# CONSIDERATIO AEQVATIONIS DIFFERENTIO-DIFFERENTIALIS

$$(a+bx)ddz+(c+ex)\frac{dxdz}{x}+(f+gx)\frac{zdx^2}{xx}=0.$$

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## I.

Primo haec aequatio ad formam differentialem simplicem reuocari potest ponendo $lz=\int v\,dx$, vt fit

$$\frac{dz}{z}=v\,dx \quad \text{et} \quad \frac{ddz}{z}-\frac{dz^2}{zz}=dx\,dv,$$

ideoque

$$\frac{ddz}{z}=dx\,dv+vv\,dx^2.$$

Diuisa enim illa aequatione per $z\,dx$ hinc orietur:

$$(a+bx)dv+(a+bx)vv\,dx+(c+ex)\frac{v\,dx}{x}+(f+gx)\frac{dx}{xx}=0$$

cuius integratio si pateret, foret pro proposita $lz=\int v\,dx$.

II. Hinc duplici modo terminus simplici quantitate $v$ affectus elidi potest. Pro altero ponamus $v=u+X$ denotante $X$ functionem ipsius $x$ mox determinandam; et facta substitutione fiet

$$\left.\begin{aligned}(a+bx)du+(a+bx)uu\,dx+(c+ex)\frac{u\,dx}{x} & +(f+gx)\frac{dx}{xx}\\ +2(a+bx)uX\,dx+(a+bx)dX & \\ +(a+bx)XX\,dx & \\ +(c+ex)\frac{X\,dx}{x} & \end{aligned}\right\}=0.$$

 Iam

Iam ſtatuatur $X = \frac{-(c+ex)}{2x(a+bx)}$, vt ſit

$$dX = \frac{(ac+2bcx+bcxx)dx}{2xx(a+bx)^2}, \text{ prodibitque}$$

$$\left.\begin{aligned}(a+bx)du+(a+bx)uudx+(f+gx)\frac{dx}{xx}&+\frac{(ac+2bcx+bexx)dx}{2xx(a+bx)}\\ &-\frac{(cc+2cex+eexx)dx}{4xx(a+bx)}\end{aligned}\right\}=0$$

ſeu hoc modo:

$$du+uudx+\frac{4(a+bx)(f+gx)+2(ac+2bcx+bexx)-(c+ex)^2}{4xx(a+bx)^2}dx=0.$$

III. Per alteram methodum cum priore coniunctam ponatur $v = Pu + X$, exiſtentibus P et X functionibus ipſius $x$, et ſubſtitutione facta obtinebitur:

$$\left.\begin{aligned}(a+bx)Pdu+(a+bx)PPuudx+(c+ex)\frac{Pudx}{x}&+\frac{(f+gx)dx}{xx}\\ +(a+bx)udP&+(a+bx)dX\\ +2(a+bx)PXudx&+(a+bx)XXdx\\ &+(c+ex)\frac{Xdx}{x}\end{aligned}\right\}=0$$

vnde fieri debet

$$\frac{(c+ex)Pdx}{x}+(a+bx)dP+2(a+bx)PXdx=0.$$

Introduxi autem hic binas functiones P et X, quo inueſtigatio latius pateat; vulgo enim hac altera methodo vtentes ponere ſolemus $v=Pu$, vt ſit $X=0$, quo caſu erit

$$\frac{dP}{P}+\frac{(c+ex)dx}{x(a+bx)}=0 \text{ ſeu } \frac{dP}{P}+\frac{c}{a}\cdot\frac{dx}{x}+\frac{ae-bc}{a}\cdot\frac{dx}{a+bx}=0$$

vnde integrando colligitur:

$$Px^{\frac{c}{a}}(a+bx)^{\frac{ae-bc}{ab}}=C \text{ et } P=Cx^{\frac{-c}{a}}(a+bx)^{\frac{c}{a}-\frac{e}{b}}$$

et aequatio noſtra differentialis primi gradus erit

C $x$

$$Cx^{-\frac{c}{a}}(a+bx)^{\frac{c}{a}-\frac{e}{b}+1}du+CCx^{-\frac{2c}{a}}(a+bx)^{\frac{2c}{a}-\frac{2e}{b}+1}uudx + \frac{(f+gx)\,dx}{xx}=0$$

fiue hoc modo

$$Cdu+CCx^{-\frac{c}{a}}(a+bx)^{\frac{c}{a}-\frac{e}{b}}uudx+x^{\frac{c}{a}}(a+bx)^{\frac{e}{b}-\frac{c}{a}-1}(f+gx)\frac{dx}{xx}=0.$$

IV. Sin autem in genere ponatur $v=Pu+X$ ftatuaturque

$$\frac{dP}{P}+\frac{(c+ex)\,dx}{x(a+bx)}+2\,X\,dx=0,$$

aequatio noftra differentialis hanc induit formam:

$$Pdu+PPuudx+dX+XXdx+\frac{(c+ex)X\,dx}{x(a+bx)}+\frac{(f+gx)\,dx}{xx(a+bx)}=0$$

in qua vel P vel X pro lubitu accipi poteft, vnde altera definietur. Veluti fi capiatur $P=\alpha\,x^n$, fiet

$$X=\frac{-n}{2x}-\frac{c-ex}{2x(a+bx)}=\frac{-na-c-(nb+e)x}{2x(a+bx)}.$$

Ex his formis cafus, quibus aequatio fit integrabilis, elicere licet, quos autem facilius ex ipfa aequatione propofita cognofcere poterimus.

V. Quodfi quaerere velimus cafus, quibus aequatio propofita integrationem admittit, in quo quidem omnis opera collocanda videtur, quamdiu integrationem in genere inftituere non licet, primum quidem ftatim fe offert forma $z=A\,x^m\,(a+bx)^n$ quae vt fatisfaciat, definiri oportet relationem quantitatum conftantium $a$, $b$, $c$, $e$, $f$, $g$. Cum igitur fit

$$\frac{dz}{z}=\frac{m\,dx}{x}+\frac{nb\,dx}{a+bx}\text{ et }\frac{ddz}{z}-\frac{dz^2}{zz}=\frac{-m\,dx^2}{xx}-\frac{nbb\,dx^2}{(a+bx)^2},$$

erit

erit

$$\frac{ddz}{z} = \frac{m(m-1)dx^2}{xx} + \frac{2mnb\,dx^2}{x(a+bx)} + \frac{n(n-1)bb\,dx^2}{(a+bx)^2},$$

hincque nascitur facta diuisione per $dx^2$ haec aequatio

$$\left.\begin{array}{l} \frac{m(m-1)(a+bx)}{xx} + \frac{2mnb}{x} + \frac{n(n-1)bb}{a+bx} \\ + \frac{f+gx}{xx} \qquad + \frac{m(c+ex)}{xx} + \frac{nb(c+ex)}{x(a+bx)} \end{array}\right\} = 0$$

quae vt subsistere possit, bini postremi termini collecti

$$\frac{nb((n-1)bx+c+ex)}{x(a+bx)}$$

denominatorem $a+bx$ amittere debent ex quo fit

$$a : c = b : (n-1)b + e, \text{ seu } n - 1 = \frac{c}{a} - \frac{e}{b},$$

et $n = 1 + \frac{c}{a} - \frac{e}{b}$,

ita vt habeatur haec aequatio

$$\frac{m(m-1)a + mc + f}{xx} + \frac{m(m-1)b + 2mnb + me + g + nbc : a}{x} = 0$$

vnde hae duae nascuntur aequationes loco $n$ valorem inuentum scribendo:

$$m(m-1)a + mc + f = 0 \text{ et}$$
$$m(m+1)b + \frac{(2m+1)bc}{a} - me - \frac{cc}{a} + \frac{bcc}{aa} + g = 0.$$

Multiplicetur haec per $a$ et illa per $-b$ fiet summa:

$$2mab + mbc - mae + bc - ce + \frac{bcc}{a} + ag - bf = 0$$

hincque $m = \frac{abf - aag - abc + ace - bcc}{2aab + abc - aae}$ et

$$(m-1)a + c = \frac{abf - aag - 2aab + aae}{2ab + bc - ae}$$

qui valores in prima substituti dant

$$\begin{array}{l} (bf-ag)^2 + f(2abb + 3bbc - 3abe - bce + aee) + c(2b-e)(ab-ac+bc) = 0 \\ \qquad + g(2aab - aae + abc - ace + bcc) \end{array}$$

quae

quae resoluta praebet,

$$ag = bf - \tfrac{1}{2}(2ab - ae + bc) + \tfrac{c}{2a}(ae - bc) \pm (2ab - ae + bc)\sqrt{\left(\tfrac{(a-c)^2}{4aa} - \tfrac{f}{a}\right)}$$

siue

$$g = \frac{c(ae - bc) + 2abf - (2ab - ae + bc)(a \pm \sqrt{((a-c)^2 - 4af)})}{2aa}.$$

