $x^2 + cy^2$ a square, we suppose, by the former method, the root to be $x + \frac{py}{q}$, and find $x^2 + cy^2 = x^2 + \frac{2pxy}{q} + \frac{p^2y^2}{q^2}$.

Expunge the x^2, divide the other terms by y, multiply by q^2, and we shall have

$$cq^2y = 2pqx + p^2y;\ \text{or}\ cq^2y - p^2y = 2pqx.$$

Lastly, dividing by $2pq$, and also by y, there results $\frac{x}{y} = \frac{cq^2 - p^2}{2pq}$. Now, as x and y, as well as p and q, are to have no common divisor, we must make x equal to the numerator, and y equal to the denominator, and hence we shall obtain the same results as we have already found, namely, $x = cq^2 - p^2$, and $y = 2pq$.

184. This solution will hold good, whether the number c be positive or negative; but, farther, if this number itself had factors, as, for instance, the formula $x^2 + acy^2$, we should not only have the preceding solution, which gives $x = acq^2 - p^2$, and $y = 2pq$, but this also, namely, $x = cq^2 - ap^2$, and $y = 2pq$; for, in this last case, we have, as in the other,

$$x^2 + acy^2 = c^2q^4 + 2acp^2q^2 + a^2p^4 = (cq^2 + ap^2)^2;$$

which takes place also when we make $x = ap^2 - cq^2$, because the square x^2 remains the same.

This new solution is also obtained from the last method, in the following manner:

If we make $x + y\sqrt{-ac} = (p\sqrt{a} + q\sqrt{-c})^2$, and $x - y\sqrt{-ac} = (p\sqrt{a} - q\sqrt{-c})^2$, we shall have $x^2 + acy^2 = (ap^2 + cq^2)^2$, and, consequently, equal to a square. Farther, because

$$x + y\sqrt{-ac} = ap^2 + 2pq\sqrt{-ac} - cq^2, \text{ and}$$
$$x - y\sqrt{-ac} = ap^2 - 2pq\sqrt{-ac} - cq^2,$$

we find $x = ap^2 - cq^2$, and $y = 2pq$.

It is farther evident, that if the number ac be resolvible into two factors, in a greater number of ways, we may also find a greater number of solutions.

185. Let us illustrate this by means of some determinate formulæ; and, first, if the formula $x^2 + y^2$ must become a square, we have $ac = 1$; so that $x = p^2 - q^2$, and $y = 2pq$; whence it follows that $x^2 + y^2 = (p^2 + q^2)^2$.

If we would have $x^2 - y^2 = \square$; we have $ac = -1$; so that we shall take $x = p^2 + q^2$, and $y = 2pq$, and there will result $x^2 - y^2 = (p^2 - q^2)^2 = \square$.

If we would have the formula $x^2 + 2y^2 = \square$, we have $ac = 2$; let us therefore take $x = p^2 - 2q^2$, or $x = 2p^2 - q^2$, and $y = 2pq$, and we shall have

$$x^2 + 2y^2 = (p^2 + q^2)^2, \text{ or } x^2 + 2y^2 = (2p^2 + q^2)^2.$$

If, in the fourth place, we would have $x^2 - 2y^2 = \square$, in which $ac = -2$, we shall have $x = p^2 + 2q^2$, and $y = 2pq$; therefore $x^2 - 2y^2 = (p^2 - 2q^2)^2$.

Lastly, let us make $x^2 + 6y^2 = \square$. Here we shall have $ac = 6$; and, consequently, either $a = 1$, and $c = 6$, or $a = 2$, and $c = 3$. In the first case, $x = p^2 - 6q^2$, and $y = 2pq$; so that $x^2 + 6y^2 = (p^2 + 6q^2)^2$; in the second, $x = 2p^2 - 3q^2$, and $y = 2pq$; whence

$$x^2 + 6y^2 = (2p^2 + 3q^2)^2.$$

186. But let the formula $ax^2 + cy^2$ be proposed to be transformed into a square. We know beforehand, that this cannot be done, except we already know a case, in which this formula really becomes a square; but we shall find this given case to be, when $x = f$, and $y = g$; so that $af^2 + cg^2 = h^2$; and we may observe, that this formula can be transformed into another of the form $t^2 + acu^2$, by making

$$t = \frac{afx + cgy}{h}, \text{ and } u = \frac{gx - fy}{h}; \text{ for if}$$

$$t^2 = \frac{a^2f^2x^2 + 2acfgxy + c^2g^2y^2}{h^2}, \text{ and}$$

$$u^2 = \frac{g^2x^2 - 2fgxy + f^2y^2}{h^2}, \text{ we have}$$

$$t^2 + acu^2 = \frac{a^2f^2x^2 + c^2g^2y^2 + acg^2x^2 + acf^2y^2}{h^2} =$$

$$\frac{ax^2(af^2 + cg^2) + cy^2(af^2 + cg^2)}{h^2};$$

also, since $af^2 + cg^2 = h^2$, we have $t^2 + acu^2 = ax^2 + cy^2$. Thus, we have given easy rules for transforming the expression $t^2 + acu^2$ into a square, to which we have now reduced the formula proposed, $ax^2 + cy^2$.

187. Let us proceed farther, and see how the formula $ax^2 + cy^2$, in which x and y are supposed to have no common divisor, may be reduced to a cube. The rules already given are by no means sufficient for this; but the method which we have last explained applies here with the greatest success: and what is particularly worthy of observation, is, that the formula may be transformed into a cube, whatever numbers a and c are; which could not take place with regard to squares, unless we already knew a case, and which does not take place with regard to any of the other even powers; but, on the contrary, the solution is always possible for the odd powers, such as the third, the fifth, the seventh, &c.

188. Whenever, therefore, it is required to reduce the formula $ax^2 + cy^2$ to a cube, we may suppose, according to the method which we have already employed, that

$$x\sqrt{a} + y\sqrt{-c} = (p\sqrt{a} + q\sqrt{-c})^3, \text{ and}$$
$$x\sqrt{a} - y\sqrt{-c} = (p\sqrt{a} - q\sqrt{-c})^3;$$

the product $(ap^2 + cq^2)^3$, which is a cube, will be equal to the formula $ax^2 + cy^2$. But it is required, also, to determine rational values for x and y, and fortunately we succeed. If we actually take the two cubes that have been pointed out, we have the two equations

$$x\sqrt{a} + y\sqrt{-c} = ap^3\sqrt{a} + 3ap^2q\sqrt{-c} - 3cpq^2\sqrt{a} - cq^3\sqrt{-c}, \text{ and}$$
$$x\sqrt{a} - y\sqrt{-c} = ap^3\sqrt{a} - 3ap^2q\sqrt{-c} - 3cpq^2\sqrt{a} + cq^3\sqrt{-c};$$

from which it evidently follows, that

$$x = ap^3 - 3cpq^2, \text{ and } y = 3ap^2q - cq^3.$$

For example, let two squares x^2, and y^2, be required, whose sum, $x^2 + y^2$, may make a cube. Here, since $a = 1$, and $c = 1$, we shall have $x = p^3 - 3pq^2$, and $y = 3p^2q - q^3$, which gives $x^2 + y^2 = (p^2 + q^2)^3$. Now, if $p = 2$, and $q = 1$, we find $x = 2$, and $y = 11$; wherefore

$$x^2 + y^2 = 125 = 5^3.$$

189. Let us also consider the formula $x^2 + 3y^2$, for the purpose of making it equal to a cube. As we have, in this case, $a = 1$, and $c = 3$, we find

$$x = p^3 - 9pq^2, \text{ and } y = 3p^2q - 3q^3,$$

whence $x^2 + 3y^2 = (p^2 + 3q^2)^3$. This formula occurs very frequently; for which reason we shall here give a Table of the easiest cases.

p	q	x	y	$x^2 + 3y^2$
1	1	8	0	$64 = 4^3$
2	1	10	9	$343 = 7^3$
1	2	35	18	$2197 = 13$
3	1	0	24	$1728 = 12^3$
1	3	80	72	$21952 = 28^3$
3	2	81	30	$9261 = 21^3$
2	3	154	45	$29791 = 31^3$

190. If the question were not restricted to the condition, that the numbers x and y must have no common divisor, it would not be attended with any difficulty; for if $ax^2 + cy^2$ were required to be a cube, we should only have to make $x = tz$, and $y = uz$, and the formula would become $at^2z^2 + cu^2z^2$; which we might make equal to the cube $\frac{z^3}{v^3}$, and should immediately find $z = v^3(at^2 + cu^2)$. Consequently, the values sought of x and y would be $x = tv^3(at^2 + cu^2)$, and $y = uv^3(at^2 + cu^2)$, which, beside the cube v^3, have also the quantity $at^2 + cu^2$ for a common divisor; so that this solution immediately gives

$$ax^2 + cy^2 = v^6(at^2 + cu^2)^2 \times (at^2 + cu^2) = v^6(at^2 + cu^2)^3,$$

which is evidently the cube of $v^2(at^2 + cu^2)$.

191. This last method, which we have made use of, is so much the more remarkable, as we are brought to solutions, which absolutely required numbers rational and integer, by means of irrational, and even imaginary quantities; and, what is still more worthy of attention, our method cannot be applied to those cases, in which the irrationality vanishes. For example, when the formula $x^2 + cy^2$ must become a cube, we can only infer from it, that its two irrational factors, $x + y\sqrt{-c}$, and $x - y\sqrt{-c}$, must likewise be cubes; and since x and y have no common divisor, these factors cannot have any. But if the radicals were to disappear, as in the case of $c = -1$, this principle would no longer exist; because the two factors, which would then be $x + y$, and $x - y$, might have common divisors, even when x and y had none; as would be the case, for example, if both these letters expressed odd numbers.

Thus, when $x^2 - y^2$ must become a cube, it is not necessary that both $x + y$, and $x - y$, should of themselves be cubes; but we may suppose $x + y = 2p^3$, and $x - y = 4q^3$; and the formula $x^2 - y^2$ will undoubtedly become a cube, since we shall find it to be $8p^3q^3$, the cube root of which is $2pq$. We shall further have $x = p^3 + 2q^3$, and $y = p^3 - 2q^3$. On the contrary, when the formula $ax^2 + cy^2$ is not resolvable into two rational factors, we cannot find any other solutions beside those which have been already given.

192. We shall illustrate the preceding investigations by some curious examples.

Question 1. Required a square, x^2, in integer numbers, and such, that, by adding 4 to it, the sum may be a cube. The condition is answered when $x^2 = 121$; but we wish to know if there are other similar cases.

As 4 is a square, we shall first seek the cases in which $x^2 + y^2$ becomes a cube. Now, we have found one case, namely, if $x = p^3 - 3pq^2$, and $y = 3p^2q - q^3$: therefore, since $y^2 = 4$, we have $y = \pm 2$, and, consequently, either $3p^2q - q^3 = +2$, or $3p^2q - q^3 = -2$. In the first case, we have $q(3p^2 - q^2) = 2$, so that q is a divisor of 2.

This being laid down, let us first suppose $q = 1$, and we shall have $3p^2 - 1 = 2$; therefore $p = 1$; whence $x = 2$, and $x^2 = 4$.

If, in the second place, we suppose $q = 2$, we have $6p^2 - 8 = \pm 2$; admitting the sign $+$, we find $6p^2 = 10$, and $p^2 = \frac{5}{3}$; whence we should get an irrational value of p, which could not apply here; but if we consider the sign $-$, we have $6p^2 = 6$, and $p = 1$; therefore $x = 11$: and these are the only possible cases; so that 4, and 121, are the only two squares, which, added to 4, give cubes.

193. *Question* 2. Required, in integer numbers, other squares, beside 25, which, added to 2, give cubes.

Since $x^2 + 2$ must become a cube, and since 2 is the double of a square, let us first determine the cases in which $x^2 + 2y^2$ becomes a cube; for which purpose we have, by Article 188, in which $a = 1$, and $c = 2$, $x = p^3 - 6pq^2$, and $y = 3p^2q - 2q^3$; therefore, since $y = \pm 1$, we must have $3p^2q - q^3$, or $q(3p^2 - 2q^2) = \pm 1$; and, consequently, q must be a divisor of 1.

Therefore let $q = 1$, and we shall have $3p^2 - 2 = \pm 1$.

If we take the upper sign, we find $3p^2 = 3$, and $p = 1$; whence $x = 5$: and if we adopt the other sign, we get a value of p, which being irrational, is of no use; it follows, therefore, that there is no square, except 25, which has the property required.

194. *Question 3.* Required squares, which, multiplied by 5, and added to 7, may produce cubes; or it is required that $5x^2 + 7$ should be a cube.

Let us first seek the cases in which $5x^2 + 7y^2$ becomes a cube. By Article 188, a being equal to 5, and c equal 7, we shall find that we must have $x = 5p^3 - 21pq^2$, and $y = 15p^2q - 7q^3$; so that in our example y being $= \pm 1$, we have $15p^2q - 7q^3 = q(15p^2 - 7q^2) = \pm 1$; therefore q must be a divisor of 1; that is to say, $q = \pm 1$; consequently, we shall have $15p^2 - 7 = \pm 1$; from which, in both cases, we get irrational values for p: but from which we must not, however, conclude that the question is impossible, since p and q might be such fractions, that $y = 1$, and that x would become an integer; and this is what really happens; for if $p = \frac{1}{2}$, and $q = \frac{1}{2}$, we find $y = 1$, and $x = 2$; but there are no other fractions which render the solution possible.

195. *Question 4.* Required squares in integer numbers, the double of which, diminished by 5, may be a cube; or it is required that $2x^2 - 5$ may be a cube.

If we begin by seeking the satisfactory cases for the formula $2x^2 - 5y^2$, we have, in the 188th Article, $a = 2$, and $c = -5$; whence $x = 2p^3 + 15pq^2$, and $y = 6p^2q + 5q^3$: so that, in this case, we must have $y = \pm 1$; consequently,

$$6p^2q + 5q^3 = q(6p^2 + 5q^2) = \pm 1;$$

and as this cannot be, either in integer numbers, or even in fractions, the case becomes very remarkable, because there is, notwithstanding, a satisfactory value of x; namely, $x = 4$; which gives $2x^2 - 5 = 27$, or equal to the cube of 3. It will be of importance to investigate the cause of this peculiarity.

196. It is not only possible, as we see, for the formula $2x^2 - 5y^2$ to be a cube; but, what is more, the root of this cube has the form $2p^2 - 5q^2$, as we may perceive by making $x = 4$, $y = 1$, $p = 2$, and $q = 1$; so that we know a case in which $2x^2 - 5y^2 = (2p^2 - 5q^2)^3$, although the two factors of $2x^2 - 5y^2$, namely, $x\sqrt{2} + y\sqrt{5}$, and $x\sqrt{2} - y\sqrt{5}$, which, according to our method, ought to be the cubes of $p\sqrt{2} + q\sqrt{5}$, and of $p\sqrt{2} - q\sqrt{5}$, are not cubes; for, in our case, $x\sqrt{2} + y\sqrt{5} = 4\sqrt{2} + \sqrt{5}$; whereas

$$(p\sqrt{2} + q\sqrt{5})^3 = (2\sqrt{2} + \sqrt{5})^3 = 46\sqrt{2} + 29\sqrt{5},$$

which is by no means the same as $4\sqrt{2} + \sqrt{5}$.

But it must be remarked, that the formula $r^2 - 10s^2$ may become 1, or -1, in an infinite number of cases; for example, if $r = 3$, and $s = 1$, or if $r = 19$, and $s = 6$: and this formula, multiplied by $2p^2 - 5q^2$, reproduces a number of this last form.

Therefore, let $f^2 - 10g^2 = 1$; and, instead of supposing, as we have hitherto done, $2x^2 - 5y^2 = (2p^2 - 5q^2)^3$, we may suppose, in a more general manner,

$$2x^2 - 5y^2 = (f^2 - 10g^2) \times (2p^2 - 5q^2)^3;$$

so that, taking the factors, we shall have

$$x\sqrt{2} \pm y\sqrt{5} = (f \pm g\sqrt{10}) \times (p\sqrt{2} \pm q\sqrt{5})^3.$$

Now, $(p\sqrt{2} \pm q\sqrt{5})^3 = (2p^3 + 15pq^2)\sqrt{2} \pm (6p^2q + 5q^3)\sqrt{5}$; and if, in order to abridge, we write $A\sqrt{2} + B\sqrt{5}$ instead of this quantity, and multiply by $f + g\sqrt{10}$, we shall have $Af\sqrt{2} + Bf\sqrt{5} + 2Ag\sqrt{5} + 5Bg\sqrt{2}$ to make equal to $x\sqrt{2} + y\sqrt{5}$; whence results $x = Af + 5Bg$, and $y = Bf + 2Ag$. Now, since we must have $y = \pm 1$, it is not absolutely necessary that $6p^2q + 5q^3 = 1$; on the contrary, it is sufficient that the formula $Bf + 2Ag$, that is to say, that $f(6p^2q + 5q^3) + 2g(2p^3 + 15pq^2)$ becomes $= \pm 1$; so that f and g may have several values. For example, let $f = 3$, and $g = 1$, the formula $18p^2q + 15q^3 + 4p^3 + 30pq^2$ must become ± 1; that is,

$$4p^3 + 18p^2q + 30pq^2 + 15q^3 = \pm 1.$$

197. The difficulty, however, of determining all the possible cases of this kind, exists only in the formula $ax^2 + cy^2$, when the number c is negative; and the reason is, that this formula, namely, $x^2 - acy^2$, which depends on it, may then become 1; which never happens when c is a positive number, because, $x^2 + cy^2$, or $x^2 + acy^2$, always gives greater numbers, the greater the values we assign to x and y. For which reason, the method we have explained cannot be successfully employed, except in those cases in which the two numbers a and c have positive values.

198. Let us now proceed to the fourth degree. Here we shall begin by observing, that if the formula $ax^2 + cy^2$ is to be changed into a biquadrate, we must have $a = 1$; for it would not be possible even to transform the formula into a square (Art. 181); and, if this were possible, we might also give it the form $t^2 + acu^2$; for which reason we shall extend the question only to this last formula, which may be reduced to the former, $x^2 + cy^2$, by supposing $a = 1$. This

being laid down, we have to consider what must be the nature of the values of x and y, in order that the formula $x^2 + cy^2$ may become a biquadrate. Now, it is composed of the two factors $(x + y\sqrt{-c}) \times (x - y\sqrt{-c})$; and each of these factors must also be a biquadrate of the same kind; therefore we must make $x + y\sqrt{-c} = (p + q\sqrt{-c})^4$, and $x - y\sqrt{-c} = (p - q\sqrt{-c})^4$, whence it follows, that the formula proposed becomes equal to the biquadrate $(p^2 + cq^2)^4$. With regard to the values of x and y, they are easily determined by the following analysis:

$x + y\sqrt{-c} = p^4 + 4p^3q\sqrt{-c} - 6cp^2q^2 + c^2q^4 - 4cpq^3\sqrt{-c}$,
$x - y\sqrt{-c} = p^4 - 4p^3q\sqrt{-c} - 6cp^2q^2 + c^2q^4 + 4cpq^3\sqrt{-c}$,
whence, $x = p^4 - 6cp^2q^2 + c^2q^4$; and $y = 4p^3q - 4cpq^3$.

199. So that when $x^2 + y^2$ is a biquadrate, because $c = 1$, we have

$$x = p^4 - 6p^2q^2 + q^4;\ \text{and}\ y = 4p^3q - 4pq^3;$$

so that $x^2 + y^2 = (p^2 + q^2)^4$.

Suppose, for example, $p = 2$, and $q = 1$; we shall then find $x = 7$, and $y = 24$; whence $x^2 + y^2 = 625 = 5^4$.

If $p = 3$, and $q = 2$, we obtain $x = 119$, and $y = 120$, which gives $x^2 + y^2 = 13^4$.

200. Whatever be the even power into which it is required to transform the formula $ax^2 + cy^2$, it is absolutely necessary that this formula be always reducible to a square; and for this purpose, it is sufficient that we already know one case in which it happens; for we may then transform the formula, as has been seen, into a quantity of the form $t^2 + acu^2$, in which the first term t^2 is multiplied only by 1; so that we may consider it as contained in the expression $x^2 + cy^2$; and in a similar manner, we may always give to this last expression the form of a sixth power, or of any higher even power.

201. This condition is not requisite for the odd powers; and whatever numbers a and c be, we may always transform the formula $ax^2 + cy^2$ into any odd power. Let the fifth, for instance, be demanded; we have only to make

$$x\sqrt{a} + y\sqrt{-c} = (p\sqrt{a} + q\sqrt{-c})^5, \text{ and}$$
$$x\sqrt{a} - y\sqrt{-c} = (p\sqrt{a} - q\sqrt{-c})^5,$$

and we shall evidently obtain $ax^2 + cy^2 = (ap^2 + cq^2)^5$. Farther, as the fifth power of $p\sqrt{a} + q\sqrt{-c}$ is $= a^2p^5\sqrt{a} + 5a^2p^4q\sqrt{-c} - 10acp^3q^2\sqrt{a} - 10acp^2q^3\sqrt{-c} + 5c^2pq^4\sqrt{a} + c^2q^5\sqrt{-c}$, we shall, with the same facility, find

$$x = a^2p^5 - 10acp^3q^2 + 5c^2pq^4, \text{ and}$$
$$y = 5a^2p^4q - 10acp^2q^3 + c^2q^5.$$

If it is required, therefore, that the sum of two squares,

such as $x^2 + y^2$, may be also a fifth power, we shall have $a = 1$, and $c = 1$; therefore, $x = p^5 - 10p^3q^2 + 5pq^4$; and $y = 5p^4q - 10p^2q^3 + q^5$; and, farther, making $p = 2$, and $q = 1$, we shall find $x = 38$, and $q = 41$; consequently,

$$x^2 + y^2 = 3125 = 5^5.$$

CHAP. XIII.

