NOVA RATIO QVANTITATES IRRATIONALES PROXIME EXPRIMENDI.

Auctore

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1.

Omnem quantitatem irrationalem ſimplicem ad hanc formam $(1+x)^n$ reduci poſſe conſtat ſiquidem exponens n numerum quemcunque fractum deſignare aſſumatur; quicunque enim numerus N ad exponentem fractum $n=\frac{\mu}{\nu}$ eleuandus proponatur eum ſemper ad hanc formam $a^\nu+b$ reuocare licet vnde formula propoſita fit $(a^\nu+b)^{\frac{\mu}{\nu}}=a^\mu\left(1+\frac{b}{a^\nu}\right)^{\frac{\mu}{\nu}}$ ſicque irrationalibus continetur in expreſſione $\left(1+\frac{b}{a^\nu}\right)^{\frac{\mu}{\nu}}$ quae cum formula propoſita $(1+x)^n$ congruit ponendo $\frac{b}{a^\nu}=x$ et $\frac{\mu}{\nu}=n$. Ac ſi pro a fractiones velimus admittere, ac b aeque negatiue ac poſitiue ſumere, quantitas $\frac{b}{a^\nu}$ hoc modo iam quouis caſu ſatis parua effici poteſt, vnde etiam more conſueto formula $(1+x)^n$ in ſeriem admodum conuergentem reſoluitur.

2. Per

2. Per euolutionem scilicet binomii Neutonianam haec formula $(1+x)^n$ duplici modo in seriem infinitam conuertitur, primum nempe directe:

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1.\,2}x^2 + \frac{n(n-1)(n-2)}{1.\,2.\,3}x^3 + \text{etc.}$$

tum vero quia est $(1+x)^n = \frac{1}{(1+x)^{-n}}$ erit quoque

$$(1+x)^n = \frac{1}{1 - \frac{n}{1}x + \frac{n(n+1)}{1.\,2}x^2 - \frac{n(n+1)(n+2)}{1.\,2.\,3}x^3 + \text{etc.}}$$

Hinc vero porro has expressiones inuicem multiplicando, et pro $2n$ scribendo n deriuabitur tertia expressio multo magis conuergens:

$$(1+x)^n = \frac{1 + \frac{n}{2}x + \frac{n(n-2)}{2.\,4}x^2 + \frac{n(n-2)(n-4)}{2.\,4.\,6}x^3 + \text{etc.}}{1 - \frac{n}{2}x + \frac{n(n+2)}{2.\,4}x^2 - \frac{n(n+2)(n+4)}{2.\,4.\,6}x^3 + \text{etc.}}$$

3. Attendenti autem facile patebit, infinitas expressiones huic postremae similes exhiberi posse, quae singulae aequales sint formulae propositae $(1+x)^n$, si enim ponamus:

$$(1+x)^n = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + \text{etc.}}{1 - \alpha x + \beta x^2 - \gamma x^3 + \delta x^4 - \varepsilon x^5 + \zeta x^6 - \text{etc.}}$$

determinatio coëfficientium praebet problema indeterminatum, atque adeo si vel numerator vel denominator ad lubitum assumitur, alterius coëfficientes inde determinantur. Hinc quaestio nascitur maximi momenti, quomodo tam numerator quam denominator determinari debeant, vt ambo simul maxime conuergant: atque hic quidem denominatori finitum terminorum numerum tribuere licet, vbi quaestio huc redit, quomodo coëfficientes denominatoris assumi oporteat, vt pro numeratore resulte series maxime conuergens.

4. Quodsi autem in denominatore datus terminorum numerus constituatur, numerator erit series maxime conuergens, si vnus pluresue eius termini se ordine excipientes plane euanescant; tum enim sequentes termini tam fient exigui, si quidem fuerit $x < 1$, vt sine notabili errore reiicquant. Atque hic notari conuenit, si pro denominatore sumatur binomium $1 - \alpha x$, quemlibet numeratoris terminum ad nihilum redigi posse; sin autem denominator statuatur trinomium, bini termini successiui numeratoris in nihilum redigi poterunt; terni vero et ita porro, si pro denominatore quadrinomium vel multinomium assumatur. Tum vero etiam perspicuum est aduergentiam eo fore maiorem, quo longius numeratoris termini euanescentes ab initio distent; vnde sequentia problemata resoluenda occurrunt.

Problema I.

5. *Binomii potestatem $(1 + x)^n$ transformare in talem expressionem maxime conuergentem:*

$$(1 + x)^n = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + etc.}{1 - \alpha x}$$

denominatore existente binomio.

Solutio.