Quare si littera $g$ hunc habeat valorem, aequatio nostra integrale habebit $z = Ax^m(a+bx)^n$ existente

$$m = \frac{a - c \pm \sqrt{((a-c)^2 - 4af)}}{2a} \text{ et } n = 1 + \frac{c}{a} - \frac{e}{b}.$$

VI. Alia via casus integrabiles reperiuntur, si valor ipsius $z$ in seriem conuertatur, quae si alicubi abrumpatur, expressionem finitam pro $z$ exhibet. Fingatur ergo:

$$z = Ax^n + Bx^{n+1} + Cx^{n+2} + Dx^{n+3} + Ex^{n+4} + \text{etc.}$$

et facta substitutione consequemur:

$$\begin{array}{lllll}
n(n-1)Aax^{n-2} & +(n+1)nBax^{n-1} & +(n+2)(n+1)Cax^{n} & +(n+3)(n+2)Dax^{n+1} & \text{etc.} \\
 & +n(n-1)Ab & +(n+1)nBb & +(n+2)(n+1)Cb & \\
nAc & +(n+1)Bc & +(n+2)Cc & +(n+3)Dc & \\
 & +nAe & +(n+1)Be & +(n+2)Ce & \\
Af & +Bf & +Cf & +Df & \\
 & +Ag & +Bg & +Dg &
\end{array}$$

quos singulos terminos ad nihilum reduci oportet. Primo ergo erit $n(n-1)a + nc + f = 0$ hincque

$$n = \frac{a - c \pm \sqrt{((a-c)^2 - 4af)}}{2a} \text{ porro vero}$$

$$B = \frac{-n(n-1)b - ne - g}{(n+1)na + (n+1)c + f}A = \frac{-n(n-1)b - ne - g}{2na + c}A$$

$$C = \frac{-(n+1)nb - (n+1)e - g}{(n+2)(n+1)a + (n+2)c + f}B = \frac{-(n+1)nb - (n+1)e - g}{2((2n+1)a + c)}B$$

$$D = \frac{-(n+2)(n+1)b - (n+2)e - g}{(n+3)(n+2)a + (n+3)c + f}C = \frac{-(n+2)(n+1)b - (n+2)e - g}{3((2n+2)a + c)}C$$

etc.

 Haec

Haec ergo ſeries alicubi abrumpitur, ſi ſumto pro $i$ numero quocunque integro poſitiuo, quo etiam cyphra referatur, fuerit

$$g = -(n+i)(n+i-1)b - (n+i)e.$$

Cum autem ſit $n+i = \frac{(2i+1)a - c \pm \sqrt{((a-c)^2 - 4af)}}{2a}$

et $n+i-1 = \frac{(2i-1)a - c \pm \sqrt{((a-c)^2 - 4af)}}{2a}$ erit

$$g = \frac{-((2i+1)a - c \pm \sqrt{((a-c)^2 - 4af)})((2i-1)ab - bc + 2ae \pm b\sqrt{((a-c)^2 - 4af)})}{4aa}$$

et euoluendo

$$g = \frac{2abf + c(ae-bc) - a(2iiab + (2i+1)(ae-bc)) \mp (2iab + ae - bc)\sqrt{((a-c)^2 - 4af)}}{2aa}$$

ſi ergo eſſet $i = -1$, quod autem hic ſumere non licet, caſus praecedens emergeret. Hinc igitur innumerabiles alii caſus ſimiles eruuntur.

VII. Poſſumus etiam ſeriem, in qua exponentes ipſius $x$ decreſcant, aſſumere, hoc modo

$$z = Ax^n + Bx^{n-1} + Cx^{n-2} + Dx^{n-3} + Ex^{n-4} + \text{etc.}$$

qua ſubſtituta noſtra aequatio fit

$$\begin{array}{llll}
+n(n-1)Abx^{n-1} & +n(n-1)Aax^{n-2} & +(n-1)(n-2)Bax^{n-3} & +(n-2)(n-3)Cax^{n-4} + \text{etc.} \\
 & +(n-1)(n-2)Bb & +(n-2)(n-3)Cb & +(n-3)(n-4)Db \\
 & +\quad nAc & +(n-1)Bc & +(n-2)Cc \\
+nAe & +(n-1)Be & +(n-2)Ce & +(n-3)De \\
 & +\quad Af & +\quad Bf & +\quad Cf \\
+Ag & +\quad Bg & +\quad Cg & +\quad Dg
\end{array}$$

hincque eſſe debet $n(n-1)b + ne + g = 0$ ſeu

$$n = \frac{b - e \pm \sqrt{((b-e)^2 - 4bg)}}{2b}, \text{ vel } g = -nnb + nb - ne.$$

Praeterea vero:

B =

$$B = \frac{+n(n-1)a+nc+f}{(2n-2)b+e}A;\quad C = \frac{(n-1)(n-2)a+(n-1)c+f}{2((2n-3)b+e)}B$$

$$D = \frac{(n-2)(n-3)a+(n-2)c+f}{3((2n-4)b+e)}C;\quad D = \frac{(n-3)(n-4)a+(n-3)c+f}{4((2n-5)b+e)}C$$

etc.

Sit vt ante $i$ numerus integer positiuus, cyphra non exclusa, et integrale finitum obtinebitur, quoties fuerit

$$(n-1)(n-i-1)a+(n-i)c+f=0 \text{ vnde fit}$$

$$n = \frac{+(2i+1)a-c\pm\sqrt{((a-c)^2-4af)}}{2a}$$

vt inuenimus; $g=-n((n-1)b+e)$, ideoque

$$g = \frac{-(2i+1)a-c\pm\sqrt{((a-c)^2-4af)})((2i-1)ab-bc+2ae\pm b\sqrt{((a-c)^2-4af)})}{4aa}$$

quae euoluta praebet vt ante

$$g = \frac{2abf+c(ae-bc)-a(2iiab+(2i+1)(ae-bc))-(2iab+ae-bc)\sqrt{((a-c)^2-4af)}}{2aa}$$

ita vt hinc iidem casus ac ante prodeant, atque adeo eadem integralia ordine retrogrado scripta obtineantur.

VIII. Verum ante quam integrale per seriem inuestigemus, nostra aequatio transformari potest in aliam eiusdem formae ponendo

$$z=(a+bx)^m v, \text{ vnde fit } \frac{dz}{z}=\frac{dv}{v}+\frac{mb\,dx}{a+bx}$$

$$\text{et } \frac{ddz}{z}=\frac{ddv}{v}-\frac{mb^2dx^2}{(a+bx)^2}+\frac{2mb\,dx\,dv}{v(a+bx)}+\frac{mmbb\,dx^2}{(a+bx)^2}$$

factaque substitutione

$$\frac{(a+bx)ddv}{v}+\frac{2mb\,dx\,dv}{v}+\frac{m(m-1)bb\,dx^2}{a+bx}=0$$
$$+\frac{(c+ex)dxdv}{xv}+\frac{mb(c+ex)dx^2}{x(a+bx)}$$
$$+\frac{(f+gx)dx^2}{xx}$$

 fiat

fiat $(m-1)bx+c+ex$ diuiſibile per $a+bx$, eritque

$$(m-1)b+e=\frac{bc}{a} \quad \text{et} \quad m=1+\frac{c}{a}-\frac{e}{b},$$

noſtraque aequatio

$$(a+bx)ddv+(c+(\frac{2bc}{a}+2b-ex)\frac{dxdv}{x}+(f+(g+\frac{bc}{a}+\frac{bcc}{aa}-\frac{ce}{a})x)\frac{vdx^2}{xx}=0.$$

Ponatur breuitatis gratia $\frac{2bc}{a}+2b-e=\varepsilon$ et

$$g+\frac{bc}{a}+\frac{bcc}{aa}-\frac{ce}{a}=\eta,$$

vt habeatur forma propoſitae ſimilis

$$(a+bx)ddv+(c+\varepsilon x)\frac{dxdv}{x}+(f+\eta x)\frac{vdx^2}{xx}=0$$

quae ergo eſt integrabilis, ſi fuerit

$$\eta=\frac{2abf+c(a\varepsilon-bc)-a(2iiab+(2i+1)(a\varepsilon-bc))\mp(2iab+a\varepsilon-bc)\sqrt{((a-c)^2-4af)}}{2aa}$$

at eſt $a\varepsilon-bc=2ab-ae+bc$, vnde habetur

$$\eta=\frac{2abf+c(2ab-ae+bc)-a(2(i+1)^2ab-(2i+1)(ae-bc))\mp(2(i+1)ab-ae+bc)\sqrt{((a-c)^2-4af)}}{2aa}$$

$$=g+\frac{c(ab-ae+bc)}{aa}, \text{ ideoque}$$

$$g=\frac{2abf+c(ac-bc)-a(2(i+1)^2ab-(2i+1)(ae-bc))\mp(2(i+1)ab-ae+bc)\sqrt{((a-c)^2-4af)}}{2aa}$$

quae expreſſio congruit cum praecedente, ſi ibi loco $i$ ponatur $-i-1$. Quare hic iam pro $i$ omnes numeros integros tam poſitiuos quam negatiuos ſumere licet.

