PROBLEMATIS CVIVSDAM GEOMETRICI PRORSVS SINGVLARIS EVOLVTIO.

Auctore

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1.

Quaestionem hic geometricam ad examen sum revocaturus quae cum ad genus adhuc nondum tractatum pertineat eo maiore attentione digna videtur, quod solutio quam Analysis suppeditat, ita sit comparata, vt ex ea vix intelligi queat, vtrum illi vnica satisfaciat linea curua an plures ac fortasse innumerabiles. Neque vero memini alia huiusmodi problemata a Geometris esse considerata quae tali incertitudini sint obnoxia, vt inuenta adeo solutione analytica ancipites haereamus, vtrum ad genus determinatarum quaestionum, an indeterminatarum sint referenda. Quare non dubito, quin euolutio quaestionis, quam hic sum instituturus, occasionem sit praebitura fines cognitionis nostrae analyticae haud mediocriter proferendi.

2. Notissima autem circuli proprietas, qua recta angulum a binis tangentibus formatum bisecans circulum normaliter traiicit, quaestionem hic tractandam suppeditauit.

Pro-

Propoſita ſcilicet recta poſitione data A B *eiuſmodi quaeritur curua* A m M, *vt ducta ad eius punctum quodcunque* M *tangente* M T *rectae illi* A B *in* T *occurrente recta* T C *angulum* A T M *biſecans eandem curuam in* m *normaliter traiiciat.* Tab. II. Fig. 2.

Manifeſtum igitur eſt huic quaeſtioni ſatisfacere circulum quemcunque rectam poſitione datam A B alicubi tangentem; et cum tam locus contactus quam magnitudo circuli arbitrio noſtro relinquatur, ſolutiones quidem innumerabiles inde naſcuntur, quae tamen omnes, quoniam eadem ſpecie continentur, vnicam quaſi ſolutionem conſtituere ſunt cenſendae. Ex quo praecipuum huius quaeſtionis momentum in hoc verſatur, num praeter circulos aliae dentur lineae curuae, in quas eadem proprietas competat.

3. Mentem ergo a circulo abſtrahentes quaeſtionem propoſitam ita aggrediamur, vt analytice generatim in eas lineas curuas inquiramus, quae proprietate memorata ſint praeditae. Methodus autem hoc problema reſoluendi ideo haud parum videtur recondita, quod bina puncta M et *m*, quorum nexus non ſatis definitur, ad eandem curuam ſint referenda, dum pro altero *M* recta tangens pro altero vero *m* normalis ad curuam conſiderari debent quas binas rectas M T et *m* T ita in recta data A B concurrere neceſſe eſt, vt iſta *m* T angulum A T M bifariam ſecet. Neque vero haec binorum punctorum M et *m* relatio eſt reciproca, vti in plerisque aliis huiusmodi quaeſtionibus vſu venit, vbi ex

certa quadam binorum punctorum relatione naturam curuae inuestigari oportet.

4. Quo igitur indolem binorum punctorum m et M ad similitudinem perducamus, ad m quoque tangentem concipiamus $m\,t$ rectae datae A B in t occurrentem, ac iam manifestum est relationem inter lineas A t et $t\,m$ eadem aequatione exprimi oportere, ac relationem inter rectas A T et T M, vbi ergo videndum est, quomodo hae binae relationes inter se connectantur. Hunc in finem consideretur angulus B $t\,m$, qui vocetur $= \omega$, et ob T $m\,t$ rectum habebimus angulum A T $m = 90° - \omega$, cui cum per conditionem problematis aequalis esse debeat angulus M T m, erit angulus M T B $= 2\,\omega$, vnde solutio eo reuocatur, vt curuae punctum M eodem modo ex angulo $2\,\omega$ determinetur, quo punctum m ex angulo ω determinatur.

5. Cum igitur posito angulo B $t\,m = \omega$ sit angulus B T M $= 2\,\omega$, vocemus rectas tangentes $m\,t = t$ et M T $=$ T, ac nunc perspicuum est, qualis fuerit functio quantitas t ipsius ω, talem functionem esse debere quantitatem T ipsius $2\,\omega$; haecque est conditio ob legem continuitatis requisita qua efficitur vt bina puncta m et M ad eandem lineam curuam continuam referantur. Simili vero modo cum etiam recta A t tanquam functio anguli B $t\,m = \omega$ spectari possit, necesse est vt recta A T similis omnino sit functio anguli dupli B T M $= 2\,\omega$. Harum autem rectarum A t et A T differentia T t, quia

quia ob triangulum $t\,m\,T$ rectangulum esse debet $T\,t = \frac{t}{cos.\,\omega}$, ex hac ipsa conditione solutio problematis erit deducenda. Ceterum hic notari conueniet angulum $B\,t\,m = \omega$ designare amplitudinem arcus $A\,m$, ita vt arcus $A\,M$ amplitudo illius sit dupla.

6. Ad quaestionem itaque resoluendam dispiciamus, quomodo per angulum $B\,t\,m = \omega$ et tangentem $t\,m = t$ interuallum $A\,t$ in recta positione data $A\,B$ determinetur, vbi quidem perinde est vbi punctum fixum A accipiatur. Ducta igitur curuae tangente proxima $\mu\,\tau$, vt sit angulus $B\,\tau\,\mu = \omega + d\omega$, ideoque angulus $t\,\mu\,\tau = d\,\omega$, vnde centro m vel μ descripto arculo elementari $t\,\theta$ radio $\mu\,\tau$ producto in θ occurrente, erit $t\,\theta = t\,d\,\omega$, hincque ob $\theta\,t\,\tau = 90^\circ - \omega$ fit $t\,\tau = \frac{t\,d\,\omega}{sin.\,\omega}$, quod est incrementum interualli $A\,t$ ita vt hinc concludamus $A\,t = \int \frac{t\,d\,\omega}{sin.\,\omega}$. Qua expressione ad priorem figuram translata, si pro t scribamus T et loco anguli ω eius duplum $2\,\omega$, lex continuitatis praebet Tab. II. Fig. 3.

$$A\,T = \int \frac{2\,T\,d\,\omega}{sin.\,2\,\omega} = \int \frac{T\,d\,\omega}{sin.\,\omega\,cos.\,\omega},$$

vnde elicitur interuallum

$$T\,t = \int \frac{T\,d\,\omega}{sin.\,\omega\,cos.\,\omega} - \int \frac{t\,d\,\omega}{sin.\,\omega}$$

quantitati $\frac{t}{cos.\,\omega}$ aequandum.