Si potestas $(1 + x)^n$ in seriem euoluatur, eaque per denominatorem $1 - \alpha x$ multiplicetur, orietur sequens aequatio conficienda:

$0 = 1$

$$0 = 1 + \tfrac{n}{1}x + \tfrac{n(n-1)}{1.\,2}x^2 + \tfrac{n(n-1)(n-2)}{1.\,2.\,3}x^3 + \tfrac{n(n-1)(n-2)(n-3)}{1.\,2.\,3.\,4}x^4 + \text{etc.}$$
$$-\alpha - \tfrac{n}{1}\alpha - \tfrac{n(n-1)}{1.\,2}\alpha - \tfrac{n(n-1)(n-2)}{1.\,2.\,3}\alpha - \text{etc.}$$
$$-1 - A - B - C - D - \text{etc.}$$

Iam prouti numeratoris terminus vel secundus vel tertius vel quartus etc. euanescere debet, sequentes coëfficientium determinationes obtinebuntur:

I. Si $A = 0$, habetur statim $\alpha = \frac{n}{1}$; et sequentes numeratoris termini erunt:

$B = -\frac{n(n+1)}{1.\,2}$; $C = -\frac{2n(n-1)(n+1)}{1.\,2.\,3}$; $D = -\frac{3n(n-1)(n-2)(n+1)}{1.\,2.\,3.\,4}$ etc.

II. Si $B = 0$, habetur statim $\alpha = \frac{n-1}{2}$, et pro numeratore:

$A = \frac{n+1}{1.\,2}$; $C = -\frac{1(n+1)n(n-1)}{2.\,1.\,2.\,3}$; $D = -\frac{2(n+1)n(n-1)(n-2)}{2.\,1.\,2.\,3.\,4}$ etc.

III. Si $C = 0$; habetur $\alpha = \frac{n-2}{3}$ et pro numeratore:

$A = \frac{2(n+1)}{3.\,1}$; $B = \frac{1(n+1)n}{3.\,1.\,2}$; $D = -\frac{1(n+1)n(n-1)(n-2)}{3.\,1.\,2.\,3.\,4}$ etc.

IV. Si $D = 0$; habetur $\alpha = \frac{n-3}{4}$ et pro numeratore:

$A = \frac{3(n+1)}{4.\,1}$; $B = \frac{2(n+1)n}{4.\,1.\,2}$; $C = \frac{1(n+1)n(n-1)}{4.\,1.\,2.\,3}$ etc.

Hinc iam in genere patet, si quilibet alius sequentium terminorum in numeratore debeat euanescere, haberi primo:

$\alpha = \frac{n-\omega}{\omega+1}$ et pro numeratore:

$A = \frac{\omega(n+1)}{(\omega+1)1}$; $B = \frac{(\omega-1)}{(\omega+1)}\frac{(n+1)n}{1.\,2}$; $C = \frac{(\omega-2)}{(\omega+1)}\frac{(n+1)n(n-1)}{1.\,2.\,3}$

$D = \frac{(\omega-3)}{(\omega+1)}\frac{(n+1)n(n-1)(n-2)}{1.\,2.\,3.\,4}$; $E = \frac{(\omega-4)}{\omega+1}.\frac{(n+1)n(n-1)(n-2)(n-3)}{1.\,2.\,3.\,4.\,5}$

 cuius

cuius progressionis lex est manifesta.

Coroll. 1.

6. Quodsi iam in numeratore termini, q[illegible] euanescentem sequuntur, omittantur, habebuntur e[illegible] pressiones finitae ac rationales continuo propius val[illegible] rem $(1+x)^n$ exhibentes; ita si primo ponatur A = [illegible], habebitur ista appropinquatio:

$$(1+x)^n = \frac{1}{1-nx}$$

quae etsi a veritate parum recedit, tamen mag[illegible] aberrat quam sequentes.

Coroll. 2.

7. Sit B = 0, et secundus casus praebebit han[illegible] appropinquationem:

$$(1+x)^n = \frac{1+\frac{n+1}{2}x}{1-\frac{(n-1)}{2}x} = \frac{1+\frac{(n+1)}{2}x}{1-\frac{(n-1)}{2}x}.$$

Hinc si sit $n = \frac{\mu}{\nu}$ erit:

$$(1+x)^{\frac{\mu}{\nu}} = \frac{1+\frac{(\mu+\nu)}{2\nu}x}{1-\frac{(\mu-\nu)}{2\nu}x}.$$

Coroll. 3.

