OBSERVATIONES CIRCA AEQVATIONEM DIFFERENTIALEM

$y\,dy + My\,dx + N\,dx = o.$

Auctore

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I.

In hac aequatione quantitates M et N vt functiones quaecunque variabilis x ſpectantur; et cum haec aequatio ita ſit comparata, vt in genere nullo modo integrari queat, methodis ſaltem etiamnunc cognitis; praecipua quaeſtio in ea indole binarum functionum M et N inueſtiganda verſatur, qua integratio abſolui queat; vnde quidem caſus per ſe obvios, veluti quando vel altera euaneſcit, vel ambae in ratione conſtante conſiſtunt, excludi conuenit.

II. Cum forma huius aequationis ſatis ſit ſimplex, vt mirari liceat, quod eius integratio vires Analyſeos adhuc eluſerit, eius certe conſideratio eo maiori attentione digna videtur; idque potiſſimum quod in forma generaliori, a Comite Riccatio olim tractata

$$dz + Pz^n\,dx + Q\,dx = o,$$

continetur, ad quam adeo duplici modo reuocari poteſt. Primo enim per y diuiſa praebet

$$dy + Ny^{-1}dx + Mdx = 0,$$

ita vt $n = -1$; tum vero posito $yy = z$, prodit

$$dz + 2Mz^{\frac{1}{2}}dx + 2Ndx = 0,$$

ita vt sit $n = \frac{1}{2}$. Vbi in genere obseruasse iuuabit, posito $z = y^{\frac{-1}{n-1}}$ formam generalem in hanc mutari

$$dy + (1-n)Qy^{\frac{n}{n-1}} + (1-n)Pdx = 0,$$

ita vt in hoc negotio exponentes n et $\frac{n}{n-1}$ pro aequiualentibus sint habendi.

III. Patet ergo, quod tantum in transitu monuerim, casum $n = 2$, quem Ricatus olim imprimis est contemplatus, hac proprietate prae reliquis esse praeditum, vt hac reductione ad se ipsum reuocetur, dum $\frac{n}{n-1}$, iterum dat binarium. Tum vero casus $n = 1$, quo aequatio

$$dz + Pzdx + Qdx = 0$$

generatim est integrabilis, perducitur ad alterum $\frac{n}{n-1} = \infty$; vnde cum potestati y^{∞} aequiualeat quasi forma exponentialis e^{y}, etiam haec aequatio

$$dy + Pe^{y}dx + Qdx = 0$$

pro integrabili est habenda, cuius integratio ponendo $e^{y} = \frac{1}{v}$, vt fiat $y = -lv$ et $dy = -\frac{dv}{v}$, hincque prodeat:

$$dv - Qvdx - Pdx = 0,$$

per se est manifesta.

IV. Vt autem ad ipſam aequationem propoſitam reuertar, ante omnia obſeruo me ad ſimilem formam, cum olim aequationem differentialem tertii gradus hanc:

$$d^3 v + A\, dt\, ddv + B\, dt^2\, dv + C\, v\, dt^3 = 0$$

tractaſſem, vbi quidem A, B, C et dt ſunt conſtantes, eſſe perductum. Poſito ſcilicet $v = e^{\int x\, dt}$ obtinueram

$$ddx + 3x\, dt\, dx + A\, dt\, dx + (x^3 + A\, xx + Bx + C)\, dt^2 = 0$$

ſeu rationem elementi conſtantis dt exuendo:

$$d.\frac{dx}{dt} + (3x + A)\, dx + (x^3 + A\, xx + Bx + C)\, dt = 0.$$

Nunc poſueram $\frac{dx}{dt} = y$ hincque $dt = \frac{dx}{y}$, ex quo nata erat haec aequatio

$$dy + (3x + A)\, dx + (x^3 + A\, xx + Bx + C)\frac{dx}{y} = 0$$

ſiue $y\, dy + (3x + A)\, y\, dx + (x^3 + A\, xx + Bx + C)\, dx = 0$

quae vtique in forma propoſita comprehenditur.

V. Cum igitur aequatio differentialis tertii ordinis, vnde hanc deriuauimus, ſit integrabilis, ac poſito

$$x^3 + A\, xx + Bx + C = (x - \alpha)(x - \beta)(x - \gamma)$$

eius integrale completum ſit

$$v = F\, e^{\alpha t} + G\, e^{\beta t} + H\, e^{\gamma t}$$

hinc ſequitur, etiam noſtrae aequationis

$$y\, dy + (3x + A)\, y\, dx + (x^3 + A\, xx + Bx + C)\, dx = 0$$

 inte-

integrale in genere exhiberi posse; quod quidem adeo algebraice ita expressum inde elicui

$$(y+(x-\alpha)(x-\beta))^{\beta-\alpha}(y+(x-\beta)(x-\gamma))^{\gamma-\beta}(y+(x-\gamma)(x-\alpha))^{\alpha-\gamma}=E$$

vbi notandum est, esse

$$A=-\alpha-\beta-\gamma;\quad B=\alpha\beta+\alpha\gamma+\beta\gamma \text{ et } C=-\alpha\beta\gamma.$$

