tam certam condensationem quam certum motum produci debere, ita vt hic non amplius infinita varietas locum inuenire queat: huicque causae qualitas illa qua soni tibiarum a reliquis instrumentis discrepant, sine dubio est tribuenda. Vnde si forte quocunque modo aeri in tubis vtrinque clausis agitatio induci posset, sonus quidem ratione grauitatis et acuminis calculo foret consentaneus, sed a sono tibiarum maxime differre posset.

CAPVT IV.

DE AGITATIONE MINIMA AERIS IN TVBIS INAEQVALITER AMPLIS.

Problema 81.

76. Dum aer quomodocunque in tubo inaequaliter amplo mouetur, eius motum ad formulas analyticas reuocare, quibus eius determinatio ad quoduis tempus contineatur.

Solutio.

Initio statum aeris tanquam cognitum spectamus; tum igitur pro loco tubi quocunque S vocata distantia $AS = S$ ponamus fuisse densitatem $= Q$, et celeritatem secundum tubi directionem $AB = \Upsilon$, ita vt Q et Υ sint functiones datae quantitatis S; cuius etiam functio erit amplitudo tubi in hoc loco, quae sit $=\Omega$, perinde atque eius altitudo super plano quodam horizontali fixo, quae sit $=Z$, siquidem et grauitatis rationem in motu aeris habere velimus. Iam elapso tempore $=t$ illam aeris particulam ponamus peruenisse in s, vt sit spatium $As=s$, et tubi amplitudo $=\omega$, eiusque altitudo super illo plano horizontali $=z$, quae quantitates ω et z sunt functiones datae ipsius s, ipsa vero haec quantitas s est functio duarum variabilium S et t; nunc vero in hoc loco s sit aeris densitas $=q$, celeritas versus B $=\Upsilon$, quae est $=(\frac{dt}{ds})$, et pressio $=p$, quam per densitatem q ita determinari nouimus, vt sit $p=\frac{aq}{b}$; densitas q est quoque functio binarum variabilium S et t. His positis principia supra stabilita pro hoc casu praebent has duas aequationes:

$$q\omega(\frac{ds}{dS})=Q\Omega \text{ et } \frac{2gdp}{q}=-2gdz-ds(\frac{dds}{dt^2})=\frac{2gadq}{bq}$$

in qua posteriori tempus t vt constans spectatur, quae propterea ita exhiberi potest

$$\frac{2ga}{bq}(\frac{dq}{dS})+2g(\frac{dz}{dS})+(\frac{ds}{dS})(\frac{dds}{dt^2})=0.$$

Ex priori autem habemus $lq=lQ+l\Omega-l\omega-l(\frac{ds}{dS})$ vnde solam S variabilem sumendo colligimus:

$$\frac{1}{q}(\frac{dq}{dS})=\frac{dQ}{QdS}+\frac{d\Omega}{\Omega dS}-\frac{1}{\omega}(\frac{d\omega}{dS})-\frac{(\frac{ddS}{dS^2})}{(\frac{ds}{dS})}.$$

Quo

Quo valore ſubſtituto aequationem habebimus a q liberam hanc:

$$\frac{2ga}{b}\left(\frac{dQ}{Q\,dS}+\frac{d\Omega}{\Omega\,dS}\right)\left(\frac{ds}{dS}\right)-\frac{2ga}{b\omega}\left(\frac{d\omega}{dS}\right)\left(\frac{ds}{dS}\right)-\frac{2ga}{b}\left(\frac{dds}{dS^2}\right)$$
$$+2g\left(\frac{dz}{dS}\right)\left(\frac{ds}{dS}\right)+\left(\frac{ds}{dS}\right)^2\left(\frac{dds}{dt^2}\right)=0$$

vbi Q et Ω ſunt functiones datae ipſius S, quantitates vero ω et z functiones ipſius s, quae ipſa eſt functio binarum variabilium S et t, eiusque natura hinc determinari debet. Quodſi ergo ponatur $d\omega = u\,ds$ et $dz = r\,ds$, vt u et r ſint functiones ipſius s datae, aequatio noſtra hanc induet formam:

$$\frac{2ga}{b}\left(\frac{dQ}{Q\,dS}+\frac{d\Omega}{\Omega\,dS}\right)\left(\frac{ds}{dS}\right)+2g\left(r-\frac{au}{b\omega}\right)\left(\frac{ds}{dS}\right)^2-\frac{2ga}{b}\left(\frac{dds}{dS^2}\right)$$
$$+\left(\frac{ds}{dS}\right)^2\left(\frac{dds}{dt^2}\right)=0$$

vnde qualis s ſit functio ipſarum S et t inueſtigari oportet, ea autem inuenta ſtatim cognoſcitur celeritas $v=\left(\frac{ds}{dt}\right)$, tum vero denſitas

$$q=\frac{Q\,\Omega}{\omega\left(\frac{ds}{dS}\right)},$$

ſimulque preſſio $p=\frac{aq}{b}$.

Scholion.

77. Ex hac aequatione generali, quae omnes motus, qui in aerem in tubis quibuscunque cadere poſſunt, in ſe complectitur, ſatis liquet quantopere etiamnunc ab eius ſolutione perfecta ſimus remoti, idque ob ſolum Analyſeos defectum. Cum enim functio s quae quaeritur praeter formulas differentia-

rentiales etiam in litteris ω, u et r inuoluatur, haec tanta complexio in caussa est, quod generalem nostri problematis solutionem nullo modo sperare queamus. Quare vt supra iam obseruauimus, omnia quae hic praestare licet, ad solos motus minimos restringuntur, cuiusmodi sunt generatio et propagatio soni, dum interea casus, qui alias simplicissimi videantur, veluti si aer in tubo comprimitur vel relaxatur, prorsus intactos relinquere cogimur. Mirum igitur est quod ii casus, qui primo intuitu maxime ardui sunt visi, et ab Auctoribus vix suscepti, nunc soli nostrae inuestigationi permittuntur, reliquis omnibus exclusis. Quocirca aequationem generalem hic inuentam ad eum casum accommodabo, quo agitatio aeris vt minima spectari potest.

Problema 82.

78. Si aeris agitatio in tubo inaequaliter amplo excitata fuerit quam minima, aequationem invenire qua huius motus continuatio continetur.

Solutio.