IX. Fieri autem poteſt, vt caſus, qui per priorem ſeriem ſunt integrabiles, iidem quoque per poſteriorem integrari ſicque pro eadem aequatione gemina integralia exhiberi queant. Ponamus enim numerum $i$ pro hac poſteriori forma ſuperare numerum integrum $i$ praecedentis formae exceſſu $\alpha-1$, ita

ita vt hic pro $i$ fcribamus $i + \alpha - 1$. Quo facto vt ambo valores ipfius $g$ congruant fieri neceffe eft

$$2(i+\alpha)^2 ab - (2i+2\alpha-1)(ae-bc) = 2iiab + (2i+1)(ae-bc)$$

et $2(i+\alpha)ab - ae + bc = 2iab + ae - bc$

ex qua fequitur $\alpha ab = ae - bc$. In priori autem fcribendo $\alpha ab$ loco $ae - bc$, prodit per $ab$ diuidendo

$$2(i+\alpha)^2 - 2\alpha i - 2\alpha\alpha + \alpha = 2ii + 2\alpha i + \alpha$$

quae cum fit identica pro omnibus valoribus ipfius $i$, habebimus $\alpha = \frac{ae-bc}{ab}$; quae expreffio debet effe numerus integer.

X. Quoniam igitur infinitos valores pro littera $g$ eruimus, quibus aequatio propofita integrationem admittit, atque adeo formula algebraica pro $z$ fatisfaciens affignari poteft; operae pretium eft, vt hos cafus accuratius perpendamus. Denotante ergo $i$ numerum quemcunque integrum fiue pofitiuum fiue negatiuum, euolutio prior §. 7. facta has duas conditiones poftulat:

$$n(n-1)b + ne + g = 0 \quad \text{et}$$

$$(n-i)(n-i-1)a + (n-i)c + f = 0$$

ex quibus deducitur:

$$n = \frac{b - e + \surd((b-e)^2 - 4bg)}{2b} \quad \text{et}$$

$$n - i = \frac{a - c + \surd((a-c)^2 - 4af)}{2a} \quad \text{vnde fit}$$

$$i = \frac{bc - ae + a\surd((b-e)^2 - 4bg) - b\surd((a-c)^2 - 4af)}{2ab}$$

Quoties ergo haec formula

$$\frac{qc - ae + a\surd((b-e)^2 - 4bg) - b\surd((a-c)^2 - 4af)}{2ab}$$

 vbi

vbi partes irrationales tam positiue quam negatiue accipi possunt, aequatur numero integro siue positiuo siue negatiuo, toties proposita aequatio integrationem admittit.

XI. Si coefficientes $a$, $b$, $c$, $e$, $f$, $g$ sint rationales, vt hoc fieri possit vel vtrumque signum radicale fieri rationale debet, vel se mutuo destruere. Pro hoc casu fit

$$aa(b-e)^2 - 4aabg = bb(a-c)^2 - 4abbf \text{ seu}$$
$$4ab(ag-bf) = (ae-bc)(ae+bc-2ab)$$

tum vero fit necesse est $i = \frac{bc-ae}{2ab}$.

Pro illo vero casu si statuamus:

$$\sqrt{((a-c)^2 - 4af)} = b \quad \text{et} \quad \sqrt{((b-e)^2 - 4bg)} = k \text{ erit}$$
$$f = \frac{(a+c)^2 - bb}{4a} \quad \text{et} \quad g = \frac{(b-e)^2 - kk}{4b}$$

Tales igitur valores si habeant litterae $f$ et $g$, dispiciatur an haec expressio $\frac{bc-ae+ak-bb}{2ab}$ sit numerus integer? Tum enim si sit numerus integer positiuus, valor ipsius $z$ per seriem priorem, sin autem negatiuus, per posteriorem exhiberi poterit. Ac si insuper $\frac{ae-ac}{ab}$ fuerit numerus integer, vtroque modo integratio absolui poterit, vnde integrale completum algebraicum obtinebitur.

XII. Casus etiam integrabiles inuestigari possunt quaerendo factorem, per quem aequatio multiplicata fiat integrabilis. In hunc finem consideremus aequationem huius formae:

$$ddz + Q\,dx\,dz + Rz\,dx^2 = 0$$

ita

ita vt sit

$$Q = \frac{c + e x}{x(a + b x)} \quad \text{et} \quad R = \frac{f + g z}{x x (a + b x)}.$$

Sitque multiplicator $2 p\, d z + q\, z\, d x$, ideoque aequatio integrabilis:

$$2 p\, d z\, d d z + q z d x d d z + 2 p Q d x d z^2 + Q q z d x^2 d z + 2 p R z d x^2 d z + q R z z d x^3 = 0.$$

Statuatur aequatio integralis:

$$p d z^2 + q z d x d z + z z d x^2 \int R q d x = C d x^2$$

cuius differentiali inde ablato fieri debet:

$$\left.\begin{array}{l} + 2 Q p d x d z^2 - d p d z^2 - q d x d z^2 \\ + Q q z d x^2 d z + 2 R p z d x^2 d z - z d x d q d z - 2 z d x^2 d z \int R q d x \end{array}\right\} = 0$$

vnde hae duae aequationes existunt:

$$d p + q d x = 2 Q p d x \quad \text{seu} \quad \frac{d p}{p} + \frac{q d x}{p} = 2 Q d x$$

et $Q q d x + 2 R p d x - d q - 2 d x \int R q d x = 0.$

Ponatur $\int R\, q\, d\, x = S$, erit $R\, d\, x = \frac{d S}{q}$ et

$$Q q d x + \frac{2 p d S}{q} - d q - 2 S d x = 0 \quad \text{seu} \quad d S - \frac{S q d x}{p} = \frac{q d q}{2 p} - \frac{Q q q d x}{2 p}$$

et pro $Q\, d\, x$ scripto superiori valore

$$d S - \frac{S q d x}{p} = \frac{q d q}{2 p} - \frac{q q d p}{4 p p} - \frac{q^3 d x}{4 p p}.$$

Sit $\varkappa$ numerus cuius logarithmus $= 1$ et integrando eruitur

$$\varkappa^{-\int \frac{q d x}{p}} S = \int \varkappa^{-\int \frac{q d x}{p}} \left(\frac{q d q}{2 p} - \frac{q q d p}{4 p p} - \frac{q^3 d x}{4 p p}\right) \quad \text{seu}$$

$$\varkappa^{-\int \frac{q d x}{p}} S = \tfrac{1}{4} C + \varkappa^{-\int \frac{q d x}{p}} \frac{q q}{4 p}, \text{ vnde fit}$$

S = ¼

$$S = \tfrac{1}{4} C \varkappa^{\frac{\int q\,dx}{p}} + \tfrac{qq}{4p} = \int R\,q\,dx\text{, hincque porro}$$

$$R = \frac{C\varkappa^{\frac{\int q\,dx}{p}}}{4p} + \frac{2p\,dq - q\,dp}{2pp\,dx} \text{ et } Q = \frac{dp}{2p\,dx} + \frac{q}{2p}.$$

Ex quibus colligimus, propoſita hac aequatione

$$ddz + \tfrac{(dp+q\,dx)dz}{2p} + \left(C\varkappa^{\frac{\int q\,dx}{2p}} p\,dx + 2p\,dq - q\,dp\right)\tfrac{zz\,dx}{4pp} = 0$$

ſi ea ducatur in $2p\,dz + qz\,dx$ fore integrale

$$p\,dz^2 + qz\,dx\,dz + \left(C\varkappa^{\frac{\int q\,dx}{p}} p + qq\right)\tfrac{zz\,dx^2}{4p} = A\,dx^2.$$

XIII. His ad propoſitum accommodatis primo obtinemus:

$\frac{(c+ex)\,dx}{x(a+bx)} = \frac{dp}{2p} + \frac{q\,dx}{2p}$ vnde colligimus

$\frac{\int q\,dx}{p} = -lp + \frac{2c}{a}lx + \frac{2(ae-bc)}{ab}l(a+bx)$, ideoque

$$\varkappa^{\frac{\int q\,dx}{p}} = \frac{x^{\frac{2c}{a}}(a+bx)^{\frac{2(ae-bc)}{ab}}}{p} \text{ hincque}$$

$$\frac{f+gx}{xx(a+bx)} = \frac{C\,x^{\frac{2c}{a}}(a+bx)^{\frac{2(ac-bc)}{ab}}\,dx + 2p\,dq - q\,dp}{4pp\,dx}.$$

At eſt $q = \frac{2p(c+ex)}{x(a+bx)} - \frac{dp}{dx}$ indeque

$$dq = \tfrac{2(c+ex)\,dp}{x(a+bx)} - \tfrac{2p\,dx(ac+2bcx+bexx)}{xx(a+bx)^2} - \tfrac{ddp}{dx}$$

quibus ſubſtitutis aequatio reſoluenda erit

$$\frac{4(f+gx)pp\,dx}{xx(a+bx)} = Cx^{\frac{2c}{a}}(a+bx)^{\frac{2(ae-bc)}{ab}}dx + \frac{2(c+ex)p\,dp}{x(a+bx)}$$
$$- \frac{4pp\,dx(ac+2bcx+bcxx)}{xx(a+bx)^2} - \frac{2p\,ddp}{dx} + \frac{dp^2}{dx}.$$

XIV.