Of some Expressions of the Form $ax^4 + by^4$, which are not reducible to Squares.

202. Much labor has been formerly employed by some mathematicians to find two biquadrates, whose sum or difference might be a square, but in vain; and at length it has been demonstrated, that neither the formula $x^4 + y^4$, nor the formula $x^4 - y^4$, can become a square, except in these evident cases; first, when $x = 0$, or $y = 0$, and, secondly, when $y = x$. This circumstance is the more remarkable, because it has been seen, that we can find an infinite number of answers, when the question involves only simple squares.

203. We shall give the demonstration to which we have just alluded; and, in order to proceed regularly, we shall previously observe, that the two numbers x and y may be considered as prime to each other: for, if these numbers had a common divisor, so that we could make $x = dp$, and $y = dq$, our formulæ would become $d^4p^4 + d^4q^4$, and $d^4p^4 - d^4q^4$: which formulæ, if they were squares, would remain squares after being divided by d^4; therefore, the formulæ $p^4 + q^4$, and $p^4 - q^4$, also, in which p and q have no longer any common divisor, would be squares; consequently, it will be sufficient to prove, that our formulæ cannot become squares in the case of x and y being prime to each other, and our demonstration will, consequently, extend to all the cases, in which x and y have common divisors.

204. We shall begin, therefore, with the sum of two biquadrates; that is, with the formula $x^4 + y^4$, considering x and y as numbers that are prime to each other: and we have to prove, that this formula becomes a square only in the cases above-mentioned; in order to which, we shall enter

upon the analysis and deductions which this demonstration requires.

If any one denied the proposition, it would be maintaining that there may be such values of x and y, as will make $x^4 + y^4$ a square, in great numbers, notwithstanding there are none in small numbers.

But it will be seen, that if x and y had satisfactory values, we should be able, however great those values might be, to deduce from them less values equally satisfactory, and from these, others still less, and so on. Since, therefore, we are acquainted with no value in small numbers, except the two cases already mentioned, which do not carry us any farther, we may conclude, with certainty, from the following demonstration, that there are no such values of x and y as we require, not even among the greatest numbers. The proposition shall afterwards be demonstrated, with respect to the difference of two biquadrates, $x^4 - y^4$, on the same principle.

205. The following consideration, however, must be attended to at present, in order to be convinced that $x^4 + y^4$ can only become a square in the self evident cases which have been mentioned.

1. Since we suppose x and y prime to each other, that is, having no common divisor, they must either both be odd, or one must be even, and the other odd.

2. But they cannot both be odd, because the sum of two odd squares can never be a square; for an odd square is always contained in the formula $4n + 1$; and, consequently, the sum of two odd squares will have the form $4n + 2$, which being divisible by 2, but not by 4, cannot be a square. Now, this must be understood also of two odd biquadrate numbers.

3. If, therefore, $x^4 + y^4$ must be a square, one of the terms must be even and the other odd; and we have already seen, that, in order to have the sum of two squares a square, the root of one must be expressible by $p^2 - q^2$, and that of the other by $2pq$; therefore, $x^2 = p^2 - q^2$, and $y^2 = 2pq$; and we should have $x^4 + y^4 = (p^2 + q^2)^2$.

4. Consequently, y would be even, and x odd; but since $x^2 = p^2 - q^2$, the numbers p and q must also be the one even, and the other odd. Now, the first, p, cannot be even; for if it were, $p^2 - q^2$ would be a number of the form $4n - 1$, or $4n + 3$, and could not become a square: therefore p must be odd, and q even, in which case it is evident, that these numbers will be prime to each other.

5. In order that $p^2 - q^2$ may become a square, or

$p^2 - q^2 = x^2$, we must have, as we have already seen, $p = r^2 + s^2$, and $q = 2rs$; for then $x^2 = (r^2 - s^2)^2$, and $x = r^2 - s^2$.

6. Now, y^2 must likewise be a square; and since, we had $y^2 = 2pq$, we shall now have $y^2 = 4rs(r^2 + s^2)$; so that this formula must be a square; therefore $rs(r^2 + s^2)$ must also be a square: and let it be observed, that r and s are numbers prime to each other; so that the three factors of this formula, namely, r, s, and $r^2 + s^2$, have no common divisor.

7. Again, when a product of several factors, that have no common divisor, must be a square, each factor must itself be a square; so that making $r = t^2$, and $s = u^2$, we must have $t^4 + u^4 = \square$.

If, therefore, $x^4 + y^4$ were a $\square$, our formula $t^4 + u^4$, which is, in like manner, the sum of two biquadrates, would also be a $\square$. And it is proper to observe here, that since $x^2 = t^4 - u^4$, and $y^2 = 4t^2u^2(t^4 + u^4)$ the numbers t and u will evidently be much smaller than x and y, since x and y are even determined by the fourth powers of t and u, and must therefore become much greater than these numbers.

8. It follows, therefore, that if we could assign, in numbers however great, two biquadrates, such as x^4 and y^4, whose sum might be a square, we could deduce from it a number, formed by the sum of two much less biquadrates, which would also be a square; and this new sum would enable us to find another of the same nature, still less, and so on, till we arrived at very small numbers. Now, such a sum not being possible in very small numbers, it evidently follows, that there is not one which we can express by very great numbers.

9. It might indeed be objected, that such a sum does exist in very small numbers; namely, in the case which we have mentioned, when one of the two biquadrates becomes nothing: but we answer, that we shall never arrive at this case, by coming back from very great numbers to the least, according to the method which has been explained; for if in the small sum, or the reduced sum, $t^4 - u^4$, we had $t = 0$, or $u = 0$, we should necessarily have $y^2 = 0$ in the great sum; but this is a case which does not here enter into consideration.

206. Let us proceed to the second proposition, and prove also that the difference of two biquadrates, or $x^4 - y^4$, can never become a square, except in the cases of $y = 0$, and $y = x$.

1. We may consider the numbers x and y as prime to each other, and consequently, as being either both odd, or

the one even and the other odd: and as in both cases the difference of two squares may become a square, we must consider these two cases separately.

2. Let us, therefore, first begin by supposing both the numbers x and y odd, and that $x = p + q$, and $y = p - q$; then one of the two numbers p and q must necessarily be even, and the other odd. We have also $x^2 - y^2 = 4pq$, and $x^2 + y^2 = 2p^2 + 2q^2$; therefore our formula $x^4 - y^4 = 4pq(2p^2 + 2q^2)$; and as this must be a square, its fourth part, $pq(2p^2 + 2q^2) = 2pq(p^2 + q^2)$, must also be a square. Also, since the factors of this formula have no common divisor (because if p is even, q must be odd), each of these factors, $2p$, q, and $p^2 + q^2$, must be a square. In order, therefore, that the first two may become squares, let us suppose $2p = 4r^2$, or $p = 2r^2$, and $q = s^2$; in which s must be odd, and the third factor, $4r^4 + s^4$, must likewise be a square.

3. Now, since $s^4 + 4r^4$ is the sum of two squares, the first of which, s^4, is odd, and the other, $4r^4$, is even, let us make the root of the first $s^2 = t^2 - u^2$, in which let t be odd, and u even; and the root of the second, $2r^2 = 2tu$, or $r^2 = tu$, where t and u are prime to each other.

4. Since $tu = r^2$ must be a square, both t and u must be squares also. If, therefore, we suppose $t = m^2$, and $u = n^2$, (representing an odd number by m, and an even number by n), we shall have $s^2 = m^4 - n^4$; so that here, also, it is required to make the difference of two biquadrates, namely, $m^4 - n^4$, a square. Now, it is obvious, that these numbers would be much less than x and y, since they are less than r and s, which are themselves evidently less than x and y. If a solution, therefore, were possible in great numbers, and $x^4 - y^4$ were a square, there must also be one possible for numbers much less; and this last would lead us to another solution for numbers still less, and so on.

5. Now, the least numbers for which such a square can be found, are in the case where one of the biquadrates is 0, or where it is equal to the other biquadrate. In the first case, we must have $n = 0$; therefore $u = 0$, and also $r = 0$, $p = 0$, and, lastly, $x^4 - y^4 = 0$, or $x^4 = y^4$; which is a case that does not belong to the present question; if $n = m$, we shall find $t = u$, then $s = 0$, $q = 0$, and, lastly, also $x = y$, which does not here enter into consideration.

207. It might be objected, that since m is odd, and n even, the last difference is no longer similar to the first; and that, therefore, we can form no analogous conclusions from it with respect to smaller numbers. But it is sufficient that the first difference has led us to the second; and we shall

shew, that $x^4 - y^4$ can no longer become a square, when one of the biquadrates is even, and the other odd.

1. We may observe, if the first term, x^4, were even, and y^4 odd, the impossibility of the thing would be self-evident, since we should have a number of the form $4n + 3$; which cannot be a square: therefore, let x be odd, and y even; then $x^2 = p^2 + q^2$, and $y = 2pq$; whence $x^4 - y^4 = p^4 - 2p^2q^2 + q^4 = (p^2 - q^2)^2$, where one of the two numbers p and q must be even, and the other odd.

2. Now, as $p^2 + q^2 = x^2$ must be a square, we have $p = r^2 - s^2$, and $q = 2rs$; whence $x = r^2 + s^2$: but from that results $y^2 = 2(r^2 - s^2) \times 2rs$, or $y^2 = 4rs \times (r^2 - s^2)$, and as this must be a square, its fourth part, $rs(r^2 - s^2)$, whose factors are prime to each other, must likewise be a square.

3. Let us, therefore, make $r = t^2$, and $s = u^2$, and we shall have the third factor $r^2 - s^2 = t^4 - u^4$, which must also be a square. Now, as this factor is equal to the difference of two biquadrates, which are much less than the first, the preceding demonstration is fully confirmed; and it is evident, that, if the difference of two biquadrates could become equal to the square of a number (however great we may suppose it), we could, by means of this known case, arrive at differences less and less, which would also be reducible to squares, without our being led back to the two evident cases mentioned at first. It is impossible, therefore, for the thing to take place even with respect to the greatest numbers.

208. The first part of the preceding demonstration, namely, where x and y are supposed odd, may be abridged as follows: if $x^4 - y^4$ were a square, we must have $x^2 = p^2 + q^2$, and $y^2 = p^2 - q^2$, representing by p and q numbers, the one of which is even and the other odd; and by these means we should obtain $x^2y^2 = p^4 - q^4$; and, consequently, $p^4 - q^4$ must be a square. Now, this is a difference of two biquadrates, the one of which is even and the other odd; and it has been proved, in the second part of the demonstration, that such a difference cannot become a square.

209. We have therefore proved these two principal propositions; that neither the sum, nor the difference, of two biquadrates, can become a square number, except in a very few self-evident cases.

Whatever formulæ, therefore, we wish to transform into squares, if those formula require us to reduce the sum, or the difference of two biquadrates to a square, it may be pronounced that the given formulæ are likewise impossible;

which happens with regard to those that we shall now point out.

1. It is not possible for the formula $x^4 + 4y^4$ to become a square; for since this formula is the sum of two squares, we must have $x^2 = p^2 - q^2$, and $2y^2 = 2pq$, or $y^2 = pq$; now p and q being numbers prime to each other, each of them must be a $\square$. If we therefore make $p = r^2$, and $q = s^2$, we shall have $x^2 = r^4 - s^4$; that is to say, the difference of two biquadrates must be a square, which is impossible.

2. Nor is it possible for the formula $x^4 - 4y^4$ to become a square; for in this case we must make $x^2 = p^2 + q^2$, and $2y^2 = 2pq$, that we may have $x^4 - 4y^4 = (p^2 - q^2)^2$; but, in order that $y^2 = pq$, both p and q must be squares: and if we therefore make $p = r^2$, and $q = s^2$, we have $x^2 = r^4 + s^4$; that is to say, the sum of two biquadrates must be reducible to a square, which is impossible.

3. It is impossible also for the formula $4x^4 - y^4$ to become a square, because in this case y must necessarily be an even number. Now, if we make $y = 2z$, we conclude that $4x^4 - 16z^4$, and consequently, also, its fourth part, $x^4 - 4z^4$, must be reducible to a square; which we have just seen is impossible.

4. The formula $2x^4 + 2y^4$ cannot be transformed into a square; for since that square would necessarily be even, and consequently, $2x^4 + 2y^4 = 4z^2$, we should have $x^4 + y^4 = 2z^2$, or $2z^2 + 2x^2y^2 = x^4 + 2x^2y^2 + y^4 = \square$; or, in like manner, $2z^2 - 2x^2y^2 = x^4 - 2x^2y^2 + y^4 = \square$. So that, as both $2z^2 + 2x^2y^2$, and $2z^2 - 2x^2y^2$, would become squares, their product, $4z^4 - 4x^4y^4$, as well as the fourth of that product, or $z^4 - x^4y^4$, must be a square. But this last is the difference of two biquadratics; and is therefore impossible.

5. Lastly, I say also that the formula $2x^4 - 2y^4$ cannot be a square; for the two numbers x and y cannot both be even, since, if they were, they would have a common divisor; nor can they be the one even and the other odd, because then one part of the formula would be divisible by 4, and the other only by 2; and thus the whole formula would only be divisible by 2; therefore these numbers x and y must both be odd. Now, if we make $x = p + q$, and $y = p - q$, one of the numbers p and q will be even and the other will be odd; and, since $2x^4 - 2y^4 = 2(x^2 + y^2) \times (x^2 - y^2)$, and $x^2 + y^2 = 2p^2 + 2q^2 = 2(p^2 + q^2)$, and $x^2 - y^2 = 4pq$, our formula will be expressed by $16pq(p^2 + q^2)$, the sixteenth part of which, or $pq(p^2 + q^2)$, must likewise be a square. But these factors are prime to each other, so that each of

them must be a square. Let us, therefore, make the first two $p = r^2$, and $q = s^2$, and the third will become $r^4 + s^4$, which cannot be a square, therefore the given formula cannot become a square.

210. We may likewise demonstrate, that the formula $x^4 + 2y^4$ can never become a square: the *rationale* of this demonstration being as follows:

1. The number x cannot be even, because in that case y must be odd; and the formula would only be divisible by 2, and not by 4; so that x must be odd.

2. If, therefore, we suppose the square root of our formula to be $x^2 + \frac{2py^2}{q}$, in order that it may become odd, we shall have $x^4 + 2y^4 = x^4 + \frac{4px^2y^2}{q} + \frac{4p^2y^4}{q^2}$, in which the terms x^4 are destroyed; so that if we divide the other terms by y^2, and multiply by q^2, we find $4pqx^2 + 4p^2y^2 = 2q^2y^2$, or $4pqx^2 = 2q^2y^2 - 4p^2y^2$, whence we obtain $\frac{x^2}{y^2} = \frac{q^2 - 2p^2}{2pq}$; that is, $x^2 = q^2 - 2p^2$, and $y^2 = 2pq$*, which are the same formulæ that have been already given.

3. So that $q^2 - 2p^2$ must be a square, which cannot happen, unless we make $q = r^2 + 2s^2$, and $p = 2rs$, in order to have $x^2 = (r^2 - 2s^2)^2$; now, this will give us $4rs(r^2 + 2s^2) = y^2$; and its fourth part, $rs(r^2 + 2s^2)$ must also be a square: consequently r and s must respectively be each a square. If, therefore, we suppose $r = t^2$, and $s = u^2$, we shall find the third factor $r^2 + 2s^2 = t^4 + 2u^4$, which ought to be a square.

4. Consequently, if $x^4 + 2y^4$ were a square, $t^4 + 2u^4$ must also be a square; and as the numbers t and u would be much less than x and y, we should always come, in the same manner, to numbers successively less: but as it is easy from trials to be convinced, that the given formula is not a square in any small number; it cannot therefore be the square of a very great number.

211. On the contrary, with regard to the formula $x^4 - 2y^4$, it is impossible to prove that it cannot become a square; and, by a process of reasoning similar to the foregoing, we even find that there are an infinite number of cases in which this formula really becomes a square.

In fact, if $x^4 - 2y^4$ must become a square, we shall see

* Because x and y are prime to each other.

that, by making $x^2 = p^2 + 2q^2$, and $y^2 = 2pq$, we find $x^4 - 2y^4 = (p^2 - 2q^2)^2$. Now, $p^2 + 2q^2$ must in that case evidently become a square; and this happens when $p = r^2 - 2s^2$, and $q = 2rs$; since we have, in this case, $x^2 = (r^2 + 2s^2)^2$; and farther, it is to be observed, that, for the same purpose, we may take $p = 2s^2 - r^2$, and $q = 2rs$. We shall therefore consider each case separately.

1. First, let $p = r^2 - 2s^2$, and $q = 2rs$; we shall then have $x = r^2 + 2s^2$; and, since $y^2 = 2pq$, we shall thus have $y^2 = 4rs(r^2 - 2s^2)$; so that r and s must be squares: making, therefore, $r = t^2$, and $s = u^2$, we shall find $y^2 = 4t^2u^2(t^4 - 2u^4)$. So that $y = 2tu\sqrt{(t^4 - 2u^4)}$, and $x = t^4 + 2u^4$; therefore, when $t^4 - 2u^4$ is a square, we shall also find $x^4 - 2y^4 = \square$; but although t and u are numbers less than x and y, we cannot conclude that it is impossible for $x^4 - 2y^4$ to become a square, from our arriving at a similar formula in smaller numbers; since $x^4 - 2y^4$ may become a square, without our being brought to the formula $t^4 - 2y^4$, as will be seen by considering the second case.

2. For this purpose, let $p = 2s^2 - r^2$, and $q = 2rs$. Here, indeed, as before, we shall have $x = r^2 + 2s^2$; but then we shall find $y^2 = 2pq = 4rs(2s^2 - r^2)$: and if we suppose $r = t^2$, and $s = u^2$, we obtain $y^2 = 4t^2u^2(2u^4 - t^4)$; consequently, $y = 2tu\sqrt{(2u^4 - t^4)}$, and $x = t^4 + 2u^4$, by which means it is evident that our formula $x^4 - 2y^4$ may also become a square, when the formula $2u^4 - t^4$ becomes a square. Now, this is evidently the case, when $t = 1$, and $u = 1$; and we from that obtain $x = 3$, $y = 2$, and, lastly,

$$x^4 - 2y^4 = 81 - (2 \times 16) = 49.$$

3. We have also seen, Art. 140, that $2u^4 - t^4$ becomes a square, when $u = 13$, and $t = 1$; since then $\sqrt{(2u^4 - t^4)} = 239$. If we substitute these values instead of t and u, we find a new case for our formula; namely, $x = 1 + 2 \times 13^4 = 57123$, and $y = 2 \times 13 \times 239 = 6214$.

4. Farther, since we have found values of x and y, we may substitute them for t and u in the foregoing formulæ, and shall obtain by these means new values of x and y.

Now, we have just found $x = 3$, and $y = 2$; let us, therefore, in the formulæ, (No. 1.) make $t = 3$, and $u = 2$; so that $\sqrt{(t^4 - 2u^4)} = 7$, and we shall have the following new values; $x = 81 + (2 \times 16) = 113$, and $y = 2 \times 3 \times 2 \times 7 = 84$; so that $x^2 = 12769$, and $x^4 = 163047361$. Farther, $y^2 = 7056$, and $y^4 = 49787136$; therefore $x^4 - 2y^4 = 63473089$: the square root of which number is 7967, and it agrees perfectly with the formula which was adopted at first, $p^2 - 2q^2$; for since $t = 3$, and $u = 2$, we have $r = 9$, and $s = 4$; wherefore $p = 81 - 32 = 49$, and $q = 72$; whence $p^2 - 2q^2 = 2401 - 10368 = -7967$.

CHAP. XIV.

Solution *of some* Questions *that belong to this part of* Algebra.

212. We have hitherto explained such artifices as occur in this part of Algebra, and such as are necessary for resolving any question belonging to it: it remains to make them still more clear, by adding here some of those questions with their solutions.

213. *Question* 1. To find such a number, that if we add unity to it, or subtract unity from it, we may obtain in both cases a square number.

Let the number sought be x; then both $x + 1$, and $x - 1$ must be squares. Let us suppose for the first case $x + 1 = p^2$, we shall have $x = p^2 - 1$, and $x - 1 = p^2 - 2$, which must likewise be a square. Let its root, therefore, be represented by $p - q$; and we shall have $p^2 - 2 = p^2 - 2pq + q^2$; consequently, $p = \frac{q^2 + 2}{2q}$. Hence we obtain

$x = \frac{q^4 + 4}{4q^2}$, in which we may give q any value whatever, even a fractional one.

If we therefore make $q = \frac{r}{s}$, so that $x = \frac{r^4 + 4s^4}{4r^2s^2}$, we shall have the following values for some small numbers:

If $r =$	1,	2,	1,	3,	4,
and $s =$	1,	1,	2,	1,	1,
we have $x =$	$\frac{5}{4}$,	$\frac{5}{4}$,	$\frac{65}{16}$,	$\frac{85}{36}$,	$\frac{65}{16}$.