7. Sumtis ergo differentialibus hanc adipiscimur aequationem:

$$\frac{T\,d\,\omega}{sin.\,\omega\,cos.\,\omega} - \frac{t\,d\,\omega}{sin.\,\omega} = \frac{d\,t}{cos.\,\omega} + \frac{t\,d\,\omega\,sin.\,\omega}{cos.\,\omega^2}$$

quae

quae reducitur ad hanc

$$\frac{T\,d\,\omega}{\text{fin.}\,\omega\,\text{cof.}\,\omega} = \frac{d\,t}{\text{cof.}\,\omega} + \frac{t\,d\,\omega}{\text{fin.}\,\omega\,\text{cof.}\,\omega^2}$$

indeque ad istam:

$$T\,\text{cof.}\,\omega = t + \frac{d\,t}{d\,\omega}\,\text{fin.}\,\omega\,\text{cof.}\,\omega.$$

Quocirca quaestio iam huc est perducta, vt pro t eiusmodi functio ipsius ω sit inuestiganda vt si pro T similis functio ipsius $2\,\omega$ scribatur, aequationi modo inuentae satisfiat. Quemadmodum autem eiusmodi conditiones sint adimplendae, methodus certa adhuc desideratur, imprimis si solutiones generales postulentur. Quamobrem haud inutile erit aequationem inuentam in plurimas alias formas transfundere, ex quibus quae fuerit simplicissima, facillime quoque poterit tractari.

8. Angulum quidem ω perpetuo in calculo conseruari oportet quia per eum quantitates ad bina puncta m et M referendae commodissime definiuntur. Ponamus igitur ipsa interualla $At = x$ et $AT = X$, ita vt iam simili modo X talis debeat esse functio ipsius $2\,\omega$, qualis x fuerit ipsius ω. Quia ergo habemus $\int \frac{T\,d\,\omega}{\text{fin.}\,\omega\,\text{cof.}\,\omega} = X$ et $\int \frac{t\,d\,\omega}{\text{fin.}\,\omega} = x$, hinc fit $t = \frac{d\,x\,\text{fin.}\,\omega}{d\,\omega}$ et nunc solutio problematis perducitur ad hanc aequationem:

$$X - x = \frac{d\,x}{d\,\omega}\,\text{tang.}\,\omega \text{ feu}$$

$$X\,\text{cof.}\,\omega = \frac{d\,x}{d\,\omega}\,\text{fin.}\,\omega + x\,\text{cof.}\,\omega.$$

Quod

Quodſi porro ponamus x ſin. $\omega = y$ ſimilique modo X ſin. $2\,\omega = \mathrm{Y}$ vt ſit X coſ. $\omega = \frac{\mathrm{Y}}{2\, \text{ſin.}\, \omega}$, haec aequatio prodit ſimplicior:

$$\frac{\mathrm{Y}}{2\, \text{ſin.}\, \omega} = \frac{d\,y}{d\,\omega} \text{ ſeu } \mathrm{Y} = \frac{2\,d\,y}{d\,\omega} \text{ ſin. } \omega$$

cui aequationi ita ſatisfieri oportet, vt qualis y fuerit functio ipſius ω, talis Y ſit functio ipſius $2\,\omega$.

9. Poſſumus quoque pro puncto m ipſam normalem m T in calculum introducere, poſita enim $m\,\mathrm{T} = p$, et normali puncto M conueniente $= \mathrm{P}$, quia eſt $p = t$ tang. ω, erit ſimili modo P $=$ T tang. $2\,\omega$ et ob $t = \frac{p\, \text{coſ.}\, \omega}{\text{ſin.}\, \omega}$ et $\mathrm{T} = \frac{\mathrm{P}\, \text{coſ.}\, 2\,\omega}{\text{ſin.}\, 2\,\omega}$ aequatio ſolutionem continens abit in hanc formam:

$$\int \frac{2\,\mathrm{P}\,d\,\omega\, \text{coſ.}\, 2\,\omega}{(\text{ſin.}\, 2\,\omega)^2} - \int \frac{p\,d\,\omega\, \text{coſ.}\, \omega}{(\text{ſin.}\, \omega)^2} = \frac{p}{\text{ſin.}\, \omega}$$

quae differentiata dat:

$$\frac{2\,\mathrm{P}\,d\,\omega\, \text{coſ.}\, 2\,\omega}{(\text{ſin.}\, 2\,\omega)^2} - \frac{p\,d\,\omega\, \text{coſ.}\, \omega}{(\text{ſin.}\, \omega)^2} = \frac{d\,p}{\text{ſin.}\, \omega} - \frac{p\,d\,\omega\, \text{coſ.}\, \omega}{(\text{ſin.}\, \omega)^2}$$

$$\text{ſeu } \mathrm{P}\,d\,\omega\, \text{coſ.}\, 2\,\omega = 2\,dp\, \text{ſin.}\, \omega\, \text{coſ.}\, \omega^2 = dp\, \text{coſ.}\, \omega\, \text{ſin.}\, 2\,\omega$$

ita vt ſit $\mathrm{P} = \frac{d\,p}{d\,\omega}$ coſ. ω tang. $2\,\omega$.