Maneant omnes denominationes, vti in praecedente problemate sunt constitutae, et quia ibi erat $u = \frac{d\omega}{ds}$ et $r = \frac{dz}{ds}$ restituantur hi valores, insuperque breuitatis gratia ponatur $\frac{2ga}{b} = cc$ vt c denotet spatium, per quod sonus vno minuto secundo propagatur; quo facto aequatio inuenta hanc induet formam:

$$\frac{1}{cc}\left(\frac{ds}{dS}\right)^2\left(\frac{dds}{dt^2}\right) = \left(\frac{dds}{dS^2}\right) + \frac{d\omega}{\omega dS}\left(\frac{ds}{dS}\right)^2 - \frac{d\Omega}{\omega ds}\left(\frac{ds}{dS}\right) - \frac{b\,dz}{a\,ds}\left(\frac{ds}{dS}\right)^2 - \frac{dQ}{Q\,dS}\left(\frac{ds}{dS}\right)$$

qua

qua resoluta porro habebitur pro tempore elapso $= t$:

$$\text{densitas } q = \frac{Q\,\Omega}{\omega\left(\frac{ds}{dS}\right)};\ \text{celeritas } \upsilon = \left(\frac{ds}{dt}\right) \text{ et pressio } p = \frac{a\,q}{b}.$$

Iam ponamus $s = S + v$ vt v sit eiusmodi functio ipsarum S et t quae euanescat posito tempore $t = 0$, quandoquidem tum fieri debet $s = S$; $q = Q$; $\omega = \Omega$ et $\upsilon = \Upsilon$, hicque spectamus v vt quantitatem valde paruam prae S, vel saltem talem vt $\left(\frac{dv}{ds}\right)$ prae vnitate negligi queat. Cum igitur sit

$$\left(\frac{ds}{dt}\right) = \left(\frac{dv}{dt}\right);\ \left(\frac{dds}{dt^2}\right) = \left(\frac{ddv}{dt^2}\right);\ \left(\frac{ds}{dS}\right) = 1 + \left(\frac{dv}{dS}\right) \text{ et } \left(\frac{dds}{dS^2}\right) = \left(\frac{ddv}{dS^2}\right);$$

vbi in nostra aequatione occurrit $\left(\frac{ds}{dS}\right)$ eius loco vnitatem scribere licet praeterquam in duobus terminis

$$\frac{d\omega}{\omega\, ds}\left(\frac{ds}{dS}\right)^2 - \frac{d\Omega}{\Omega\, dS}\left(\frac{ds}{dS}\right),$$

quia hic ob differentiam inter s et S minimam fere est

$$\frac{d\omega}{\omega\, ds} = \frac{d\Omega}{\Omega\, dS},$$

ideoque prae differentia non amplius particula $\left(\frac{dv}{dS}\right)$ vt euanescens spectari potest. Qua circumstantia obseruata habebimus

$$\frac{1}{cc}\left(\frac{ddv}{dt^2}\right) = \left(\frac{ddv}{dS^2}\right) + \frac{d\omega}{\omega\, ds} - \frac{d\Omega}{\Omega\, dS} + \frac{d\Omega}{\Omega\, dS}\left(\frac{dv}{dS}\right) - \frac{b\, dz}{a\, ds} - \frac{dQ}{Q\, ds}$$

vbi cum puncta S et s sibi sint proxima, loco $\frac{dz}{ds}$ scribere licet $\frac{dZ}{dS}$ functionem ipsius S tantum. Deinde ob eandem rationem quia formula $\frac{d\omega}{\omega\, ds}$ nasci-

 tur

tur ex formula $\frac{d\Omega}{\Omega d S}$ si hic loco S scribatur $S + v$, erit

$$\frac{d\omega}{\omega d s} = \frac{d\Omega}{\Omega d S} + \frac{v}{d S} d. \frac{d\Omega}{\Omega d S}.$$

Ponamus ergo breuitatis gratia $\frac{d\Omega}{\Omega d S} = U$, quae erit functio data ipsius S, vnde fit amplitudo $\Omega = A e^{\int U d S}$; fietque nostra aequatio:

$$\frac{1}{c c}\left(\frac{d d v}{d t^2}\right) = \left(\frac{d d v}{d S^2}\right) + U\left(\frac{d v}{d S}\right) + \frac{v d U}{d S} - \frac{b d Z}{a d S} - \frac{d Q}{Q d S}$$

qua resoluta erit pro motus determinatione:

$$q = \frac{Q\Omega}{\omega\left(1 + \left(\frac{d v}{d S}\right)\right)} = \frac{Q}{(1 + U v)\left(1 + \left(\frac{d v}{d S}\right)\right)} = Q\left(1 - U v - \left(\frac{d v}{d S}\right)\right)$$

quia est $\omega = \Omega + \frac{v d\Omega}{d S} = A e^{\int U d S}(1 + U v)$

deinde celeritas $\mathfrak{v} = \left(\frac{d v}{d t}\right)$ et pressio $p = \frac{\alpha q}{q}$.

Coroll. 1.

79. Totum ergo negotium huc redit vt ex aequatione differentiali secundi gradus inuenta inuestigetur, qualis functio sit quantitas v binarum variabilium S et t, vbi quidem obseruo binos terminos postremos $-\frac{b d Z}{a d S} - \frac{d Q}{Q d S}$ resolutionem non impedire quoniam functionem solius variabilis S continent.

Coroll. 2.

80. Conditiones autem, sub quibus aequationis inuentae integrale completum eruere licet ope methodorum quidem adhuc cognitarum, ab indole functionis U, qua ea per variabilem S definitur pendent.

Pro

Pro eiusque natura fieri potest, vt integratio modo succedat modo calculi vires superet.

Scholion.

81. Hic igitur confugiendum est ad ea analyseos sublimioris, quae circa functiones duarum variabilium versatur artificia, quibus huiusmodi aequationes

$$\frac{1}{cc}\left(\frac{ddv}{dt^2}\right) = \left(\frac{ddv}{dS^2}\right) + U\left(\frac{dv}{dS}\right) + vT$$

tractare docui; vbi in eos casus functionum U et T, quae solam variabilem S inuoluere assumuntur, inquiri oportet quibus integrale completum exhibere licet. Ante omnia autem hic obseruari oportet, quoties haec integratio succedit per methodos quidem cognitas, integrale semper huiusmodi forma exprimi vt sit

$$v = Lf:(S \pm ct) + Mf':(S \pm ct) + Nf'':(S \pm ct) \text{ etc.}$$

vbi circa haec functionum signa tenendum est; si fuerit

$$f:u = V \text{ esse } f':u = \frac{dV}{du};\ f'':u = \frac{ddV}{du^2}, \text{ etc.}$$

Hinc patet infinitas solutiones locum habere posse, prout huius formae progressio vlterius continuetur. Primam ergo solutionem seu primos integrabilitatis casus ex solo primo termino huius progressionis inuestigabo, deinde duos eius terminos in subsidium vocando secundos integrabilitatis casus eliciam, sicque porro ad altiores ascendere licebit, plures continuo terminos accipiendo.

Problema 83.

82. Inuenire rationem amplitudinis tubi, in quo aer minimas peragit agitationes, vt motus determinatio per primam soluendi methodum succedat.

Solutio.