214. *Question* 2. To find such a number x, that if we add to it any two numbers, for example, 4 and 7, we obtain in both cases a square.

According to this enunciation, the two formulæ, $x + 4$ and $x + 7$, must become squares. Let us therefore suppose the first $x + 4 = p^2$, which gives us $x = p^2 - 4$, and the second will become $x + 7 = p^2 + 3$; and, as this last formula must also be a square, let its root be represented by $p + q$, and we shall have $p^2 + 3 = p^2 + 2pq + q^2$; whence we obtain $p = \frac{3-q^2}{2q}$, and, consequently, $x = \frac{9-22q^2+q^4}{4q^2}$; and if we also take a fraction $\frac{r}{s}$ for q, we find

$x = \frac{9s^4-22r^2s^2+r^4}{4r^2s^2}$, in which we may substitute for r and s any integer numbers whatever.

If we make $r = 1$, and $s = 1$, we find $x = -3$; therefore $x + 4 = 1$, and $x + 7 = 4$.

If x were required to be a positive number, we might make $s = 2$, and $r = 1$; we should then have $x = \frac{57}{16}$, whence $x + 4 = \frac{121}{16}$, and $x + 7 = \frac{169}{16}$.

If we make $s = 3$, and $r = 1$, we have $x = \frac{133}{9}$; whence $x + 4 = \frac{169}{9}$, and $x + 7 = \frac{196}{9}$.

In order that the last term of the formula, which expresses x, may exceed the middle term, let us make $r = 5$, and $s = 1$, and we shall have $x = \frac{21}{25}$; consequently, $x + 4 = \frac{121}{25}$, and $x + 7 = \frac{196}{25}$.

215. *Question* 3. Required such a fractional value of x, that if added to 1, or subtracted from 1, it may give in both cases a square.

Since the two formulæ $1 + x$, and $1 - x$, must become squares, let us suppose the first $1 + x = p^2$, and we shall have $x = p^2 - 1$; also, the second formula will then be $1 - x = 2 - p^2$. As this last formula must become a square, and neither the first nor the last term is a square, we must endeavour to find a case, in which the formula does become a □, and we soon perceive one, namely, when $p = 1$. If we therefore make $p = 1 - q$, so that $x = q^2 - 2q$, we have $2 - p^2 = 1 + 2q - q^2$; and supposing its root to be $1 - qr$, we shall have $1 + 2q - q^2 = 1 - 2qr + q^2r^2$; so that $2 - q = -2r + qr^2$, and $q = \frac{2r+2}{r^2+1}$; whence results

$x = \frac{4r-4r^3}{(r^2+1)^2}$; and since r is a fraction, if we make $r = \frac{t}{u}$, we shall have $x = \frac{4tu^3-4t^3u}{(t^2+u^2)^2} = \frac{4tu(u^2-t^2)}{(t^2+u^2)^2}$, where it is evident that u must be greater than t.

Let therefore $u = 2$, and $t = 1$, and we shall find $x = \frac{24}{25}$.

Let $u = 3$, and $t = 2$; we shall then have $x = \frac{120}{169}$; and the formulæ $1 + x = \frac{289}{169}$, and $1 - x = \frac{49}{169}$, will both be squares.

216. *Question* 4. To find such numbers x, that whether they be added to 10, or subtracted from 10, the sum and the difference may be squares.

It is required therefore, to transform into squares the formulæ $10 + x$, and $10 - x$, which might be done by the method that has just been employed; but let us explain another mode of proceeding. It will be immediately perceived, that the product of these two formulæ, or $100 - x^2$, must likewise become a square. Now, its first term being already a square, we may suppose its root to be $10 - px$, by which means we shall have $100 - x^2 = 100 - 20px + p^2x^2$;

therefore $p^2x + x = 20p$, and $x = \frac{20p}{p^2+1}$; now, from this it is only the product of the two formulæ which becomes a square, and not each of them separately. But provided one becomes a square, the other will necessarily be also a square.

Now $10 + x = \frac{10p^2 + 20p + 10}{p^2+1} = \frac{10(p^2 + 2p + 1)}{p^2+1}$, and since $p^2 + 2p + 1$ is already a square, the whole is reduced to making the fraction $\frac{10}{p^2+1}$, or $\frac{10p^2+10}{(p^2+1)^2}$, a square also. For this purpose we have only to make $10p^2 + 10$ a square, and here it is necessary to find a case in which that takes place. It will be perceived that $p = 3$ is such a case; for which reason we shall make $p = 3 + q$, and shall have $100 + 60q + 10q^2$. Let the root of this be $10 + qt$, and we shall have the final equation,

$$100 + 60q + 10q^2 = 100 + 20qt + q^2t^2,$$

which gives $q = \frac{60-20t}{t^2-10}$, by which means we shall determine $p = 3 + q$, and $x = \frac{20p}{p^2+1}$.

Let $t = 3$, we shall then find $q = 0$, and $p = 3$; therefore $x = 6$, and our formulæ $10 + x = 16$, and $10 - x = 4$.

But if $t = 1$, we have $q = -\frac{40}{9}$, and $p = -\frac{13}{9}$, so that $x = -\frac{234}{25}$; now it is of no consequence if we also make $x = +\frac{234}{25}$; therefore $10 + x = \frac{484}{25}$, and $10 - x = \frac{16}{25}$, which quantities are both squares.

217. *Remark.* If we wished to generalise this question,

by demanding such numbers, x, for any number, a, that both $a + x$, and $a - x$ may be squares, the solution would frequently become impossible; namely, in all cases in which a was not the sum of two squares. Now, we have already seen, that, between 1 and 50, there are only the following numbers that are the sums of two squares, or that are contained in the formula $x^2 + y^2$:

1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32, 34, 36, 37, 40, 41, 45, 49, 50.

So that the other numbers, comprised between 1 and 50, which are,

3, 6, 7, 11, 12, 14, 15, 19, 21, 22, 23, 24, 27, 28, 30, 31, 33, 35, 38, 39, 42, 43, 44, 46, 47, 48, cannot be resolved into two squares; consequently, whenever a is one of these last numbers, the question will be impossible; which may be thus demonstrated: Let $a + x = p^2$, and $a - x = q^2$, then the addition of the two formulæ will give $2a = p^2 + q^2$; therefore $2a$ must be the sum of two squares. Now, if $2a$ be such a sum, a will be so likewise *; consequently, when a is not the sum of two squares, it will always be impossible for $a + x$, and $a - x$, to be each squares at the same time.

218. As 3 is not the sum of two squares, it follows, from what has been said, that, if $a = 3$, the question is impossible. It might, however, be objected, that there are, perhaps, two fractional squares whose sum is 3; but we answer that this also is impossible: for if $\frac{p^2}{q^2} + \frac{r^2}{s^2} = 3$, and we were to multiply by q^2s^2, we should have $3q^2s^2 = p^2s^2 + q^2r^2$; and the second side of this equation, which is the sum of two squares, would be divisible by 3; but we have already seen (Art. 170) that the sum of two squares, that are prime to each other, can have no divisors, except numbers, which are themselves sums of two squares.

The numbers 9 and 45, it is true, are divisible by 3, but they are also divisible by 9, and even each of the two squares that compose both the one and the other, is divisible by 9, since $9 = 3^2 + 0^2$, and $45 = 6^2 + 3^2$; which is therefore a different case, and does not enter into consideration here. We may rest assured, therefore, of this conclusion; that if a number, a, be not the sum of two squares in integer numbers, it will not be so in fractions.

* For, let $x^2 + y^2 = 2a$; and put $x = s + d$, and $y = s - d$; then $(s+d)^2 + (s-d)^2 = 2s^2 + 2d^2$: that is, $x^2 + y^2 = 2s^2 + 2d^2 = 2a$, or $s^2 + d^2 = a$. B.

On the contrary, when the number a is the sum of two squares in fractional numbers, it is also the sum of two squares in integer numbers an infinite number of ways: and this we shall illustrate.

219. *Question 5.* To resolve, in as many ways as we please, a number, which is the sum of two squares, into another, that shall also be the sum of two squares.

Let $f^2 + g^2$ be the given number, and let two other squares, x^2 and y^2, be required, whose sum $x^2 + y^2$ may be equal to the number $f^2 + g^2$. Here it is evident, that if x is either greater or less than f, y, on the other hand, must be either less or greater than g: if, therefore, we make $x = f + pz$, and $y = g - qz$, we shall have

$$f^2 + 2fpz + p^2z^2 + g^2 - 2gqz + q^2z^2 = f^2 + g^2,$$

where the two terms f^2 and g^2 are destroyed; after which there remain only terms divisible by z. So that we shall have $2fp + p^2z - 2gq + q^2z = 0$, or $p^2z + q^2z = 2gq - 2fp$; therefore $z = \frac{2gq - 2fp}{p^2 + q^2}$, whence we get the following values for x and y, namely, $x = \frac{2gpq + f(q^2 - p^2)}{p^2 + q^2}$, and . . . . . . $y = \frac{2fpq + g(p^2 - q^2)}{p^2 + q^2}$; in which we may substitute all possible numbers for p and q.

If 2, for example, be the number proposed, so that $f = 1$, and $g = 1$, we shall have $x^2 + y^2 = 2$; and because $x = \frac{2pq + q^2 - p^2}{p^2 + q^2}$, and $y = \frac{2pq + p^2 - q^2}{p^2 + q^2}$, if we make $p = 2$, and $q = 1$, we shall find $x = \frac{1}{5}$, and $y = \frac{7}{5}$.

220. *Question 6.* If a be the sum of two squares, to find such a number, x, that $a + x$, and $a - x$, may become squares.

Let $a = 13 = 9 + 4$, and let us make $13 + x = p^2$, and $13 - x = q^2$. Then we shall first have, by addition, $26 = p^2 + q^2$; and, by subtraction, $2x = p^2 - q^2$; consequently, the values of p and q must be such, that $p^2 + q^2$ may become equal to the number 26, which is also the sum of two squares, namely, of $25 + 1$. Now, since the question in reality is to resolve 26 into two squares, the greater of which may be expressed by p^2, and the less by q^2, we shall immediately have $p = 5$, and $q = 1$; so that $x = 12$.

But we may resolve the number 26 into two squares in an infinite number of other ways: for, since $p = 5$, and $q = 1$, if we write t and u, instead of p and q, and p and q, instead of x and y, in the formulæ of the foregoing example, we shall find

$$p = \frac{2tu+5(u^2-t^2)}{t^2+u^2}, \text{ and } q = \frac{10tu+t^2-u^2}{t^2+u^2}.$$

Here we may now substitute any numbers for t and u, and by those means determine p and q, and, consequently, also the value of $x = \frac{p^2-q^2}{2}$.

For example, let $t = 2$, and $u = 1$; we shall then have $p = \frac{11}{5}$, and $q = \frac{23}{5}$; wherefore $p^2 - q^2 = \frac{408}{25}$, and $x = \frac{204}{25}$.

221. But, in order to resolve this question generally, let $a = c^2 + d^2$, and put z for the unknown quantity; that is to say, the formulæ, $a+z$, and $a-z$, must become squares.

Let us therefore make $a + z = x^2$, and $a - z = y^2$, and we shall thus have first $2a = 2(c^2 + d^2) = x^2 + y^2$, then $2z = x^2 - y^2$. Therefore the squares x^2 and y^2 must be such, that $x^2 + y^2 = 2(c^2 + d^2)$; where $2(c^2 + d^2)$ is really the sum of two squares, namely, $(c + d)^2 + (c - d)^2$; and, in order to abbreviate, let us suppose $c + d = f$, and $c - d = g$; then we must have $x^2 + y^2 = f^2 + g^2$; and this will happen, according to what has been already said, when

$$x = \frac{2gpq+f(q^2-p^2)}{p^2+q^2}, \text{ and } y = \frac{2fpq+g(p^2-q^2)}{p^2+q^2};$$

from which we obtain a very easy solution, by making $p = 1$, and $q = 1$; for we find $x = \frac{2g}{2} = g = c - d$, and $y = f = c + d$; consequently, $z = 2cd$; and it is evident that $a + z = c^2 + 2cd + d^2 = (c + d)^2$, and

$$a - z = c^2 - 2cd + d^2 = (c - d)^2.$$

Let us attempt another solution, by making $p = 2$, and $q = 1$; we shall then have $x = \frac{c-7d}{5}$, and $y = \frac{7c+d}{5}$, where c and d, as well as x and y, may be taken *minus*, because we have only to consider their squares. Now, since x must be greater than y, let us make d negative, and we shall have $x = \frac{c+7d}{5}$, and $y = \frac{7c-d}{5}$: hence

$z = \frac{24d^2+14cd-24c^2}{25}$; and this value being added to $a = c^2 + d^2$, gives $\frac{c^2+14cd+49d^2}{25}$, the square root of which is $\frac{c+7d}{5}$. If we then subtract z from a, there remains $\frac{49c^2-14cd+d^2}{25}$, which is the square of $\frac{7c-d}{5}$; the former of these two square roots being x, and the latter y.

222. *Question* 7. Required such a number, x, that whether we add unity to itself, or to its square, the result may be a square.

It is here required to transform the two formulæ $x + 1$, and $x^2 + 1$, into squares. Let us therefore suppose the first $x + 1 = p^2$; and, because $x = p^2 - 1$, the second, $x^2 + 1 = p^4 - 2p^2 + 2$, must be a square: which last formula is of such a nature as not to admit of a solution, unless we already know a satisfactory case; but such a case readily occurs, namely, that of $p=1$: therefore let $p=1+q$, and we shall have $x^2+1 = 1 + 4q^2 + 4q^3 + q^4$, which may become a square in several ways.

1. If we suppose its root to be $1 + q^2$, we shall have $1 + 4q^2 + 4q^3 + q^4 = 1 + 2q^2 + q^4$; so that $4q + 4q^2 = 2q$, or $4 + 4q = 2$, and $q = -\frac{1}{2}$; therefore $p = \frac{1}{2}$, and $x = -\frac{3}{4}$.

2. Let the root be $1 - q^2$, and we shall find $1 + 4q^2 + q^3 + q^4 = 1 - 2q^2 + q^4$; consequently $q = -\frac{3}{2}$, and $p = -\frac{1}{2}$, which gives $x = -\frac{3}{4}$, as before.

3. If we represent the root by $1 + 2q + q^2$, in order to destroy the first, and the last two terms, we have

$$1 + 4q^2 + 4q^3 + q^4 = 1 + 4q + 6q^2 + 4q^3 + q^4,$$

whence we get $q = -2$, and $p = -1$; and therefore $x = 0$.

4. We may also adopt $1 - 2q - q^2$ for the root, and in this case shall have

$$1 + 4q^2 + 4q^3 + q^4 = 1 - 4q + 2q^2 + 4q^3 + q^4;$$

but we find, as before, $q = -2$.

5. We may, if we choose, destroy the first two terms, by making the root equal to $1 + 2q^2$; for we shall then have $1 + 4q^2 + 4q^3 + q^4 = 1 + 4q^2 + 4q^4$; also, $q = \frac{4}{3}$, and $p = \frac{7}{3}$; consequently, $x = \frac{40}{9}$; lastly, $x+1 = \frac{49}{9} = (\frac{7}{3})^2$, and $x^2 + 1 = \frac{1681}{81} = (\frac{41}{9})^2$.

A greater number of values will be found for q, by making use of those which we have already determined. Thus, having found $q = -\frac{1}{2}$; let now $q = -\frac{1}{2} + r$, and we shall have $p = \frac{1}{2} + r$; also, $p^2 = \frac{1}{4} + r + r^2$, and $p^4 = \frac{1}{16} + \frac{1}{2}r + \frac{3}{2}r^2 + 2r^3 + r^4$; whence the expression

$$p^4 - 2p^2 + 2 = \tfrac{25}{16} - \tfrac{3}{2}r - \tfrac{1}{2}r^2 + 2r^3 + r^4,$$

to which our formula, $x^2 + 1$, is reduced, must be a square, and it must also be so when multiplied by 16; in which case, we have $25 - 24r - 8r^2 + 32r^3 + 16r^4$ to be a square. For which reason, let us now represent

1. The root by $5 + fr \pm 4r^2$; so that

$$25 - 24r - 8r^2 + 32r^3 + 16r^4 =$$
$$25 + 10fr \pm 40r^2 + f^2r^2 \pm 8fr^3 + 16r^4.$$

The first and the last terms destroy each other; and we may destroy the second also, if we make $10f = -24$, and, consequently, $f = -\frac{12}{5}$; then dividing the remaining terms by r^2, we have $-8 + 32r = \pm 40 + f^2 \pm 8fr$; and, admitting the upper sign, we find $r = \dfrac{48 + f^2}{32 - 8f}$. Now, because $f = -\frac{12}{5}$, we have $r = \frac{21}{20}$; therefore $p = \frac{31}{20}$, and $x = \frac{561}{400}$; so that $x + 1 = (\frac{31}{20})^2$, and $x^2 + 1 = (\frac{689}{400})^2$.

2. If we adopt the lower sign, we have

$$-8 + 32r = -40 + f^2 - 8fr,$$

whence $r = \dfrac{f^2 - 32}{32 + 8f}$; and since $f = -\frac{12}{5}$, we have $r = -\frac{41}{20}$; therefore $p = \frac{31}{20}$, which leads to the preceding equation.

3. Let $4r^2 + 4r \pm 5$ be the root; so that

$$16r^4 + 32r^3 - 8r^2 - 24r + 25 =$$
$$16r^4 + 32r^3 \pm 40r^2 + 16r^2 \pm 40r + 25:$$

and as on both sides the first two terms and the last destroy each other, we shall have

$$-8r - 24 = \pm 40r + 16r \pm 40, \text{ or}$$
$$-24r - 24 = \pm 40r \pm 40.$$

Here, if we admit the upper sign, we shall have

$$-24r - 24 = 40r + 40, \text{ or } 0 = 64r + 64, \text{ or}$$

$0 = r + 1$, that is $r = -1$, and $p = -\frac{1}{2}$; but this is a case already known to us, and we should not have found a different one by making use of the other sign.

4. Let now the root be $5 + fr + gr^2$, and let us determine f and g so, that the first three terms may vanish: then, since

$$25 - 24r - 8r^2 + 32r^3 + 16r^4 =$$
$$25 + 10fr + 10f^2r^2 + 10gr^2 + 2fgr^3 + g^2r^4,$$

we shall first have $10f = -24$, so that $f = -\frac{12}{5}$; then

$$10g + f^2 = -8, \text{ or } g = \frac{-8 - f^2}{10} = \frac{-344}{250} = \frac{-172}{125}.$$

When, therefore, we have substituted and divided the remaining terms by r^3, we shall have

$$32 + 16r = 2fg + g^2r, \text{ and } r = \frac{2fg - 32}{16 - g^2}.$$

Now, the numerator $2fg - 32$ becomes here

$$\frac{+24 \times 172 - 32 \times 625}{5 \times 125} = \frac{-32 \times 496}{625} = \frac{-16 \times 32 \times 31}{625}, \text{ and}$$

the denominator

$$16 - g^2 = (4 - g) \times (4 + g) = \tfrac{328}{125} \times \tfrac{672}{125} = \frac{8 \times 32 \times 41 \times 21}{25 \times 625};$$

so that $r = -\frac{1550}{861}$; and hence we conclude that $p = -\frac{2239}{1722}$, by means of which we obtain a new value of $x = p^2 - 1$.

223. *Question* 8. To find a number, x, which, added to each of the given numbers, a, b, c, produces a square.

Since here the three formulæ $x + a$, $x + b$, and $x + c$, must be squares, let us make the first $x + a = z^2$, and we shall have $x = z^2 - a$, and the two other formulæ will be changed into $z^2 + b - a$, and $z^2 + c - a$.

It is now required for each of these to be a square; but this does not admit of a general solution: the problem is frequently impossible, and its possibility entirely depends on the nature of the numbers $b - a$, and $c - a$. For example, if $b - a = 1$, and $c - a = -1$, that is to say, if $b = a + 1$, and $c = a - 1$, it would be required to make $z^2 + 1$, and $z^2 - 1$ squares, and, consequently, that z should be a fraction; so that we should make $z = \dfrac{p}{q}$, and it would be necessary that the two formulæ $p^2 + q^2$, and $p^2 - q^2$, should be squares, and, consequently, that their product also, $p^4 - q^4$, should be a square. Now, we have already shewn (Art. 202) that this is impossible.

Were we to make $b - a = 2$, and $c - a = -2$, that is, $b = a + 2$, and $c = a - 2$; and also, if $z = \dfrac{p}{q}$, we should have the two formulæ, $p^2 + 2q^2$, and $p^2 - 2q^2$, to transform into squares; consequently, it would also be necessary for their product, $p^4 - 4q^4$, to become a square; but this we have likewise shewn to be impossible. (Art. 209.)

In general, let $b - a = m$, $c - a = n$, and $z = \frac{p}{q}$: then the formulæ $p^2 + mq^2$, and $p^2 + nq^2$, must become squares; but we have seen that this is impossible, both when $m = +1$, and $n = -1$, and when $m = +2$, and $n = -2$.

It is also impossible, when $m = f^2$, and $n = -f^2$; for, in that case, we should have two formulæ, whose product would be $= p^4 - f^4q^4$, that is to say, the difference of two biquadrates; and we know that such a difference can never become a square.