VI. Cum enim ob $v=e^{\int x\,dt}$ sit $x=\frac{dv}{v\,dt}$, fiet.

$$x=\frac{\alpha F e^{\alpha t}+\beta G e^{\beta t}+\gamma H e^{\gamma t}}{v} \text{ et ob } y=\frac{dx}{dt}$$

$$y=\frac{(\beta-\alpha)^2 FGe^{(\alpha+\beta)t}+(\gamma-\alpha)^2 FHe^{(\alpha+\gamma)t}+(\gamma-\beta)^2 GHe^{(\beta+\gamma)t}}{vv}$$

Inde porro colligitur:

$$x-\alpha=\frac{(\beta-\alpha)Ge^{\beta t}+(\gamma-\alpha)He^{\gamma t}}{v} \text{ et } x-\beta=\frac{(\alpha-\beta)Fe^{\alpha t}+(\gamma-\beta)He^{\gamma t}}{v}$$

sicque conficitur

$$y+(x-\alpha)(x-\beta)=\frac{(\gamma-\alpha)(\gamma-\beta)He^{\gamma t}}{v}$$

similique modo

$$y+(x-\beta)(x-\gamma)=\frac{(\alpha-\beta)(\alpha-\gamma)Fe^{\alpha t}}{v}$$

$$\text{et } y+(x-\gamma)(x-\alpha)=\frac{(\beta-\gamma)(\beta-\alpha)Ge^{\beta t}}{v}$$

vnde veritas integralis exhibiti fit manifesta, et quia id inuoluit quantitatem constantem E per arbitrarias F, G, H definitam, pro completo erit habendum.

VII. Haec consideratio viam nobis aperit, a priori ad aequationes formae propositae perueniendi. Sumtis

Sumtis enim tribus functionibus ipsius x, quae sint P, Q, R statuatur aequatio integralis:

$$(y+P)^{\lambda}(y+Q)^{\mu}(y+R)^{\nu} = \text{Const.}$$

vnde haec nascitur aequatio differentialis:

$$\frac{\lambda dy + \lambda dP}{y+P} + \frac{\mu dy + \mu dQ}{y+Q} + \frac{\nu dy + \nu dR}{y+R} = 0.$$

quae a fractionibus liberata hanc induit formam:

$$\begin{aligned}&(\lambda+\mu+\nu)yydy + ydy(\lambda(Q+R)+\mu(P+R)+\nu(P+Q))\\ &\quad + dy(\lambda QR + \mu PR + \nu PQ) + yy(\lambda dP + \mu dQ + \nu dR)\\ &\quad + y(\lambda(Q+R)dP + \mu(P+R)dQ + \nu(P+Q)dR)\\ &\qquad + \lambda QRdP + \mu PRdQ + \nu PQdR = 0\end{aligned}$$

ex qua forma proposita resultat statuendo:

1°. $\lambda + \mu + \nu = 0$

2°. $\lambda QR + \mu PR + \nu PQ = 0$

3°. $\lambda dP + \mu dQ + \nu dR = 0$ seu $\lambda P + \mu Q + \nu R = \text{Const.}$

VIII. Si hic ponamus $\lambda P + \mu Q + \nu R = a$ et $P + Q + R = S$, vt singulas litteras P, Q, R hinc definire valeamus, ratione habita primae conditionis, qua esse oportet $\lambda + \mu + \nu = 0$, reperiemus hos valores:

$$P = \frac{3\lambda\mu\nu S + (\lambda\lambda + 2\mu\nu)a + (\mu - \nu)\sqrt{((\lambda\lambda - \mu\nu)aa - 3\lambda\mu\nu a S)}}{9\lambda\mu\nu}$$

$$Q = \frac{3\lambda\mu\nu S + (\mu\mu + 2\lambda\nu)a + (\nu - \lambda)\sqrt{((\mu\mu - \lambda\nu)aa - 3\lambda\mu\nu a S)}}{9\lambda\mu\nu}$$

$$R = \frac{3\lambda\mu\nu S + (\nu\nu + 2\lambda\mu)a + (\lambda - \mu)\sqrt{(\nu\nu - \lambda\mu)aa - 3\lambda\mu\nu a S}}{9\lambda\mu\nu}$$

vbi signa radicalia ob $\lambda\lambda - \mu\nu = \mu\mu - \nu\nu - \lambda\mu$ inter se conueniunt.