Cum posita tubi amplitudine $\Omega = A\, e^{\int U\, dS}$ haec aequatio integrari debeat:

$$\frac{1}{cc}\left(\frac{ddv}{dt^2}\right) = \left(\frac{ddv}{dS^2}\right) + U\left(\frac{dv}{dS}\right) + \frac{v\,dU}{dS} - \frac{b\,dZ}{a\,dS} - \frac{dQ}{Q\,dS}$$

statuamus $v = Lf:(S + ct) + O$ vbi L et O tantum sint functiones ipsius S, et facta substitutione obtinebimus:

$$\left(\frac{ddv}{dS^2}\right) = Lf'':(S+ct) + \frac{2\,dL}{dS}f':(S+ct) + \frac{ddL}{dS^2}f:(S+ct) + \frac{ddO}{dS^2}$$

$$U\left(\frac{dv}{dS}\right) = \quad ULf':(S+ct) + \frac{U\,dL}{dS}f:(S+ct) + \frac{U\,dO}{dS}$$

$$\frac{v\,dU}{dS} = \quad + \frac{L\,dU}{dS}f:(S+ct) + \frac{O\,dU}{dS}$$

$$-\frac{b\,dZ}{a\,dS} - \frac{dQ}{Q\,dS} \quad -\frac{b\,dZ}{a\,dS} - \frac{dQ}{Q\,dS}$$

quae iunctim sumta ipsi $\frac{1}{cc}\left(\frac{ddv}{dt^2}\right) = Lf'':(S + ct)$ aequari oportet: vnde nascuntur hae aequationes:

I. $\frac{2\,dL}{dS} + LU = 0$

II. $\frac{ddL}{dS^2} + \frac{U\,dL + L\,dU}{dS} = 0$

III. $\frac{ddO}{dS^2} + \frac{U\,dO + O\,dU}{dS} - \frac{b\,dZ}{a\,dS} - \frac{dQ}{Q\,dS} = 0$

quarum secunda integrata dat $\frac{dL}{dS} + LU = C$, quae cum prima collata praebet $-\frac{dL}{dS} = C$, hincque $L = \alpha S + \mathfrak{G}$ et $U = \frac{-2\alpha}{\alpha S + \mathfrak{G}} = \frac{d\Omega}{\Omega\, dS}$. Quare cum sit fit $\frac{d\Omega}{\Omega} = \frac{-2\alpha dS}{\alpha S + \beta}$ fiet amplitudo tubi $\Omega = \frac{ff}{(\alpha S + \beta)^2}$, folutionem ope primae methodi admittens. Tertia vero aequatio integrata dat:

$$\frac{dO}{dS} + UO - \frac{b}{a}Z - lQ = C \text{ feu ob } U = \frac{-2\alpha}{\alpha S + \beta}$$

$$dO - \frac{2\alpha O dS}{\alpha S + \beta} - \frac{b}{a}Z dS - dS\, lQ = C dS$$

quae per $(\alpha S + \beta)^2$ diuifa et integrata producit

$$\frac{O}{(\alpha S + \beta)^2} = \int \frac{dS(\frac{b}{a}Z + lQ + C)}{(\alpha S + \beta)^2}.$$

Quare fi amplitudo tubi ita fit variabilis, vt longitudini $AS = S$ refpondeat amplitudo $\Omega = \frac{ff}{(\alpha S + \beta)^2}$, tum aequationis motum determinantis integrale completum erit

$$v = (\alpha S + \beta)^2 \int \frac{dS(C + \frac{b}{a}Z + lQ)}{(\alpha S + \beta)^2} + (\alpha S + \beta)\Gamma:(S + ct)$$
$$+ (\alpha S + \beta)\Delta:(S - ct)$$

quandoquidem functionem affumtam $f:(S + ct)$ geminare licet introducendo tam $-c$ quam $+c$.

Coroll. 1.

83. In hac folutione ftatim continetur cafus tuborum aeque ampiorum conftantes ita definiendo vt fit $\alpha = 0$ et $\beta = 1$. Haec autem folutio multo latius patet, cum eius ope agitationes aeris in tubis eiusmodi inaequaliter amplis quoque definiri queant, quorum amplitudo in hac formula continetur $\Omega = \frac{ff}{(\alpha S + \beta)^2}$.

Coroll. 2.

84. Inuenta autem hac functione v elapso tempore t aer qui initio erat ad S translatus erit per internallum $Ss = v$ tum vero eius densitas ob $U = \frac{-2\alpha}{\alpha S + \beta}$ erit

$$q = Q\left(+\frac{2\alpha v}{\alpha S + \beta} - \left(\frac{dv}{dS}\right)\right) \text{ et celeritas } \mathfrak{v} = \left(\frac{dv}{dt}\right).$$

Coroll. 3.

85. Cum autem differentiando sit:

$$\left(\frac{dv}{dS}\right) = 2\alpha(\alpha S + \beta)\int\frac{dS(C + \frac{b}{a}Z + lQ)}{(\alpha S + \beta)^2} + C + \tfrac{b}{a}Z + lQ + \alpha\Gamma:(S + ct) + \alpha\Delta:(S - ct)$$
$$+ (\alpha S + \beta)\Gamma':(S + ct) + (\alpha S + \beta)\Delta':(S - ct)$$

habebitur pro densitate:

$$\frac{q}{Q} = 1 - C - \tfrac{b}{a}Z - lQ + \alpha\Gamma:(S + ct) + \alpha\Delta:(S - ct)$$
$$-(\alpha S + \beta)\Gamma':(S + ct) - (\alpha S + \beta)\Delta':(S - ct)$$

et pro celeritate:

$$\mathfrak{v} = c(\alpha S + \beta)\Gamma':(S + ct) - c(\alpha S + \beta)\Delta':(S - ct).$$

Scholion.

85. Si tubos, ad quos haec solutio est accommodata, rotundos statuamus, vt omnes sectiones ad eius directionem normaliter factae sint circuli, eorum figura est conoidica hyperbolica conuersione hyperbolae aequilaterae KCM circa alteram assymtotam IB nata. Cum enim in hac hyperbola sit SM. IS $= aa$, erit amplitudo in S $= \pi SM^2 = \frac{\pi aa}{IS^2}$, sumto

Tab. VII. Fig. 80.

ſumto ergo interuallo $IA = \frac{\beta}{\alpha}$, et poſito $AS = S$ ob $IS = \frac{\alpha S + \beta}{\alpha}$, fiet amplitudo $\Omega = \frac{\pi \alpha \alpha a a}{(\alpha S + \beta)^2}$, ideoque $ff = \pi \alpha \alpha a a$. Quoties ergo tubus habuerit huiusmodi figuram conoidicam hyperbolicam, aeris agitationes in huiusmodi tubis, dummodo ſint minimae, perinde definiri poterunt, atque in tubis aequaliter amplis. Interim tamen ipſa motus determinatio aliquanto erit operoſior. Ceterum hic obſeruari conuenit aeris grauitatem, quam in ſuperioribus capitibus neglexinus, inueſtigationem plane non turbare: id quod etiam de aliis viribus, quae forte aerem ſollicitarent eſt tenendum.

Problema 84.

86. Inueſtigare rationem qua tubi amplitudo debet eſſe comparata vt aequationis differentiodifferentialis integrale completum per ſecundam formam exhiberi poſſit.

Solutio.