Likewise, when $m = 2f^2$, and $n = -2f^2$, we have the two formulæ $p^2 + 2f^2q^2$, and $p^2 - 2f^2q^2$, which cannot both become squares, because their product $p^4 - 4f^4q^4$ must become a square. Now, if we make $fq = r$, this product is changed into $p^4 - 4r^4$, a formula, the impossibility of which has been already demonstrated.

If we suppose $m = 1$, and $n = 2$, so that it is required to reduce to squares the formulæ $p^2 + q^2$, and $p^2 + 2q^2$, we shall make $p^2 + q^2 = r^2$, and $p^2 + 2q^2 = s^2$; the first equation will give $p^2 = r^2 - q^2$, and the second will give $r^2 + q^2 = s^2$; and therefore both $r^2 - q^2$, and $r^2 + q^2$, must be squares: but the impossibility of this is proved, since the product of these formulæ, or $r^4 - q^4$, cannot become a square.

These examples are sufficient to shew, that it is not easy to choose such numbers for m and n as will render the solution possible. The only means of finding such values of m and n, is to imagine them, or to determine them by the following method.

Let us make $f^2 + mg^2 = h^2$, and $f^2 + ng^2 = k^2$; then we have, by the former equation, $m = \frac{h^2 - f^2}{g^2}$, and, by the latter, $n = \frac{k^2 - f^2}{g^2}$; this being done, we have only to take for f, g, h, and k, any numbers at pleasure, and we shall have values of m and n that will render the solution possible.

For example, let $h = 3$, $k = 5$, $f = 1$, and $g = 2$, we shall have $m = 2$, and $n = 6$; and we may now be certain that it is possible to reduce the formulæ $p^2 + 2q^2$ and $p^2 + 6q^2$ to squares, since it takes place when $p = 1$, and $q = 2$. But the former formula generally becomes a square, if $p = r^2 - 2s^2$, and $q = 2rs$; for then $p^2 + 2q^2 =$

$(r^2 + 2s^2)^2$. The latter formula also becomes $p^2 + 6q^2 =$ $r^4 + 20r^2s^2 + 4s^4$; and we know a case in which it becomes a square, namely, when $p = 1$, and $q = 2$, which gives $r = 1$, and $s = 1$; or, generally, $r = s$; so that the formula is $25s^4$. Knowing this case, therefore, let us make $r = s + t$; and we shall then have $r^2 = s^2 + 2st + t^2$, or $r^4 = s^4 +$ $4s^3t + 6s^2t^2 + 4st^3 + t^4$; so that our formula will become $25s^4 + 44s^3t + 26s^2t^2 + 4st^3 + t^4$: and, supposing its root to be $5s^2 + fst + t^2$, we shall make it equal to the square $25s^4 + 10fs^3t + f^2s^2t^2 + 10s^2t^2 + 2fst^3 + t^4$, by which means the first and last terms will be destroyed. Let us likewise make $2f = 4$, or $f = 2$, in order to remove the last
likewise make $2f = 4$, or $f = 2$, in order to remove the last
terms but one, and we shall obtain the equation
$44s + 26t = 10fs + 10t + f^2t = 20s + 14t$, or $2s = -t$,
and $\frac{s}{t} = -\frac{1}{2}$; therefore $s = -1$, and $t = 2$, or $t = -2s$;
and, consequently, $r = -s$, also $r^2 = s^2$, which is nothing
more than the case already known.

Let us rather, therefore, determine f in such a manner, that the second terms may vanish. We must make $10f = 44$, or $f = \frac{22}{5}$; and then dividing the other terms by st^2, we shall have $26s + 4t = 10s + f^2s + 2ft$, that is, $-\frac{84}{25}s = \frac{24}{5}t$; which gives $t = -\frac{7}{10}s$, and $r = s + t = \frac{3}{10}s$, or $\frac{r}{s} = \frac{3}{10}$; so that $r = 3$, and $s = 10$; by which means we find $p = 2s^2 - r^2 = 191$, and $q = 2rs = 60$, and our formulæ will be

$$p^2 + 2q^2 = 43681 = (209)^2 \text{ and}$$
$$p^2 + 6q^2 = 58081 = (241)^2.$$

224. *Remark.* In the same manner, other numbers may be found for m and n, that will make our formulæ squares; and it is proper to observe, that the ratio of m to n is arbitrary.

Let this ratio be as a to b, and let $m = az$, and $n = bz$; it will be required to know how z is to be determined, in order that the two formulæ $p^2 + azq^2$, and $p^2 + bzq^2$, may be transformed into squares: the method of doing which we shall explain in the solution of the following problem.

225. *Question* 9. Two numbers, a and b, being given, to find the number z such, that the two formulæ, $p^2 + azq^2$, and $p^2 + bzq^2$, may become squares; and, at the same time, to determine the least possible values of p and q.

Here, if we make $p^2 + azq^2 = r^2$, and $p^2 + bzq^2 = s^2$, and multiply the first equation by a, and the second by b, the difference of the two products will furnish the equation

$(b-a)p^2=br^2-as^2$, and, consequently, $p^2=\frac{br^2-as^2}{b-a}$; which formula must be a square: now, this happens when $r=s$. Let us, therefore, in order to remove the fractions, suppose $r=s+(b-a)t$, and we shall have

$$p^2=\frac{br^2-as^2}{b-a}=\frac{bs^2+2b(b-a)st+b(b-a)^2t^2-as^2}{b-a}=$$
$$\frac{(b-a)s^2+2b(b-a)st+b(b-a)^2t^2}{b-a}=$$
$$s^2+2bst+b(b-a)t^2.$$

Let us now make $p=s+\frac{x}{y}t$, and we shall have

$$p^2=s^2+\frac{2x}{y}st+\frac{x^2}{y^2}t^2=s^2+2bst+b(b-a)t^2,$$

in which the terms s^2 destroy each other; so that the other terms being divided by t, and multiplied by y^2, give $2sxy+tx^2=2bsy^2+b(b-a)ty^2$; whence

$$t=\frac{2sxy-2sby^2}{b(b-a)y^2-x^2}, \text{ and } \frac{t}{s}=\frac{2xy-2by^2}{b(b-a)y^2-x^2}.$$

So that $t=2xy-2by^2$, and $s=b(b-a)y^2-x^2$; farther, $r=2(b-a)xy-b(b-a)y^2-x^2$; and, consequently,

$$p=s+\frac{x}{y}t=b(b-a)y^2+x^2-2bxy=(x-by)^2-aby^2.$$

Having therefore found p, r, and s, it remains to determine z; and, for this purpose, let us subtract the first equation, $p^2+azq^2=r^2$, from the second, $p^2+bzq^2=s^2$; the remainder will be $zq^2(b-a)=s^2-r^2=(s+r)\times(s-r)$.

Now, $s+r=2(b-a)xy-2x^2$, and
$s-r=2b(b-a)y^2-2(b-a)xy$, or
$s+r=2x((b-a)y-x)$, and
$s-r=2(b-a)\times(by-x)y$; so that
$(b-a)zq^2=2x((b-a)y-x)\times 2(b-a)\times(by-x)y$, or
$zq^2=2x((b-a)y-x)\times(by-x)2y$, or
$zq^2=4xy((b-a)y-x)\times(by-x)$;

consequently, $z=\frac{4xy((b-a)y-x)\times(by-x)}{q^2}$.

We must therefore take the greatest square for q^2, that will divide the numerator; but let us observe, that we have already found $p=b(b-a)y^2+x^2-2bxy=(x-by)^2-aby^2$; and therefore we may simplify, by making $x=v+by$, or $x-by=v$; for then $p=v^2-aby^2$, and

$z=\frac{4(v+by)\times vy\times(v+ay)}{q^2}$, or $z=\frac{4vy(v+ay)\times(v+by)}{q^2}$.

By these means we may take any numbers for v and y, and assuming for q^2 the greatest square contained in the numerator, we shall easily determine the value of z; after which, we may return to the equations $m=az$, $n=bz$, and $p=v^2-aby^2$, and shall obtain the formulæ required.

1. $p^2+azq^2=(v^2-aby^2)^2+4avy(v+ay)\times(v+by)$, which is a square, whose root is $r=-v^2-2avy-aby^2$.

2. The second formula becomes

$p^2+bzq^2=(v^2-aby^2)^2+4bvy(v+ay)\times(v+by)$, which is also a square, whose root is $s=-v^2-2bvy-aby^2$, and the values both of r and s may be taken positive.

It may be proper to analyse these results in some examples.

226. *Example* 1. Let $a=-1$, and $b=+1$, and let us endeavour to seek such a number for z, that the two formulæ p^2-zq^2, and p^2+zq^2, may become squares; namely, the first r^2, and the second s^2.

We have therefore $p=v^2+y^2$; and, in order to find z, we have only to consider the formula

$z=\frac{4vy(v-y)\times(v+y)}{q^2}$; and, by giving different values to v and y, we shall see those that result for z.

	1	2	3	4	5	6
v	2	3	4	5	16	8
y	1	2	1	4	9	1
$v-y$	1	1	3	1	7	7
$v+y$	3	5	5	9	25	9
zq^2	4×6	4×30	16×15	$9\times16\times5$	$36\times25\times16\times7$	$16\times9\times14$
q^2	4	4	16	9×16	$36\times25\times16$	16×9
z	6	30	15	5	7	14
p	5	13	17	41	337	65

And by means of these values, we may resolve the following formulæ, and make squares of them:

1. We may transform into squares the formulæ p^2-6q^2, and p^2+bq^2; which is done by supposing $p=5$, and $q=2$; for the first becomes $25-24=1$, and the second

$$25+24=49.$$

2. Likewise, the two formulæ p^2-30q^2, and p^2+30q^2;

namely, by making $p = 13$, and $q = 2$; for the first becomes $169 - 120 = 49$, and the second $169 + 120 = 289$.

3. Likewise the two formulæ $p^2 - 15q^2$, and $p^2 + 15q^2$; for if we make $p = 17$, and $q = 4$, we have, for the first, $289 - 240 = 49$, and for the second $289 + 240 = 529$.

4. The two formulæ $p^2 - 5q^2$, and $p^2 + 5q^2$, become likewise squares: namely, when $p = 41$, and $q = 12$; for then $p^2 - 5q^2 = 1681 - 720 = 961 = 31^2$, and

$p^2 + 5q^2 = 1681 + 720 = 2401 = 49^2$.

5. The two formulæ $p^2 - 7q^2$, and $p^2 + 7q^2$, are squares, if $p = 337$, and $q = 120$; for the first is then $113569 - 100800 = 12769 = 113^2$, and the second is $113569 + 100800 = 214369 = 463^2$.

6. The formulæ $p^2 - 14q^2$, and $p^2 + 14q^2$, become squares in the case of $p = 65$, and $q = 12$; for then

$$p^2 - 14q^2 = 4225 - 2016 = 2209 = 47^2, \text{ and}$$
$$p^2 + 14q^2 = 4225 + 2016 = 6241 = 79^2.$$

227. *Example 2.* When the two numbers m and n are in the ratio of 1 to 2; that is to say, when $a = 1$, and $b = 2$, and therefore $m = z$, and $n = 2z$, to find such values for z, that the formulæ $p^2 + zq^2$ and $p^2 + 2zq^2$ may be transformed into squares.

Here it would be superfluous to make use of the general formulæ already given, since this example may be immediately reduced to the preceding. In fact, if $p^2 + zq^2 = r^2$, and $p^2 + 2zq^2 = s^2$, we have, from the first equation, $p^2 = r^2 - zq^2$; which being substituted in the second, gives $r^2 + zq^2 = s^2$; so that the question only requires, that the two formulæ, $r^2 - zq^2$, and $r^2 + zq^2$, may become squares; and this is evidently the case of the preceding example. We shall consequently have for z the following values: 6, 30, 15, 5, 7, 14, &c.

We may also make a similar transformation in a general manner. For, supposing that the two formulæ $p^2 + mq^2$, and $p^2 + nq^2$, may become squares, let us make $p^2 + mq^2 = r^2$, and $p^2 + nq^2 = s^2$; the first equation gives $p^2 = r^2 - mq^2$; the second will become

$s^2 = r^2 - mq^2 + nq^2$, or $r^2 + (n - m)q^2 = s^2$: if, therefore, the first formulæ are possible, these last $r^2 - mq^2$, and $r^2 + (n - m)q^2$, will be so likewise; and as m and n may be substituted for each other, the formulæ $r^2 - nq^2$, and $r^2 + (m - n)q^2$, will also be possible: on the contrary, if the first are impossible, the others will be so likewise.

228. *Example 3.* Let m be to n as 1 to 3, or let $a = 1$, and $b = 3$, so that $m = z$, and $n = 3z$, and let it be required to transform into squares the formulæ $p^2 + zq^2$, and $p^2 + 3zq^2$.

Since $a = 1$, and $b = 3$, the question will be possible in all the cases in which $zq^2 = 4xy(v + y) \times (v + 3y)$, and $p = v^2 - 3y^2$. Let us therefore adopt the following values for v and y:

v	1	3	4	1	16
y	1	2	1	8	9
$v+y$	2	5	5	9	25
$v+3y$	4	9	7	25	43
zq^2	16 × 2	4 × 9 × 30	4 × 4 × 35	4 × 9 × 25 × 4 × 2	4 × 9 × 16 × 25 × 43
q^2	16	4 × 9	4 × 4	4 × 4 × 9 × 25	4 × 9 × 16 × 25
z	2	30	35	2	43
p	2	3	13	191	13

Now, we have here two cases for $z = 2$, which enables us to transform, in two ways, the formulæ $p^2 + 2q^2$, and $p^2 + 6q^2$.

The first is, to make $p = 2$, and $q = 4$, and consequently also $p = 1$, and $q = 2$; for we have then from the last $p^2 + 2q^2 = 9$, and $p^2 + 6q^2 = 25$.

The second is, to suppose $p = 191$, and $q = 60$, by which means we shall have $p^2 + 2q^2 = (209)^2$, and $p^2 + 6q^2 = (241)^2$.

It is difficult to determine whether we cannot also make $z = 1$; which would be the case, if zq^2 were a square: but, in order to determine the question, whether the two formulæ $p^2 + q^2$, and $p^2 + 3q^2$, can become squares, the following process is necessary.

229. It is required to investigate, whether we can transform into squares the formulæ $p^2 + q^2$, and $p^2 + 3q^2$, with the same values of p and q. Let us here suppose $p^2 + q^2 = r^2$, and $p^2 + 3q^2 = s^2$, which leads to the investigation of the following circumstances.

1. The numbers p and q may be considered as prime to each other; for if they had a common divisor, the two formulæ would still continue squares, after dividing p and q by that divisor.

2. It is impossible for p to be an even number; for in that case q would be odd; and, consequently, the second formula would be a number of the class $4n + 3$, which cannot become a square; wherefore p is necessarily odd, and p^2 is a number of the class $8n + 1$.

3. Since p therefore is odd, q must in the first formula not only be even, but divisible by 4, in order that q^2 may become a number of the class $16n$, and that $p^2 + q^2$ may be of the class $8n + 1$.

4. Farther, p cannot be divisible by 3; for in that case, p^2 would be divisible by 9, and q^2 not; so that $3q^2$ would

only be divisible by 3, and not by 9; consequently, also, $p^2 + 3q^2$ could only be divisible by 3, and not by 9, and therefore could not be a square; so that p cannot be divisible by 3, and p^2 will be a number of the class $3n + 1$.

5. Since p is not divisible by 3, q must be so; for otherwise q^2 would be a number of the class $3n + 1$, and consequently $p^2 + q^2$ a number of the class $3n + 2$, which cannot be a square: therefore q must be divisible by 3.

6. Nor is p divisible by 5; for if that were the case, q would not be so, and q^2 would be a number of the class $5n + 1$, or $5n + 4$; consequently, $3q^2$ would be of the class $5n + 3$, or $5n + 2$; and as $p^2 + 3q^2$ would belong to the same classes, this formula therefore could not in that case become a square; consequently p must not be divisible by 5, and p^2 must be a number of the class $5n + 1$, or of the class $5n + 4$.

7. But since p is not divisible by 5, let us see whether q is divisible by 5, or not; since if q were not divisible by 5, q^2 must be of the class $5n + 2$, or $5n + 3$, as we have already seen; and since p^2 is of the class $5n + 1$, or $5n + 4$, $p^2 + 3q^2$ must be the same; namely, $5n + 1$, or $5n + 4$; and therefore, of one of the forms $5n + 3$, or $5n + 2$. Let us consider these cases separately.

If we suppose p^2 (F) $5n + 1$ *, then we must have q^2 (F) $5n + 4$, because otherwise $p^2 + q^2$ could not be a square; but we should then have $3q^2$ (F) $5n + 2$ and $p^2 + 3q^2$ (F) $5n + 3$, which cannot be a square.

In the second place, let p^2 (F) $5n + 4$; in this case we must have q^2 (F) $5n + 1$, in order that $p^2 + q^2$ may be a square, and $3q^2$ (F) $5n + 3$; therefore $p^2 + 3q^2$ (F) $5n + 2$, which cannot be a square. It follows, therefore, that q^2 must be divisible by 5.

8. Now, q being divisible first by 4, then by 3, and in the third place by 5, it must be such a number as $4 \times 3 \times 5m$, or $q = 60m$; so that our formulæ would become $p^2 + 3600m^2 = r^2$, and $p^2 + 10800m^2 = s^2$: this being established, the first, subtracted from the second, will give $7200m^2 = s^2 - r^2 = (s + r) \times (s - r)$; so that $s + r$ and $s - r$ must be factors of $7200m^2$, and at the same time

* In the former editions of this work, the sign = is used to express the words, "*of the form*." This was adopted in order to save the repetition of these words; but as it may occasionally produce ambiguity, or confusion, it was thought proper to substitute (F) instead of =, which is to be read thus: p^2 (F) $5n + 1$, *of the form* $5n + 1$.

it should be observed, that s and r must be odd numbers, and also prime to each other*.

9. Farther, let $7200m^2 = 4fg$, or let its factors be $2f$ and $2g$, supposing $s + r = 2f$, and $s - r = 2g$, we shall have $s = f + g$, and $r = f - g$; f and g, also, must be prime to each other, and the one must be odd and the other even. Now, as $fg = 1800m^2$, we may resolve $1800m^2$ into two factors, the one being even and the other odd, and having at the same time no common divisor.

10. It is to be farther remarked, that since $r^2 = p^2 + q^2$, and since r is a divisor of $p^2 + q^2$, $r = f - g$ must likewise be the sum of two squares (Art. 170); and as this number is odd, it must be contained in the formula $4n + 1$.

11. If we now begin with supposing $m = 1$, we shall have $fg = 1800 = 8 \times 9 \times 25$, and hence the following results: $f = 1800$, and $g = 1$, or $f = 200$, and $g = 9$, or $f = 72$, and $g = 25$, or $f = 225$, and $g = 8$.

The 1st, 2d, 3d, 4th gives:

- 1st: $r = f - g = 1799$ (F) $4n + 3$;
- 2d: $r = f - g = 191$ (F) $4n + 3$;
- 3d: $r = f - g = 47$ (F) $4n + 3$;
- 4th: $r = f - g = 217$ (F) $4n + 1$;

So that the first three must be excluded, and there remains only the fourth: from which we may conclude, generally, that the greater factor must be odd, and the less even; but even the value, $r = 217$, cannot be admitted here, because that number is divisible by 7, which is not the sum of two squares†.

12. If $m = 2$, we shall have $fg = 7200 = 32 \times 225$; for which reason we shall make $f = 225$, and $g = 32$, so that $r = f - g = 193$; and this number being the sum of two squares, it will be worth while to try it. Now, as $q = 120$, and $r = 193$, and $p^2 = r^2 - q^2 = (r + q) \times (r - q)$, we shall have $r + q = 313$, and $r - q = 73$; but since these factors are not squares, it is evident that p^2 does not become a square. In the same manner, it would be in vain to substitute any other numbers for m, as we shall now shew.

230. *Theorem.* It is impossible for the two formulæ $p^2 + q^2$, and $p^2 + 3q^2$, to be both squares at the same time; so that in the cases where one of them is a square, it is certain that the other is not.

* Because p is odd and q is even; therefore $p^2 + q^2 = r^2$, and $p^2 + 3q^2 = s^2$, must be both odd. B.

† Because the sum of two squares, prime to each other, can only be divided by numbers of the same form. B.

Demonstration. We have seen that p is odd, and q even, because $p^2 + q^2$ cannot be a square, except when $q = 2rs$, and $p = r^2 - s^2$; and $p^2 + 3q^2$ cannot be a square, except when $q = 2tu$, and $p = t^2 - 3u^2$, or $p = 3u^2 - t^2$. Now, as in both cases q must be a double product, let us suppose for both, $q = 2abcd$; and, for the first formula, let us make $r = ab$, and $s = cd$; for the second, let $t = ac$, and $u = bd$. We shall have for the former $p = a^2b^2 - c^2d^2$, and for the latter $p = a^2c^2 - 3b^2d^2$, or $p = 3b^2d^2 - a^2c^2$, and these two values must be equal; so that we have either $a^2b^2 - c^2d^2 = a^2c^2 - 3b^2d^2$, or $a^2b^2 - c^2d^2 = 3b^2d^2 - a^2c^2$; and it will be perceived that the numbers a, b, c, and d, are each less than p and q. We must however consider each case separately: the first gives $a^2b^2 + 3b^2d^2 = c^2d^2 + a^2c^2$, or $b^2(a^2 + 3d^2) = c^2(a^2 + d^2)$, whence $\frac{b^2}{c^2} = \frac{a^2+d^2}{a^2+3d^2}$ a fraction that must be a square.