10. In computum etiam duci poteſt ipſe arcus curuae A m, qui ponatur $= s$, ita vt relatio inter hunc arcum A $m = s$ et amplitudinem eius ω ſit definienda. Ex figura autem 2, hoc facillime praeſtabitur, cum enim poſita tangente $m\,t = t$ ſit $\mu\,\tau = t + d\,t$ et $m\,\mu = d\,s$, ob $t\,\theta = t\,d\,\omega$ hincque $\tau\,\theta = \frac{t\,d\,\omega\, \text{coſ.}\, \omega}{\text{ſin.}\, \omega}$ erit $\mu\,\theta = t + d\,t + \frac{t\,d\,\omega\, \text{coſ.}\, \omega}{\text{ſin.}\, \omega}$, Tab. II. Fig. 3.

quae linea ipsi $\mu t = t + ds$ aequalis ſtatui debet vnde colligitur:

$$ds = dt + \frac{t\, d\omega \operatorname{cof.} \omega}{\operatorname{fin.} \omega} = \frac{d.\, t \operatorname{fin.} \omega}{\operatorname{fin.} \omega}.$$

Tab. II. Fig. 2. ideoque $t = \frac{1}{\operatorname{fin.} \omega} \int ds \operatorname{fin.} \omega$. Quare ſi ſimili modo ponatur arcus $AM = S$ tangente exiſtente $MT = T$, habebimus

$$T = \frac{2}{\operatorname{fin.} 2\omega} \int dS \operatorname{fin.} \omega = \frac{1}{\operatorname{fin.} \omega \operatorname{cof.} \omega} \int dS \operatorname{fin.} \omega \operatorname{cof.} \omega,$$

quos valores in aequatione ſupra inuenta $T \operatorname{cof.} \omega = t + \frac{dt}{d\omega} \operatorname{fin.} \omega \operatorname{cof.} \omega$ ſubſtitui oportet, ex quo conficitur

$$\frac{1}{\operatorname{fin.} \omega} \int dS \operatorname{fin.} \omega \operatorname{cof.} \omega = \frac{1}{\operatorname{fin.} \omega} \int ds \operatorname{fin.} \omega + \frac{ds}{d\omega} \operatorname{fin.} \omega \operatorname{cof.} \omega - \frac{\operatorname{cof.} \omega^2}{\operatorname{fin.} \omega} \int ds \operatorname{fin.} \omega$$

ſeu $\int dS \operatorname{fin.} \omega \operatorname{cof.} \omega = \operatorname{fin.} \omega^2 \int ds \operatorname{fin.} \omega + \frac{ds}{d\omega} \operatorname{fin.} \omega^2 \operatorname{cof.} \omega$.

11. Hac aequatione differentiata prodit

$$dS \operatorname{fin.} \omega \operatorname{cof.} \omega = 2 d\omega \operatorname{fin.} \omega \operatorname{cof.} \omega \int ds \operatorname{fin.} \omega + ds \operatorname{fin.} \omega^3 + \frac{dds}{d\omega} \operatorname{fin.} \omega^2 \operatorname{cof.} \omega + 2 ds \operatorname{fin.} \omega \operatorname{cof.} \omega^2 - ds \operatorname{fin.} \omega^3$$

ac facta diuiſione per $\operatorname{fin.} \omega \operatorname{cof.} \omega$ fit

$$dS = 2 d\omega \int ds \operatorname{fin.} \omega + 2 ds \operatorname{cof.} \omega + \frac{dds}{d\omega} \operatorname{fin.} \omega$$

Denuo ergo differentiatione inſtituta nanciſcimur:

$$ddS = 2 d\omega ds \operatorname{fin.} \omega + 2 dds \operatorname{cof.} \omega - 2 d\omega ds \operatorname{fin.} \omega + \frac{d^3 s}{d\omega} \operatorname{fin.} \omega + dds \operatorname{cof.} \omega$$

ideoque $ddS = 3 dds \operatorname{cof.} \omega + \frac{d^3 s}{d\omega} \operatorname{fin.} \omega$

vbi quidem differentiale $d\omega$ pro conſtanti eſt ſumtum. Huic aequationi ſatisfieri ſtatim liquet, ſumendo $\frac{dds}{d\omega^2} = 0$ quia tum quoque fit $\frac{ddS}{d\omega^2} = 0$; hinc autem colli-

colligitur $s = a + b\,\omega$ ita vt arcus $Am = s$ ſit amplitudini proportionalis, quae eſt notiſſima circuli proprietas.

12. Quodſi porro radius oſculi curuae in m ſtatuatur $= r$, in M vero $= R$, quoniam tum ſit $ds = r\,d\omega$ et $dS = 2\,R\,d\omega$, aequationem modo inventam facile ad radios curuaturae traducimus, vnde reſultat

$$2\,dR = 3\,dr\,\text{coſ.}\,\omega + \frac{ddr}{d\omega}\,\text{ſin.}\,\omega$$

vbi multo magis perſpicuum eſt circulos ſatisfacere, cum ſumto radio oſculi conſtante conditio iſtius aequationis impleatur. Vlterius vero curuae $A\,m\,M$ euoluta induci poteſt, cuius radius oſculi pro puncto ex m nato ſi ponatur $= r'$, pro puncto autem ab M deriuato $= R'$, quia conſtat eſſe $dr = r'\,d\omega$ et $dR = 2\,R'\,d\omega$ prodit haec aequatio:

$$4\,R' = 3\,r'\,\text{coſ.}\,\omega + \frac{dr'}{d\omega}\,\text{ſin.}\,\omega.$$

Quaeſtio igitur noſtra iam huc eſt perducta vt iudicetur, num huic aequationi per alios valores praeter $r' = 0$ et $R' = 0$ ſatisfieri queat.