Hic ſcilicet quaeritur indoles functionis U a ſola variabili S pendentis, vt aequationis noſtrae integrale completum huiusmodi forma exprimi queat:

$$v = O + Mf' : (S + ct) + Lf : (S + ct)$$

vbi L, M, O ſint functiones ſolius variabilis S. Faciamus ergo ſubſtitutionem in noſtra aequatione, ac reperiemus:

0 =

$$\begin{array}{llll}
0 = \frac{ddO}{dS^2} & + Mf''':(S+ct) & + \frac{2dM}{dS}f'':(S+ct) & + \frac{ddM}{dS^2}f':(S+ct) \\
+ \frac{UdO}{dS} & + L & + \frac{2dL}{dS} & + \frac{ddL}{dS^2}f:(S+ct) \\
+ \frac{OdU}{dS} & + UM & + \frac{UdM}{dS} & \\
- \frac{bdZ}{adS} & & + UL & + \frac{UdL}{dS} \\
- \frac{dQ}{QdS} & - Mf''':(S+ct) & - Lf'':(S+ct) + \frac{MdU}{dS} & + \frac{LdU}{dS}
\end{array}$$

cuius ſingula membra, quatenus diuerſas functiones complectuntur ſeorſim ad nihilum redigi oportet: ex quo ſequentes quatuor aequationes naſcuntur.

$$\text{I. } \frac{2dM}{dS} + UM = 0$$

$$\text{II. } \frac{ddM}{dS^2} + \frac{2dL}{dS} + \frac{UdM + MdU}{dS} + UL = 0$$

$$\text{III. } \frac{ddL}{dS^2} + \frac{UdL + LdU}{dS} = 0$$

$$\text{IV. } \frac{ddO}{dS} + UdO + OdU - \frac{b}{a}dZ - \frac{dQ}{Q} = 0$$

Tertia integrata dat $\frac{dL}{dS} + UL = A$, quae cum prima combinata eliminando U praebet $MdL - 2LdM = AMdS$, vnde colligitur $L = AMM\int\frac{dS}{MM}$, et $U = -\frac{2dM}{MdS}$. His valoribus in ſecunda aequatione ſubſtitutis peruenitur ad hanc aequationem,

$$-\frac{ddM}{dS^2} + 2A + \frac{2AMdM}{dS}\int\frac{dS}{MM} = 0$$

Ad hanc reſoluendam ſit $\int\frac{dS}{MM} = R$, hincque $dS = MMdR$, et quia eſt $\frac{ddM}{dS} = d.\frac{dM}{dS} = d.\frac{dM}{MMdR}$, illa aequatio per dS multiplicata praebet

$$-d.\frac{dM}{MMdR} + 2AMMdR + 2AMRdM = 0$$

qua

qua resoluta habebitur $dS = MMdR$; $L = AMMR$ et $U = -\frac{2\,dM}{M^3\,dR}$. Verum ista aequatio per R multiplicata integrabilis redditur cum sit

$\int R\,d.\frac{dM}{MM\,dR} = \frac{R\,dM}{MM\,dR} + \frac{1}{M}$, ideoque habebimus:

$$\frac{R\,dM}{MM\,dR} + \frac{1}{M} = AMMRR + B = \frac{M\,dR + R\,dM}{MM\,dR}$$

Ponatur denique $MR = x$, seu $M = \frac{x}{R}$ erit $Axx + B = \frac{RR\,dx}{xx\,dR}$ hincque $\frac{dR}{RR} = \frac{dx}{xx(Axx+B)}$ seu $\frac{B\,dR}{RR} = \frac{dx}{xx} - \frac{A\,dx}{Axx+B}$; ideoque $\frac{B}{R} = \frac{1}{x} + \int\frac{A\,dx}{Axx+B}$. vnde R datur per x, tum ob $M = \frac{x}{R}$ reliquae quantitates omnes per x dabuntur: fit autem $dS = \frac{xx\,dR}{RR} = \frac{dx}{Axx+B}$ et $U\,dS = \frac{d\Omega}{\Omega} = -\frac{2\,dM}{M}$ ita vt fit amplitudo $\Omega = \frac{C}{MM} = \frac{CRR}{xx}$. Euoluamus hinc casus quibus amplitudo Ω per variabilem S algebraice definitur, quod fit si constans $B = o$, tum enim erit $\frac{dR}{RR} = \frac{dx}{Ax^4}$, ideoque $\frac{1}{R} = \frac{1+Dx^3}{3Ax^3}$, et $R = \frac{3Ax^3}{1+Dx^3}$, atque $M = \frac{1+Dx^3}{3Axx}$. Cum autem fit $dS = \frac{dx}{Axx}$ erit $\frac{1}{x} = -AS - E$, seu $x = -\frac{1}{AS+E}$ hincque

$$M = \frac{(AS+E)^3 - D}{3A(AS+E)},\ \text{et}\ R = -\frac{3A}{(AS+E)^3 - D};$$

vnde porro colligimus

$$\Omega = \frac{9AAC(AS+E)^2}{((AS+E)^3 - D)^2},\ \text{et}\ L = -\frac{(AS+E)^3 + D}{3(AS+E)^2}$$

Denique pro quantitate O obtinemus:

$$\frac{dO}{dS} + UO - \frac{b}{a}Z - lQ + lF = o \ \text{seu}$$

$$dO + \frac{O\,d\Omega}{\Omega} - \frac{b}{a}Z\,dS - dS\,l\frac{Q}{F} = o.\ \text{Ergo}$$

$$O = \frac{1}{\Omega}\int\Omega\,dS\left(\frac{b}{a}Z + l\frac{Q}{F}\right)$$

Immutemus parumper conſtantes; et quando amplitudo tubi Ω pro abſciſſa $AS = S$ ita fuerit comparata, vt ſit $\Omega = \frac{ff(\alpha S + \beta)^2}{((\alpha S + \beta)^3 - \gamma)^2}$, tum ſumtis

$$O = \frac{1}{\Omega} \int \Omega\, dS \left(\frac{b}{a} Z + l \frac{Q}{B}\right)$$

$$M = \frac{(\alpha S + \beta)^3 - \gamma}{3\alpha(\alpha S + \beta)} \text{ et } L = -\frac{(\alpha S + \beta)^3 + \gamma}{3(\alpha S + \beta)^2}$$

erit noſtrae aequationis integrale completum,

$$v = O + M\Gamma' : (S + ct) + L\Gamma : (S + ct)$$
$$+ M\Delta' : (S - ct) + L\Delta : (S - ct)$$

vnde porro definitur denſitas $q = Q\left(1 - \frac{v\, d\Omega}{\Omega\, dS} - \left(\frac{dv}{dS}\right)\right)$ et celeritas $u = \left(\frac{dv}{dt}\right)$.

Coroll. 1.

87. Hic primum obſeruandum eſt quantitates pro L et M inuentas per quantitatem quamcunque conſtantem multiplicari poſſe, quam functiones indefinitae complecti ſunt cenſendae, hinc ſumi poterit

$$M = \frac{(\alpha S + \beta)^3 - \gamma}{\alpha\delta(\alpha S + \beta)} \text{ et } L = -\frac{(\alpha S + \beta)^3 + \gamma}{\delta(\alpha S + \beta)^2}$$

quod tenendum eſt ſi forte illi valores fiant infiniti.

Coroll. 2.