Now, the numerator and denominator can here have no other common divisor than 2, because their difference is $2d^2$. If, therefore, 2 were a common divisor, both $\frac{a^2+d^2}{2}$, and $\frac{a^2+3d^2}{2}$, must be a square; but the numbers a and d are in this case both odd, so that their squares have the form $8n + 1$, and the formula $\frac{a^2+3d^2}{2}$ is contained in the expression $4n + 2$, and cannot be a square; wherefore 2 cannot be a common divisor; the numerator $a^2 + d^2$, and the denominator $a^2 + 3d^2$ are therefore prime to each other, and each of them must of itself be a square.

But these formulæ are similar to the former, and if the last were squares, similar formulæ, though composed of the smallest numbers, would have also been squares; so that we conclude, reciprocally, from our not having found squares in small numbers, that there are none in great.

This conclusion however is not admissible, unless the second case, $a^2b^2 - c^2d^2 = 3b^2d^2 - a^2c^2$, furnishes a similar one. Now, this equation gives $a^2b^2 + a^2c^2 = 3b^2d^2 + c^2d^2$, or $a^2(b^2 + c^2) = d^2(3b^2 + c^2)$; and, consequently, $\frac{a^2}{d^2} = \frac{b^2+c^2}{3b^2+c^2} = \frac{c^2+b^2}{c^2+3b^2}$; so that as this fraction ought to be a square, the foregoing conclusion is fully confirmed; for, if in great numbers there were cases in which $p^2 + q^2$, and $p^2 + 3q^2$, were squares, such cases must have also existed with regard to smaller numbers; but this is not the fact.

231. *Question* 10. To determine three numbers, x, y, and z, such, that multiplying them together two and two, and adding 1 to the product, we may obtain a square each time; that is, to transform into squares the three following formulæ:

$$xy + 1,\ xz + 1,\ \text{and } yz + 1.$$

Let us suppose one of the last two, as $xz + 1, = p^2$, and the other $yz + 1 = q^2$, and we shall have $x = \frac{p^2-1}{z}$, and $y = \frac{q^2-1}{z}$. The first formula is now transformed to $\frac{(p^2-1)\times(q^2-1)}{z^2} + 1$, which must consequently be a square, and will be no less so, if multiplied by z^2; so that $(p^2 - 1) \times (q^2 - 1) + z^2$, must be a square, which it is easy to form. For, let its root be $z + r$, and we shall have

$$(p^2 - 1) \times (q^2 - 1) = 2rz + r^2, \text{ and}$$

$z = \frac{(p^2-1)\times(q^2-1)-r^2}{2r}$, in which any numbers may be substituted for p, q, and r.

For example, if $r = (pq + 1)$, we shall have

$r^2 = p^2q^2 + 2qp + 1$, and $z = \frac{p^2+2pq+q^2}{2pq+2}$; wherefore

$x = \frac{(p^2-1)\times(2pq+2)}{p^2+2pq+q^2} = \frac{2(pq+1)\times(p^2-1)}{(p+q)^2}$, and

$y = \frac{2(pq+1)\times(q^2-1)}{(p+q)^2}$.

But if whole numbers be required, we must make the first formula $xy + 1 = p^2$, and suppose $z = x + y + q$; then the second formula becomes $x^2 + xy + xq + 1 = x^2 + qx + p^2$, and the third will be $xy + y^2 + qy + 1 = y^2 + qy + p^2$. Now, these evidently become squares, if we make $q = \pm 2p$; since in that case the second is $x^2 \pm 2px + p^2$, the root of which is $x \pm p$, and the third is $y^2 \pm 2py + p^2$, the root of which is $y \pm p$. We have consequently this very elegant solution: $xy + 1 = p^2$, or $xy = p^2 - 1$, which applies easily to any value of p; and from this the third number also is found, in two ways, since we have either $z = x + y + 2p$, or $z = x + y - 2p$. Let us illustrate these results by some examples.

1. Let $p = 3$, and we shall have $p^2 - 1 = 8$; if we make $x = 2$, and $y = 4$, we shall have either $z = 12$, or $z = 0$; so that the three numbers sought are 2, 4, and 12.

2. If $p = 4$, we shall have $p^2 - 1 = 15$. Now, if $x = 5$, and $y = 3$, we find $z = 16$, or $z = 0$; wherefore the three numbers sought are 3, 5, and 16.

3. If $p = 5$, we shall have $p^2 - 1 = 24$; and if we farther make $x = 3$, and $y = 8$, we find $z = 21$, or $z = 1$; whence the following numbers result; 1, 3, and 8; or 3, 8, and 21.

232. *Question* 11. Required three whole numbers x, y, and z, such, that if we add a given number, a, to each product of these numbers, multiplied two and two, we may obtain a square each time.

Here we must make squares of the three following formulæ,

$$xy + a,\ xz + a, \text{ and } yz + a.$$

Let us therefore suppose the first $xy + a = p^2$, and make $z = x + y + q$; then we shall have, for the second formula, $x^2 + xy + xq + a = x^2 + xq + p^2$; and, for the third, $xy + y^2 + yq + a = y^2 + qy + p^2$; and these both become squares by making $q = \pm 2p$: so that $z = x + y \pm 2p$; that is, to say, we may find two different values for z.

233. *Question* 12. Required four whole numbers, x, y, z, and v, such, that if we add a given number, a, to the products of these numbers, multiplied two by two, each of the sums may be a square.

Here, the six following formulæ must become squares:

$$1.\ xy + a,\quad 2.\ xz + a,\quad 3.\ yz + a,$$
$$4.\ xv + a,\quad 5.\ yv + a,\quad 6.\ zv + a.$$

If we begin by supposing the first $xy + a = p^2$, and take $z = x + y + 2p$, the second and third formulæ will become squares. If we farther suppose $v = x + y - 2p$, the fourth and fifth formulæ will likewise become squares; there remains therefore only the sixth formula, which will be $x^2 + 2xy + y^2 - 4p^2 + a$, and which must also become a square. Now, as $p^2 = xy + a$, this last formula becomes $x^2 - 2xy + y^2 - 3a$; and, consequently, it is required to transform into squares the two following formulæ:

$$xy + a = p^2, \text{ and } (x - y)^2 - 3a.$$

If the root of the last be $(x - y) - q$, we shall have $(x - y)^2 - 3a = (x - y)^2 - 2q(x - y) + q^2$; so that

$$-3a = -2q(x - y) + q^2, \text{ and } x - y = \frac{q^2 + 3a}{2q}, \text{ or}$$

$$x = y + \frac{q^2 + 3a}{2q};\ \text{consequently, } p^2 = y^2 + \frac{q^2 + 3a}{2q}y + a.$$

If $p = y + r$, we shall have

$$2ry + r^2 = \frac{q^2 + 3a}{2q}y + a, \text{ or}$$

$$4qry + 2qr^2 = (q^2 + 3a)y + 2aq, \text{ or}$$
$$2qr^2 - 2aq = (q^2 + 3a)y - 4qry, \text{ and}$$
$$y = \frac{2qr^2 - 2aq}{q^2 + 3a - 4qr},$$

where q and r may have any values, provided x and y become whole numbers; for since $p = y + r$, the numbers, z and v, will likewise be integers. The whole depends therefore chiefly on the nature of the number a, and it is true that the condition which requires integer numbers might cause some difficulties; but it must be remarked, that the solution is already much restricted on the other side, because we have given the letters, z and v, the values $x + y \pm 2p$, notwithstanding they might evidently have a great number of other values. The following observations, however, on this question, may be useful also in other cases.

1. When $xy + a$ must be a square, or $xy = p^2 - a$, the numbers x and y must always have the form $r^2 - as^2$ (Art. 176); if, therefore, we suppose

$$x = b^2 - ac^2, \text{ and } y = d^2 - ae^2,$$

we find $xy = (bd - ace)^2 - a(be - cd)^2$.

If $be - cd = \pm 1$, we shall have $xy = (bd - ace)^2 - a$, and, consequently, $xy + a = (bd - ace)^2$.

2. If we farther suppose $z = f^2 - ag^2$, and give such values to f and g, that $bg - cf = \pm 1$, and also $dg - ef = \pm 1$, the formulæ $xz + a$, and $yz + a$, will likewise become squares. So that the whole consists in giving such values to b, c, d, and e, and also to f and g, that the property which we have supposed may take place.

3. Let us represent these three couples of letters by the fractions $\frac{b}{c}$, $\frac{d}{e}$, and $\frac{f}{g}$; now, they ought to be such, that the difference of any two of them may be expressed by a fraction, whose numerator is 1. For since

$\frac{b}{c} - \frac{d}{e} = \frac{be - dc}{ce}$, this numerator, as has been seen, must be equal to ± 1. Besides, one of these fractions is arbitrary; and it is easy to find another, in order that the given condition may take place. For example, let the first

$\frac{b}{c} = \frac{3}{2}$, the second $\frac{d}{e}$ must be nearly equal to it; if, therefore, we make $\frac{d}{e} = \frac{4}{3}$, we shall have the difference $z = \frac{1}{6}$. We may also determine this second fraction by means of the first, generally; for since $\frac{3}{2} - \frac{d}{e} = \frac{3e-2d}{2e}$, we must have $3e - 2d = 1$, and, consequently, $2d = 3e - 1$, and $d = e + \frac{e-1}{2}$. So that making $\frac{e-1}{2} = m$, or $e = 2m+1$, we shall have $d = 3m + 1$, and our second fraction will be $\frac{d}{e} = \frac{3m+1}{2m+1}$. In the same manner, we may determine the second fraction for any first whatever, as in the following Table of examples:

$\frac{b}{c} = \frac{3}{2}$	$\frac{5}{3}$	$\frac{7}{3}$	$\frac{8}{5}$	$\frac{11}{4}$	$\frac{13}{8}$	$\frac{17}{7}$
$\frac{d}{e} = \frac{3m+1}{2m+1}$	$\frac{5m+1}{3m+1}$	$\frac{7m+2}{3m+1}$	$\frac{8m+3}{5m+2}$	$\frac{11m+3}{4m+1}$	$\frac{13m+5}{8m+3}$	$\frac{17m+5}{7m+2}$

4. When we have determined, in the manner required, the two fractions, $\frac{b}{c}$, and $\frac{d}{e}$, it will be easy to find a third also analogous to these. We have only to suppose $f = b + d$, and $g = c + e$, so that $\frac{f}{g} = \frac{b+d}{c+e}$; for the two first giving $be - cd = \pm 1$, we have $\frac{f}{g} - \frac{b}{c} = \frac{\pm 1}{c^2 + ce}$; and subtracting likewise the second from the third, we shall have

$$\frac{f}{g} - \frac{d}{e} = \frac{be - cd}{e^2 + ce} = \frac{\pm 1}{ce + e^2}.$$

5. After having determined in this manner the three fractions, $\frac{b}{c}$, $\frac{d}{e}$, and $\frac{f}{g}$, it will be easy to resolve our question for three numbers, x, y, and z, by making the three formulæ $xy + a$, $xz + a$, and $yz + a$, become squares: since we have only to make $x = b^2 - ac^2$, $y = d^2 - ae^2$, and $z = f^2 - ag^2$. For example, in the foregoing Table,

let us take $\frac{b}{c} = \frac{5}{3}$, and $\frac{d}{e} = \frac{7}{4}$, we shall then have $\frac{f}{g} = \frac{12}{7}$; whence $x = 25 - 9a$, $y = 49 - 16a$, and $z = 144 - 49a$; by which means we have

1. $xy + a = 1225 - 840a + 144a^2 = (35 - 12a)^2$;
2. $xz + a = 3600 - 2520a + 441a^2 = (60 - 21a)^2$;
3. $yz + a = 7056 - 4704a + 784a^2 = (84 - 28a)^2$.

234. In order now to determine, according to our question, four letters, x, y, z, and v, we must add a fourth fraction to the three preceding: therefore let the first three be $\frac{b}{c}$, $\frac{d}{e}$, $\frac{f}{g} = \frac{b+d}{c+e}$, and let us suppose the fourth fraction $\frac{h}{k} = \frac{b+d}{e+g} = \frac{2d+b}{2e+c}$, so that it may have the given relation with the third and second; if after this we make $x = b^2 - ac^2$, $y = d^2 - ae^2$, $z = f^2 - ag^2$, and $v = h^2 - ak^2$, we shall have already fulfilled the following conditions:

$$xy + a = \square,\quad xz + a = \square,\quad yz + a = \square,$$
$$yv + a = \square,\quad zy + a = \square.$$

It therefore only remains to make $xv + a$ become a square, which does not result from the preceding conditions, because the first fraction has not the necessary relation with the fourth. This obliges us to preserve the indeterminate number m in the three first fractions; by means of which, and by determining m, we shall be able also to transform the formula $xv + a$ into a square.

6. If we therefore take the first case from our small Table, and make $\frac{b}{c} = \frac{3}{2}$, and $\frac{d}{e} = \frac{3m+1}{2m+1}$; we shall have $\frac{f}{g} = \frac{3m+4}{2m+3}$, and $\frac{h}{k} = \frac{6m+5}{4m+4}$, whence $x = 9 - 4a$, and $v = (6m+5)^2 - a(4m+4)^2$;

$$\text{so that } xv + a = \begin{cases} 9(6m+5)^2 - 4a(6m+5)^2 \\ -9a(4m+4)^2 + 4a^2(4m+4)^2 \end{cases}$$

$$\text{or } xv + a = \begin{cases} 9(6m+5)^2 + 4a^2(4m+4)^2 \\ -a(288m^2 + 528m + 244), \end{cases}$$

which we can easily transform into a square, since m^2 will be found to be multiplied by a square; but on this we shall not dwell.

7. The fractions, which have been found to be neces-

sary, may also be represented in a more general manner; for if $\frac{b}{c} = \frac{\beta}{1}$, $\frac{d}{e} = \frac{n\beta - 1}{n}$, we shall have

$\frac{f}{g} = \frac{n\beta+\beta-1}{n+1}$, and $\frac{g}{h} = \frac{2n\beta+\beta-2}{2n+1}$; if in this last fraction we suppose $2n + 1 = m$, it will become $\frac{\beta m - 2}{m}$; consequently, the first gives $x = \beta^2 - a$, and the last furnishes $v = (\beta m - 2)^2 - am^2$. The only question therefore is, to make $xv + a$ a square. Now, because

$$v = (\beta^2 - a)m^2 - 4\beta m + 4, \text{ we have}$$

$xv + a = (\beta^2 - a)^2m^2 - 4(\beta^2 - a)\beta m + 4\beta^2 - 3a$; and since this must be a square, let us suppose its root to be $(\beta^2 - a)m - p$; the square of which quantity being $(\beta^2 - a)^2m^2 - 2(\beta^2 - a)mp + p^2$, we shall have $-4(\beta^2 - a)\beta m + 4\beta^2 - 3a = -2(\beta^2 - a)mp + p^2$; wherefore $m = \frac{p^2 - 4\beta^2 + 3a}{(\beta^2 - a) \times (2p - 4\beta)}$. If $p = 2\beta + q$, we shall find $m = \frac{4\beta q + q^2 + 3a}{2q(\beta^2 - a)}$, in which we may substitute any numbers whatever for β and q.

For example, if $a = 1$, let us make $\beta = 2$: we shall then have $m = \frac{4q + q^2 + 3}{6q}$; and making $q = 1$, we shall find $m \frac{4}{3}$; farther, $m = 2n + 1$; but without dwelling any longer on this question, let us proceed to another.

235. *Question* 12. Required three such numbers, x, y, and z, that the sums and differences of these numbers, taken two by two, may be squares.

The question requiring us to transform the six following formulæ into squares, *viz.*

$$x + y, \quad x + z, \quad y + z,$$
$$x - y, \quad x - z, \quad y - z,$$

let us begin with the last three, and suppose $x - y = p^2$, $x - z = q^2$, and $y - z = r^2$; the last two will furnish $x = q^2 + z$, and $y = r^2 + z$; so that we shall have $q^2 = p^2 + r^2$, because $x - y = q^2 - r^2 = p^2$; hence, $p^2 + r^2$, or the sum of two squares, must be equal to a square q^2; now, this happens, when $p = 2ab$, and $r = a^2 - b^2$, since then $q = a^2 + b^2$. But let us still preserve the letters p, q, and r, and consider also the first three formulæ. We shall have,

1. $x + y = q^2 + r^2 + 2z$;
2. $x + z = q^2 + 2z$;
3. $y + z = r^2 + 2z$.

Let the first $q^2 + r^2 + 2z, = t^2$, by which means $2z = t^2 - q^2 - r^2$; we must also have $t^2 - r^2 = \Box$, and $t^2 - q^2 = \Box$; that is to say, $t^2 - (a^2 - b^2)^2 = \Box$, and $t^2 - (a^2 + b^2)^2 = \Box$; we shall have to consider the two formulæ $t^2 - a^4 - b^4 + 2a^2b^2$, and $t^2 - a^4 - b^4 - 2a^2b^2$. Now, as both $c^2 + d^2 + 2cd$, and $c^2 + d^2 - 2cd$, are squares, it is evident that we shall obtain what we want by comparing $t^2 - a^4 - b^4$, with $c^2 + d^2$, and $2a^2b^2$ with $2cd$. With this view, let us suppose $cd = a^2b^2 = f^2g^2h^2k^2$, and take $c = f^2g^2$, and $d = h^2k^2$; $a^2 = f^2h^2$, and $b^2 = g^2k^2$, or $a = fh$, and $b = gk$; the first equation $t^2 - a^4 - b^4 = c^2 + d^2$, will assume the form $t^2 - f^4h^4 - g^4k^4 = f^4g^4 + h^4k^4$; whence $t^2 = f^4g^4 + f^4h^4 - g^4k^4 + h^4k^4$, or $t^2 = (f^4 + k^4) \times (g^4 + h^4)$; consequently, this product must be a square; but as the resolution of it would be difficult, let us consider the subject under a different point of view.

If from the first three equations $x - y = p^2$, $x - z = q^2$, $y - z = r^2$, we determine the letters y and z, we shall find $y = x - p^2$, and $z = x - q^2$; whence it follows that $q^2 = p^2 + r^2$. Our first formulæ now become $x + y = 2x - p^2$, $x + z = 2x - q^2$, and $y + z = 2x - p^2 - q^2$. Let us make this last $2x - p^2 - q^2 = t^2$, so that $2x = t^2 + p^2 + q^2$, and there will only remain the formulæ $t^2 + q^2$, and $t^2 + p^2$, to transform into squares. But since we must have $q^2 = p^2 + r^2$, let $q = a^2 + b^2$, and $p = a^2 - b^2$; and we shall then have $r = 2ab$, and, consequently, our formulæ will be:

1. $t^2 + (a^2 + b^2)^2 = t^2 + a^4 + b^4 + 2a^2b^2 = \Box$;
2. $t^2 + (a^2 - b^2)^2 = t^2 + a^4 + b^4 - 2a^2b^2 = \Box$.

In order to accomplish our purpose, we have only to compare again $t^2 + a^4 + b^4$ with $c^2 + d^2$, and $2a^2b^2$, with $2cd$. Therefore, as before, let $c = f^2g^2$, $d = h^2k^2$, $a = fh$, and $b = gk$; we shall then have $cd = a^2b^2$, and we must again have

$t^2 + f^4h^4 + g^4k^4 = c^2 + d^2 = f^4g^4 + h^4k^4$; whence $t^2 = f^4g^4 - f^4h^4 + h^4k^4 - g^4k^4 = (f^4 - k^4) \times (g^4 - h^4)$.

So that the whole is reduced to finding the differences of two pair of biquadrates, namely, $f^4 - k^4$, and $g^4 - h^4$, which, multiplied together, may produce a square.

For this purpose, let us consider the formula $m^4 - n^4$; let us see what numbers it furnishes, if we substitute given numbers for m and n, and attend to the squares that will be found among those numbers; the property of $m^4 - n^4 = (m^2 + n^2) \times (m^2 - n^2)$, will enable us to construct for our purpose the following Table:

A Table of Numbers contained in the Formula $m^4 - n^4$.

m^2	n^2	m^2-n^2	m^2+n^2	m^4-n^4
4	1	3	5	3×5
9	1	8	10	16×5
9	4	5	13	5×13
16	1	15	17	$3\times5\times17$
16	9	7	25	25×7
25	1	24	26	$16\times3\times13$
25	9	16	34	$16\times2\times17$
49	1	48	50	$25\times16\times2\times3$
49	16	33	65	$3\times5\times11\times13$
64	1	63	65	$9\times5\times7\times13$
81	49	32	130	$64\times5\times13$
121	4	117	125	$25\times9\times5\times13$
121	9	112	130	$16\times2\times5\times7\times13$
121	49	72	170	$144\times5\times17$
144	25	119	169	$169\times7\times17$
169	1	168	170	$16\times3\times5\times7\times17$
169	81	88	250	$25\times16\times5\times11$
225	64	161	289	$289\times7\times23$

We may already deduce some answers from this. For, if $f^2 = 9$, and $k^2 = 4$, we shall have $f^4 - k^4 = 13 \times 5$; farther, let $g^2 = 81$, and $h^2 = 49$, we shall then have $g^4 - h^4 = 64 \times 5 \times 13$; therefore $t^2 = 64 \times 25 \times 169$, and $t = 520$. Now, since $t^2 = 270400$, $f = 3$, $g = 9$, $k = 2$, $h = 7$, we shall have $a = 21$, and $b = 18$; so that $p = 117$, $q = 765$, and $r = 756$; from which results $2x = t^2 + p^2 + q^2 = 869314$; consequently, $x = 434657$; then $y = x - p^2 = 420968$, and lastly, $z = x - q^2 = -150568$. This last number may also be taken positively; the difference then becomes the sum, and, reciprocally, the sum becomes the difference. Since therefore the three numbers sought are:

$$x = 434657$$
$$y = 420968$$
$$z = 150568$$

we have $x + y = 855625 = (925)^2$
$x + z = 585225 = (765)^2$
and $y + z = 571536 = (756)^2$

also, $x - y = 13689 = (117)^2$
$x - z = 284089 = (533)^2$
and $y - z = 270400 = (520)^2$.