XIV. Verum hoc modo haud minoribus difficultatibus implicamur, quam si ipsam aequationem propositam resoluere vellemus. Aliam ergo viam magis particularem ingrediamur, et quaeramus conditiones coefficientium A, B, C vt haec aequatio:

$$A x^{\lambda} ddz + B x^{\lambda-1} dx dz + C x^{\lambda-2} z dx^2 = 0$$

si multiplicetur per $2 x dz + \alpha z dx$, fiat integrabilis. Cum igitur productum sit

$$\begin{aligned} &+2Ax^{\lambda+1}dzddz + 2Bx^{\lambda}dxdz^2 + \alpha Bx^{\lambda-1}zdx^2dz + \alpha Cx^{\lambda-2}zzdx^3 = 0 \\ &+\alpha A x^{\lambda} zdxddz \qquad\qquad + 2Cx^{\lambda-1}zdx^2dz \end{aligned}$$

integrale sit necesse est

$$A x^{\lambda+1} dz^2 + \alpha A x^{\lambda} z dx dz + \frac{\alpha}{\lambda-1} C x^{\lambda-1} zz dx^2 = E dx^2$$

cuius differentiale si inde auferatur, prodibit haec aequatio:

$$\left.\begin{aligned} &+x^{\lambda} dx dz^2 (2B - (\lambda+1)A - \alpha A) \\ &+x^{\lambda-1} z dx^2 dz (\alpha B + 2C - \alpha\lambda A - \tfrac{2\alpha C}{\lambda-1}) \end{aligned}\right\} = 0.$$

Vnde vtroque membro seorsim annihilato fit primo

$$B = \frac{\alpha+\lambda+1}{2} A \text{ hincque}$$

$$\frac{\alpha(\alpha-\lambda+1)}{2} A = \frac{2(\alpha-\lambda+1)}{\lambda-1} C$$

ex qua duplici modo eruitur:

$$\text{vel } \lambda = \alpha + 1 \quad \text{vel } C = \frac{\alpha(\lambda-1)}{4} A.$$

Duae ergo aequationes oriuntur

altera $A x^{\alpha+1} ddz + (\alpha+1) A x^{\alpha} dx dz + C x^{\alpha-1} z dx^2 = 0$

altera $A x^{\lambda} ddz + \frac{1}{2}(\alpha+\lambda+1) A x^{\lambda-1} dx dz + \frac{1}{4}\alpha(\lambda-1) A x^{\lambda-2} z dx^2 = 0$

 qua-

quarum vtraque per $2\,x\,dz + \alpha\,z\,dx$ multiplicata fit integrabilis; illius enim integrale erit:

$$A\,x^{\alpha+2}\,dz^2 + \alpha\,A\,x^{\alpha+1}\,z\,dx\,dz + C\,x^{\alpha}\,zz\,dx^2 = E\,dx^2$$

huius vero

$$A\,x^{\lambda+1}\,dz^2 + \alpha\,A\,x^{\lambda}\,z\,dx\,dz + \tfrac{1}{4}\alpha\alpha\,A\,z^{\lambda-1}\,zz\,dx^2 = F\,dx^2.$$

XV. Summa ergo harum duarum aequationum eodem factore integrabilis reddetur. Scilicet haec aequatio:

$$(A\,x^{\alpha+1} + D\,x^{\lambda})\,ddz + ((\alpha+1)A\,x^{\alpha} + \tfrac{1}{2}(\alpha+\lambda+1)D\,x^{\lambda-1})\,dx\,dz$$
$$+ (C\,x^{\alpha-1} + \tfrac{1}{4}\alpha(\lambda-1)D\,x^{\lambda-2})\,z\,dx^2 = 0$$

multiplicata per $2\,x\,dz + \alpha\,z\,dx$ integrale praebet:

$$(A\,x^{\alpha+2} + D\,x^{\lambda+1})\,dz^2 + \alpha(A\,x^{\alpha+1} + D\,x^{\lambda})\,z\,dx\,dz$$
$$+ (C\,x^{\alpha} + \tfrac{1}{4}\alpha\alpha\,D\,x^{\lambda-1})\,zz\,dx^2 = E\,dx^2$$

quod isto modo repraesentari potest:

$$(A\,x^{\alpha} + D\,x^{\lambda-1})(x\,dz + \tfrac{1}{2}\alpha\,z\,dx)^2 = dx^2((\tfrac{1}{4}\alpha\alpha\,A - C)\,x^{\alpha}\,zz + E)$$

ita vt fit

$$x\,dz + \tfrac{1}{2}\alpha\,z\,dx = \tfrac{1}{2}\,dx\,\sqrt{\frac{4\,E + (\alpha\alpha\,A - 4\,C)\,x^{\alpha}\,zz}{A\,x^{\alpha} + D\,x^{\lambda-1}}}.$$

Ponatur

$$x^{\alpha}\,zz = v \text{ erit } x^{\alpha-1}\,z\,(2\,x\,dz + \alpha\,z\,dx) = 2\,v\,dv,$$

ideoque

$$2\,x\,dz + \alpha\,z\,dx = \frac{dv}{x^{\alpha-1}\,z} = \frac{2\,dv}{x^{\frac{1}{2}\alpha-1}}; \text{ vnde fit}$$

$$2\,x^{1-\frac{1}{2}\alpha}\,dv = dx\,\sqrt{\frac{4\,E + (\alpha\alpha\,A - 4\,C)\,vv}{A\,x^{\alpha} + D\,x^{\lambda-1}}} \text{ seu}$$

2 dv

$$\frac{2\,dv}{\sqrt{(4E+(\alpha\alpha A-4C)vv)}}=\frac{x^{\frac{1}{2}\alpha-1}dx}{\sqrt{(Ax^{\alpha}+Dx^{\lambda-1})}}.$$

XVI. Quo nunc hanc aequationem ad nostram formam perducamus, quod duplici modo fieri potest, ponamus primo $\lambda=\alpha$, vt facta diuisione per $x^{\alpha}$ habeatur haec aequatio:

$$(D+Ax)ddz+((\alpha+\tfrac{1}{2})D+(\alpha+1)Ax)\frac{dxdz}{x}$$
$$+(\tfrac{1}{4}\alpha(\alpha-1)D+Cx)\frac{zdx^2}{xx}=0$$

quae multiplicata per $x^{\alpha}(2xdz+\alpha zdx)$ fit integrabilis, existente integrali posito

$x^{\alpha}zz=vv$ seu $z=x^{-\frac{1}{2}\alpha}v$

$$\frac{2\,dv}{\sqrt{(4E+(\alpha\alpha A-4C)vv)}}=\frac{x^{\frac{1}{2}\alpha-1}dx}{\sqrt{x^{\alpha-1}(D+Ax)}}=\frac{dx}{\sqrt{x(D+Ax)}}.$$

Sit iam $D=a$; $A=b$; $(\alpha+\frac{1}{2})a=c$, seu $\alpha=\frac{c}{a}-\frac{1}{2}$, et $C=g$ et prodibit haec aequatio:

$$(a+bx)ddz+(c+\tfrac{b(a+2c)}{2a}x)\frac{dxdz}{x}+(\tfrac{(2c-a)(2c-3a)}{16a}+gz)\frac{zdx^2}{xx}=0$$

ita vt pro forma proposita sit

$e=\frac{b(a+2c)}{2a}$ et $f=\frac{(2c-a)(2c-3a)}{16a}$

huiusque aequationis posito $z=x^{\frac{1}{4}-\frac{c}{2a}}v$ integrale erit:

$$\frac{2\,dv}{\sqrt{(4E+(\frac{b(2c-a)^2}{4aa}-4g)vv)}}=\frac{dx}{\sqrt{x(a+bx)}}.$$