88. Si enim capiatur $\gamma = \infty$, et $f = n\gamma$, vt amplitudo fiat $\Omega = nn(\alpha S + \beta)^2$, ſumatur quoque $\delta = \gamma$, eritque

$$M = \frac{-1}{\alpha(\alpha S + \beta)} \text{ et } L = \frac{1}{(\alpha S + \beta)^2}.$$

Hoc

Hoc autem casu tubi habebuntur conici vel pyramidici, in quibus ergo pariter agitationes aeris minimas definire licebit.

Coroll. 3.

89. Si sumatur $\gamma = 0$, tubus ita erit formatus vt fit $\Omega = \frac{ff}{(\alpha S + \beta)^4}$ seu amplitudines erunt reciproce vt biquadrata abscissarum: tum autem fit

$$M = \frac{1}{\alpha}(\alpha S + \beta)^2 \text{ et } L = -(\alpha S + \beta).$$

Coroll. 4.

90. In genere autem haec solutio ad eiusmodi tubos conoidicos applicari potest qui oriuntur ex conversione huiusmodi curuae hyperbolicae circa axem I B, pro qua posita abscissa $AS = x$, et applicata $SM = y$, fit $y = \frac{a^3 x}{x^3 + b^3}$: quae curua axem in A secans duas habet asymtotas GH et KL inter se normales. Tab. VII. Fig. 90.

Scholion 1.

91. Si etiam figuras tubi transcendentes cognoscere velimus faciamus $B = Amm$ et $A = \frac{\alpha}{m}$ et aequatio $dS = \frac{dx}{Axx + B}$ seu $\alpha dS = \frac{m dx}{xx + mm}$ dabit $x = m \text{ tang.} (\alpha S + \beta)$, vnde porro colligitur $\frac{\alpha mm}{R} = \alpha S + \gamma + \text{cot.} (\alpha S + \beta)$ seu $R = \frac{\alpha m m}{\alpha S + \gamma + \text{cot.}(\alpha S + \beta)}$ hincque

$$M = \frac{1 + (\alpha S + \gamma)\,\text{tang.}(\alpha S + \beta)}{\alpha m} \text{ et } L = \frac{1 + (\alpha S + \gamma)\,\text{tang.}(\alpha S + \beta)}{m \,\text{cot.}(\alpha S + \beta)}$$

Deinde fit amplitudo $\Omega = \frac{c}{(1 + (\alpha S + \gamma) \, tang. \, (\alpha S + \zeta))^2}$, quantitas vero O hinc definitur vt ante

$$O = \frac{1}{\Omega} \int \Omega \, dS \left(\frac{b}{a} Z + \iota \frac{Q}{B}\right).$$

Ex his autem formis colligimus methodum aequationes primum erutas multo facilius resoluendi. Cum enim prima det $U = -\frac{2 \, dM}{M \, dS}$, tertia vero integrata $\frac{dL}{dS} + UL = A$, erit $dL - \frac{2 L \, dM}{M} = A \, dS$, ponamus $L = My$, fietque $M \, dy - y \, dM = A \, dS$. Nunc secunda quod fieri potest integrata praebet

$$\frac{dM}{dS} + 2L + UM + \int UL \, dS = 0$$

quae ob

$$UM = -\frac{2 \, dM}{dS} \text{ et } \int UL dS = -2\int \frac{L dM}{M} = -2\int y dM$$

transit in:

$$\frac{dM}{dS} + 2 My - \frac{2 \, dM}{dS} - 2 \int y \, dM = 0,$$

quae contrahitur in hanc $2 \int M \, dy - \frac{dM}{dS} = 0$. Ex illa autem fit

$$2 M \, dy = A \, dS + y \, dM + M \, dy \text{ ita vt fit}$$

$$2 \int M \, dy = AS + My + B = \frac{dM}{dS}$$

vnde cum parum colligere liceat, consideremus has duas aequationes:

$$M \, dy - y \, dM = A \, dS \text{ et } 2 \int M \, dy = \frac{dM}{dS}$$

quas ambas differentiemus sumto elemento dS constante

$$M \, ddy - y \, ddM = 0 \text{ et } 2 M \, dy = \frac{ddM}{dS},$$

vnde

vnde fit

$$\frac{ddM}{M} = \frac{ddy}{y} = 2\,dy\,dS \text{ seu } ddy = 2\,y\,dy\,dS$$

cuius integrale est $dy = dS(yy + mm)$, hincque porro Ang. tang. $\frac{y}{m} = mS + n$ et conuertendo $y = m$ tang. $(mS + n)$. Iam ex aequatione

$$\frac{M\,dy - y\,dM}{yy} = \frac{A\,dS}{yy} = \frac{A\,dS}{mm\,tang.(mS+n)^2} \text{ fit}$$

$$-\frac{M}{y} = -\frac{M}{m\,tang.(mS+n)} = \frac{A}{mm}\int\frac{dS\,cos.(mS+n)^2}{sin.(mS+n)^2}$$

$$= \frac{A}{mm}\int\frac{dS}{sin.(mS+n)^2} - \frac{AS}{mm} = -\frac{A}{m^3\,tang.(mS+n)} - \frac{AS}{mm} - \frac{Ak}{m^3}$$

vnde fit

$$M = \frac{A}{mm}(1 + (mS + k)\,tang.(mS + n)),$$

hincque porro

$$L = \frac{A(1 + (mS + k)\,tang.(mS + n))}{m\,cos.(mS + n)}$$

ac denique prodit amplitudo $\Omega = \frac{Const.}{MM}$ seu

$$\Omega = \frac{ff}{(1 + (mS + k)\,tang.(mS + n))^2}.$$

Scholion 2.

92. Adhuc facilius hanc resolutionem instituere licet hoc modo: cum ex prima aequatione sit $U = -\frac{2\,dM}{M\,dS}$ ponatur statim $L = My$, eritque

$$MU = -\frac{2\,dM}{dS} \text{ et } LU = -\frac{2\,y\,dM}{dS},$$

ex quo secunda aequatio fiet

$$\frac{ddM}{dS} + 2\,d.My - 2\,d.\frac{dM}{dS} - 2\,y\,dM = 0 \text{ seu}$$

$$-\frac{ddM}{dS} + 2\,M\,dy = 0 \text{ seu } \frac{ddM}{M} = 2\,dy\,dS$$

 simili

simili modo tertia abit in hanc formam:

$$\frac{d\,d.\,M\,y}{d\,S} - 2\,d.\frac{y\,d\,M}{d\,S} = 0.$$

factaque euolutione

$$M\,ddy + 2\,dM\,dy + y\,ddM - 2\,dy\,dM - 2\,y\,ddM = 0$$

hoc est

$$M\,d\,dy - y\,d\,d\,M = 0 \text{ seu } \frac{d\,d\,M}{M} = \frac{d\,d\,y}{y}$$

Ex his coniunctis fit vt ante $d\,dy = 2\,y\,dy\,d\,S$. Quod si hinc statim tubi formas algebraicas elicere velimus sumatur superior constans $m = 0$, vt fit $d\,y = y\,y\,d\,S$ ideoque $\frac{1}{y} = -S - \frac{\beta}{\alpha}$, seu $y = \frac{-\alpha}{\alpha\,S + \beta}$. Tum aequatio $M\,dy - y\,d\,M = A\,d\,S$ dat

$$-\frac{M}{y} = \frac{A}{\alpha\,\alpha}\int dS(\alpha S + \beta)^2 = \frac{A}{3\,\alpha^3}((\alpha S + \beta)^3 - \gamma),$$

ideoque

$$M = \frac{A((\alpha\,S + \beta)^3 - \gamma)}{3\,\alpha\,\alpha(\alpha\,S + \beta)} \text{ et } L = -\frac{A((\alpha\,S + \beta)^3 - \gamma)}{3\,\alpha(\alpha\,S + \beta)^2},$$

ac denique amplitudo $\Omega = \frac{C\,(\alpha\,S + \beta)^2}{((\alpha\,S + \beta)^3 - \gamma)^2}$ prorsus vt ante.