The Table which has been given, would enable us to find other numbers also, by supposing $f^2 = 9$, and $k^2 = 4$, $g^2 = 121$, and $h^2 = 4$; for then $t^2 = 13 \times 5 \times 5 \times 13 \times 9 \times 25 = 9 \times 25 \times 25 \times 169$, and

$$t = 3 \times 5 \times 5 \times 13 = 975.$$

Now, as $f = 3$, $g = 11$, $k = 2$, and $h = 2$, we have $a = fh = 6$, and $b = gk = 22$; consequently, $p = a^2 - b^2 = -448$, $q = a^2 + b^2 = 520$, and $r = 2ab = 264$; whence $2x = t^2 + p^2 + q^2 = 950625 + 200704 + 270400 = 1421729$, and $x = \frac{1421729}{2}$; wherefore $y = x - p^2 = \frac{1020321}{2}$, and $z = x - q^2 = \frac{880929}{2}$.

Now, it is to be observed, that if these numbers have the property required, they will preserve it by whatever square they are multiplied. If, therefore, we take them four times greater, the following numbers must be equally satisfactory: $x = 2843458$, $y = 2040642$, and $z = 1761858$; and as these numbers are greater than the former, we may consider the former as the least which the question admits of.

236. *Question* 14. Required three such squares, that the difference of every two of them may be a square.

The preceding solution will serve to resolve the present question. In fact, if x, y, and z, are such numbers that the following formulæ, namely,

$$x + y = \square, \quad x - y = \square, \quad x + z = \square,$$
$$x - z = \square, \quad y + z = \square, \quad y - z = \square,$$

may become squares; it is evident, likewise, that the product $x^2 - y^2$ of the first and second, the product $x^2 - z^2$ of the third and fourth, and the product $y^2 - z^2$ of the fifth and sixth, will be squares; and, consequently, x^2, y^2, and z^2, will be three such squares as are sought. But these numbers would be very great, and there are, doubtless, less numbers that will satisfy the question; since, in order that $x^2 - y^2$ may become a square, it is not necessary that $x + y$, and $x - y$, should be squares: for example, $25 - 9$ is a square, although neither $5+3$, nor $5-3$, are squares. Let us, therefore, resolve the question independently of this consideration, and remark, in the first place, that we may take 1 for one of the squares sought: the reason for which is, that if the formulæ x^2-y^2, x^2-z^2, and y^2-z^2, are squares, they will continue so, though divided by z^2; consequently, we may suppose that the question is to transform $\left(\frac{x^2}{z^2}-\frac{y^2}{z^2}\right)$, $\left(\frac{x^2}{z^2}-1\right)$, and $\left(\frac{y^2}{z^2}-1\right)$ into squares, and it then refers only to the two fractions $\frac{x}{z}$, and $\frac{y}{z}$.

If we now suppose $\frac{x}{z}=\frac{p^2+1}{p^2-1}$, and $\frac{y}{z}=\frac{q^2+1}{q^2-1}$, the last two conditions will be satisfied; for we shall then have $\frac{x^2}{z^2}-1=\frac{4p^2}{(p^2-1)^2}$, and $\frac{y^2}{z^2}-1=\frac{4q^2}{(q^2-1)^2}$. It only remains, therefore, to consider the first formula

$$\frac{x^2}{z^2}-\frac{y^2}{z^2}=\frac{(p^2+1)^2}{(p^2-1)^2}-\frac{(q^2+1)^2}{(q^2-1)^2}=$$
$$\left(\frac{p^2+1}{p^2-1}+\frac{q^2+1}{q^2-1}\right)\times\left(\frac{p^2+1}{p^2-1}-\frac{q^2+1}{q^2-1}\right).$$

Now, the first factor here is $\frac{2(p^2q^2-1)}{(p^2-1)\times(q^2-1)}$; the second is $\frac{2(q^2-p^2)}{(p^2-1)\times(q^2-1)}$, and the product of these two factors is $=\frac{4(p^2q^2-1)\times(q^2-p^2)}{(p^2-1)\times(q^2-1)^2}$. It is evident that the denominator of this product is already a square, and that the numerator contains the square 4; therefore it is only required to transform into a square the formula $(p^2q^2-1)\times(q^2-p^2)$, or $(p^2q^2-1)\times(\frac{q^2}{p^2}-1)$; and this is done by making $pq=\frac{f^2+g^2}{2fg}$, and $\frac{q}{p}=\frac{h^2+k^2}{2hk}$, because then each factor separately becomes a square. We may also be convinced of this, by remarking that $pq\times\frac{q}{p}=q^2=\frac{f^2+g^2}{2fg}\times\frac{h^2+k^2}{2hk}$; and, consequently, the product of these two fractions must be a square; as it must also be when multiplied by

$4f^2g^2\times h^2k^2$, by which means it becomes equal to $fg(f^2+g^2)\times hk(h^2+k^2)$. Lastly, this formula becomes precisely the same as that before found, if we make $f=a+b$, $g=a-b$, $h=c+d$, and $k=c-d$; since we have then

$$2(a^4-b^4)\times 2(c^4-d^4)=4\times(a^4-b^4)\times(c^4-d^4),$$

which takes place, as we have seen, when $a^2=9$, $b^2=4$, $c^2=81$; and $d^2=49$, or $a=3$, $b=2$, $c=9$, and $d=7$. Thus, $f=5$, $g=1$, $h=16$, and $k=2$, whence $pq=\frac{13}{5}$, and $\frac{q}{p}=\frac{260}{64}=\frac{65}{16}$; the product of these two equations gives $q=\frac{65\times13}{16\times5}=\frac{13\times13}{16}$; wherefore $q=\frac{13}{4}$, and it follows that $p=\frac{4}{5}$, by which means we have $\frac{x}{z}=\frac{p^2+1}{p^2-1}=-\frac{41}{9}$, and $\frac{y}{z}=\frac{q^2+1}{q^2-1}=\frac{185}{153}$; therefore, since $x=-\frac{41z}{9}$, and $y=\frac{185z}{153}$, in order to obtain whole numbers, let us make $z=153$, and we shall have $x=-697$, and $y=185$.

Consequently, the three square numbers sought are,

$$\left.\begin{array}{l}x^2=485809\\ y^2=\ \ 34225\\ z^2=\ \ 23409\end{array}\right\}\text{ and }\left\{\begin{array}{l}x^2-y^2=451584=(672)^2\\ y^2-z^2=\ \ 10816=(104)^2\\ x^2-z^2=462400=(680)^2.\end{array}\right.$$

It is farther evident, that these squares are much less than those which we should have found, by squaring the three numbers x, y, and z of the preceding solution.

237. Without doubt it will here be objected, that this solution has been found merely by trial, since we have made use of the Table in Article 235. But in reality we have only made use of this, to get the least possible numbers; for if we were indifferent with regard to brevity in the calculation, it would be easy, by means of the rules above given, to find an infinite number of solutions; because, having found $\frac{x}{z}=\frac{p^2+1}{p^2-1}$, and $\frac{y}{z}=\frac{q^2+1}{q^2-1}$, we have reduced the question to that of transforming the product $(p^2q^2-1)\times(\frac{q^2}{p^2}-1)$ into a square. If we therefore make $\frac{q}{p}=m$, or $q=mp$, our formula will become $(m^2p^4-1)\times(m^2-1)$, which is evidently a square, when $p=1$; but we shall farther see,

that this value will lead us to others, if we write $p = 1 + s$; in consequence of which supposition, we have to transform the formula

$$(m^2 - 1) \times (m^2 - 1 + 4ms^2 + 6m^2s^2 + 4m^2s^3 + m^2s^4)$$

into a square; it will be no less a square, if we divide it by $(m^2 - 1)^2$; this division gives us

$$1 + \frac{4m^2s}{m^2-1} + \frac{6m^2s^2}{m^2-1} + \frac{4m^2s^3}{m^2-1} + \frac{m^2s^4}{m^2-1};$$

and if to abridge we make $\frac{m^2}{m^2-1} = a$, we shall have to reduce the formula $1 + 4as + 6as^2 + 4as^3 + as^4$ to a square. Let its root be $1 + fs + gs^2$, the square of which is $1 + 2fs + 2gs^2 + f^2s^2 + 2fgs^3 + g^2s^4$, and let us determine f and g in such a manner, that the first three terms may vanish; namely, by making $2f = 4a$, or $f = 2a$, and $6a = 2g + f^2$, or $g = \frac{6a - f^2}{2} = 3a - 2a^2$, the last two terms will furnish the equation $4a + as = 2fg + g^2s$; whence $s = \frac{4a - 2fg}{g^2 - a} = \frac{4a - 12a + 8a^3}{4a^4 - 12a^3 + 9a^2 - a} = \frac{4 - 12a + 8a^2}{4a^3 - 12a^2 + 9a - 1}$, or, dividing by $a - 1$, $s = \frac{4(2a-1)}{4a^2 - 8a + 1}$.

This value is already sufficient to give us an infinite number of answers, because the number m, in the value of a, $= \frac{m^2}{m^2-1}$, may be taken at pleasure. It will be proper to illustrate this by some examples.

1. Let $m = 2$, we shall have $a = \frac{4}{3}$; so that

$s = 4 \times \frac{\frac{5}{3}}{-\frac{23}{9}} = -\frac{60}{23}$; whence $p = -\frac{37}{23}$, and $q = -\frac{74}{23}$;

lastly, $\frac{x}{z} = \frac{949}{420}$, and $\frac{y}{z} = \frac{6005}{4947}$.

2. If $m = \frac{3}{2}$, we shall have $a = \frac{9}{5}$, and

$s = 4 \times \frac{\frac{13}{5}}{-\frac{11}{25}} = -\frac{260}{11}$; consequently, $p = -\frac{249}{11}$, and $q = -\frac{747}{22}$, by which means we may determine the fractions

$$\frac{x}{z}, \text{ and } \frac{y}{z}.$$

There is here a particular case that deserves to be at-

tended to; which is that in which a is a square, and takes place, for example, when $m = \frac{5}{3}$; since then $a = \frac{25}{16}$. If here again, in order to abridge, we make $a = b^2$, so that our formula may be $1 + 4b^2s + 6b^2s^2 + 4b^2s^3 + b^2s^4$, we may compare it with the square of $1 + 2b^2s + bs^2$, that is to say, with $1 + 4b^2s + 2bs^2 + 4b^4s^2 + 4b^3s^3 + b^2s^4$; and expunging on both sides the first two terms and the last, and dividing the rest by s^2, we shall have $6b^2 + 4b^2s = 2b + 4b^4 + 4b^3s$, whence $s = \frac{6b^2 - 2b - 4b^4}{4b^3 - 4b^2} = \frac{3b - 1 - 2b^3}{2b^2 - 2b}$; but this fraction being still divisible by $b - 1$, we shall, at last, have $s = \frac{1 - 2b - 2b^2}{2b}$, and $p = \frac{1 - 2b^2}{2b}$.

We might also have taken $1 + 2bs + bs^2$ for the root of our formula; the square of this trinomial being $1 + 4bs + 2bs^2 + 4b^2s^2 + 4b^2s^3 + b^2s^4$, we should have destroyed the first, and the last two terms; and dividing the rest by s, we should have been brought to the equation $4b^2 + 6b^2s = 4b + 2bs + 4b^2s$. But as $b^2 = \frac{25}{16}$, and $b = \frac{5}{4}$, this equation would have given us $s = -2$, and $p = -1$; consequently, $p^2 - 1 = 0$, from which we could not have drawn any conclusion, since we should have had $z = 0$.

To return then to the former solution, which gave $p = \frac{1 - 2b^2}{2b}$; as $b = \frac{5}{4}$, it shews us that if $m = \frac{5}{3}$, we have $p = \frac{17}{20}$, and $q = mp = \frac{17}{12}$; consequently, $\frac{x}{z} = \frac{689}{111}$, and $\frac{y}{x} = \frac{433}{143}$.

238. *Question* 15. Required three square numbers such, that the sum of every two of them may be a square.

Since it is required to transform the three formulæ $x^2 + y^2$, $x^2 + z^2$, and $y^2 + z^2$ into squares, let us divide them by z^2, in order to have the three following,

$$\frac{x^2}{z^2} + \frac{y^2}{z^2} = \square, \quad \frac{x^2}{z^2} + 1 = \square, \quad \frac{y^2}{z^2} + 1 = \square.$$

The last two are answered, by making $\frac{x}{z} = \frac{p^2 - 1}{2p}$, and $\frac{y}{z} = \frac{q^2 - 1}{2q}$, which also changes the first formula into this, $\frac{(p^2-1)^2}{4p^2} + \frac{(q^2-1)^2}{4q^2}$, which ought also to continue a square after being multiplied by $4p^2q^2$; that is, we must have $q^2(p^2-1)^2+p^2(q^2-1)^2=\square$. Now, this can scarcely be obtained, unless we previously know a case in which this formula becomes a square: and as it is also difficult to find such a case, we must have recourse to other artifices, some of which we shall now explain.

1. As the formula in question may be expressed thus, $q^2(p+1)^2\times(p-1)^2+p^2(q+1)^2\times(q-1)^2=\square$, let us make it divisible by the square $(p+1)^2$; which may be done by making $q-1=p+1$, or $q=p+2$; for then $q+1=p+3$, and the formula becomes $(p+2)^2\times(p+1)^2\times(p-1)^2+p^2(p+3)^2\times(p+1)^2=\square$; so that dividing by $(p+1)^2$, we have $(p+2)^2\times(p-1)^2+p^2(p+3)^2$, which must be a square, and to which we may give the form $2p^4+8p^3+6p^2-4p+4$. Now, the last term here being a square, let us suppose the root of the formula to be $2+fp+gp^2$, or gp^2+fp+2, the square of which is $g^2p^4+2fgp^3+4gp^2+f^2p^2+4fp+4$, and we shall destroy the last three terms, by making $4f=-4$, or $f=-1$, and $4g+1=6$, or $g=\frac{5}{4}$; also the first terms being divided by p^3, will give $2p+8=g^2p+2fg=\frac{25}{16}p-\frac{5}{2}$;

or $p=-24$, and $q=-22$; whence $\dfrac{x}{z}=\dfrac{p^2-1}{2p}=-\frac{575}{48}$;

or $x=-\frac{575}{48}z$, and $\dfrac{y}{z}=\dfrac{q^2-1}{2q}=-\frac{483}{44}$, or $y=-\frac{483}{44}z$.

Let us now make $z=16\times3\times11$; we shall then have $x=575\times11$, and $y=483\times12$; and, consequently, the roots of the three squares sought will be:

$$x=6325=11\times23\times25;$$
$$y=5796=12\times21\times23;$$
$$\text{and } z=528=3\times11\times16;$$

for from these result,

$$x^2+y^2=23^2(275^2+252^2)=23^2\times373^2.$$
$$x^2+z^2=11^2(575^2+48^2)=11^2\times577^2.$$
$$\text{and } y^2+z^2=12^2(483^2+44^2)=12^2\times485^2.$$

2. We may also make our formula divisible by a square, in an infinite number of ways; for example, if we suppose $(q+1)^2=4(p+1)^2$, or $q+1=2(p+1)$, that is to say, $q=2p+1$, and $q-1=2p$, the formula will become $(2p+1)^2\times(p+1)^2\times(p-1)^2+p^2\times4(p+1)^2\times4p^2=\square$; which may be divided by $(p+1)^2$, by which means we have $(2p+1)^2\times(p-1)^2+16p^4=\square$, or $20p^4-4p^3-3p^2+2p+1=\square$; but from this we derive nothing.

3. Let us then rather make $(q-1)^2=4(p+1)^2$, or $q-1=2(p+1)$; we shall then have $q=2p+3$, and $q+1=2p+4$, or $q+1=2(p+2)$, and after having divided our formula by $(p+1)^2$, we shall obtain the following; $(2p+3)^2\times(p-1)^2+16p^2(p+2)^2$, or $9-6p+53p^2+68p^3+20p^4$. Let its root be $3-p+gp^2$, the square of which is $9-6p+6gp^2+p^2-2gp^3+g^2p^4$; the first two terms vanish, and we may destroy the third by making $6g+1=53$, or $g=\frac{26}{3}$; so that the other terms are divisible by p, and give $20p+68=g^2p-2g$, or $\frac{496}{9}p=\frac{256}{3}$; therefore $p=\frac{48}{31}$, and $q=\frac{189}{31}$, by which means we obtain a new solution.

4. If we make $q-1=\frac{4}{3}(p-1)$, we have $q=\frac{4}{3}p-\frac{1}{3}$, and $q+1=\frac{4}{3}p+\frac{2}{3}=\frac{2}{3}(2p+1)$, and the formula, after being divided by $(p-1)^2$, becomes

$\left(\dfrac{4p-1}{9}\right)^2\times(p+1)^2+\frac{64}{81}p^2(2p+1)^2$; multiplying by 81, we have $9(4p-1)^2\times(p+1)^2+64p^2(2p+1)^2=$

$$400p^4+472p^3+73p^2-54p+9,$$

in which the first and last terms are both squares. If, therefore, we suppose the root to be $20p^2-9p+3$, the square of which is $400p^4-360p^3+120p^2+81p^2-54p+9$, we shall have $472p+73=-360p+201$; wherefore $p=\frac{2}{13}$, and $q=\frac{8}{39}-\frac{1}{3}=-\frac{5}{39}$.

We might likewise have taken for the root $20p^2+9p-3$, the square of which is $400p^4+360p^3-120p^2+81p^2-54p+9$; but comparing this square with our formula, we should have found $472p+73=360p-39$, and consequently $p=-1$, a value which can be of no use to us.

5. We may also make our formula divisible by the two squares, $(p+1)^2$, and $(p-1)^2$, at the same time. For this purpose, let us make $q=\dfrac{pt+1}{p+t}$; so that

$$q+1=\frac{pt+p+t+1}{p+t}=\frac{(p+1)\times(t+1)}{p+t}, \text{ and}$$

$$q-1=\frac{pt-p-t+1}{p+t}=\frac{(p-1)\times(t-1)}{p+t}:$$

this formula will be divisible by $(p+1)^2\times(p-1)^2$, and will be reduced to $\dfrac{(pt+1)^2}{(p+t)^2}+\dfrac{(t+1)^2\times(t-1)^2}{(p+t)^4}\times p^2$. If we multiply by $(p+t)^4$, the formula, as before, must be transformable into a square, and we shall have

$$(pt+1)^2\times(p+t)^2+p^2(t+1)^2\times(t-1)^2, \text{ or}$$

$t^2p^4+2t(t^2+1)p^3+2t^2p^2+(t^2+1)^2p^2+(t^2-1)^2p^2+2t(t^2+1)p+t^2$
in which the first and the last terms are squares. Let us therefore take for the root $tp^2+(t^2+1)p-t$, the square of which is

$$t^2p^4+2t(t^2+1)p^3-2t^2p^2+(t^2+1)^2p^2-2t(t^2+1)p+t^2,$$

and we shall have, by comparing,

$$2t^2p+(t^2+1)^2p+2t(t^2+1)+(t^2-1)^2p=$$
$$-2t^2p+(t^2+1)^2p-2t(t^2+1), \text{ or, by subtraction,}$$
$$4t^2p+4t(t^2+1)+(t^2-1)^2p=0, \text{ or}$$
$$(t^2+1)^2p^*+4t(t^2+1)=0,$$

that is to say, $t^2+1=\frac{-4t}{p}$; whence $p=\frac{-4t}{t^2+1}$; consequently, $pt+1=\frac{-3t^2+1}{t^2+1}$, and $p+t=\frac{t^3-3t}{t^2+1}$; lastly, $q=\frac{-3t^2+1}{t^3-3t}$, where the value of the letter t is arbitrary.

For example, let $t=2$; we shall then have $p=\frac{-8}{5}$ and $q=\frac{-11}{2}$; so that $\frac{x}{z}=\frac{p^2-1}{2p}=+\frac{39}{80}$, and $\frac{y}{z}=\frac{q^2-1}{2q}=\frac{117}{44}$, or $x=\frac{3\times13}{4\times4\times5}z$, and $y=\frac{9\times13}{4\times11}z$.