XVII. Statuatur nunc secundo $\lambda=\alpha+2$, vt facta diuisione per $x^{\alpha+1}$ oriatur haec aequatio:

$$(A+Dx)\,ddz+((\alpha+1)A+(\alpha+\tfrac{3}{2})Dx)\frac{dx\,dz}{x}$$
$$+(C+\tfrac{1}{4}\alpha(\alpha+1)Dx)\frac{z\,dx^2}{xx}=0$$

quae multiplicata per $x^{\alpha+1}(2x\,dz+\alpha z\,dx)$, posito $x^{\alpha}zz=vv$ seu $z=x^{-\frac{1}{2}\alpha}v$ habebit integrale:

$$\frac{2\,dv}{\sqrt{(4E+(\alpha\alpha A-4C)vv)}}=\frac{x^{\frac{1}{2}\alpha-1}dx}{\sqrt{x^{\alpha}(A+Dx)}}=\frac{dx}{\sqrt{x(A+Dx)}}.$$

Sit iam $A=a$; $D=b$; $(\alpha+1)a=c$, seu $\alpha=\frac{c}{a}-1$ et $C=f$ vt obtineatur haec aequatio:

$$(a+bx)\,ddz+(c+\tfrac{b(a+2c)}{2a}x)\frac{dx\,dz}{x}+(f+\tfrac{bc(c-a)}{4aa}x)\frac{z\,dx^2}{xx}=0$$

et pro forma proposita fit:

$$e=\frac{b(a+2c)}{2a}\quad\text{et}\quad g=\frac{bc(c-a)}{4aa}$$

cuius posito $z=x^{\frac{1}{2}-\frac{c}{2a}}v$ integrale est:

$$\frac{2\,dv}{\sqrt{(4E+(\frac{(c-a)^2}{a}-4f)vv)}}=\frac{dx}{\sqrt{x(a+bx)}}.$$

XVIII. Non solum autem quoties ipsa aequatio proposita:

$$(a+bx)\,ddz+(c+ex)\frac{dx\,dz}{x}+(f+gx)\frac{z\,dx^2}{xx}=0$$

in altera harum formarum est contenta, quod euenit si fuerit vel $e=\frac{b(a+2c)}{2a}$ et $f=\frac{(2c-a)(2c-3a)}{16a}$

vel $e=\frac{b(a+2c)}{2a}$ et $g=\frac{bc(c-a)}{4aa}$

integrationem admittit, sed etiam quoties eadem transformata in alterutra continetur. Transformatio autem vt supra §. 8. vidimus fit substitutione

$z=$

$$z = (a+bx)^{1+\frac{c}{a}-\frac{e}{b}}\, v,$$

vnde oritur

$$(a+bx)ddv + (c+\varepsilon x)\frac{dx\,dv}{x} + (f+\eta x)\frac{v\,dx^2}{xx} = 0$$

existente

$$\varepsilon = \frac{2b(a+c)}{a} - e \text{ et } \eta = g - \frac{ce}{a} + \frac{bc(a+c)}{aa}.$$

Haec autem ponendo $v = x^n s$, ob

$$\frac{dv}{v} = \frac{n\,dx}{x} + \frac{ds}{s} \text{ et } \frac{ddv}{v} = \frac{n(n-1)dx^2}{xx} + \frac{2n\,dx\,ds}{xs} - \frac{dds}{s},$$

transformatur in hanc:

$$\left.\begin{array}{l} (a+bx)\frac{dds}{s} + 2n(a+bx)\frac{dx\,ds}{xs} + n(n-1)(a+bx)\frac{dx^2}{xx} \\ \qquad + (c+\varepsilon x)\frac{dx\,ds}{xs} + n(c+\varepsilon x)\frac{dx^2}{xx} \\ \qquad\qquad + (f+\eta x)\frac{dx^2}{xx} \end{array}\right\} = 0$$

vnde hi bini casus integrabiles eruuntur.

Primus si $D = a$; $A = b$; $(\alpha + \frac{1}{2})a = 2na + c$

$$(\alpha+1)b = 2nb + \frac{2b(a+c)}{a} - e$$

$$\tfrac{1}{4}\alpha(\alpha-1)a = n(n-1)a + nc + f$$

$$C = n(n-1)b + \frac{2nb(a+c)}{a} - ne + g - \frac{ce}{a} + \frac{bc(a+c)}{aa}$$

hincque $\alpha = 2n - \frac{1}{2} + \frac{c}{a}$, et $e = \frac{3}{2}b + \frac{bc}{a} = \frac{b(3a+2c)}{2a}$

atque $(n - \frac{1}{2} + \frac{c}{2a})(n - \frac{3}{4} + \frac{c}{2a})a = n(n-1)a + nc + f$

vnde $f = \frac{cc}{4a} - \frac{c}{2} + \frac{3a}{16} = \frac{(2c-a)(2c-3a)}{16a}$.

Alter casus his conditionibus continetur:

$A = a$; $D = b$; $(\alpha+1)a = 2na + c$; $(\alpha+\frac{3}{2})b = 2nb + \frac{2b(a+c)}{a} - e$

$C = n(n-1)a + nc + f$;

$\frac{1}{4}\alpha(\alpha+1)b = n(n-1)b + \frac{2nb(a+c)}{a} - ne + g - \frac{ce}{a} + \frac{bc(a+c)}{aa}$

 vnde

vnde fit $\alpha = 2n - 1 + \frac{c}{a}$; $e = \frac{b(3a+2c)}{2a}$ atque

$$g = \frac{bc}{4a} + \frac{bcc}{4aa} = \frac{bc(a+c)}{4aa}$$

vbi constat numerum $n$ nihil conferre.

XIX. Quatuor ergo hinc nacti sumus casus integrabiles, qui sunt:

$$1^\circ.\ e = \frac{b(a+2c)}{2a};\quad f = \frac{(2c-a)(2c-3a)}{16a}$$

$$2^\circ.\ e = \frac{b(a+2c)}{2a};\quad g = \frac{bc(c-a)}{4aa}$$

$$3^\circ.\ e = \frac{b(3a+2c)}{2a};\quad f = \frac{(2c-a)(2c-3a)}{4aa}$$

$$4^\circ.\ e = \frac{b(3a+2c)}{2a};\quad g = \frac{bc(a+c)}{4a}$$

quibus adeo integrale completum exhibuimus. Videamus ergo quomodo hi casus se habeant ad conditionem, quam supra ex serie deduximus, vtrum in ea contineantur nec ne?

Pro primo igitur habemus

$$bc - ae = \frac{-ab}{2};\quad b - e = \frac{b(a-2c)}{2a}$$

et $\sqrt{((a-c)^2 - 4af)} = \pm \frac{a}{2}$, vnde haec formula

$$-\tfrac{1}{4} \mp \tfrac{1}{4} + \frac{\sqrt{(bb(a-2c)^2 - 16aabg)}}{4aa}$$

numerus integer esse deberet.

Pro secundo est

$$bc - ae = \frac{-ab}{2};\ \text{et}\ \sqrt{((b-e)^2 - 4bg)} = \pm \tfrac{b}{2},$$

ergo haec formula

$$-\tfrac{1}{4} \pm \tfrac{1}{4} - \frac{\sqrt{((a-c)^2 - 4af)}}{2a}$$

numerus integer esse deberet.

Pro

Pro tertio eſt

$$bc - ae = \frac{-3ab}{2}; \quad b - e = \frac{-b(a + 2c)}{2a}, \text{ et}$$

$$\surd((a - c)^2 - 4af) = \pm \frac{a}{2};$$

vnde haec formula

$$-\frac{3}{4} \mp \frac{1}{4} + \frac{\surd(bb(a + 2c)^2 - 16aabg)}{4ab}$$

numerus integer eſſe deberet.

Pro quarto eſt

$$bc - ae = \frac{-3ab}{2} \text{ et } \surd((b - e)^2 - 4bg) = \pm \frac{b}{2}$$

vnde haec formula

$$-\frac{3}{4} \mp \frac{1}{4} - \frac{\surd((a - c)^2 - 4af)}{2a}$$

numerus integer eſſe deberet.

Vnde perſpicitur hos quatuor caſus in ſuperiori conditione non contineri, ideoque hinc omnino nouos caſus integrabilitatis erui.

XX. Cum igitur hi caſus, quibus integrale completum eruimus, omnino diſcrepent ab iis, quibus ſupra integrale particulare exhibuimus, iuuabit oſtendiſſe, quomodo etiam his caſibus integrale completum obtineri poſſit; quod ſequenti modo facillime praeſtari videtur.