13. Cum ſit $ds = dt + \frac{t\,d\omega\,\text{coſ.}\,\omega}{\text{ſin.}\,\omega}$, erit radius oſculi curuae in m quem poſuimus

$$r = \frac{ds}{d\omega} = \frac{dt}{d\omega} + \frac{t\,\text{coſ.}\,\omega}{\text{ſin.}\,\omega},$$

et radius oſculi in M

$$R = \frac{dT}{2\,d\omega} + \frac{T\,\text{coſ.}\,2\,\omega}{\text{ſin.}\,2\,\omega},$$

 quibus

quibuscum normales ad curuam in m et M ductas et in C concurrentes comparemus. Quia vero est angulus $MTC = ATC = 90^\circ - \omega$, ob $MT = T$ erit

$$MC = \frac{T\, cos.\, \omega}{sin.\, \omega} = \frac{t}{sin.\, \omega} + \frac{d\, t}{d\, \omega} cos.\, \omega$$

in subsidium vocando relationem inter tangentes $mt = t$ et $MT = T$ supra inuentam. Tum vero hinc porro habetur:

$$CT = \frac{T}{sin.\, \omega} = \frac{t}{sin.\, \omega\, cos.\, \omega} + \frac{d\, t}{d\, \omega}:$$

vnde auferatur interuallum

$$Tm = t\, tang.\, \omega = \frac{t\, sin.\, \omega}{cos.\, \omega},$$ vt remaneat

$$m\, C = \frac{t\, cos.\, \omega}{sin.\, \omega} + \frac{d\, t}{d\, \omega},$$

ex quo perspicuum est hanc rectam m C ipsi radio osculi in m aequari, ideoque punctum C reperiri in euoluta curuae quaesitae.

14. Cum normalis m C vtpote radius osculi euolutam curuae quaesitae in C tangat, normalis vero M C per idem punctum transeat, ea hic euolutam secabit, nisi euoluta in vnicum punctum concentretur, vti euenit in circulo. Quare si praeter circulum aliae curuae nostrae quaestioni satisfaciant, earum euolutas ita comparatas esse oportet, vt si normalis m C euolutam tangat in C, normalis M C eandem hic secet producta vero alibi tangat, sicque necesse est, omnes euolutae tangentes eandem simul alicubi intersecare, quod quomodo in omnibus tangentibus contingere possit, haud perspicitur, nisi forte

forte spirales lineae miris spiris contortae, sint adhibendae. Summa autem difficultas huiusmodi mirabiles figuras concipiendi earum impossibilitatem minime arguit, neque hinc concludere licet praeter circulos nullas alias lineas curuas quaestioni satisfacere posse.

15. Quanquam autem vix vlla patet via aequationes inuentas resoluendi, tamen ea, quae videtur simplicissima $Y = \frac{2\,dy}{d\omega}\,\text{sin.}\,\omega$, pro qua est $x = \frac{y}{\text{sin.}\,\omega}$ et

$$mt = t = \frac{dx\,\text{sin.}\,\omega}{d\omega} = \frac{dy}{d\omega} - \frac{y\,\text{cos.}\,\omega}{\text{sin.}\,\omega}$$

vnde fit $ds = \frac{d\,dy}{d\omega} + y\,d\omega$ haud leue suppeditat argumentum, quo praeter circulos omnes aliae figurae excludi videntur. Si enim solutionem generalem quaerentes statuamus:

$$y = A - B\omega^2 + C\omega^4 - D\omega^6 + E\omega^8 - \text{etc.}$$
$$+ \mathfrak{A}\omega - \mathfrak{B}\omega^3 + \mathfrak{C}\omega^5 - \mathfrak{D}\omega^7 + \mathfrak{E}\omega^9 - \text{etc.}$$

vt sit ob principium continuitatis:

$$Y = A - 2^2 B\omega^2 + 2^4 C\omega^4 - 2^6 D\omega^6 + 2^8 E\omega^8 + \text{etc.}$$
$$+ 2\mathfrak{A}\omega - 2^3\mathfrak{B}\omega^3 + 2^5\mathfrak{C}\omega^5 - 2^7\mathfrak{D}\omega^7 + 2^9\mathfrak{E}\omega^9 - \text{etc.}$$

euidens est, tam potestates pares, quam impares seorsim definiri posse.

16. Ponamus breuitatis gratia

$\text{sin.}\,\omega = \omega - \alpha\omega^3 + \beta\omega^5 - \gamma\omega^7 + \delta\omega^9 - \text{etc.}$ vt sit

$$\alpha = \frac{1}{1 \cdot 2 \cdot 3};\ \beta = \frac{\alpha}{4 \cdot 5};\ \gamma = \frac{\beta}{6 \cdot 7};\ \delta = \frac{\gamma}{8 \cdot 9};\ \text{etc.}$$

et pro potestatibus paribus, ob

$$\frac{dy}{d\omega} = -2B\omega + 4C\omega^3 - 6D\omega^5 + 8E\omega^7 - \text{etc.}$$

aequatio $\frac{2\,dy}{d\omega}$ fin. $\omega - Y = 0$ praebet hanc aequationem

$$\begin{array}{llllll} 0 = & -4B\omega^2 & +8C\omega^4 & -12D\omega^6 & +16E\omega^8 & -20F\omega^{10} \\ & & +4\alpha B & -8\alpha C & +12\alpha D & -16\alpha E \\ & & & -4\beta B & +8\beta C & -12\beta D \\ & & & & +4\gamma B & -8\gamma C \\ & & & & & -4\delta B \\ -A & +2^2 B & -2^4 C & +2^6 D & -2^8 E & +2^{10} F \text{ etc.} \end{array}$$

vnde colligitur $A = 0$, tum vero per 4 diuidendo prodeunt hae aequalitates:

$$\begin{array}{l} (2^2 - 2)\, C = \alpha B \\ (2^4 - 3)\, D = 2\alpha C + \beta B \\ (2^6 - 4)\, E = 3\alpha D + 2\beta C + \gamma B \\ (2^8 - 5)\, F = 4\alpha E + 3\beta D + 2\gamma C + \delta B \\ \quad\quad \text{etc.} \end{array}$$