Farther, if $x=3\times11\times13$, we have
$y=4\times5\times9\times13$, and
$z=4\times4\times5\times11$,

and the roots of the three squares sought are
$x=3\times11\times13=429$,
$y=4\times5\times9\times13=2340$, and
$z=4\times4\times5\times11=880$:

where it is evident that these are still less than those found above, from which we derive

$$x^2+y^2=3^2\times13^2(121+3600)=3^2\times13^2\times61^2,$$
$$x^2+z^2=11^2\times(1521+6400)=11^2\times89^2,$$
$$y^2+z^2=20^2\times(13689+1936)=20^2\times125^2.$$

6. The last remark we shall make on this question is, that each answer easily furnishes a new one; for when we have found three values, $x=a$, $y=b$, and $z=c$, so that $a^2+b^2=\square$, $a^2+c^2=\square$, and $b^2+c^2=\square$, the three following values will likewise be satisfactory, namely, $x=ab$, $y=bc$, and $z=ac$. Then we must have

$$x^2+y^2=a^2b^2+b^2c^2=b^2(a^2+c^2)=\square,$$
$$x^2+z^2=a^2b^2+a^2c^2=a^2(b^2+c^2)=\square,$$
$$y^2+z^2=a^2c^2+b^2c^2=c^2(a^2+b^2)=\square.$$

Now, as we have just found
$x=a=3\times11\times13$,
$y=b=4\times5\times9\times13$, and
$z=c=4\times4\times5\times11$,

we have, therefore, according to the new solution,
$x=ab=3\times4\times5\times9\times11\times13\times13$,
$y=bc=4\times4\times4\times5\times5\times9\times11\times13$,
$z=ac=3\times4\times4\times5\times11\times11\times13$.

And all these three values being divisible by
$3\times4\times5\times11\times13$,

are reducible to the following,
$x=9\times13$, $y=3\times4\times4\times5$, and $z=4\times11$; or
$x=117$, $y=240$, and $z=44$,

which are still less than those which the preceding solution gave, and from them we deduce

$$x^2+y^2=71289=267^2,$$
$$x^2+z^2=15625=125^2,$$
$$y^2+z^2=59536=244^2.$$

239. *Question* 16. Required two such numbers, x and y, that each being added to the square of the other, may make a square; that is, that $x^2+y=\square$, and $y^2+x=\square$.

If we begin with supposing $x^2+y=p^2$, and from that deduce $y=p^2-x^2$, we shall have for the other formula $p^4-2p^2x^2+x^4+x=\square$, which it would be difficult to resolve.

Let us, therefore, suppose one of the formulæ $x^2+y=(p-x)^2=p^2-2px+x^2$; and, at the same time, the other $y^2+x=(q-y)^2=q^2-2qy+y^2$, and we shall thus obtain the two following equations,

$$y+2px=p^2, \text{ and } x+2py=q^2,$$

from which we easily deduce

$$x=\frac{2qp^2-q^2}{4pq-1}, \text{ and } y=\frac{2pq^2-q^2}{4pq-1},$$

in which p and q are indeterminate. Let us, therefore, suppose, for example, $p=2$, and $y=3$, then we shall have

* Thus, $(t^2-1)^2=t^4-2t^2+1$, which multiplied by p becomes
pt^4-2pt^2+p,
Then adding $4pt^2$
We have $pt^4+2pt^2+p=(t^2+1)^2p$, as above.

for the two numbers sought $x = \frac{15}{23}$, and $y = \frac{32}{23}$, by which means $x^2 + y = \frac{225}{529} + \frac{32}{23} = \frac{961}{529} = (\frac{31}{23})^2$, and $y^2 + x = \frac{1024}{529} + \frac{15}{23} = \frac{1369}{529} = (\frac{37}{23})^2$. If we made $p = 1$, and $q = 3$, we should have $x = -\frac{3}{11}$, and $y = \frac{17}{11}$, an answer which is inadmissible, since one of the numbers sought is negative.

But let $p = 1$, and $q = \frac{3}{2}$, we shall then have $x = \frac{3}{20}$, and $y = \frac{7}{10}$, whence we derive

$$x^2 + y = \frac{9}{400} + \frac{7}{10} = \frac{289}{400} = (\tfrac{17}{20})^2, \text{ and}$$
$$y^2 + x = \frac{49}{100} + \frac{3}{20} = \frac{64}{100} = (\tfrac{8}{10})^2.$$

240. *Question* 17. To find two numbers, whose sum may be a square, and whose squares added together may make a biquadrate.

Let us call these numbers x and y; and since $x^2 + y^2$ must become a biquadrate, let us begin with making it a square: in order to which, let us suppose $x = p^2 - q^2$, and $y = 2pq$, by which means, $x^2 + y^2 = (p^2 + q^2)^2$. But, in order that this square may become a biquadrate, $p^2 + q^2$ must be a square; let us therefore make $p = r^2 - s^2$, and $q = 2rs$, in order that $p^2 + q^2 = (r^2 + s^2)^2$; and we immediately have $x^2 + y^2 = (r^2 + s^2)^4$, which is a biquadrate. Now, according to these suppositions, we have $x = r^4 - 6r^2s^2 + s^4$, and $y = 4r^3s - 4rs^3$; it therefore remains to transform into a square the formula

$$x + y = r^4 + 4r^3s - 6r^2s^2 - 4rs^3 + s^4.$$

Supposing its root to be $r^2 + 2rs + s^2$, or the formula equal to the square of this, $r^4 + 4r^3s + 6r^2s^2 + 4rs^3 + s^4$, we may expunge from both the first two terms and also s^4, and divide the rest by rs^2, so that we shall have

$$6r + 4s = -6r - 4s, \text{ or } 12r + 8s = 0; \text{ so that}$$

$s = -\dfrac{12r}{8} = -\frac{3}{2}r$. We might also suppose the root to be $r^2 - 2rs + s^2$, and make the formula equal to its square $r^4 - 4r^3s + 6r^2s^2 - 4rs^3 + s^4$; the first and the last two terms being thus destroyed on both sides, we should have, by dividing the other terms by r^2s, $4r - 6s = -4r + 6s$, or $8r = 12s$; consequently, $r = \frac{3}{2}s$; so that by this second supposition, if $r = 3$, and $s = 2$, we shall find $x = -119$, or a negative value.

But let us make $r = \frac{3}{2}s + t$, and we shall have for our formula

$$r^2 = \tfrac{9}{4}s^2 + 3st + t^2;\ r^3 = \tfrac{27}{8}s^3 + \tfrac{27}{4}s^2t + \tfrac{9}{2}st^2 + t^3.$$

$$\begin{aligned}
\text{Therefore } r^4 &= \tfrac{81}{16}s^4 + \tfrac{27}{2}s^3t + \tfrac{27}{2}s^2t^2 + 6st^3 + t^4\\
+ 4r^3s &= \tfrac{27}{2}s^4 + 27s^3t + 18s^2t^2 + 4st^3\\
- 6r^2s^2 &= -\tfrac{27}{2}s^4 - 18s^3t - 6s^2t^2\\
- 4rs^3 &= -6s^4 - 4s^3t\\
+ s^4 &= + s^4;
\end{aligned}$$

and, consequently, the formula will be $\dfrac{1}{16}s^4 + \dfrac{37}{2}s^3t + \dfrac{51}{2}s^2t^2 + 10st^3 + t^4$.

This formula ought also to be a square, if multiplied by 16, by which means it becomes

$$s^4 + 296s^3t + 408s^2t^2 + 160st^3 + 16t^4.$$

Let us make this equal to the square of $s^2 + 148st - 4t^2$, that is, to $s^4 + 296s^3t + 21896s^2t^2 - 1184st^3 + 16t^4$; the first two terms, and the last, are destroyed on both sides, and we thus obtain the equation

$$21896s - 1184t = 408s + 160t, \text{ which gives}$$

$$\frac{s}{t} = \frac{1344}{21488} = \frac{336}{5372} = \frac{84}{1343}.$$

Therefore, since $s = 84$, and $t = 1343$, we shall have $r = \frac{3}{2}s + t = 1469$, and, consequently,

$$x = r^4 - 6r^2s^2 + s^4 = 4565486027761, \text{ and}$$
$$y = 4r^3s - 4rs^3 = 1061652293520.$$

CHAP. XV.

Solutions *of some* Questions, *in which* Cubes *are required.*

241. In the preceding chapter, we have considered some questions, in which it was required to transform certain formulæ into squares, and they afforded an opportunity of explaining several artifices requisite in the application of the rules which have been given. It now remains, to consider questions, which relate to the transformation of certain formulæ into cubes; and the following solutions will throw some light on the rules, which have been already explained for transformations of this kind.

242. *Question* 1. It is required to find two cubes, x^3, and y^3, whose sum may be a cube.

Since $x^3 + y^3$ must be a cube, if we divide this formula by y^3, the quotient ought likewise to be a cube, or $\dfrac{x^3}{y^3} + 1 = c$. If, therefore, $\dfrac{x}{y} = z - 1$, we shall have

$z^3 - 3z^2 + 3z - 1^3 = c$. If we should here, according to the rules already given, suppose the cube root to be $z - u$, and, by comparing the formula with the cube $z^3 - 3uz^2 + 3u^2z - u^3$, determine u so, that the second term may also vanish, we should have $u = 1$; and the other terms forming the equation $3z = 3u^2z - u^3 = 3z - 1$, we should find $z = \infty$, from which we can draw no conclusion. Let us therefore rather leave u undetermined, and deduce z from the quadratic equation $-3z^2 + 3z = -3uz^2 + 3u^2z - u^3$, or $3uz^2 - 3z^2 = 3u^2z - 3z - u^3$, or $3(u-1)z^2 = 3(u^2-1)z - u^3$, or

$z^2 = (u+1)z - \frac{u^3}{3(u-1)}$; from this we shall find

$$z = \frac{u+1}{2} \pm \sqrt{\left(\frac{u^2+2u+1}{4} - \frac{u^3}{3(u-1)}\right)}$$

or $z = \frac{u+1}{2} \mp \sqrt{\left(\frac{-u^3+3u^2-3u-3}{12(u-1)}\right)}$; so that the question is reduced to transforming the fraction under the radical sign into a square. For this purpose, let us first multiply the two terms by $3(u-1)$, in order that the denominator becoming a square, namely, $36(u-1)^2$, we may only have to consider the numerator $-3u^4 + 12u^3 - 18u^2 + 9$: and, as the last term is a square, we shall suppose the formula, according to the rule, equal to the square of $gu^2 + fu + 3$, that is, to $g^2u^4 + 2fgu^3 + f^2u^2 + 6gu^2 + 6fu + 9$. We may make the last three terms disappear, by putting $6f = 0$, or $f = 0$, and $6g + f^2 = -18$, or $g = -3$; and the remaining equation, namely,

$$-3u + 12 = g^2u + 2fu = 9u,$$

will give $u = 1$. But from this value we learn nothing; so that we shall proceed by writing $u = 1 + t$. Now, as our formula becomes in this case $-12t - 3t^4$, which cannot be a square, unless t be negative, let us at once make $t = -s$; by these means we have the formula $12s - 3s^4$, which becomes a square in the case of $s = 1$. But here we are stopped again; for when $s = 1$, we have $t = -1$, and $u = 0$, from which we can draw no conclusion, except that in whatever manner we set about it, we shall never find a value that will bring us to the end proposed; and hence we may already infer, with some degree of certainty, that it is impossible to find two cubes whose sum is a cube. But we shall be fully convinced of this from the following demonstration.

243. *Theorem.* It is impossible to find any two cubes, whose sum, or difference, is a cube.

We shall begin by observing, that if this impossibility applies to the sum, it applies also to the difference, of two cubes. In fact, if it be impossible for $x^3 + y^3 = z^3$, it is also impossible for $z^3 - y^3 = x^3$. Now, $z^3 - y^3$ is the difference of two cubes; therefore, if the one be possible, the other is so likewise. This being laid down, it will be sufficient, if we demonstrate the impossibility either in the case of the sum, or difference; which demonstration requires the following chain of reasoning.

1. We may consider the numbers x and y as prime to each other; for if they had a common divisor, the cubes would also be divisible by the cube of that divisor. For example, let $x = ma$, and $y = mb$, we shall then have $x^3 + y^3 = m^3a^3 + m^3b^3$; now if this formula be a cube, $a^3 + b^3$ is a cube also.

2. Since, therefore, x and y have no common factor, these two numbers are either both odd, or the one is even and the other odd. In the first case, z would be even, and in the other that number would be odd. Consequently, of these three numbers x, y, and z, there is always one which is even, and two that are odd; and it will therefore be sufficient for our demonstration to consider the case in which x and y are both odd: because we may prove the impossibility in question either for the sum, or for the difference; and the sum only happens to become the difference, when one of the roots is negative.

3. If therefore x and y are odd, it is evident that both their sum and their difference will be an even number.

Therefore let $\frac{x+y}{2} = p$, and $\frac{x-y}{2} = q$, and we shall have $x = p + q$, and $y = p - q$; whence it follows, that one of the two numbers, p and q, must be even, and the other odd. Now, we have, by adding $(p+q)^3 = x^3$, to $(p-q)^3 = y^3$, $x^3 + y^3 = 2p^3 + 6pq^2 = 2p(p^2 + 3q^2)$; so that it is required to prove that this product $2p(p^2+3q^2)$ cannot become a cube; and if the demonstration were applied to the difference, we should have $x^3 - y^3 = 6p^2q + 2q^3 = 2q(q^2 + 3p^2)$, a formula precisely the same as the former, if we substitute p and q for each other. Consequently, it is sufficient for our purpose to demonstrate the impossibility of the formula $2p(p^2 + 3q^2)$, since it will necessarily follow, that neither the sum nor the difference of two cubes can become a cube.

4. If therefore $2p(p^2 + 3q^2)$ were a cube, that cube would be even, and, consequently, divisible by 8: con-

sequently, the eighth part of our formula, or $\frac{1}{4}p(p^2+3q^2)$, would necessarily be a whole number, and also a cube. Now, we know that one of the numbers p and q is even, and the other odd; so that p^2+3q^2 must be an odd number, which not being divisible by 4, p must be so, or $\frac{p}{4}$ must be a whole number.

5. But in order that the product $\frac{1}{4}p(p^2+3q^2)$ may be a cube, each of these factors, unless they have a common divisor, must separately be a cube; for if a product of two factors, that are prime to each other, be a cube, each of itself must necessarily be a cube; and if these factors have a common divisor, the case is different, and requires a particular consideration. So that the question here is, to know if the factors p, and p^2+3q^2, might not have a common divisor. To determine this, it must be considered, that if these factors have a common divisor, the numbers p^2, and p^2+3q^2, will have the same divisor; that the difference also of these numbers, which is $3q^2$, will have the same common divisor with p^2; and that, since p and q are prime to each other, these numbers p^2, and $3q^2$, can have no other common divisor than 3, which is the case when p is divisible by 3.

6. We have consequently two cases to examine: the one is, that in which the factors p, and p^2+3q^2, have no common divisor, which happens always, when p is not divisible by 3; the other case is, when these factors have a common divisor, and that is when p may be divided by 3; because then the two numbers are divisible by 3. We must carefully distinguish these two cases from each other, because each requires a particular demonstration.

7. *Case* 1. Suppose that p is not divisible by 3, and, consequently, that our two factors $\frac{p}{4}$, and p^2+3q^2, are prime to each other; so that each must separately be a cube. Now, in order that p^2+3q^2 may become a cube, we have only, as we have seen before, to suppose $p+q\sqrt{-3}=(t+u\sqrt{-3})^3$, and $p-q\sqrt{-3}=(t-u\sqrt{-3})^3$, which gives $p^2+3q^2=(t^2+3u^2)^3$, which is a cube, and gives us $p=t^3-9tu^2=t(t^2-9u^2)$, also $q=3t^2u-3u^3=3u(t^2-u^2)$. Since therefore q is an odd number, u must also be odd; and, consequently, t must be even, because otherwise t^2-u^2 would be even.

8. Having transformed p^2+3q^2 into a cube, and having found $p=t(t^2-9u^2)=t(t+3u)\times(t-3u)$, it is also required that $\frac{p}{4}$, and consequently $2p$, be a cube; or, which comes to the same, that the formula $2t(t+3u)\times(t-3u)$ be a cube. But here it must be observed that t is an even number, and not divisible by 3; since otherwise p would be divisible by 3, which we have expressly supposed not to be the case: so that the three factors, $2t$, $t+3u$, and $t-3u$, are prime to each other; and each of them must separately be a cube. If, therefore, we make $t+3u=f^3$, and $t-3u=g^3$, we shall have $2t=f^3+g^3$. So that, if $2t$ is a cube, we shall have two cubes f^3, and g^3, whose sum would be a cube, and which would evidently be much less than the cubes x^3 and y^3 assumed at first; for as we first made $x=p+q$, and $y=p-q$, and have now determined p and q by the letters t and u, the numbers x and y must necessarily be much greater than t and u.

9. If, therefore, there could be found in great numbers two such cubes as we require, we should also be able to assign in less numbers two cubes whose sum would make a cube, and in the same manner we should be led to cubes always less. Now, as it is very certain that there are no such cubes among small numbers, it follows that there are not any among the greater numbers. This conclusion is confirmed by that which the second case furnishes, and which will be seen to be the same.

10. *Case* 2. Let us now suppose, that p is divisible by 3, and that q is not so, and let us make $p=3r$; our formula will then become $\frac{3r}{4}\times(9r^2+3q^2)$, or $\frac{9}{4}r(3r^2+q^2)$; and these two factors are prime to each other, since $3r^2+q^2$ is neither divisible by 2 nor by 3, and r must be even as well as p; therefore each of these two factors must separately be a cube.

11. Now, by transforming the second factor $3r^2+q^2$, or q^2+3r^2, we find, in the same manner as before, $q=t(t^2-9u^2)$, and $r=3u(t^2-u^2)$; and it must be observed, that since q was odd, t must be here likewise an odd number, and u must be even.

12. But $\frac{9r}{4}$ must also be a cube; or multiplying by the cube $\frac{8}{27}$, we must have $\frac{2r}{3}$, or

$2u(t^2 - u^2) = 2u(t + u) \times (t - u)$ a cube; and as these three factors are prime to each other, each must of itself be a cube. Suppose therefore $t + u = f^3$, and $t - u = g^3$, we shall have $2u = f^3 - g^3$; that is to say, if $2u$ were a cube, $f^3 - g^3$ would be a cube. We should consequently have two cubes, f^3 and g^3, much smaller than the first, whose difference would be a cube, and that would enable us also to find two cubes whose sum would be a cube; since, we should only have to make $f^3 - g^3 = h^3$, in order to have $f^3 = h^3 + g^3$, or a cube equal to the sum of two cubes. Thus, the foregoing conclusion is fully confirmed; for as we cannot assign, in great numbers, two cubes whose sum or difference is a cube, it follows from what has been before observed, that no such cubes are to be found among small numbers.

244. Since it is impossible, therefore, to find two cubes, whose sum or difference is a cube, our first question falls to the ground: and, indeed, it is more usual to enter on this subject with the question of determining three cubes, whose sum may make a cube; supposing, however, two of those cubes to be arbitrary, so that it is only required to find the third. We shall therefore proceed immediately to this question.

245. *Question 2.* Two cubes a^3, and b^3, being given, required a third cube, such, that the three cubes added together may make a cube.

It is here required to transform into a cube the formula $a^3 + b^3 + x^3$; which cannot be done unless we already know a satisfactory case; but such a case occurs immediately; namely, that of $x = -a$. If therefore we make $x = y - a$, we shall have $x^3 = y^3 - 3ay^2 + 3a^2y - a^3$; and, consequently, it is the formula $y^3 - 3ay^2 + 3a^2y + b^3$ that must become a cube. Now, the first and the last term here being cubes, we immediately find two solutions.

1. The first requires us to represent the root of the formula by $y + b$, the cube of which is $y^3 + 3by^2 + 3b^2y + b^3$; and we thus obtain $-3ay + 3a^2 = 3by + 3b^2$; and, consequently, $y = \dfrac{a^2 - b^2}{a + b} = a - b$; but $x = -b$, so that this solution is of no use.

2. But we may also represent the root by $fy + b$, the cube of which is $f^3y^3 + 3bf^2y^2 + 3b^2fy + b^3$, and then determine f in such a manner, that the third terms may be destroyed, namely, by making $3a^2 = 3b^2f$, or $f = \dfrac{a^2}{b^2}$; for

we thus arrive at the equation

$y - 3a = f^3y + 3bf^2 = \dfrac{a^6y}{b^6} + \dfrac{3a^4}{b^3}$, which multiplied by b^6, becomes $b^6y - 3ab^6 = a^6y + 3a^4b^3$. This gives

$$y = \frac{3a^4b^3 + 3ab^6}{b^6 - a^6} = \frac{3ab^3(a^3 + b^3)}{b^6 - a^6} = \frac{3ab^3}{b^3 - a^3};$$ and, consequently,

$$x = y - a = \frac{2ab^3 + a^4}{b^3 - a^3} = a \times \frac{2b^3 + a^3}{b^3 - a^3}.$$ So that the two cubes a^3 and b^3 being given, we know also the root of the third cube sought; and if we would have that root positive, we have only to suppose b^3 to be greater than a^3. Let us apply this to some examples:

1. Let 1 and 8 be the two given cubes, so that $a = 1$, and $b = 2$; the formula $9 + x^3$ will become a cube, if $x = \frac{17}{7}$; for we shall have $9 + x^3 = \frac{8000}{343} = (\frac{20}{7})^3$.