IX. Irrationalitate harum formularum sublata ad eandem aequationem peruenitur, cuius integrale supra exhibui (§. V.) vnde hanc euolutionem vlterius non prosequor. Interim tamen maximi momenti esse arbitror, obseruasse aequationem differentialem generalem §. VII. expositam per se reddi integrabilem, si ea diuidatur per

$$(y+P)(y+Q)(y+R),$$

quod si ad aequationem superiorem

$$y\,dy+(3x+A)y\,dx(x^3+Axx+Bx+C)dx=0,$$

attendamus; reperiemus, eam per se integrabilem reddi, si diuidatur per hanc formam:

$$y^3+yy(3xx+2Ax+B)+y(3x+A)(x^3+Axx+Bx+C)$$
$$+(x^3+Axx+Bx+C)^2$$

etiamsi hinc minus pateat, integrale adeo algebraice exhiberi posse. Quae obseruatio me deducit ad methodum illam generalem iam dudum a me expositam, qua ostendi, omnium aequationum differentialium integrationes commodissime per multiplicatores absolui posse.

X. Cum igitur hic multiplicator seu diuisor ita comparatus esse debeat, vt formula per se fiat integrabilis, vtique necesse est, vt criteria huiusmodi formularum perspecta habeamus, quae integrabilitatem certo indicent, etiamsi forte ipsa integratio difficulter ac nonnisi per quadraturas satis complicatas confici queat. Omnium autem formularum integrabilium, cuiuscunque gradus differentialia implicent,

cent, hanc esse proprietatem nuper demonstraui, vt positis

$$dy = p\,dx, \quad dp = q\,dx, \quad dq = r\,dx \text{ etc.}$$

quo pacto eae semper ad talem formam $V\,dx$ reducuntur, in qua littera V vtcunque quantitates x, y, p, q, r etc implicabit, tam futurum sit

$$\left(\frac{dV}{dy}\right) - \frac{1}{dx}\,d\left(\frac{dV}{dp}\right) + \frac{1}{dx^2}\,dd.\left(\frac{dV}{dq}\right) - \frac{1}{dx^3}\,d^3.\left(\frac{dV}{dr}\right) + \text{etc} = 0$$

ac viciſſim quoties haec conditio locum habeat, toties quoque formulam $V\,dx$ esse integrabilem.

XI. Hanc igitur quaestionem nunc euoluendam suscipio.

Inuestigare eiusmodi functionem Z binarum variabilium x *et* y, *per quam aequatio nostra:*

$$y\,dy + My\,dx + N\,dx = 0$$

diuisa fiat integrabilis!

Hoc ergo casu erit $V = \frac{yp + My + N}{Z}$ vnde colligitur

$$\left(\frac{dV}{dy}\right) = \frac{p}{Z} - \frac{yp}{ZZ}\left(\frac{dZ}{dy}\right) + \frac{M}{Z} - \frac{(My+N)}{ZZ}\left(\frac{dZ}{dy}\right)$$

tum vero ob $\left(\frac{dV}{dp}\right) = \frac{y}{Z}$ porro differentiando reperitur

$$\frac{1}{dx}d.\left(\frac{dV}{dp}\right) = \frac{p}{Z} - \frac{yp}{ZZ}\left(\frac{dZ}{dy}\right) - \frac{y}{ZZ}\left(\frac{dZ}{dx}\right) \text{ ob } \frac{dZ}{dx} = \left(\frac{dZ}{dx}\right) + p\left(\frac{dZ}{dy}\right).$$

Cum itaque fieri oporteat

$$\left(\frac{dV}{dy}\right) - \frac{1}{dx}d.\left(\frac{dV}{dp}\right) = 0$$

habebimus hanc aequationem pro definienda functione Z

$$\frac{M}{Z} - \frac{(My+N)}{ZZ}\left(\frac{dZ}{dy}\right) + \frac{y}{ZZ}\left(\frac{dZ}{dx}\right) = 0$$

seu $MZ - My\left(\frac{dZ}{dy}\right) - N\left(\frac{dZ}{dy}\right) + y\left(\frac{dZ}{dx}\right) = 0.$

XII.

XII. Si loco diuiforis Z fumatur poteftas quaecunque Z^n, vt integrabilis reddi debeat haec forma:

$$\frac{y\,dy + My\,dx + N\,dx}{Z^n}$$

functionem Z ex hac aequatione definiri oportebit

$$MZ - n(My + N)\left(\frac{dZ}{dy}\right) + ny\left(\frac{dZ}{dx}\right) = 0$$

vnde viciffim inueftigatio ita inftitui poterit, vt fumta pro lubitu forma functionis Z, inde indoles quantitatum M et N, quae per folam variabilem x determinantur, quaeratur, vt aequatio propofita hoc modo integrabilis reddi queat. Quamobrem his veftigiis infiftens fequentes cafus euoluam, vbi quidem litteris P, Q, R etc. functiones folius variabilis x indicari moneo.

Cafus I.

Quo integrabilis reddi debet haec forma:

$$\frac{y\,dy + My\,dx + N\,dx}{(y + P)^n}.$$

XIII. Cum ergo fit $Z = y + P$ erit

$$\left(\frac{dZ}{dy}\right) = 1 \text{ et } \left(\frac{dZ}{dx}\right) = \frac{dP}{dx}$$

vnde §. praec. hanc fuppeditat aequationem:

$$0 = My + MP - nMy - nN + \frac{ny\,dP}{dx}$$

quoniam igitur M, N et P funt functiones folius x, feorfim effe debet:

1°. $n\,dP = (n-1)M\,dx$ et 2°. $nN\,dx = MP\,dx$.