Si aequationi

$$P\,ddz + Q\,dx\,dz + R\,z\,dx^2 = 0$$

ſatisfaciat valor $z = V$, vt ſit

$$P\,ddV + Q\,dx\,dV + R\,V\,dx^2 = 0,$$

illa aequatio reddetur integrabilis ducta in $\frac{V}{P(V\,dz - z\,dV)}$.

Poſita

Posito enim

$$\int \frac{PVddz + QVdxdz + RVzdx^2}{P(Vdz - zdV)} = Sdx, \text{ erit}$$

$$Sdx - lP(Vdz - zdV) = \int \frac{QVdxdz + RVzdx^2 - VdPdz + PzddV + zdVdP}{P(Vdz - zdV)}$$

$$= \int \frac{Qdx}{P} - lP + \int \frac{z(PddV + QdxdV + RVdx^2)}{P(Vdz - zdV)}.$$

At $PddV + QdxdV + RVdx^2 = 0$,

ideoque habetur:

$$Sdx = l(Vdz - zdV) + \int \frac{Qdx}{P} + \text{Const.} = \text{Const.}$$

vnde erit

$$Vdz - zdV = C\varkappa^{\frac{-\int Qdx}{P}} dx \quad \text{et}$$

$$z = CV \int \frac{dx}{VV} \varkappa^{\frac{-\int Qdx}{P}}$$

quod est integrale completum ex particulari $z = V$ erutum.

XXI. Quoniam vtriusque generis casus ex aequatione proposita eliciuntur, si ea per formam $2pdz + qdzdx$ multiplicata integrabilis efficiatur, posito $p = uu$, vt sit

$$q = \frac{2uu(c + ex)}{x(a + bx)} - \frac{2udu}{dx},$$

erit aequatio integralis:

$$uudz^2 + qzdxdz + (C\varkappa^{\frac{\int qdx}{uu}} + qq)\frac{zzdx^2}{4uu} = Adx^2$$

vbi est

$$\varkappa^{\frac{\int qdx}{uu}} = \frac{x^{\frac{2c}{a}}(a + bx)^{\frac{2(ae - bc)}{ab}}}{uu}$$

ideo-

ideoque

$$\left(u\,dz + \frac{q\,z\,dx}{2\,u}\right)^2 = A\,dx^2 - \frac{C x^{\frac{2c}{a}} (a+bx)^{\frac{2(ae-bc)}{ab}} zz\,dx^2}{4\,u^4}$$

verum quantitatem $u$ ex hac aequatione elici oportet.

$$\frac{ddu}{dx} - \frac{(c+ex)\,du}{x(a+bx)} + \frac{(f+gx)\,u\,dx}{xx(a+bx)} + \frac{(ac+2bcx+bexx)\,u\,dx}{xx(a+bx)^2}$$
$$= \frac{C x^{\frac{2c}{a}} (a+bx)^{\frac{2(ae-bc)}{ab}} dx}{4\,u^3}$$

et prioris quidem generis casus hinc sumta constante $C = 0$ sunt deducti. Verum haec aequatio posito

$u = x^{\frac{c}{a}} (a+bx)^{\frac{ae-bc}{ab}} v$ abit in hanc

$$\frac{C x^{\frac{-2c}{a}} (a+bx)^{\frac{2(bc-ae)}{ab}} dx^2}{4\,v^3} = ddv + \frac{(c+ex)\,dx\,dv}{x(a+bx)} + \frac{(f+gx)\,v\,dx^2}{xx(a+bx)}$$

cuius applicatio est facilior, vnde si $C = 0$, quantitas $v$ satisfacere debet huic aequationi

$$(a+bx)\,ddv + \frac{(c+ex)\,dx\,dv}{x} + \frac{(f+gx)\,v\,dx^2}{xx} = 0$$

ita vt hinc ex valore particulari obtineatur completus. At si ponamus $u = x^m (a+bx)^n$ erit

$$\tfrac{1}{4} C x^{\frac{2c}{a}-4m} (a+bx)^{\frac{2(ae-bc)}{ab}-4n} = \frac{m(m-1)}{xx} + \frac{(f-mc+(g-me+mnb)x)}{xx(a+bx)}$$
$$+ \frac{a(c+(2-n)bcx+(n-1)(nbb-be)\,xx}{xx(a+bx)^2}$$

ideoque tam exponentes $m$ et $n$ cum constante $C$, quam relatio coefficientium $a, b, c, e, f, g$ ex hac aequatione definiri debet

$$\tfrac{1}{4} C x^{\frac{2c}{a}-4m+2} (a+bx)^{\frac{2(ae-bc)}{ab}-4n+2} = m(m-1)(a+bx)^2$$

$+(a+bx)(f-mc+(g-me+2mnb)x)+ac(2-n)bcx$
$+(n-1)(nb-e)bxx.$

XXII. Hinc plures casus resultant, quos euolvamus:

*Primus*. Si exponens

$$\frac{2(ae-bc)}{ab}-4n+2=2 \text{ seu } n=\frac{ae-bc}{2ab},$$

quo esse debet

$$\frac{2c}{a}-4m+2=0 \text{ seu } m=\frac{a+c}{2a},$$

vt habeatur

$$\tfrac{1}{4}C(a+bx)^2=\frac{cc-aa}{4aa}(a+bx)^2+(f-\frac{c(a+c)}{2a}+(g-\frac{bc(a+c)}{2aa})x)(a+bx)$$
$$+ac+(2-n)bcx+(n-1)(nb-e)bxx$$

vbi postremum membrum per $a+bx$ diuisibile esse debet, id quod duplici modo fieri potest.

1°. Vel est $n=1$; ideoque $e=\frac{2ab+bc}{a}=\frac{b(2a+c)}{a}$, sicque erit

$$\tfrac{1}{4}C(a+bx)=\frac{cc-aa}{4aa}(a+bx)+f-\frac{c(a+c)}{2a}+(g-\frac{bc(a+c)}{2aa})x+a$$

vnde fit

$$\tfrac{1}{4}Ca=\frac{cc-aa}{4a}+f+\frac{c(a-c)}{2a}=f-\frac{(a-c)^2}{4a}$$

et $\tfrac{1}{4}Cb=\frac{b(cc-aa)}{4aa}+g-\frac{bc(a+c)}{2aa}=g-\frac{b(a+c)^2}{4aa}$.

Ergo

$$bf-ag-\frac{b(a-c)^2}{4a}+\frac{b(a+c)^2}{4a}=0 \text{ seu } g=\frac{bf}{a}+\frac{bc}{a}=\frac{b(c+f)}{a}$$

et. $\frac{q}{2u}=\frac{u(c+ex)}{x(a+bx)}-\frac{du}{dx}=x^{\frac{c-a}{2a}}(c+ex)-\frac{(a+c)}{2a}x^{\frac{c-a}{2a}}(a+bx)-bx^{\frac{a+c}{2a}}$

seu $\frac{q}{2u}=\frac{a(c-a)+b(a+c)x}{2a}x^{\frac{c-c}{2a}}$. Consequenter aequatio integralis

$(x^{\frac{a+c}{2a}}$

$$(x^{\frac{a+c}{2a}}(a+bx)dz+\frac{a(c-a)+b(a+c)x}{2a}x^{\frac{c-a}{2a}}zdx)^2=Adx^2$$
$$-(\frac{f}{a}-\frac{(a-c)^2}{4aa})\frac{zzdx^2}{xx}.$$

2°. Vel eſt $n=\frac{ae-bc}{ab}=\frac{ae-bc}{ab}$, ideoque $e=\frac{bc}{a}$ et $n=0$; vnde fit

$$\tfrac{1}{4}C(a+bx)=\frac{cc-aa}{4aa}(a+bx)+f-\frac{c(a+c)}{2a}+(g-\frac{bc(a+c)}{2aa})x+c+\frac{bc}{a}x$$

ergo $\frac{1}{4}Ca=\frac{cc-aa}{4a}+f+\frac{c(a-c)}{2a}=f-\frac{(a-c)^2}{4a}$

et $\frac{1}{4}Cb=\frac{b(cc-aa)}{4aa}+g+\frac{bc(a-c)}{2aa}=g-\frac{b(a-c)^2}{4aa}$

vnde colligitur $bf=ag$ ſeu $g=\frac{bf}{a}$; qui eſt caſus, quo aequatio propoſita per $a+bx$ diuiſibilis exiſtit, ſicque nihil habet difficultatis.