8. Sit C = 0, et tertius casus dabit:

$$(1+x)^n = \frac{1+\frac{2(n+1)}{4}x+\frac{(n+1)n}{4\cdot 3}xx}{[illegible]\ (n-2)\ [illegible]}$$

vnd

vnde si fuerit $n = \frac{\mu}{\nu}$ erit:

$$(1+x)^{\frac{\mu}{\nu}} = \frac{1 + \frac{2(\mu+\nu)}{3 \cdot 1 \cdot \nu} x + \frac{1(\mu+\nu)\mu}{3 \cdot 1 \cdot 2 \cdot \nu^2} x^2}{1 - \frac{(\mu - 2\nu)}{3\nu} x}.$$

Coroll. 4.

9. Sit $D = 0$, et quartus casus dat:

$$(1+x)^n = \frac{1 + \frac{3(n+1)}{4 \cdot 1} x + \frac{2(n+1)n}{4 \cdot 1 \cdot 2} x^2 + \frac{1(n+1)n(n-1)}{4 \cdot 1 \cdot 2 \cdot 3} x^3}{1 - \frac{(n-3)}{4} x}$$

ideoque si $n = \frac{\mu}{\nu}$ erit:

$$(1+x)^{\frac{\mu}{\nu}} = \frac{1 + \frac{3(\mu+\nu)}{4 \cdot 1 \cdot \nu} x + \frac{2(\mu+\nu)\mu}{4 \cdot 1 \cdot 2 \cdot \nu^2} x^2 + \frac{1(\mu+\nu)\mu(\mu-\nu)}{4 \cdot 1 \cdot 2 \cdot 3 \cdot \nu^3} x^3}{1 - \frac{(\mu - 3\nu)}{4\nu} x}$$

vnde perspicuum est, quomodo huiusmodi formulae vlterius continuari debent; quamobrem plures hic non exhibeo.

Coroll. 5.

10. In genere autem habebitur haec forma:

$$(1+x)^n = \frac{1 + \frac{(\omega-1)(n+1)}{\omega \cdot 1} x + \frac{(\omega-2)(n+1)n}{\omega \cdot 1 \cdot 2} x^2 + \frac{(\omega-3)(n+1)n(n-1)}{\omega \cdot 1 \cdot 2 \cdot 3} x^3 + \text{etc.}}{1 - \frac{(n - \omega + 1)}{\omega} x}$$

vbi pro ω sumi potest numerus quicunque; haecque expressio si in infinitum continuetur, non solum ad veritatem appropinquat, sed ipsum verum valorem formulae $(1+x)^n$ exhibebit.

Coroll. 6.

11. Si sumatur $\omega = n + 1$ denominator in vnitatem abibit, orieturque nota series Neutoniana:

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1.\ 2}x^2 + \frac{n(n-1)(n-2)}{1.\ 2.\ 3}x^3 + \text{etc.}$$

Sin autem pro ω capiatur numerus infinitus, erit:

$$(1+x)^n = \frac{1 + \frac{(n+1)}{1}x + \frac{(n+1)n}{1.\ 2}x^2 + \frac{(n+1)n(n-1)}{1.\ 2.\ 3}x^3 + \text{etc.}}{1+x}$$

cuius ratio quoque ex binomio Neutoniano est manifesta.

Coroll. 7.

12. Si ponatur $\omega = n$ habebitur: $(1+x)^n =$

$$\frac{1 + \frac{(n+1)(n-1)}{1.\ n}x + \frac{(n+1)(n-2)}{1.\ 2}x^2 + \frac{(n+1)(n-1)(n-3)}{1.\ 2.\ 3}x^3 + \frac{(n+1)(n-1)(n-2)(n-4)}{1.\ 2.\ 3.\ 4}x^4 + \text{etc.}}{1 - \frac{1}{n}x}$$

vel numeratorem et denominatorem per n multiplicando:

$$(1+x)^n = \frac{n + \frac{(n+1)(n-1)}{1}x + \frac{(n+1)n(n-2)}{1.\ 2}x^2 + \frac{(n+1)n(n-1)(n-3)}{1.\ 2.\ 3}x^3 + \text{etc}}{n-x}$$

Coroll. 8.

13. Si ponatur $\omega = x$, fiet denominator $= x - n$ et obtinetur: $(1+x)^n =$

$$\frac{1 + \frac{(n+1)}{1}(x-1) + \frac{(n+1)n}{1.\ 2}x(x-2) + \frac{(n+1)n(n-1)}{1.\ 2.\ 3}x^2(x-3) + \frac{(n+1)n(n-1)(n-2)}{1.\ 2.\ 3.\ 4}x^3(x-4) + \text{etc.}}{x-n}$$

similique modo ex hac expressione innumerabiles series deduci possunt, quarum ratio aliunde non tam facile perspici poterit; vnde haec inuestigatio doctrinam serierum non mediocriter amplificare videtur.