17. Etfi hinc vix vllam legem progreffionis concinnam fperare licet, tamen praeter expectationem euenit, vt fumto $B = \frac{a}{1.2}$ fequentes eliciantur valores:

$$B = \frac{a}{1.2};\; C = \frac{a}{1.2.3.4};\; D = \frac{a}{1\ldots\ldots 6};\; E = \frac{a}{1\ldots\ldots 8} \text{ etc.}$$

fimilique modo pro poteftatibus imparibus reperitur:

$$\mathfrak{A} = b;\; \mathfrak{B} = \frac{b}{1.2.3};\; \mathfrak{C} = \frac{b}{1\ldots\ldots 5};\; \mathfrak{D} = \frac{b}{1\ldots\ldots 7} \text{ etc.}$$

quae feries cum per finus et cofinus fint fummabiles, colligitur fore $y = a$ cof. $\omega - a + b$ fin. ω; vnde concluditur

$$t = \frac{-a(1 - \text{cof.}\,\omega)}{\text{fin.}\,\omega} = -a \text{ tang.} \tfrac{1}{2}\omega \text{ et } \frac{ds}{d\omega} = r = -a,$$

ita vt curuae quaefitae radius ofculi vbique fit eiusdem magnitudinis ideoque curua circulus.

18.

18. Cum autem haec lex progreſſionis ex calculo perquam operoſo et moleſto per inductionem ſit concluſa, ne huic concludendi generi ſaepenumero fallaci nimium confidere videar, idem alio adhuc modo facilius euinci poteſt. Statuatur enim

$$y = A + B\,\text{coſ.}\,\omega + C\,\text{coſ.}\,3\,\omega + D\,\text{coſ.}\,5\,\omega + E\,\text{coſ.}\,7\,\omega + \text{etc.}$$

vbi quidem multipla paria omitto, vt quantitas $\frac{2\,dy}{d\omega}\,\text{ſin.}\,\omega$ tantum multipla ſimilia anguli ω complectatur, qualia valor:

$$Y = A + B\,\text{coſ.}\,2\,\omega + C\,\text{coſ.}\,6\,\omega + D\,\text{coſ.}\,10\,\omega + E\,\text{coſ.}\,14\,\omega \text{ etc.}$$

continet. Cum igitur ſit

$$\frac{dy}{d\omega} = -B\,\text{ſin.}\,\omega - 3\,C\,\text{ſin.}\,3\,\omega - 5\,D\,\text{ſin.}\,5\,\omega - 7\,E\,\text{ſin.}\,7\,\omega - \text{etc.}$$

per -2 ſin. ω multiplicando abit $Y - \frac{2\,dy}{d\omega}\,\text{ſin.}\,\omega = 0$ in

$$\begin{array}{llllll}
B & -B\,\text{coſ.}\,2\,\omega & -3\,C\,\text{coſ.}\,4\,\omega & -5\,D\,\text{coſ.}\,6\,\omega & -7\,E\,\text{coſ.}\,8\,\omega & -9\,F\,\text{coſ.}\,10\,\omega \text{ etc.} = 0 \\
 & +3\,C & +5\,D & +7\,E & +9\,F & +11\,G \\
+A & +B & & +C & & +D
\end{array}$$

vnde primo patet eſſe $B = -A$, tum vero $C = 0$, $D = 0$, $E = 0$, ſequentesque litteras omnes euaneſcere; ita vt ſit

$$y = A\,(1 - \text{coſ.}\,\omega).$$

19. Simili modo ſi ſtatuamus:

$$y = A + B\,\text{ſin.}\,\omega + C\,\text{ſin.}\,3\,\omega + D\,\text{ſin.}\,5\,\omega + E\,\text{ſin.}\,7\,\omega + F\,\text{ſin.}\,9\,\omega \text{ etc.}$$

hincque

$$\frac{dy}{d\omega} = B\,\text{coſ.}\,\omega + 3\,C\,\text{coſ.}\,3\,\omega + 5\,D\,\text{coſ.}\,5\,\omega + 7\,E\,\text{coſ.}\,7\,\omega \text{ etc.}$$

aequatio noſtra induet hanc formam:

$$
\begin{array}{llllll}
0 = B\,\text{fin.}\,2\omega & +3C\,\text{fin.}\,4\omega & +5D\,\text{fin.}\,6\omega & +7E\,\text{fin.}\,8\omega & +9F\,\text{fin.}\,10\omega & +\text{etc.} \\
\quad -3C & -5D & -7E & -9F & -11G & \\
-A-B & & -C & & -D &
\end{array}
$$

vbi fit $A = 0$, tum vero vt ante $C = 0$, $D = 0$, $E = 0$, $F = 0$ etc. ita vt his valoribus coniunctis habeatur

$$y = a(1 - \text{cof.}\,\omega) + b\,\text{fin.}\,\omega$$

quae eft ipfa folutio iam ante inuenta.