Quare

Quare pro lubitu ſumta functione P habebimus

$$M\,dx = \frac{n}{n-1}\,dP \quad \text{et} \quad N\,dx = \frac{1}{n-1}\,P\,dP$$

vnde diſcimus hanc aequationem:

$$y\,dy + \frac{n}{n-1}\,y\,dP + \frac{1}{n-1}\,P\,dP = 0$$

integrabilem reddi, ſi diuidatur per formam $(y+P)^n$.

XIV. Haec autem aequatio nullam plane habet difficultatem, quoniam eſt homogenea, atque adeo per hanc formam $(n-1)yy + nyP + PP$ diuiſa integrabilis euadat, ex quo diuiſore, quia conſtat factoribus $(y+P)((n-1)y+P)$ deducitur aequatio:

$$\frac{1}{n-2}\cdot\frac{dy+dP}{y+P} - \frac{1}{(n-1)(n-2)}\cdot\frac{(n-1)dy+dP}{(n-1)y+P} = 0$$

cuius integralis manifeſto eſt

$$(y+P)^{n-1} = A((n-1)y+P)$$

quae etiam ex illo diuiſore concluditur. Tantum obſeruo caſu $n=2$, quo haec forma fit incongrua, integrale fore

$$l(y+P) - \frac{y}{y+P} = \text{Conſt.}$$

quippe cuius differentiatio praebet

$$\frac{y\,dy + 2y\,dP + P\,dP}{(y+P)^2} = 0.$$

XV. Singularis hic ſe obtulit caſus, quo aequatio:

$$(y+P)^{n-1} = A((n-1)y+P),$$

quae ob conſtantem arbitrariam A eſt indefinita, ſumto $n=2$ hac indole penitus priuatur. Vt autem tum eius vera forma eliciatur, ſtatuatur more ſolito $n = 2 + \omega$, denotante ω fractionem infinite parvam, vt fit

$$(y + P)^\omega = 1 + \omega\, l(y + P);$$

fic illa aequatio hanc induet formam:

$$y + P + \omega(y + P)\, l(y + P) = A(y + P) + A\omega y$$

fit nunc $A = 1 + B\omega$ ac prodibit

$$(y + P)\, l(y + P) = y + B(y + P)$$

ficque loco conftantis A alia arbitraria B eft introducta.

XVI. Vt aequationem inuentam elegantiorem reddamus, ponamus $P = x^{n-1}$, vt prodeat

$$y\,dy + n\,x^{n-2}\,y\,dx + x^{2n-3}\,dx = 0$$

quam ergo integrabilem fieri nouimus, fi diuidatur per $(y + x^{n-1})^n$, et integrale eius completum erit

$$(y + x^{n-1})^{n-1} = A((n-1)y + x^{n-1})$$

dum obferuetur cafu $n = 2$ integrale effe

$$l(y + x^{n-1}) - \frac{y}{y + x^{n-1}} = \text{Conft.} = l(y + x) - \frac{y}{y + x}.$$

Hic vero nouus cafus fingularis occurrit $n = 1$, aequationem praebens

$$y\,dy + \frac{y\,dx}{x} + \frac{dx}{x} = 0,$$

quae per $y + 1$ diuifa integrale dat $y + l\frac{x}{y + 1} = \text{Conft.}$ Quod vt ex forma illa generali eruatur, pofito $n = 1 + \omega$, erit

$$(y + 1 + \omega l x)^\omega = 1 + \omega l(y + 1) = A(\omega y + 1 + \omega l x)$$

fit

fit ergo $A = 1 + B\,\omega$ prodibitque

$$l(y+1) = y + lx + B \text{ feu } y + l\frac{x}{y+1} = \text{Conft. vt ante.}$$

Cafus II.

Quo integrabilis reddenda eft haec forma

$$\frac{y\,dy + My\,dx + N\,dx}{(yy + Py + Q)^n}.$$

XVII. Quia hic eft $Z = yy + Py + Q$ pro §. 12. habebimus:

$$\left(\frac{dZ}{dy}\right) = 2y + P \text{ et } \left(\frac{dZ}{dx}\right) = \frac{y\,dP}{dx} + \frac{dQ}{dx}$$

hincque iftam aequationem refoluendam

$$\begin{array}{rlll} 0 = & Myy & + MPy & + MQ \\ & - 2nMP & - nMP & - nNP \\ & + \frac{n\,dP}{dx} & + 2nN & \\ & & + \frac{n\,dQ}{dx}, & \end{array}$$

vnde refultant hae tres:

$$n\,dP = (2n-1)M\,dx \quad \text{feu} \quad M\,dx = \frac{n\,dP}{2n-1}$$

$$n\,dQ = (n-1)MP\,dx + 2nN\,dx$$

$$nNP = MQ, \quad \text{feu} \quad N\,dx = \frac{MQ\,dx}{nP} = \frac{Q\,dP}{(2n-1)P},$$