Scholion 3.

93. Praeter hos duos casus autem, quibus forma tubi vel est algebraica, vel a circulo pendet, tertium non omitti conuenit, quo ea per quantitates exponentiales determinatur. Pro eo autem aequationis $d\,dy = 2\,y\,dy\,d\,S$ integrale sumitur $dy = dS(yy - mm)$ vnde fit $2\,m\,S + 2\,n = l\frac{y - m}{y + m}$ vnde fit

$$y = \frac{1 + e^{2\,m\,S + 2\,n}}{1 - e^{2\,m\,S + 2\,n}}\,m,$$

tum

tum vero porro

$$-\frac{M}{y}=\frac{A}{mm}\int\frac{dS(1-e^{2mS+2n})^2}{(1+e^{2mS+2n})^2}=\frac{A}{m^3}\cdot\frac{e^{2mS+2n}(mS+k-1)+mS+k+1}{1+e^{2mS+2n}}$$

ideoque

$$M=-\frac{A}{mm}\cdot\frac{e^{2mS+2n}(mS+k-1)+mS+k+1}{1-e^{2mS+2n}}$$

et

$$L=-\frac{A}{m}\cdot\frac{e^{2mS+2n}(mS+k-1)+mS+k+1}{(1-e^{2mS+2n})^2}(1+e^{2mS+2n})$$

atque amplitudo

$$\Omega=\frac{C(1-e^{2mS+2n})^2}{(e^{2mS+2n}(mS+k-1)+mS+k+1)^2}.$$

Certum eſt ex hoc caſu perinde atque ex priori a circulo pendente caſum algebraicum prodire debere, ſi ſtatuatur $m=0$ quae tamen euolutio minus eſt obuia, et haud exiguam induſtriam Analyſtae poſtulat.

Problema 85.

92. Inueſtigare caſus pro tubi amplitudine, vt aequatio differentio-differentialis inuenta per operationes tertii ordinis integrari queat, vel eius integrale completum per tertiam formam ſupra explicatam exhiberi queat.

Solutio.

Hoc ergo caſu integrale quaeſitum talem formam habere debebit:

$$v=O+Nf'':(S+ct)+Mf':(S+ct)+Lf:(S+ct)$$

et

et quia iam patet quantitatem O vt ante definitum iri, tantum diuersas functionum species notemus:

$f'''':(S+ct)$	$f''':(S+ct)$	$f'':(S+ct)$	$f':(S+ct)$	$f:(S+ct)$
$+N$	$+\frac{2dN}{dS}$	$+\frac{ddN}{dS^2}$		
	$+M$	$+\frac{2dM}{dS}$	$+\frac{ddM}{dS^2}$	
		$+L$	$+\frac{2dL}{dS}$	$+\frac{ddL}{dS^2}$
	$+UN$	$+\frac{UdN}{dS}$		
		$+UM$	$+\frac{UdM}{dS}$	
			$+UL$	$+\frac{UdL}{dS}$
		$+\frac{NdU}{dS}$	$+\frac{MdU}{dS}$	$+\frac{LdU}{dS}$
$-N$	$-M$	$-L$		

vnde deducimus quatuor sequentes aequationes:

$$\text{I.}\quad \frac{2dN}{dS}+UN=0$$

$$\text{II.}\quad \frac{ddN}{dS^2}+\frac{2dM}{dS}+\frac{UdN+NdU}{dS}+UM=0$$

$$\text{III.}\quad \frac{ddM}{dS^2}+\frac{2dL}{dS}+\frac{UdM+MdU}{dS}+UL=0$$

$$\text{IV.}\quad \frac{ddL}{dS^2}+\frac{UdL+LdU}{dS}=0.$$

Cum nunc ex prima sit $U=\frac{-2dN}{NdS}$, ponamus $M=Ny$ et $L=Nz$, vt fiat

$$UN=\frac{-2dN}{dS};\ UM=\frac{-2ydN}{dS};\ \text{et}\ UL=\frac{-2zdN}{dS};$$

quibus valoribus substituendis aequatio II. fit

$$\frac{ddN}{dS^2}+\frac{2Ndy+2ydN}{dS}-\frac{2ddN}{dS^2}-\frac{2ydN}{dS}=0$$

quae contrahitur in hanc formam:

$$\text{II.}\quad \frac{-ddN}{dS}+2Ndy=0$$

Tertia

Tertia aequatio vero hinc euadit:

$$\frac{Nddy + 2dNdy + yddN}{dS^2} + \frac{2Ndz + 2zdN}{dS} - \frac{2yddN - 2dydN}{dS^2} - \frac{2zdN}{dS} = 0$$

quae contrahitur in hanc formam:

III. $\frac{Nddy - yddN}{dS} + 2Ndz = 0.$

Quarta denique ita euoluitur:

$$\frac{Nddz + 2dNdz + zddN}{dS^2} - \frac{2zddN - 2dzdN}{dS^2} = 0$$

quae ergo praebet hanc formam:

IV. $\frac{Nddz - zddN}{dS} = 0.$

Ex his triplici modo fit:

$$\frac{ddN}{N} = 2\,dy\,dS = \frac{ddy + 2dzdS}{y} = \frac{ddz}{z}$$

vnde hae duae aequationes resoluendae proponuntur:

$$2ydydS = ddy + 2dzdS \quad \text{et} \quad yddz - zddy = 2zdzdS$$

quarum posterior integrata statim dat

$$ydz - zdy = dS(zz + A)$$

prior vero pariter integrata praebet:

$$dy + 2z\,dS = dS(yy + B)$$

seu $dy = dS(yy - 2z + B).$

Iam illa per hanc diuisa suppeditat hanc aequationem ab elemento dS liberam

$$\frac{ydz - zdy}{dy} = \frac{zz + A}{yy - 2z + B}$$

fit $z = uy$ et haec aequatio fit

$$\frac{yydu}{dy} = \frac{uuyy + A}{yy - 2uy + B}$$

seu $y^4du - 2uy^3du + Byydu - uuyydy - Ady = 0.$

Necessitas hic vrget, vt faciamus $A = 0$ et $B = 0$; quo facto habebimus:

$$yydu - 2uydu - uudy = 0 \text{ seu } yydu = d.\,uuy$$

diuidamus per $u^4 y^2$ erit $\frac{du}{u^4} = \frac{d.\,uuy}{u^4 yy}$, eiusque integrale:

$$\frac{1}{3u^3} = \frac{1}{uuy} - \frac{1}{3m^3} \text{ seu } y = \frac{3m^3 u}{m^3 + u^3} \text{ et } z = \frac{3m^3 uu}{m^3 + u^3},$$

vnde fit

$$dS = \frac{dy}{yy - 2uy} = \frac{ydz - zdy}{zz}, \text{ siue}$$

$$dS = \frac{du}{uu} = -d.\,\frac{y}{z}, \text{ ita vt fit } \frac{1}{u} = -S - \frac{\beta}{\alpha} \text{ et } u = \frac{-\alpha}{\alpha S + \beta}.$$