20. Haec autem folutio merito non fatis generalis videtur, cum ex valoribus ipfius y multipla paria anguli ω praeter neceffitatem reiecerimus; iis enim admiffis fi ftatuamus:

$$y = A + B\,\text{cof.}\,\omega + C\,\text{cof.}\,2\omega + D\,\text{cof.}\,3\omega + E\,\text{cof.}\,4\omega + F\,\text{cof.}\,5\omega \text{ etc.}$$

vt fit

$$\frac{dy}{d\omega} = -B\,\text{fin.}\,\omega - 2C\,\text{fin.}\,2\omega - 3D\,\text{fin.}\,3\omega - 4E\,\text{fin.}\,4\omega - 5F\,\text{fin.}\,5\omega - \text{etc.}$$

fequens refultat aequatio:

$$
\begin{array}{lllllll}
0 = B & +2C\,\text{cof.}\,\omega & -B\,\text{cof.}\,2\omega & -2C\,\text{cof.}\,3\omega & -3D\,\text{cof.}\,4\omega & -4E\,\text{cof.}\,5\omega & -5F\,\text{cof.}\,6\omega \\
 & +3D & +4E & +5F & +6G & +7H & \\
+A & +B & & +C & & +D &
\end{array}
$$

vbi iterum eft

$$B = -A;\ C = 0,\ D = 0,\ E = 0,\ F = 0,\ G = 0 \text{ etc.}$$

ita vt nunc quidem certum videatur, folutionem praecedentem latiffime patere, neque praeter circulos vllas alias curuas quaeftioni fatisfacere poffe. Quod enim hic de cofinibus eft oftenfum, idem fimili modo de finibus demonftratur.

21.

21. Cum igitur superior inductio sit penitus confirmata, illae determinationes eo magis sunt notatu dignae quae huc redeunt, vt si fuerit:

$$2^2 C = 2C + \frac{B}{1.2.3}$$
$$2^4 D = 3D + \frac{2C}{1.2.3} + \frac{B}{1\ldots5}$$
$$2^6 E = 4E + \frac{3D}{1.2.3} + \frac{2C}{1\ldots5} + \frac{B}{1\ldots7}$$
$$2^8 F = 5F + \frac{4E}{1.2.3} + \frac{3D}{1\ldots5} + \frac{2C}{1\ldots7} + \frac{B}{1\ldots9}$$
etc.

tum istae quantitates ita determinentur, vt sit

$$B = \frac{\alpha}{1.2};\ C = \frac{\alpha}{1\ldots4};\ D = \frac{\alpha}{1\ldots6};\ E = \frac{\alpha}{1\ldots8} \text{ etc.}$$

qui valores, quemadmodum ex illis relationibus commode deriuentur, omnino accuratiorem disquisitionem mereri videntur.

22. Simili vero modo si ibi §. 15. statuatur:

$$y = \mathfrak{A}\omega - \mathfrak{B}\omega^3 + \mathfrak{C}\omega^5 - \mathfrak{D}\omega^7 + \mathfrak{E}\omega^9 - \text{etc. vt sit}$$
$$\frac{dy}{d\omega} = \mathfrak{A} - 3\mathfrak{B}\omega^2 + 5\mathfrak{C}\omega^4 - 7\mathfrak{D}\omega^6 + 9\mathfrak{E}\omega^8 - \text{etc.}$$

obtinetur ista aequatio

$$\begin{array}{rrrrrl}
0 = 2\mathfrak{A}\omega & -6\mathfrak{B}\omega^3 & +10\mathfrak{C}\omega^5 & -14\mathfrak{D}\omega^7 & +18\mathfrak{E}\omega^9 & -\text{etc.}\\
-2\alpha\mathfrak{A} & +6\alpha\mathfrak{B} & -10\alpha\mathfrak{C} & +14\alpha\mathfrak{D} & & \\
 & +2\beta\mathfrak{A} & -6\beta\mathfrak{B} & +10\beta\mathfrak{C} & & \\
 & & -2\gamma\mathfrak{A} & +6\gamma\mathfrak{B} & & \\
 & & & +2\delta\mathfrak{A} & & \\
-2\mathfrak{A} & +2^3\mathfrak{B} & -2^5\mathfrak{C} & +2^7\mathfrak{D} & -2^9\mathfrak{E} & \text{etc.}
\end{array}$$

 vnde

vnde nafcuntur hae determinationes:

$$2^2\mathfrak{B} = 3\mathfrak{B} + \frac{\mathfrak{A}}{1.\,2.\,3}$$
$$2^4\mathfrak{C} = 5\mathfrak{C} + \frac{3\,\mathfrak{B}}{1.\,2.\,3} + \frac{\mathfrak{A}}{1\ldots 5}$$
$$2^6\mathfrak{D} = 7\mathfrak{D} + \frac{5\,\mathfrak{C}}{1.\,2.\,3} + \frac{3\,\mathfrak{B}}{1\ldots 5} + \frac{\mathfrak{A}}{1\ldots 7}$$
$$2^8\mathfrak{E} = 9\mathfrak{E} + \frac{7\,\mathfrak{D}}{1.\,2.\,3} + \frac{5\,\mathfrak{C}}{1\ldots 5} + \frac{3\,\mathfrak{B}}{1\ldots 7} + \frac{\mathfrak{A}}{1\ldots 9}$$
etc.

quae euolutae fequentes fuppeditant valores:

$$\mathfrak{A} = b;\ \mathfrak{B} = \frac{b}{1.\,2.\,3};\ \mathfrak{C} = \frac{b}{1\ldots 5};\ \mathfrak{D} = \frac{b}{1\ldots 7}\ \text{etc.}$$

quae fimplex progreffionis lex nonnifi per multas ambages ex illis formulis deducitur, ideoque diftinctior euolutio maxime defideratur.