XXIII. *Secundus* caſus eſt quo

$$\frac{2(ae-bc)}{ab}-4n+2=1 \quad \text{et } \frac{2c}{a}-4m+2=0$$

ideoque $m=\frac{a+c}{2a}$ et $n=\frac{ab+2ae-2bc}{4ab}$, ita vt habeamus

$$\tfrac{1}{4}C(a+bx)=\frac{cc-aa}{4aa}(a+bx)^2+(f-\frac{c(a+c)}{2a}+(g+\frac{b(a+c)(a-2c)}{4aa})x(a+bx)$$
$$+ac+(2-n)bcx+(n-1)(nb-e)bxx$$

qui caſus iterum in duos diſpertitur:

1°. Vel eſt $n=1$, ideoque $2ae-2bc=3ab$ et $e=\frac{3ab+2bc}{2a}=\frac{b(3a+2c)}{2a}$

vnde fit

$$\tfrac{1}{4}C=\frac{cc-aa}{4aa}(a+bx)+f+\frac{c(a-c)}{2a}+gx+\frac{b(a+c)(a-2c)}{4aa}x$$

hincque

$$\tfrac{1}{4}C=\frac{cc-aa}{4a}+f+\frac{c(a-c)}{2a}=f-\frac{(a-c)^2}{4aa}$$

 et

et $0 = \frac{b(cc-aa)}{4aa} + g + \frac{b(a+c)(a-2c)}{4aa}$ seu $g = \frac{bc(a+c)}{4aa}$

qui est casus quartus in §. 19.

2°. Vel est $n = \frac{ae-bc}{ab} = \frac{ab+2ae-2bc}{4ab}$, ideoque $e = \frac{b(a+2c)}{2a}$

et $n = \frac{1}{2}$; vnde aequatio per $a+bx$ diuisa fit

$$\tfrac{1}{4}C = \frac{cc-aa}{4aa}(a+bx) + f + \frac{c(a-c)}{2a} + \left(g + \frac{b(a+c)(a-2c)}{4aa}\right)x + \frac{bc}{2a}x$$

ita vt sit

$$\tfrac{1}{4}C = \frac{cc-aa}{4a} + f + \frac{c(a-c)}{2a} = f - \frac{(a-c)^2}{4a}$$

et $\frac{b(cc-a)}{4aa} + \frac{bc}{2a} + \frac{b(a+c)(a-2c)}{4aa} + g = 0$ seu $g = \frac{bc(c-a)}{4aa}$

qui erat casus 2°. in §. 19.

XXIV. *Tertius* casus est quo

$$\frac{2(ae-bc)}{ab} - 4n + 2 = 1 \quad \text{et} \quad \frac{2c}{a} - 4m + 2 = 1$$

ideoque $m = \frac{a+2c}{4a}$ et $n = \frac{ab+2ae-2bc}{4ab}$

sicque habebimus:

$$\tfrac{3}{4}Cx(a+bx) = \frac{(a+2c)(2c-3a)}{16aa}(a+bx)^2 + \left(f - \frac{c(a+2c)}{4a} + \left(g + \frac{b(a+2c)(a-2c)}{8aa}\right)x(a+bx)\right.$$
$$+ ac + (2-n)bcx + (n-1)(nb-1)bxx$$

cuius vltimum membrum duplici modo per $a+bx$ redditur diuisibile.

1°. Si $n = 1 = \frac{ab+2ae-2bc}{4ab}$; ideoque $e = \frac{b(3a+2c)}{2a}$

vnde oritur

$$\tfrac{1}{4}Cx = \frac{(a+2c)(2c-3a)}{16aa}(a+bx) + f + \frac{c(3a-2c)}{4a} + gx + \frac{b(a+2c)(a-2c)}{8aa}x$$

ita vt fieri oporteat

$$\frac{(a+2c)(2c-3a)}{16a} + f - \frac{c(2c-3a)}{4a} = 0 \text{ seu } f = \frac{(2c-a)(2c-3a)}{16a}$$

et

et $\frac{1}{4}Cx = \frac{b(a+2c)(2c-3a)}{16aa} + g + \frac{b(a+2c)(a-2c)}{8aa} = g - \frac{b(a+2c)^2}{16aa}$

qui erat casus 3° in §. 19.

2°. Si $n = \frac{ae-bc}{ab} = \frac{ab+2ae-2bc}{4ab}$ seu $e = \frac{b(a+2c)}{2a}$ et $n = \frac{1}{2}$

$$\frac{1}{4}Cx = \frac{(a+2c)(2c-3a)}{16aa}(a+bx) + f - \frac{c(a+2c)}{4a} + gx + \frac{b(a+2c)(a-2c)}{8aa}x$$
$$+ c + \frac{bc}{2a}x$$

ideoque

$$f + \frac{(a+2c)(2c-3a)}{16a} + \frac{c(3a-2c)}{4a} = 0 \text{ seu } f = \frac{(2c-a)(2c-3a)}{16a}$$

et $\frac{1}{4}C = \frac{b(a+2c)(2c-3a)}{16aa} + g + \frac{b(a+2c)(a-2c)}{8aa} + \frac{bc}{2a} = g - \frac{b(a-2c)}{16aa}$

qui erat casus 1°. in §. 19.

XXV. *Quartus* casus est quo

$$\frac{2(ae-bc)}{ab} - 4n + 2 = 0 \quad \text{et} \quad \frac{2c}{a} - 4m + 2 = 0$$

ideoque $m = \frac{a+c}{2a}$ et $n = \frac{ab+ae-bc}{2ab}$, vt habeamus:

$$\frac{1}{4}C = \frac{cc-aa}{4aa}(a+bx)^2 + (a+bx)(f - \frac{c(a+c)}{2a} + (g + \frac{b(aa-cc)}{2aa})x)$$
$$+ ac + \frac{(3ab-ae+bc)}{2a}cx - \frac{(ab-ae+bc)(ab-ae-bc)}{4aa}xx$$

vnde singulas potestates seorsim tollendo colligimus

$$\frac{bb(cc-aa)}{4aa} + bg + \frac{bb(aa-cc)}{2aa} - \frac{(ab-ae+bc)(ab-ae-bc)}{4aa} = 0$$
$$\frac{b(cc-aa)}{2aa} + bf - \frac{bc(a+c)}{2a} + ag + \frac{b(aa-cc)}{2aa} + \frac{c(3ab-ae+bc)}{2a} = 0$$

ex illa fit $g = \frac{e(e-2b)}{4b}$ ex hac vero $bf + ag = \frac{c(e-2b)}{2}$ ideoque $f = \frac{(e-2b)(2bc-ae)}{4bb}$, quae sunt binae conditiones; tum vero capi debebit

$$\frac{1}{4}C = \frac{ac-aa}{4} + af - \frac{c(a+c)}{2} + ac = af - \frac{1}{4}(a-c)^2.$$

XXVI. *Quintus* caſus quo

$\frac{2(ae-bc)}{ab}-4n+2=0$; et $\frac{2c}{a}-4m+2=1$, ideoque

$m=\frac{2c+a}{4a}$ et $n=\frac{ab+ae-bc}{2ab}$ vt habeamus:

$$\tfrac{1}{4}Cx=\frac{(2c+a)(2c-3a)}{16aa}(a+bx)^2+(a+bx)\left(f-\frac{c(2c+a)}{2a}+\left(g+\frac{b(a-c)(2c+a)}{4aa}\right)x\right)$$
$$+ac+\frac{3ab-ae+bc}{2a}cx-\frac{(ab-ae+bc)(ab-ae-bc)}{4aa}xx$$

hincque:

$$\frac{bb(2c+a)(2c-3a)}{16aa}+bg+\frac{bb(a-c)(2c+a)}{4aa}-\frac{(ab-ae+bc)(ab-ae-bc)}{4aa}=0$$

ſeu $g=\frac{(b-2e)(3b-2e)}{16b}$

$\frac{(2c+a)(2c-3a)}{16}+af-\frac{c(2c+a)}{4}+ac=0$ ſeu $f=\frac{(2c-a)(2c-3a)}{16a}$ et

$\frac{1}{4}C=\frac{(ab-ae+bc)^2}{4ab}$.