Ergo iam per S habebimus hos valores:

$$y = \frac{-3\alpha m^3 (\alpha S + \beta)^2}{m^3 (\alpha S + \beta)^3 - \alpha^3} \quad \text{et} \quad z = \frac{3\alpha^2 m^3 (\alpha S + \beta)}{m^3 (\alpha S + \beta)^3 - \alpha^3}.$$

Ponamus breuitatis gratia $\frac{\alpha^3}{m^3} = \gamma^3$ vt fiat

$$y = \frac{-3\alpha(\alpha S + \beta)^2}{(\alpha S + \beta)^3 - \gamma^3} \quad \text{et} \quad z = \frac{3\alpha\alpha(\alpha S + \beta)}{(\alpha S + \beta)^3 - \gamma^3}.$$

Nunc porro ex aequatione IV colligimus:

$$N\,dz - z\,dN = A\,dS = \frac{A\,du}{uu}$$

quae ob $z = \frac{3\alpha^3 uu}{\alpha^3 + \gamma^3 u^3}$ diuisa per zz dat

$$-d.\,\frac{N}{z} = \frac{B\,du}{u^6}(\alpha^3 + \gamma^3 u^3)^2 \text{ et integrando}$$

$$-\frac{N}{z} = C + B\left(\gamma^6 u - \frac{\alpha^3\gamma^3}{u^2} - \frac{\alpha^6}{5u^5}\right)$$

hinc $N = \frac{Bz}{5u^5}(\alpha^6 + 5\alpha^3\gamma^3 u^3 - 5\gamma^6 u^6) - Cz$ et euoluendo

$$N = \frac{A(\alpha S + \beta)^6 - 5A\gamma^3(\alpha S + \beta)^3 - 5A\gamma^6 + B(\alpha S + \beta)}{(\alpha S + \beta)^3 - \gamma^3}$$

mutatis scilicet constantibus, vnde statim definitur amplitudo tubi $\Omega = \frac{C}{NN}$, tum vero erit

$$M = \frac{-3\alpha(\alpha S + \beta)^2}{(\alpha S + \beta)^3 - \gamma^3} N \text{ et } L = \frac{3\alpha\alpha(\alpha S + \beta)}{(\alpha S + \beta)^3 - \gamma^3} N.$$

Pro

Pro quantitate autem O vt ante eſt

$$O = \frac{1}{\Omega} \int \Omega\, dS \left(\frac{b}{a} Z + l \frac{\Omega}{B}\right).$$

Quoties ergo amplitudo Ω hoc modo definitur, ſolutio completa noſtri problematis ita ſe habebit vt ſit:

$$v = O + N\Gamma'' : (S+ct) + M\Gamma' : (S+ct) + L\Gamma : (S+ct)$$
$$+ N\Delta'' : (S-ct) + M\Delta' : (S-ct) + L\Delta : (S-ct).$$

Deinde vero eſt denſitas $q = Q\left(1 - \frac{v\, d\Omega}{\Omega\, dS} - \left(\frac{dv}{dS}\right)\right)$ et celeritas $u = \left(\frac{dv}{dt}\right)$.

Coroll. 1.

95. Determinatio haec amplitudinis tubi reſolutionem propoſitam admittens duplici modo eſt particularis, dum duas conſtantes arbitrarias in aequatione inter y et z inuenta nihilo aequales poſuimus; interim tamen forma pro Ω eruta plures adhuc conſtantes arbitrarias complectitur, ideoque latius patet quam caſus praecedens in omni extenſione acceptus.

Coroll. 2.

96. Caſus ſimpliciores in hac ſolutione contenti oriuntur ſi fiat $\gamma = 0$, tum enim erit:

$$N = \frac{A(\alpha S + \beta)^5 + B(\alpha S + \beta)}{(\alpha S + \beta)^3}$$

prout igitur fuerit vel $A = 0$ vel $B = 0$, reperitur vel

$$N = (\alpha S + \beta)^{-2} \text{ vel } N = (\alpha S + \beta)^2;$$

 illo

illo igitur casu amplitudo ita definitur vt sit

$\Omega = C(\alpha S + \mathfrak{C})^4$, hoc vero $\Omega = C(\alpha S + \mathfrak{C})^{-6}$.

Coroll. 3.

97. Solutio per operationem primi ordinis inuenta etiam duos casus simpliciores dederat hos:

$$\Omega = C(\alpha S + \mathfrak{C})^0 \text{ et } \Omega = C(\alpha S + \mathfrak{C})^{-2}$$

operatio vero secundi ordinis praebuerat hos duos

$$\Omega = C(\alpha S + \mathfrak{C})^2 \quad \text{et } \Omega = C(\alpha S + \mathfrak{C})^{-4}$$

nunc haec tertii ordinis suppeditat:

$$\Omega = C(\alpha S + \mathfrak{C})^4 \quad \text{et } \Omega = C(\alpha S + \mathfrak{C})^{-6},$$

vnde concludere licet hanc methodum ad omnes tubos applicari posse hac formula $\Omega = C(\alpha S + \mathfrak{C})^{\pm 2i}$ contentos; exclusis exponentibus imparibus.

Scholion 1.

98. Omnes plane tuborum formae hoc modo resolutionem admittentes repeti debent ex hac aequatione

$$y^4 du - 2uy^3 du - uuyy dy + Byy du - A dy = 0$$

vbi A et B constantes ab arbitrio nostro pendentes denotant, quas tamen ambas in praecedente solutione euanescentes facere sum coactus, vt integratio succederet. Nunc autem obseruo dummodo sit $A = 0$ ideoque

$$dS = \frac{y dz - z dy}{zz} = \frac{du}{uu},$$

inte-

integrationem etiamnunc adminiſtrari poſſe, quaecunque conſtans pro B aſſumatur. Cum enim tum ſit

$$yy\,du - 2uy\,du - uu\,dy + B\,du = 0$$

ponamus $uuy = x$ ſeu $y = \frac{x}{uu}$ erit

$$\frac{xx\,du}{u^4} - dx + B\,du = 0 \text{ ſeu } d.\frac{1}{x} + B\,du\left(\frac{1}{x}\right)^2 + \frac{du}{u^4} = 0$$

qui eſt caſus integrabilis aequationis Riccatianae, cui ſatisfacit forma $\frac{1}{x} = \frac{\alpha}{u} + \frac{\beta}{uu}$; ſit enim