23. Neque vero hoc argumentum fatis validum videtur ad euincendum praeter circulum nullas alias lineas curuas quaeftioni noftrae fatisfacere. Cum enim aequationem $Y = \frac{2\,d\,y}{d\,\omega}$ fin. ω ita refolui oporteat, vt Y fiat talis functio ipfius $2\,\omega$, qualis y eft ipfius ω, refolutio hic adhibita naturam iftius aequationis minime exhaurire videtur, propterea quod pro y eiusmodi feriem affumfimus, in qua nonnifi poteftates ipfius ω, quarum exponentes funt numeri integri pofitiui, occurrunt; dum fortaffe etiam exponentes vel fracti vel negatiui, vel etiam irrationales atque adeo imaginarii locum habere poffent; imaginariis fcilicet fe mutuo deftruentibus. Quod num fieri poffit ftatuamus:

$$y = A\,\omega^\alpha + B\,\omega^6 + C\,\omega^\gamma\ \text{etc.}$$

quorum

quorum exponentium primus α ſit minimus. Erit ergo

$$\frac{dy}{d\omega} = \alpha A \omega^{\alpha-1} + \beta B \omega^{\beta-1} \text{ etc.}$$

quae ſeries per

$$2 \sin. \omega = 2\omega - \frac{2}{1.2.3}\omega^3 + \frac{2}{1.2\ldots5}\omega^5 - \text{etc.}$$

multiplicata praebet hanc formam

$$\begin{aligned} &2\alpha A \omega^\alpha + 2\beta B \omega^\beta \text{ etc.} \\ &\quad - \frac{2\alpha A}{1.2.3}\omega^{\alpha+2} - \frac{2\beta B}{1.2.3}\omega^{\beta+2} - \text{etc.} \\ &\qquad + \frac{2\alpha A}{1\ldots5}\omega^{\alpha+4} + \text{etc.} \end{aligned}$$

ipſi $Y = 2^\alpha A \omega^\alpha + 2^\beta B \omega^\beta + 2^\gamma C \omega^\gamma$ etc. aequandam. Neceſſe ergo eſt, vt infimae poteſtates vtrinque ſint aequales, vnde fit $2^\alpha = 2\alpha$, quod realiter duobus tantum modis fieri poteſt ſumendo vel $\alpha = 1$ vel $\alpha = 2$, ex quo ſeries ſupra aſſumtae totum negotium conficere videntur.

24. Cum autem nullum ſit dubium, quin aequatio $2^\alpha = 2\alpha$ praeter has duas radices innumeras alias imaginarias complectatur, merito naſcitur ſuſpicio, ex iis quoque ſolutiones vtiles naſci poſſe. Neque igitur abs re fore arbitror methodum indicaſſe, qua etiam radices imaginarias huius aequationis proxime ſaltem definire liceat. Cum ergo α ſit numerus imaginarius, eius forma certo eſt huiusmodi

$$\alpha = \mu + \nu\sqrt{-1},$$

vnde fit

$$2^\alpha = 2^\mu . 2^{\nu\sqrt{-1}} = 2^\mu(\cos.(\nu l 2) + \sqrt{-1}.\ \sin.(\nu l 2)),\ \text{cui}$$

$$2\alpha = 2\mu + 2\nu\sqrt{-1}$$

 ita

ita aequale statui debet vt partes reales et imaginariae seorsim aequentur: hinc ergo fit

$$2\mu = 2^{\mu}\,\mathrm{cof.}\,(\nu\, l\, 2) \quad \text{et} \quad 2\nu = 2^{\mu}\,\mathrm{fin.}\,(\nu\, l\, 2)$$

ideoque

$$4(\mu\mu + \nu\nu) = 2^{2\mu} \quad \text{et} \quad \nu = \sqrt{(2^{2\mu-2} - \mu\mu)}.$$

Ponamus $\frac{\mu}{2^{\mu-1}} = \mathrm{cof.}\Phi$, et oritur $\nu = 2^{\mu-1}\,\mathrm{fin.}\Phi = \mu\,\mathrm{tang}\,\Phi$.

Cum igitur fit $\mathrm{cof.}\,(\nu\, l\, 2) = \frac{\mu}{2^{\mu-1}} = \mathrm{cof}\,\Phi$, sequitur fore $\nu\, l\, 2 = 2\,n\,\pi \pm \Phi$, denotante $2\,n\,\pi$ multiplum quodcunque totius circumferentiae. Quare habetur $\mu\, l\, 2.\,\mathrm{tang.}\,\Phi = 2\,n\,\pi \pm \Phi$ et $\mu = \frac{2\,n\,\pi \pm \Phi}{l\,2.\,\mathrm{tang.}\,\Phi}$, qui valor cum illo ex formula $\mu = 2^{\mu-1}\,\mathrm{cof.}\,\Phi$ oriundo conuenire debet Hoc autem ob n numerum arbitrarium dummodo integrum infinitis modis obtineri posse euidens est. Scilicet sumendo $n = 1$ et ponendo $\mu = \frac{2\pi - \Phi}{l\,2.\,\mathrm{tang.}\,\Phi}$, approximationibus aliquot institutis colligitur angulus $\Phi = 61^{\circ}, 24', 24''$, hincque

$$\mu = 4,0980836 \quad \text{et} \quad \nu = 7,51850, \text{ ita vt fit}$$

$$\alpha = \mu \pm \nu\sqrt{-1} = 4,0980836 \pm 7,51850.\sqrt{-1}.$$

25. Inuentis ergo huiusmodi valoribus imaginariis pro α, vt fit $2^{\alpha} = 2\,\alpha$, talis forma pro y fingi debet

$$y = A\,\omega^{\alpha} - B\,\omega^{\alpha+2} + C\,\omega^{\alpha+4} - D\,\omega^{\alpha+6} + \text{etc.}$$

vnde fit

$$Y = 2\,\alpha A\,\omega^{\alpha} - 2^{3}\alpha B\omega^{\alpha+2} + 2^{5}\alpha C\omega^{\alpha+4} - 2^{7}\alpha D\omega^{\alpha+6} + \text{etc.}$$

quae

quae ſeries ipſi

$$\frac{2\,dy}{d\omega}\,\text{ſin.}\,\omega = \frac{dy}{d\omega}\left(2\,\omega - \frac{2}{1\cdot2\cdot3}\,\omega^3 + \frac{2}{1\cdot\cdot\cdot\cdot5}\,\omega^5 - \text{etc.}\right)$$