Ergo $n\,dQ = \frac{n(n-1)P\,dP}{2n-1} + \frac{2nQ\,dP}{(2n-1)P}$

feu $dQ - \frac{2Q\,dP}{(2n-1)P} = \frac{n-1}{2n-1}P\,dP$

quae aequatio per $P^{\frac{-2}{2n-1}}$ multiplicata et integrata dat

$$P^{\frac{-2}{2n-1}} Q = \tfrac{1}{4} P^{\frac{4n-4}{2n-1}} + A,$$ hincque fit

$$Q = \tfrac{1}{4} PP + A P^{\frac{2}{2n-1}}$$ et per P resolutio erit

$$M\,dx = \frac{n\,dP}{2n-1}, \text{ et } N\,dx = \frac{dP}{2n-1}\left(\tfrac{1}{4} P + A P^{\frac{3-2n}{2n-1}}\right).$$

XVIII. Sit $P = 2x^{2n-1}$; vt fiat
$Q = x^{4n-2} + Bxx$; $M\,dx = 2nx^{2n-2}dx$ et $N\,dx = x^{4n-3}dx + Bx\,dx$
atque hinc intelligimus hanc aequationem

$$y\,dy + 2nx^{2n-2}y\,dx + x^{4n-3}dx + Bx\,dx = 0$$

integrabilem reddi, si diuidatur per

$$(yy + 2x^{2n-1}y + x^{4n-2} + Bxx)^n.$$

Euidens est hanc aequationem ad simpliciorem formam perduci ponendo $y = z - x^{2n-1}$, tum enim prodit

$$z\,dz + Bx\,dx + x^{2n-2}(z\,dx - x\,dz) = 0$$

quae per $(zz + Bxx)^n$ diuisa vtique fit integrabilis integrali existente:

$$\frac{-1}{2(n-1)(zz+Bxx)^{n-1}} + \int \frac{x^{2n-2}(z\,dx - x\,dz)}{(zz+Bxx)^n} = 0$$

cuius posterius membrum posito $z = vx$ abit in

$$-\int \frac{dv}{(vv+B)^n},$$

ita vt nulla supersit difficultas.

Casus III.

Quo integrabilis reddenda est haec forma

$$\frac{y\,dy + M\,y\,dx + N\,dx}{(y^3 + P\,y^2 + Q\,y + R)^n}.$$

XIX. Ob $Z = y^3 + P\,y^2 + Q\,y + R$ erit $\left(\frac{dZ}{dy}\right) = 3yy + 2Py + Q$ et $\left(\frac{dZ}{dx}\right) = \frac{y^2 dP + y\,dQ + dR}{dx}$ ex quo sequenti aequationi est satisfaciendum

$$\begin{aligned} 0 = M\,y^3 &+ M\,P\,y^2 + M\,Q\,y + M\,R \\ - 3nM &- 2n\,M\,P - n\,M\,Q - n\,N\,Q \\ + \frac{n\,dP}{dx} &- 3\,n\,N - 2n\,N\,P \\ &+ \frac{n\,dQ}{dx} + \frac{n\,dR}{dx} \end{aligned}$$

quae suppeditat has quatuor determinationes:

1°. $n\,dP = (3\,n - 1)\,M\,dx$

2°. $n\,dQ = (2\,n - 1)\,M\,P\,dx + 3\,n\,N\,dx$

3°. $n\,dR = (n - 1)\,M\,Q\,dx + 2\,n\,N\,P\,dx$

4°. $n\,N\,Q\,dx = M\,R\,dx.$

XX. Ex vltima colligimus $M : N = n\,Q : R$, vnde ex prioribus litteras M et N elidendo obtinemus:

$$(3n - 1)\,Q\,dQ = (2n - 1)\,P\,Q\,dP + 3\,R\,dP \text{ et}$$
$$(3\,n - 1)\,Q\,dR = (n - 1)\,Q\,Q\,dP + 2\,P\,R\,dP$$

hincque $(2n - 1)PQdR + 3RdR = (n - 1)QQdQ + 2PRdQ$. Vnde si quantitates Q et R per P definire licuerit, tum erit

$$M\,dx = \frac{n}{3\,n - 1}\,dP \text{ et } N\,dx = \frac{R\,dP}{(3\,n - 1)\,Q}.$$

Primum autem obferuo illis aequationibus fatisfieri poffe ponendo

$$Q = \alpha P^2 \quad \text{et} \quad R = \beta P^3,$$

hosque coefficientes α et β duplicem determinationem fortiri, fcilicet

vel $\alpha = \frac{1}{3}$ et $\beta = \frac{1}{27}$

vel $\alpha = \frac{2n-1}{(3n-1)^2}$ et $\beta = \frac{-(n-1)(2n-1)^2}{(3n-1)^3}$

quo quidem cafu aequatio noftra fit homogenea.