$$-\frac{\alpha}{uu} - \frac{2\beta}{u^3} + \frac{B\alpha\alpha}{uu} + \frac{2B\alpha\beta}{u^3} + \frac{B\beta\beta}{u^4} + \frac{1}{u^4} = 0$$

vnde ſumi debet

$$B = -\frac{1}{\beta\beta} \text{ et } \alpha = \frac{1}{B} = -\beta\beta,$$

ita vt ſit

$$\frac{1}{x} = \frac{-\beta\beta}{u} + \frac{\beta}{uu}.$$

Ponatur ergo

$$\frac{1}{x} = \frac{-\beta\beta}{u} + \frac{\beta}{uu} + \frac{1}{p},$$

vt aequationis

$$d.\frac{1}{x} - \frac{du}{\beta\beta}\left(\frac{1}{x}\right)^2 + \frac{du}{u^4} = 0$$

integrale completum eruamus, orieturque:

$$dp - \frac{2p\,du}{u} + \frac{2p\,du}{\beta uu} + \frac{du}{\beta\beta} = 0$$

quae per $\frac{e^{-\frac{2}{\beta u}}}{uu}$ multiplicata et integrata praebet:

$$\frac{e^{-\frac{2}{\beta u}}p}{uu} + \frac{e^{-\frac{2}{\beta u}}}{2\beta} = \frac{C}{2\beta} \text{ ſeu } p = \frac{Ce^{\frac{2}{\beta u}}uu}{2\beta} - \frac{uu}{2\beta}$$

 ita

ita vt sit $\frac{1}{x} = -\frac{\beta\beta}{u} + \frac{\beta}{uu} + \frac{2\beta}{uu\left(Ce^{\frac{2}{\beta u}} - 1\right)}$

hincque $x = \frac{uu\left(Ce^{\frac{2}{\beta u}} - 1\right)}{\beta(1+\beta u) + C\beta e^{\frac{2}{\beta u}}(1-\beta u)} = uuy.$

Ergo $y = \frac{Ce^{\frac{2}{\beta u}} - 1}{\beta(1+\beta u) + C\beta e^{\frac{2}{\beta u}}(1-\beta u)}$ et $z = \frac{u\left(Ce^{\frac{2}{\beta u}} - 1\right)}{\beta(1+\beta u) + C\beta e^{\frac{2}{\beta u}}(1-\beta u)}$ existente $u = \frac{-m}{mS+n}$; tum vero est $N\,dz - z\,dN = D\,dS = \frac{D\,du}{uu}$ ex qua per zz diuisa reperitur N indeque reliqua.

Si hic faceremus $\beta = \infty$ praecedens solutio resultaret.

Scholion 2.

Si hoc modo vlterius progredi placeat, calculus quidem fit laboriosior, sed tamen artificio hic vsitato praecipuas difficultates superare licebit. Veluti si pro integralis forma quarti ordinis ponamus:

$$v = O + Nf''':(S+ct) + Mf'':(S+ct) + Lf':(S+ct) + Kf:(S+ct)$$

et ob

$$U\,dS = \frac{d\Omega}{\Omega} = \frac{-2\,dN}{N} \text{ ideoque } \Omega = \frac{ff}{NN} \text{ ponatur}$$

$$M = Nx,\ L = Ny \text{ et } K = Nz;$$

perueniturque ad has aequationes:

$$\frac{ddN}{N} = 2\,dx\,dS = \frac{ddx + 2\,dy\,dS}{x} = \frac{ddy + 2\,dz\,dS}{y} = \frac{ddz}{z}$$

vnde

vnde deducimus:

I. $2xdxdS = ddx + 2dydS$

II. $yddx - xddy + 2ydydS - 2xdzdS = 0$

III. $ddz - 2zdxdS = 0$

IV. $zddy - yddz + 2zdzdS = 0$

hincque integrando

I. $xxdS = dx + 2ydS + \text{Const.}\, dS$

II+III. $ydx - xdy + dz + yydS - 2xzdS = \text{Const.}\, dS$

IV. $zdy - ydz + zzdS = \text{Const.}\, dS.$

Omnes has tres constantes nihilo sumamus aequales, et vltima praebet $\frac{y}{z} = -S$, ita vt sit $y = -zS$, vbi constantis additione non est opus, quia loco S scriptum concipere licet $S + \frac{\varepsilon}{\alpha}$ vti supra. Posito igitur $y = -zS$ ex prima fit $dx = dS(xx + 2zS)$, et ex secunda

$$-Szdx + Sxdz - xzdS + dz + SSzzdS = 0$$

quae per zz diuisa et integrata edit

$$-\frac{Sx}{z} - \frac{1}{z} + \frac{1}{3}S^3 + \frac{1}{3}A = 0 \text{ ideoque } z = \frac{3(1 + Sx)}{A + S^3}$$

hocque valore in prima substituto fit

$$dx = dS\left(xx + \frac{6S(1 + Sx)}{A + S^3}\right).$$

cui cum satisfaciat $x = -\frac{1}{S}$, ponatur $x = -\frac{1}{S} + \frac{1}{v}$ eritque $0 = dv - \frac{2vdS}{S} + \frac{6vSSdS}{A + S^3} + dS$, cuius integrale,

$$v = \frac{BSS + AAS - AS^4 - \frac{1}{5}S^7}{(A + S^3)^2} \text{ suppeditat}$$

$x =$

$$x = \frac{-B + 3ASS + \frac{6}{5}S^5}{BS + AA - AS^3 - \frac{1}{5}S^6}, \text{ hincque porro}$$

$$z = \frac{3(A + S^3)}{BS + AA - AS^3 - \frac{1}{5}S^6} \text{ et } y = \frac{-3S(A + S^3)}{BS + AA - AS^3 - \frac{1}{5}S^6}.$$

Supereſt $z\,dN - N\,dz = C\,dS$, ideoque $\frac{N}{z} = C\int\frac{dS}{zz}$ vnde reliqua facile expediuntur.

Scholion 3.

100. Hoc modo progrediendo continuo plures reperiuntur tuborum figurae, in quibus aeris agitationes minimas definire licebit, et quaelibet adeo operatio infinities maiorem multitudinem ſuppeditat, quam praecedens. Interim tamen infinitae tuborum figurae manent excluſae, pro quibus etiamnum aeris motum determinari non licet. Poſt figuram ergo cylindricam, quam felici ſucceſſu expediuimus, ſequitur conoidica hyperbolica, vix difficiliorem ſolutionem poſtulans quam illa, cum ab eadem operatione prima ſit ſuppeditata, ita vt hae duae figurae ad primum ordinem ſint referendae. Ex ordine ſecundo imprimis notatu eſt digna figura conica aequatione $\Omega = Ass$ contenta, tum vero etiam hyperbolica $\Omega = \frac{A}{s^4}$, nec non haec algebraica multo latius patens $\Omega = \frac{Ass}{(s^3 + B)^2}$, praeter quas innumerabiles aliae tranſcendentes ſunt erutae; pro altioribus autem ordinibus motus determinatio continuo fit operoſior, vt operae non ſit pretium tantos labores ſuſcipere. Species itaque tantum ſimpliciſſimas examini ſum ſubiecturus.

CAPVT