aequalis eſt ſtatuenda, ſcilicet huic expreſſioni

$$\begin{array}{llll} 2\,\alpha\,A\,\omega^{\alpha} & -\,2\,(\alpha+2)\,B\,\omega^{\alpha+2} & +\,2\,(\alpha+4)\,C\,\omega^{\alpha+4} & -\ \text{etc.} \\ & -\,\frac{2\,\alpha\,A}{1\cdot2\cdot3} & +\,\frac{2\,(\alpha+2)\,B}{1\cdot2\cdot3} & \\ & & +\,\frac{2\,\alpha\,A}{1\cdot\cdot\cdot\cdot5} & \end{array}$$

vnde ſequentes determinationes colliguntur:

$$2^2\,\alpha\,B = (\alpha+2)\,B + \frac{\alpha\,A}{1\cdot2\cdot3}$$

$$2^4\,\alpha\,C = (\alpha+4)\,C + \frac{(\alpha+2)\,B}{1\cdot2\cdot3} + \frac{\alpha\,A}{1\cdot\cdot\cdot\cdot5}$$

$$2^6\,\alpha\,D = (\alpha+6)\,D + \frac{(\alpha+4)\,C}{1\cdot2\cdot3} + \frac{(\alpha+2)\,B}{1\cdot2\cdot\cdot\cdot\cdot5} + \frac{\alpha\,A}{1\cdot\cdot\cdot\cdot7}$$

etc.

26. Hinc ergo ſine dubio coefficientes B, C, D etc. definire licet, at ne vllum dubium ſuperſit, an hoc modo pro y valores reales obtineri queant, notandum eſt binos valores idoneos pro y inuentos ſemper inter ſe coniungi poſſe. Hoc obſeruato ponamus ex valore $\alpha = \mu + \nu\sqrt{-1}$ prodiiſſe $y = P$, ex valore autem $\alpha = \mu - \nu\sqrt{-1}$ fieri $y = Q$, quae binae expreſſiones vtcunque fuerint imaginariae certum eſt ex iis realem huiusmodi $y = m\,P + n\,Q$ conflari poſſe, idque duplici modo. Atque ex forma aſſumta quidem $\alpha = \mu + \nu\sqrt{-1}$ ſeries pro y ita erunt comparatae:

$$y = \omega^{\mu}\,\text{cof.}\,(\nu\,l\,\omega).\,(A + B\,\omega^2 + C\,\omega^4 + D\,\omega^6 + \text{etc.}) \quad \text{vel}$$

$$y = \omega^{\mu}\,\text{ſin.}\,(\nu\,l\,\omega).\,(\mathfrak{A} + \mathfrak{B}\,\omega^2 + \mathfrak{C}\,\omega^4 + \mathfrak{D}\,\omega^6 + \text{etc.})$$

 quarum

quarum ſumma vel differentia itidem praebet valorem ſatisfacientem pro y. Euidens autem eſt hos valores minime in ſeries ſecundum poteſtates ſimplices ipſius ω progredientes conuerti poſſe. Huiusmodi autem formae ſtatim ab initio pro y fingi poſſunt; vbi cum in Y contineatur coſ. $(\nu l 2 \omega)$ notandum eſt eſſe:

$$\text{coſ.}(\nu l 2 \omega) = \text{coſ.}(\nu l \omega).\, \text{coſ.}(\nu l 2) - \text{ſin.}(\nu l \omega).\, \text{ſin.}(\nu l 2).$$

27. Ne autem hunc calculum nimis moleſtum quo infinitae aliae problematis ſolutiones declarari videntur ſuſcipiam idem ex relatione inter radios oſculi euolutae curuae quaeſitae facilius oſtendi poſſe videtur. Cum enim §. 12. inuenerimus

$$4 R' = 3 r' \,\text{coſ.}\,\omega + \frac{d r'}{d \omega} \,\text{ſin.}\,\omega\,, \text{ ſi fingamus:}$$

$$r' = A \omega^n + B \omega^{n+2} + C \omega^{n+4} + \text{etc. vt ſit}$$

$$R' = 2^n A \omega^n + 2^{n+2} B \omega^{n+2} + 2^{n+4} C \omega^{n+4} + \text{etc.}$$

pro poteſtate infima ω^n debet eſſe

$$4.\, 2^n A = 3 A + n A \text{ ſeu } 4.\, 2^n = 3 + n = 2^{n+2}$$

cui conditioni duplici modo ſatisfit, altero $n = -2$, altero $n = -1$. Statuamus ergo

$$r' = \frac{A}{\omega} + B \omega + C \omega^3 + D \omega^5 + \text{etc. erit}$$

$$\frac{d r'}{d \omega} = \frac{-A}{\omega^2} + B + 3 C \omega^2 + 5 D \omega^4 + \text{etc.}$$

$$R' = \frac{A}{2 \omega} + 2 B \omega + 8 C \omega^3 + 32 D \omega^5 + \text{etc.}$$

vnde

vnde colligitur esse debere:

$$\frac{2A}{\omega} + 8B\omega + 32C\omega^3 + 128D\omega^5 + \text{etc.} =$$

$$\begin{array}{llll}
3A & +3B & +3C & +3D \\
 & -\frac{3A}{1.2} & -\frac{3B}{1.2} & -\frac{3C}{1.2} \\
 & & +\frac{3A}{1\ldots4} & +\frac{3B}{1\ldots4} \\
 & & & -\frac{3A}{1\ldots6} \\
-A & +B & -3C & +5D \\
 & +\frac{A}{1\ldots3} & -\frac{B}{1\ldots3} & -\frac{3C}{1\ldots3} \\
 & & -\frac{A}{1\ldots5} & +\frac{B}{1\ldots5} \\
 & & & +\frac{A}{1\ldots7}
\end{array}$$

hincque eiusmodi eruuntur valores, qui vtique solutionem realem praebere videntur. Interim tamen desideratur adhuc methodus hoc problema perfectius resoluendi

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