XXI. Confideremus primo cafum quo $n = 1$, quippe quem iam fupra aliunde elicuimus; eruntque noftrae aequationes:

$$1^\circ.\ 2QdQ = PQdP + 3RdP \quad \text{et} \quad 2^\circ.\ QdR = PRdP$$

hincque $3^\circ.\ PQdR + 3RdR = 2PRdQ$.

Iam ex 2°. fit $Q = \frac{PRdP}{dR}$, fumtoque dR conftante

$$dQ = \frac{PRddP}{dR} + PdP + \frac{RdP^2}{dR},$$

vnde prima abit in hanc:

$$\frac{2PPRRdPddP}{dR^2} + \frac{2PPRdP^2}{dR} + \frac{2PRRdP^3}{dR^2} - \frac{PPRdP^2}{dR} = 3RdP$$

feu $\frac{2PPRRddP + 2PRRdP^2}{dR} + PPRdP = 3RdR$

diuidatur per $PR\sqrt{R}$ vt prodeat

$$\frac{2PddP\sqrt{R}}{dR} + \frac{2dP^2\sqrt{R}}{dR} + \frac{PdP}{\sqrt{R}} = \frac{3dR}{P\sqrt{R}}$$

cuius prius membrum integratum dat

$$\frac{2PdP\sqrt{R}}{dR} = 3\int\frac{dR}{P\sqrt{R}} \quad \text{hincque} \quad dP = \tfrac{3}{2}\cdot\frac{dR}{P\sqrt{R}}\int\frac{dR}{P\sqrt{R}}$$

quae quidem forma parum lucri attuliffe videtur.

XXII.

XXII. Ponamus autem $\int \frac{dR}{P\sqrt{R}} = u$, vt fit $P = \frac{dR}{du\sqrt{R}}$ ac postrema aequatio $dP = \frac{3}{2}u\,du$ dabit $P = \frac{3}{4}uu + A$ vnde fit $\frac{dR}{\sqrt{R}} = \frac{3}{4}uu\,du + A\,du$ hincque integrando

$$2\sqrt{R} = \frac{1}{4}u^3 + Au + 2B \text{ et } R = (\tfrac{1}{8}u^3 + \tfrac{1}{2}Au + B)^2$$

et ob $Q = \frac{PR\,dP}{dR} = \frac{P\,dP\sqrt{R}}{P\,du} = \frac{3}{2}u\sqrt{R}$ erit

$$Q = \tfrac{3}{2}u(\tfrac{1}{8}u^3 + \tfrac{1}{2}Au + B) \text{ ac denique}$$

$$M\,dx = \tfrac{1}{2}dP = \tfrac{3}{4}u\,du \text{ et}$$

$$N\,dx = \tfrac{1}{2}dP.\tfrac{R}{Q} = \tfrac{1}{2}\tfrac{dR}{P} = \tfrac{1}{2}du\sqrt{R} = \tfrac{1}{2}du(\tfrac{1}{8}u^3 + \tfrac{1}{2}Au + B)$$

statuamus nunc $u = 2x + 2f$ vt fiat

$$P = 3xx + 6fx + 3ff + A;$$

$$Q = 3(x+f)\left(x^3 + 3fxx + 3ffx + f^3 + Ax + Af + B\right)$$

$$R = \left(x^3 + 3fxx + (3ff + A)x + f^3 + Af + B\right)^2$$

$$M\,dx = 3dx(x+f) \text{ et } N\,dx = dx\left(x^3 + 3fxx + 3ffx + f^3 + Ax + Af + B\right).$$

XXIII. Ponamus porro:

$$3f = a,\ 3ff + A = b \text{ et } f^3 + Af + B = c,$$

atque obtinebimus hanc aequationem

$$y\,dy + (3x + a)y\,dx + (x^3 + axx + bx + c)\,dx = 0$$

quae ergo integrabilis redditur, si diuidatur per

$$y^3 + (3xx + 2ax + b)yy + (3x + a)(x^3 + axx + bx + c)y + (x^3 + axx + bx + c)^2$$ sicque

ſicque diuiſoris forma hoc caſu aſſumta ad eam ipſam integrationem nos deduxit, quam iam ſupra §. IX. ſumus adepti. Circa hunc ergo diuiſorem annotaſſe iuuabit quod ſupra iam vidimus, ſi ponatur

$$x^3+axx+bx+c=(x+\alpha)(x+\beta)(x+\gamma)$$

vt ſit

$$a=\alpha+\beta+\gamma;\ b=\alpha\beta+\alpha\gamma+\beta\gamma;\ \text{et}\ c=\alpha\beta\gamma,$$

diuiſorem noſtrum ſic in ternos factores reſolutum exhiberi poſſe

$$(y+(x+\alpha)(x+\beta))(y+(x+\alpha)(x+\gamma))(y+(x+\beta)(x+\gamma))$$

et aequationis noſtrae integrale completum fore

$$(y+(x+\alpha)(x+\beta))^{\alpha-\beta}(y+(x+\beta)(x+\gamma))^{\beta-\gamma}(y+(x+\gamma)(x+\alpha))^{\gamma-\alpha}=\text{Conſt.}$$