XXVII. *Sextus* caſus quo

$\frac{2(ae-bc)}{ab}-4n+2=0$ et $\frac{2c}{a}-4m+2=2$ ideoque

$m=\frac{c}{2a}$ et $n=\frac{ab+ae-bc}{2ab}$ vt habeamus:

$$\tfrac{1}{4}Cxx=\frac{c(c-2a)}{4aa}(a+bx)^2+(a+bx)\left(f-\frac{cc}{2a}+\left(g+\frac{bc(a-c)}{2aa}\right)x\right)$$
$$+ac+\frac{3ab-ae+bc}{2a}cx-\frac{(ab-ae+bc)(ab-ae-bc)}{4aa}xx$$

vnde fieri oportet:

$\frac{c(c-2a)}{4}+af-\frac{cc}{2}+ac=0$ ſeu $f=\frac{c(c-2a)}{4a}$

$\frac{bc(c-2a)}{2a}+bf-\frac{bcc}{2a}+ag+\frac{bc(a-c)}{2a}+\frac{3ab-ae+bc}{2a}c=0$

ſeu $g=\frac{-c(4ab-2ae+bc)}{4aa}$ atque

$$\tfrac{1}{4}C=\frac{-bc(4ab-2ae+bc)}{4aa}-\frac{(a-e)^2}{4}=bg-\tfrac{1}{4}(b-e)^2.$$

XXVIII. Pro his autem caſibus omnibus cum ſit $u=x^m(a+bx)^n$

erit

erit

$$\frac{q}{2u} = x^{m-1}(a+bx)^{n-1}(c+ex) - mx^{m-1}(a+bx)^{n} - nbx^{m}(a+bx)^{n-1}$$

feu $\frac{q}{2u} = x^{m-1}(a+bx)^{n-1}(c-ma+(e-(m+n)b)x)$

vnde aequatio integralis colligitur:

$$x^{2m}(a+bx)^{2n}\left(dz+\frac{c-ma+(e-(m+n)b)x}{x(a+bx)}z\,dx\right)^2 = A\,dx^2$$
$$-\tfrac{1}{4}C x^{\frac{2c}{a}-4m}(a+bx)^{\frac{2(ae-bc)}{ab}-4n}zz\,dx^2$$

vel erit $dz+\frac{c-ma+(e-(m+n)b)x}{x(a+bx)}z\,dx =$

$$\frac{dx\sqrt{\left(A-\frac{1}{4}Cx^{\frac{2c}{a}-4m}(a+bx)^{\frac{2(ae-bc)}{ab}-4n}zz\right)}}{x^{m}(a+bx)^{n}}.$$

Quare pro cafibus inuentis integralia aequationis propofitae

$$(a+bx)ddz+(c+ex)\frac{dx\,dz}{x}+\frac{(f+gx)z\,dx^2}{xx}=0$$

fequenti modo fe habebunt.

## Cafus I.

$m=\frac{a+c}{2a}$; $n=1$; $e=\frac{b(2a+c)}{a}$; $g=\frac{b(c+f)}{a}$ et $\frac{1}{4}C=\frac{f}{a}-\frac{(a-c)^2}{4aa}$.

Integrale igitur erit

$$dz+\frac{a(c-a)+(a+c)x}{2ax(a+bx)}z\,dx =$$
$$\frac{dx}{x^{\frac{a+c}{2a}}(a+bx)}\sqrt{\left(A-\frac{(4af-(a-c)^2)}{4aaxx}zz\right)}.$$

## Cafus II.

$e=\frac{bc}{a}$; $g=\frac{bf}{a}$; $m=\frac{a+c}{2a}$; $n=0$ et $\frac{1}{4}C=\frac{4af-(a-c)^2}{4aa}$.

Inte-

Integrale ergo erit

$$dz + \frac{a(c-a)+b(c-a)x}{2ax(a+bx)} z\,dx =$$

$$\frac{dx}{x^{\frac{a+c}{2a}}} \sqrt{\left(A - \frac{(4af-(a-c)^2)}{4aaxx} zz\right)} \text{ siue}$$

$$dz + \frac{(c-a)z\,dx}{2ax} = \frac{dx}{x^{\frac{a+c}{2a}}} \sqrt{\left(A + \frac{((a-c)^2-4af)}{4aaxx} zz\right)}.$$

## Casus III.

$e = \frac{b(3a+2c)}{2a}$; $g = \frac{bc(a+c)}{4aa}$; $m = \frac{a+c}{2a}$; $n = 1$ et $\frac{1}{4}C = \frac{4af-(a-c)^2}{4a}$,

vnde integrale est

$$dz + \frac{a(c-a)+bcx}{2ax(a+bx)} z\,dx = \frac{dx}{x^{\frac{a+c}{2a}}(a+bx)} \sqrt{\left(A + \frac{((a-c)^2-4af)zz}{4axx(a+bx)}\right)}$$

## Casus IV.

$e = \frac{b(a+2c)}{2a}$; $g = \frac{bc(c-a)}{4aa}$; $m = \frac{a+c}{2a}$; $n = \frac{1}{2}$ et $\frac{1}{4}C = \frac{4af-(a-c)^2}{4a}$;

vnde integrale est

$$dz + \frac{(c-a)z\,dx}{2ax} = \frac{dz}{x^{\frac{a+c}{2a}}(a+bx)^{\frac{1}{2}}} \sqrt{\left(A + \frac{((a-c)^2-4af)zz}{4axx(a+bx)}\right)}.$$

## Casus V.

$e = \frac{b(3a+2c)}{2a}$; $g = \frac{(2c-a)(2c-3a)}{16a}$; $m = \frac{a+2c}{4a}$; $n = 1$ et $\frac{1}{4}C = g - \frac{b(a+2c)^2}{16aa}$;

vnde integrale erit

$$dz + \frac{a(2c-a)+b(2c+a)x}{4ax(a+bx)} z\,dx = \frac{dx}{x^{\frac{a+2c}{4a}}(a+bx)} \sqrt{\left(A + \frac{(b(a+2c)^2-16aag)zz}{16aax(a+bx)}\right)}$$

Casus

## Casus VI.

$e = \frac{b(a+2c)}{2c}$; $f = \frac{(2c-a)(2c+7a)}{16a}$; $m = \frac{a+2c}{4a}$; $n = \frac{1}{2}$ et $\frac{1}{4}C = g - \frac{b(a-2c)^2}{16aa}$,

vnde integrale erit

$$dz + \frac{(2c-a)zdx}{4ax} = \frac{dx}{x^{\frac{a+2c}{4a}}(a+bx)^{\frac{1}{2}}}\sqrt{\left(A + \frac{(b(a-2c)^2 - 16aag)zz}{16aax(a+bx)}\right)}$$

## Casus VII.

$f = \frac{(e-2b)(2bc-ae)}{4bb}$; $g = \frac{e(e-2b)}{4b}$; $m = \frac{a+c}{2a}$; $n = \frac{ab+ae-bc}{2ab}$
et $\frac{1}{4}C = af - \frac{1}{4}(a-c)^2$,

vnde integrale erit

$$dz + \frac{c-a+(e-2b)x}{2x(a+bx)}zdx = \frac{dx}{x^m(a+bx)^n}\sqrt{\left(A + \frac{((a-c)^2-4af)zz}{4xx(a+bx)^2}\right)}.$$

## Casus VIII.

$f = \frac{(2c-a)(2c-3a)}{16a}$; $g = \frac{(b-2e)(3b-2e)}{16b}$; $m = \frac{2c+a}{4a}$; $n = \frac{ab+ae-be}{2ab}$
et $\frac{1}{4}C = \frac{(ab-ae+bc)^2}{4ab}$;

vnde integrale erit

$$dz + \frac{2c-a+(2e-3b)x}{4x(a+bx)}zdx = \frac{dx}{x^m(a+bx)^n}\sqrt{\left(A - \frac{(ab-ae+bc)^2 zz}{4abx(a+bx)^2}\right)}$$

## Casus IX.

$f = \frac{c(c-2a)}{4a}$; $g = \frac{-c(4ab-2ae+bc)}{4aa}$; $m = \frac{c}{2a}$; $n = \frac{ab+ae-bc}{2ab}$
et $\frac{1}{4}C = bg - \frac{1}{4}(b-e)^2$

vnde integrale

$$dz + \frac{c+(e-b)x}{2x(a+bx)}zdx = \frac{dx}{x^m(a+bx)^n}\sqrt{\left(A + \frac{((b-e)^2-4bg)zz}{4(a+bx)^2}\right)}.$$

XXIX. Praeter hos vero nouem casus, quibus binae relationes inter coefficientes praescribuntur, initio innumerabiles casus integrabiles duplici modo eruimus. Altero priori §. VI. integrale algebraicum huius formae:

$$z = A x^{n} + B x^{n+1} + C x^{n+2} + D x^{n+3} + \text{etc.}$$

assignari potest, si denotante $i$ numerum integrum positiuum quemcunque fuerit

$$n(n-1)a + nc + f = 0 \quad \text{et}$$
$$(n+i)(n+i-1)b + (n+i)e + g = 0.$$

Altero vero posteriori §. VIII. Integrale huius est formae

$$z = (a+bx)^{\frac{ab - ae + bc}{ab}} (Ax^{n} + Bx^{n+1} + Cx^{n+2} + \text{etc.})$$

si fuerit

$$n(n-1)a + nc + f = 0 \quad \text{et}$$
$$(n+i)(n+i-1)b + (n+i)\left(\frac{2bc}{a} + 2b - e\right) + g + \frac{bc}{a} + \frac{bcc}{aa} - \frac{ce}{a} = 0.$$

Quae integralia etsi sunt particularia, tamen ex iis completa facile determinantur.

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SOLV-