2. Let the given cubes be 8 and 27, so that $a = 2$, and $b = 3$; the formula $35 + x^3$ will be a cube, when $x = \frac{124}{19}$.

3. If 27 and 64 be the given cubes, that is, if $a = 3$, and $b = 4$, the formula $91 + x^3$ will become a cube, if $x = \frac{465}{37}$.

And, generally, in order to determine third cubes for any two given cubes, we must proceed by substituting $\dfrac{2ab^3 + a^4}{b^3 - a^3} + z$ instead of x, in the formula $a^3 + b^3 + x^3$; for by these means we shall arrive at a formula like the preceding, which would then furnish new values of z; but it is evident that this would lead to very prolix calculations.

246. In this question, there likewise occurs a remarkable case; namely, that in which the two given cubes are equal, or $a = b$; for then we have $x = \dfrac{3a^4}{0} = \infty$; that is, we have no solution; and this is the reason why we are not able to resolve the problem of transforming into a cube the formula $2a^3 + x^3$. For example, let $a = 1$, or let this formula be $2 + x^3$, we shall find that whatever forms we give it, it will always be to no purpose, and we shall seek in vain for a satisfactory value of x. Hence, we may conclude with sufficient certainty, that it is impossible to find a cube equal to the sum of a cube, and of a double cube; or that the equation $2a^3 + x^3 = y^3$ is impossible. As this equation

gives $2a^3 = y^3 - x^3$, it is likewise impossible to find two cubes having their difference equal to the double of another cube; and the same impossibility extends to the sum of two cubes, as is evident from the following demonstration.

247. *Theorem.* Neither the sum nor the difference of two cubes can become equal to the double of another cube; or, in other words, the formula $x^3 \pm y^3 = 2z^3$ is always impossible, except in the evident case of $y = x$.

We may here also consider x and y as prime to each other; for if these numbers had a common divisor, it would be necessary for z to have the same divisor; and, consequently, for the whole equation to be divisible by the cube of that divisor. This being laid down, as $x^3 \pm y^3$ must be an even number, the numbers x and y must both be odd, in consequence of which both their sum and their difference must be even. Making, therefore, $\frac{x+y}{2} = p$, and $\frac{x-y}{2} = q$, we shall have $x = p + q$ and $y = p - q$; and of the two numbers p and q, the one must be even and the other odd. Now, from this, we obtain

$$x^3 + y^3 = 2p^3 + 6pq^2 = 2p(p^2 + 3q^2),$$
$$\text{and } x^3 - y^3 = 6p^2q + 2q^3 = 2q(3p^2 + q^2),$$

which are two formulæ perfectly similar. It will therefore be sufficient to prove that the formula $2p(p^2 + 3q^2)$ cannot become the double of a cube, or that $p(p^2 + 3q^2)$ cannot become a cube: which may be demonstrated in the following manner.

1. Two different cases again present themselves to our consideration: the one, in which the two factors p, and $p^2 + 3q^2$, have no common divisor, and must separately be a cube; the other in which these factors have a common divisor, which divisor, however, as we have seen (Art. 243), can be no other than 3.

2. *Case* 1. Supposing, therefore, that p is not divisible by 3, and that thus the two factors are prime to each other, we shall first reduce $p^2 + 3q^2$ to a cube by making $p = t(t^2 - 9u^2)$, and $q = 3u(t^2 - 9u^2)$; by which means it will only be farther necessary for p to become a cube. Now, t not being divisible by 3, since otherwise p would also be divisible by 3, the two factors t, and $t^2 - 9u^2$, are prime to one another, and, consequently, each must separately be a cube.

3. But the last factor has also two factors, namely $t + 3u$, and $t - 3u$, which are prime to each other, first because t is not divisible by 3, and, in the second place, because one of

the numbers t or u is even, and the other odd; for if these numbers were both odd, not only p, but also q, must be odd, which cannot be: therefore, each of these two factors, $t + 3u$, and $t - 3u$, must separately be a cube.

4. Therefore let $t + 3u = f^3$, and $t - 3u = g^3$, and we shall then have $2t = f^3 + g^3$. Now, t must be a cube, which we shall denote by h^3, by which means we must have $f^3 + g^3 = 2h^3$; consequently, we should have two cubes much smaller, namely, f^3 and g^3, whose sum would be the double of a cube.

5. *Case* 2. Let us now suppose p divisible by 3, and, consequently, that q is not so.

If we make $p = 3r$, our formula becomes $3r(9r^2 + 3q^2) = 9r(3r^2 + q^2)$, and these factors being now numbers prime to one another, each must separately be a cube.

6. In order therefore to transform the second $q^2 + 3r^2$, into a cube, we shall make $q = t(t^2 - 9u^2)$, and $r = 3u(t^2 - u^2)$; and again one of the numbers t and u must be odd, and the other even, since otherwise the two numbers q and r would be even. Now, from this we obtain the first factor $9r = 27u(t^2 - u^2)$; and as it must be a cube, let us divide it by 27, and the formula $u(t^2 - u^2)$, or $u(t + u) \times (t - u)$, must be a cube.

7. But these three factors being prime to each other, they must all be cubes of themselves. Let us therefore suppose for the last two $t + u = f^3$, and $t - u = g^3$, we shall then have $2u = f^3 - g^3$; but as u must be a cube, we should in this way have two cubes, in much smaller numbers, whose difference would be equal to the double of another cube.

8. Since therefore we cannot assign, in small numbers, any cubes, whose sum or difference is the double of a cube, it is evident that there are no such cubes, even among the greatest numbers.

9. It will perhaps be objected, that our conclusion might lead to error; because there does exist a satisfactory case among these small numbers; namely, that of $f = g$. But it must be considered that when $f = g$, we have, in the first case, $t + 3u = t - 3u$, and therefore $u = 0$; consequently, also $q = 0$; and, as we have supposed $x = p + q$, and $y = p - q$, the first two cubes, x^3 and y^3, must have already been equal to one another, which case was expressly excepted. Likewise, in the second case, if $f = g$, we must have $t + u = t - u$, and also $u = 0$: therefore $r = 0$, and $p = 0$; so that the first two cubes, x^3 and y^3, would again

become equal, which does not enter into the subject of the problem.

248. *Question 3.* Required in general three cubes, x^3, y^3, and z^3, whose sum may be equal to a cube.

We have seen that two of these cubes may be supposed to be known, and that from them we may determine the third, provided the two are not equal; but the preceding method furnishes in each case only one value for the third cube, and it would be difficult to deduce from it any new ones.

We shall now, therefore, consider the three cubes as unknown; and, in order to give a general solution, let us make $x^3 + y^3 + z^3 = v^3$. Here, by transposing one of the terms, we have $x^3 + y^3 = v^3 - z^3$, the conditions of which equation we may satisfy in the following manner.

1. Let $x = p + q$, and $y = p - q$, and we shall have, as before, $x^3 + y^3 = 2p(p^2 + 3q^2)$. Also, let $v = r + s$, and $z = r - s$, which gives $v^3 - z^3 = 2s(s^2 + 3r^2)$; therefore we must have $2p(p^2 + 3q^2) = 2s(s^2 + 3r^2)$, or

$$p(p^2 + 3q^2) = s(s^2 + 3r^2).$$

2. We have already seen (Art. 176), that a number, such as $p^2 + 3q^2$, can have no divisors except numbers of the same form. Since, therefore, these two formulæ, $p^2 + 3q^2$, and $s^2 + 3r^2$, must necessarily have a common divisor, let that divisor be $t^2 + 3u^2$.

3. And let us, therefore, make

$$p^2 + 3q^2 = (f^2 + 3g^2) \times (t^2 + 3u^2), \text{ and}$$
$$s^2 + 3r^2 = (h^2 + 3k^2) \times (t^2 + 3u^2),$$

and we shall have $p = ft + 3gu$, and $q = gt - fu$; consequently, $p^2 = f^2t^2 + 6fgtu + 9g^2u^2$, and

$$q^2 = g^2t^2 - 2fgtu + f^2u^2; \text{ whence,}$$
$$p^2 + 3q^2 = (f^2 + 3g^2)t^2 + (3f^2 + 9g^2)u^2; \text{ or}$$
$$p^2 + 3q^2 = (f^2 + 3g^2) \times (t^2 + 3u^2).$$

4. In the same manner, we may deduce from the other formula, $s = ht + 3ku$, and $r = kt - hu$; whence results the equation,

$$(ft + 3gu) \times (f^2 + 3g^2) \times (t^2 + 3u^2) =$$
$$(ht + 3ku) \times (h^2 + 3k^2) \times (t^2 + 3u^2),$$

which being divided by $t^2 + 3u^2$, and reduced, gives

$$ft(f^2 + 3g^2) + 3gu(f^2 + 3g^2) =$$
$$ht(h^2 + 3k^2) + 3ku(h^2 + 3k^2), \text{ or}$$
$$ft(f^2 + 3g^2) - ht(h^2 + 3k^2) =$$
$$3ku(h^2 + 3k^2) - 3gu(f^2 + 3g^2),$$

by which means $t = \dfrac{3k(h^2+3k^2) - 3g(f^2+3g^2)}{f(f^2+3g^2) - h(h^2+3k^2)}u.$

5. Let us now remove the fractions, by making

$$u = f(f^2 + 3g^2) - h(h^2 + 3k^2); \text{ then}$$
$$t = 3k(h^2 + 3k^2) - 3g(f^2 + 3g^2),$$

where we may give any values whatever to the letters f, g, h, and k.

6. When therefore we have determined, from these four numbers, the values of t and u, we shall have

$$p = ft + 3gu, \qquad q = gt - fu,$$
$$r = kt - hu, \qquad s = ht + 3ku;$$

whence we shall at last arrive at the solution of the question, $x = p + q$, $y = p - q$, $z = r - s$, and $v = r + s$; and this solution is general, so far as to comprehend all the possible cases, since in the whole calculation we have admitted no arbitrary limitation. The whole artifice consisted in rendering our equation divisible by $t^2 + 3u^2$; for we have thus been able to determine the letters t and u by an equation of the first degree: and innumerable applications may be made of these formulæ, some of which we shall give for the sake of example.

1. Let $k = 0$, and $h = 1$, we shall have $t = -3g(f^2 + 3g^2)$, and $u = f(f^2 + 3g^2) - 1$; so that $p = -3fg(f^2 + 3g^2) + 3fg(f^2 + 3g^2) - 3g$, or $p = -3g$; $q = -(f^2 + 3g^2)^2 + f$; $s = -3g(f^2 + 3g^2)$; $r = -f(f^2 + 3g^2) + 1$; consequently,

$$x = -3g - (f^2 + 3g^2)^2 + f,$$
$$y = -3g + (f^2 + 3g^2)^2 - f,$$
$$z = (3g - f) \times (f^2 + 3g^2) + 1;$$

lastly, $v = -(3g + f) \times (f^2 + 3g^2) + 1$.

If we also suppose $f = -1$, and $g = +1$, we shall have $x = -20$, $y = 14$, $z = 17$, and $v = -7$; and thence results the final equation, $-20^3 + 14^3 + 17^3 = -7^3$, or $14^3 + 17^3 + 7^3 = 20^3$.

2. Let $f = 2$, $g = 1$, and consequently $f^2 + 3g^2 = 7$; farther, $h = 0$, and $k = 1$; so that $h^2 + 3k^2 = 3$; we shall then have $t = -12$, and $u = 14$; so that

$$p = 2t + 3u = 18, \qquad q = t - 2u = -40,$$
$$r = t = -12, \qquad \text{and } s = 3u = 42.$$

From this will result

$$x = p + q = -22, \qquad y = p - q = 58,$$
$$z = r - s = -54, \qquad \text{and } v = r + s = 30;$$

therefore,

$$30^3 = 22^3 + 58^3 - 54^3, \text{ or}$$
$$58^3 = 30^3 + 54^3 + 22^3;$$

and as all these roots are divisible by 2, we shall also have

$$29^3 = 15^3 + 27^3 + 11^3.$$

3. Let $f = 3$, $g = 1$, $h = 1$, and $k = 1$; so that $f^2 + 3g^2 = 12$, $h^2 + 3k^2 = 4$; also $t = -24$, and $u = 32$. Here, these two values being divisible by 8, and as we consider only their ratios, we may make $t = -3$, and $u = 4$. Whence we obtain

$$p = 3t + 3u = +3, \qquad q = t - 3u = -15,$$
$$r = t - u = -7, \quad \text{and} \quad s = t + 3u = +9;$$

consequently, $x = -12$, and $y = 18$,
$z = -16$, and $v = 2$,

whence $-12^3 + 18^3 - 16^3 = 2^3$, or $18^3 = 16^3 + 12^3 + 2^3$, or, dividing by the cube of 2, $9^3 = 8^3 + 6^3 + 1^3$.

4. Let us also suppose $g = 0$, and $k = h$, by which means we leave f and h undetermined. We shall thus have $f^2 + 3g^2 = f^2$, and $h^2 + 3k^2 = 4h^2$; so that $t = 12h^3$, and $u = f^3 - 4h^3$; also, $p = st = 12fh^3$, $q = -f^4 + 4fh^3$, $r = 12h^4 - hf^3 + 4h^4 = 16h^4 - hf^3$, and $s = 3hf^3$; lastly, $x = p + q = 16fh^3 - f^4$, $\quad y = p - q = 8fh^3 + f^4$, $z = r - s = 16h^4 - 4hf^3$, and $v = r + s = 16h^4 + 2hf^3$. If we now make $f = h = 1$, we have $x = 15$, $y = 9$, $z = 12$, and $v = 18$; or, dividing all by 3, $x = 5$, $y = 3$, $z = 4$, and $v = 6$; so that $3^3 + 4^3 + 5^3 = 6^3$. The progression of these three roots, 3, 4, 5, increasing by unity, is worthy of attention; for which reason, we shall investigate whether there are not others of the same kind.

249. *Question* 4. Required three numbers, whose difference is 1, and forming such an arithmetical progression, that their cubes added together may make a cube.

Let x be the middle number, or term, then $x - 1$ will be the least, and $x + 1$ the greatest; the sum of the cubes of these three numbers is $3x^3 + 6x = 3x(x^2 + 2)$, which must be a cube. Here, we must previously have a case, in which this property exists, and we find, after some trials, that that case is $x = 4$.

So that, according to the rules already given, we may make $x = 4 + y$; whence $x^2 = 16 + 8y + y^2$, and $x^3 = 64 + 48y + 12y^2 + y^3$, and by these means our formula becomes $216 + 150y + 36y^2 + 3y^3$, in which the first term is a cube, but the last is not.

Let us, therefore, suppose the root to be $6 + fy$, or the formula to be $216 + 108fy + 18f^2y^2 + f^3y^3$, and destroy the two second terms, by writing $108f = 150$, or $f = \frac{25}{18}$; the other terms, divided by y^2, will give

$$36 + 3y = 18f^2 + f^3y = \frac{25^2}{18} + \frac{25^3}{18^3}y, \text{ or}$$

$18^3 \times 36 + 18^3 \times 3y = 18^2 \times 25^2 + 25^3y$, or
$18^3 \times 36 - 18^2 \times 25^2 = 25^3y - 18^3 \times 3y$; therefore

$$y = \frac{18^3 \times 36 - 18^2 \times 25^2}{25^3 - 3 \times 18^3} = \frac{18^2 \times (18 \times 36 - 25^2)}{25^3 - 3 \times 18^3}; \text{ that}$$

is, $y = \dfrac{-324 \times 23}{1871} = \dfrac{-7452}{1871}$; and, consequently, $x = \frac{32}{1871}$.

As it might be difficult to pursue this reduction in cubes, it is proper to observe, that the question may always be reduced to squares. In fact, since $3x(x^2 + 2)$ must be a cube, let us suppose $3x(x^2 + 2) = x^3y^3$; dividing by x, we shall have $3x^2 + 6 = x^2y^3$; and, consequently,

$x^2 = \dfrac{6}{y^3 - 3} = \dfrac{36}{6y^3 - 18}$. Now, the numerator of this fraction being already a square, it is only necessary to transform the denominator, $6y^3 - 18$, into a square, which also requires that we have already found a case. For this purpose, let us consider that 18 is divisible by 9, but 6 only by 3, and that y therefore may be divided by 3; if we make $y = 3z$, our denominator will become $162z^3 - 18$, which being divided by 9, and becoming $18z^3 - 2$, must still be a square. Now, this is evidently true of the case $z = 1$. So that we shall make $z = 1 + v$, and we must have $16 + 54v + 54v^2 + 18v^3 = \square$. Let its root be $4 + \frac{27}{4}v$, the square of which is $16 + 54v + \frac{729}{16}v^2$, and we must have $54 + 18v = \frac{729}{16}$; or $18v = -\frac{135}{16}$, or $2v = -\frac{15}{16}$; and, consequently, $v = -\frac{15}{32}$; which produces $z = 1 + v = \frac{17}{32}$, and then $y = \frac{51}{32}$.

Let us now resume the denominator

$$6y^3 - 18 = 162z^3 - 18 = 9(18z^3 - 2);$$

and since the square root of the factor, $18z^3 - 2$, is $4 + \frac{27}{4}v = \frac{107}{128}$, that of the whole denominator is $\frac{321}{128}$: but the root of the numerator is 6; therefore $x = \dfrac{6}{\frac{321}{128}} = \frac{256}{107}$, a value quite different from that which we found before. It follows, therefore, that the roots of our three cubes sought are $x - 1 = \frac{149}{107}$, $x = \frac{256}{107}$, $x + 1 = \frac{363}{107}$: and the sum of the cubes of these three numbers will be a cube, whose root, $xy, = \frac{256}{107} \times \frac{51}{32} = \frac{13056}{3424} = \frac{408}{107}$.

250. We shall here finish this Treatise on the Indeterminate Analysis, having had sufficient occasion, in the questions which we have resolved, to explain the chief artifices that have hitherto been devised in this branch of Algebra.

QUESTIONS FOR PRACTICE.

1. To divide a square number (16) into two squares.
Ans. $\frac{256}{25}$, and $\frac{144}{25}$.

2. To find two square numbers, whose difference (60) is given.
Ans. $72\frac{1}{4}$, and $132\frac{1}{4}$.

3. From a number x to take two given numbers 6 and 7, so that both remainders may be square numbers.
Ans. $x = \frac{121}{16}$.

4. To find two numbers in proportion as 8 is to 15, and such, that the sum of their squares shall make a square number.
Ans. 576, and 1080.

5. To find four numbers such, that if the square number 100 be added to the product of every two of them, the sum shall be all squares.
Ans. 12, 32, 88, and 168.

6. To find two numbers, whose difference shall be equal to the difference of their squares, and the sum of their squares a square number.
Ans. $\frac{4}{7}$, and $\frac{3}{7}$.

7. To find two numbers, whose product being added to the sum of their squares, shall make a square number.
Ans. 5 and 3, 8 and 7, 16 and 5, &c.

8. To find two such numbers, that not only each number, but also their sum and their difference, being increased by unity, shall be square numbers.
Ans. 3024, and 5624.

9. To find three square numbers such, that the sum of their squares shall be a square number.
Ans. 9, 16, and $\frac{144}{25}$.

10. To divide the cube number 8 into three other cube numbers.
Ans. $\frac{64}{27}$, $\frac{125}{27}$, and 1.

11. Two cube numbers, 8 and 1, being given, to find two other cube numbers, whose difference shall be equal to the sum of the given cubes.
Ans. $\frac{8000}{343}$, and $\frac{4913}{343}$.

12. To find three such cube numbers, that if 1 be subtracted from every one of them, the sum of the remainders shall be a square.
Ans. $\frac{4913}{3375}$, $\frac{21952}{3375}$, and 8.

13. To find two numbers, whose sum shall be equal to the sum of their cubes.
Ans. $\frac{5}{7}$, and $\frac{8}{7}$.

14. To find three such cube numbers, that the sum of them may be both a square and a cube.
Ans. 1, $\frac{2084383}{274625}$, $\frac{15252992}{274625}$.

ADDITIONS

BY

M. DE LA GRANGE.

ADVERTISEMENT.

THE geometricians of the last century paid great attention to the Indeterminate Analysis, or what is commonly called the *Diophantine Algebra*; but Bachet and Fermat alone can properly be said to have added any thing to what Diophantus himself has left us on that subject.

To the former, we particularly owe a complete method of resolving, in integer numbers, all indeterminate problems of the first degree *: the latter is the author of some methods for the resolution of indeterminate equations, which exceed the second degree †; of the singular method, by which we demonstrate that it is impossible for the sum, or the difference of two biquadrates to be a square ‡; of the solution of a great number of very difficult problems; and of several admirable theorems respecting integer numbers, which he left without demonstration, but of which the greater part has since been demonstrated by M. Euler in the Petersburg Commentaries ‖.

* See Chap. 3, in these Additions. I do not here mention his Commentary on Diophantus, because that work, properly speaking, though excellent in its way, contains no discovery.

† These are explained in the 8th, 9th, and 10th chapters of the preceding Treatise. Père Billi has collected them from different writings of M. Fermat, and has added them to the new edition of Diophantus, published by M. Fermat, junior.

‡ This method is explained in the 13th chapter of the preceding Treatise; the principles of it are to be found in the *Remarks* of M. Fermat, on the XXVIth Question of the VIth Book of Diophantus.

‖ The problems and theorems, to which we allude, are