XXIV. Circa aequationem differentio-differentialem §. 21. quae per R diuiſa eſt:

$$\frac{2PPRddP+2PRdP^2+PPdPdR}{dR}=3dR$$

obſeruo, eam multo facilius integrari poſſe, ſi modo per $\frac{dP}{dR}$ multiplicetur, vt habeatur

$$\frac{2PPRdPddP+2PRdP^3+PPdP^2dR}{dR^2}=3dP$$

cuius integrale ſtatim eſt

$$\frac{PPRdP^2}{dR^2}=3P+A \text{ hincque } \frac{dR}{\sqrt{R}}=\frac{PdP}{\sqrt{(3P+A)}}$$

cuius integrale denuo eſt

$$2\sqrt{R}=2\left(\frac{P}{9}-\frac{2A}{27}\right)\sqrt{(3P+A)}+B.$$

XXV. Pro poſitione $n=1$ calculum in genere expedire licuit, pro aliis autem valoribus ipſius n negotium minus ſuccedit, excepto vnico caſu $n=\frac{1}{3}$,

quo

quo aequatio prima §. 19. ſtatim dat $dP = 0$ ideoque $P = a$; vnde inter Q et R haec prodit aequatio

$$-\tfrac{1}{3}aQdR + 3RdR = -\tfrac{2}{3}QQdQ + 2aRdQ$$

ſeu $9RdR + 2QQdQ = 6aRdQ + aQdR$

quam autem euoluere non licet, niſi ſumatur $a = 0$, tum autem oritur

$$\tfrac{9}{2}RR + \tfrac{2}{3}Q^3 = \text{Conſt. ſeu } R = \sqrt{}(A - \tfrac{4}{27}Q^3).$$

Deinde

$$N\,dx = \tfrac{1}{3}\,dQ \quad \text{et} \quad M\,dx = \tfrac{1}{9}\,dQ.\tfrac{Q}{R}.$$

Statuamus

$Q = 3x$, vt fiat $R = \sqrt{}(A - 4x^3)$; $Ndx = dx$ et $Mdx = \frac{x\,dx}{\sqrt{}(A-4x^3)}$

vnde deducimus hanc aequationem differentialem:

$$y\,dy + \frac{x\,y\,dx}{\sqrt{}(A - 4x^3)} + dx = 0$$

quam nunc nouimus integrabilem reddi, ſi diuidatur per

$$\sqrt[3]{}(y^3 + 3xy + \sqrt{}(A - 4x^3)).$$

Hinc autem ipſum integrale neutiquam explicite exhiberi poteſt; quin etiam aequatio iſta ita eſt comparata, vt nulla alia via ad conſtructionem perduci queat.

Caſus IV.

Quo integrabilis reddenda eſt haec forma

$$\frac{y\,dy + M\,y\,dx + N\,dx}{(y^4 + Py^3 + Qy^2 + Ry + S)^n}.$$

XXVI. Pro hoc casu methodus nostra sequentes suppeditat aequationes:

$1^\circ.\ n\,dP = (4n-1)\,M\,dx$

$2^\circ.\ n\,dQ = (3n-1)\,MP\,dx + 4nN\,dx$

$3^\circ.\ n\,dR = (2n-1)\,MQ\,dx + 3nNP\,dx$

$4^\circ.\ n\,dS = (n-1)\,MR\,dx + 2nNQ\,dx$

$5^\circ.\quad MS = nNR$ seu $M:N = nR:S$

vnde deriuantur istae

$$(4n-1)R\,dQ = (3n-1)PR\,dP + 4S\,dP$$
$$(4n-1)R\,dR = (2n-1)QR\,dP + 3PS\,dP$$
$$(4n-1)R\,dS = (n-1)\,RR\,dP + 2QS\,dP,$$

ex binis postremis elidendo Q oritur:

$$2(4n-1)RS\,dR - (2n-1)(4n-1)RR\,dS = 6PSS\,dP - (n-1)(2n-1)R^3\,dP.$$

XXVII. Quoniam hic in genere vix quicquam concludere licet, praeter casum homogeneitatis, quo fieri potest

$$Q = \alpha P^2;\ R = \beta P^3;\ \text{ et } S = \gamma P^4$$

consideremus casum $n = 1$, vt diuisor fit

$$y^4 + Py^3 + Qy^2 + Ry + S.$$

Habebimus ergo $M\,dx = \frac{1}{3}\,dP : N\,dx = \frac{S\,dP}{2R} = \frac{dS}{2Q}$ et

$1^\circ.\ 3R\,dQ = 2PR\,dP + 4S\,dP$

$2^\circ.\ 3R\,dR = QR\,dP + 3PS\,dP$

$3^\circ.\ 3R\,dS = 2QS\,dP$

vnde eliminato Q ex duabus postremis fit

$$2RS\,dR - RR\,dS = 2PSS\,dP$$

quae

quae per S S diuifa commode integrationem admittit praebetque

$$\frac{RR}{S} = PP + A \quad \text{feu} \quad S = \frac{RR}{PP + A}$$

quare ex fecunda elicitur

$$Q = \frac{3\,dR}{dP} - \frac{3PS}{R} = \frac{3\,dR}{dP} - \frac{3PR}{PP + A}.$$

XXVIII. Prima vero dat:

$$dQ = \tfrac{2}{3} P\,dP + \frac{4S\,dP}{3R} = \tfrac{2}{3} P\,dP + \frac{4R\,dP}{3(PP + A)}$$

inde vero fumendo dP conftans reperitur:

$$dQ = \frac{3\,ddR}{dP} - \frac{3P\,dR}{PP + A} + \frac{3PPR\,dP - 3AR\,dP}{(PP + A)^2}$$

ficque oritur haec aequatio differentialis fecundi gradus

$$\frac{ddR}{dP} - \frac{P\,dR}{PP + A} + \frac{5PPR\,dP - 13AR\,dP}{9(PP + A)^2} - \tfrac{2}{9} P\,dP = 0$$

quam dubito in genere refolui poffe.

XXIX. Confiderabo ergo cafum quo $A = 0$, ideoque

$$S = \frac{RR}{PP} \quad \text{et} \quad Q = \frac{3\,dR}{dP} - \frac{3R}{P};$$

ita vt haec refoluenda fit aequatio:

$$\frac{ddR}{dP} - \frac{dR}{P} + \frac{5R\,dP}{9PP} - \tfrac{2}{9} P\,dP = 0.$$

Statuamus ergo $R = \alpha P^3 + u$ fietque

$$\frac{ddu}{dP} - \frac{du}{P} + \frac{5u\,dP}{9PP} + 6\alpha P\,dP - 3\alpha P dP + \tfrac{5}{9}\alpha P dP - \tfrac{2}{9} P dP = 0$$

et fumto $\alpha = \frac{1}{16}$ erit

$$\frac{ddu}{dP} - \frac{du}{P} + \frac{5u\,dP}{9PP} = 0$$

pro qua porro $u = P^{\lambda}$ et ex aequalitate $\lambda\lambda - 2\lambda + \frac{5}{9} = 0$ colligo $\lambda = 1 \pm \frac{2}{3}$, hincque integrale completum

$$u = \alpha P^{\frac{5}{3}} + \beta P^{\frac{1}{3}}.$$

 Quo-

Quocirca habebimus:

$$R = \tfrac{1}{16} P^3 + \alpha P^{\frac{5}{3}} + \beta P^{\frac{1}{3}}$$

$$Q = \tfrac{3}{8} P^2 + 2 \alpha P^{\frac{2}{3}} - 2 \beta P^{-\frac{2}{3}}$$

$$S = (\tfrac{1}{16} P^2 + \alpha P^{\frac{2}{3}} + \beta P^{-\frac{2}{3}})^2$$

ac tandem $M\,dx = \tfrac{1}{3} dP$,

et $N\,dx = \frac{dP}{3P} (\tfrac{1}{16} P^2 + \alpha P^{\frac{2}{3}} + \beta P^{-\frac{2}{3}})$.

XXX. Statuamus nunc $P = t^3$, vt fublata irrationalitate fiat:

$$M\,dx = tt\,dt \text{ et } N\,dx = \frac{dt}{t} (\tfrac{1}{16} t^2 + \alpha t^6 + \tfrac{\beta}{tt})$$

et aequatio noftra huius fit formae:

$$y\,dy + ytt\,dt + \frac{dt}{t^3} (\tfrac{1}{16} t^8 + \alpha t^4 + \beta) = 0.$$

quam iam nouimus integrabilem reddi fi diuidatur per

$$y^4 + t^3 y^3 + (\tfrac{3}{8} t^6 + 2\alpha tt - \tfrac{2\beta}{tt}) y^2 + (\tfrac{1}{16} t^9 + \alpha t^5 + \beta t) y + (\tfrac{1}{16} t^6 + \alpha t^2 + \tfrac{\beta}{tt})^2.$$

Hic diuifor duobus conftat factoribus:

$$(yy + (\tfrac{1}{2} t^3 + \tfrac{2}{t} \sqrt{-\beta}) + \tfrac{1}{16} t^6 + \alpha tt + \tfrac{\beta}{tt}) (yy + (\tfrac{1}{2} t^3 \sqrt{-\beta}) y + \tfrac{1}{16} t^6 + \alpha tt + \tfrac{\beta}{tt})$$

fi diuifores magis complicatos adhibere vellemus, vix quicquam ad vfum inde concludere liceret.

CONSI-