Proble-

Problema II.

14. *Binomii poteſtatem* $(1+x)^n$ *transformare in huiusmodi ſeriem maxime conuergentem:*

$$(1+x)^n = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + \text{etc.}}{1 - \alpha x + \beta x^2}$$

denominatore exiſtente trinomio.

Solutio.

Reſoluta poteſtate $(1+x)^n$ in ſeriem more conſueto, confici oportebit ſequentem aequationem:

$$\begin{array}{l} 0 = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^4 + \text{etc.} \\ \quad - \alpha - \frac{n}{1}\alpha - \frac{n(n-1)}{1\cdot 2}\alpha - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\alpha - \text{etc.} \\ \quad\quad + \beta + \frac{n}{1}\beta + \frac{n(n-1)}{1\cdot 2}\beta + \text{etc.} \\ \quad -1 - A - B - C - D - \text{etc.} \end{array}$$

atque hic denominatorem $1 - \alpha x + \beta x^2$ ita definire licet, vt in numeratore bini termini ſucceſſiue euaneſcant, vnde is eo magis conuergens reddetur:

I. Sit $A = 0$ et $B = 0$, erit $\alpha = \frac{n}{1}$ et $\beta = \frac{n(n+1)}{1\cdot 2}$, vnde habetur

$$C = \frac{n}{1}\left(\frac{(n-1)(n-2)}{2\cdot 3} - \frac{(n-1)n}{2\cdot 1} + \frac{(n+1)n}{1\cdot 2}\right) = \frac{1(n+2)(n+1)n}{3\cdot 1\cdot 2\cdot 1}$$

$$D = \frac{n(n-1)}{1\cdot 2}\left(\frac{(n-1)(n-3)}{3\cdot 4} - \frac{(n-2)n}{3\cdot 1} + \frac{n(n+1)}{1\cdot 2}\right) = \frac{2(n+2)(n+1)n(n-1)}{4\cdot 1\cdot 2\cdot 3}$$

$$E = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{(n-3)(n-4)}{4\cdot 5} - \frac{(n-3)n}{4\cdot 1} + \frac{(n+1)n}{1\cdot 2}\right) = \frac{3(n+2)(n+1)n(n-1)(n-2)}{5\cdot 1\cdot 1\cdot 2\cdot 3}$$

et in genere erit:

$$N = \ldots\ldots\left(\frac{(n-\nu)(n-\nu-1)}{(\nu+1)(\nu+2)} - \frac{(n-\nu)n}{(\nu+1)1} + \frac{(n+1)n}{1\cdot 2}\right) = \ldots\ldots\frac{\nu(n+2)(n+1)}{1\cdot 2\cdot(\nu+2)}$$

ex quo generali valore illi ſpeciales facile deriuantur.

II.

II. Sit B = o et C = o, erit pro α et $\mathfrak{C}$:

$$\mathfrak{C} - \frac{n}{1}\alpha + \frac{n(n-1)}{1\cdot 2} = 0, \quad \text{hinc} \quad \alpha = \frac{2(n-1)}{2}$$
$$\mathfrak{C} - \frac{(n-1)}{2}\alpha + \frac{(n-1)(n-2)}{2\cdot 3} = 0, \qquad \mathfrak{C} = \frac{n(n+1)}{2\cdot 3}.$$

Pro numeratore vero habebitur:

$$A = \frac{n}{1} - \frac{2(n-1)}{2} = \frac{n+2}{2}$$
[illegible]
$$D = \frac{n(n-1)}{1\cdot 2}\left(\frac{(n-2)(n-3)}{3\cdot 4} - \frac{2(n-2)(n-1)}{3\cdot 2} + \frac{n(n-1)}{[illegible]}\right) = \frac{[illegible](n+2)(n+1)n(n-1)}{[illegible]\ 3\cdot 1\cdot 2}$$
$$E = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{(n-3)(n-4)}{4\cdot 5} - \frac{2(n-3)(n-1)}{[illegible]} + \frac{n(n-1)}{[illegible]\cdot 3}\right) = \frac{2\cdot 3(n+2)(n+1)n(n-1)(n-2)}{4\cdot 5\ 2\cdot 3\cdot 1\cdot 2\cdot 3}$$
$$F = \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}\left(\frac{(n-4)(n-5)}{5\cdot 6} - \frac{2(n-4)(n-1)}{5\cdot 2} + \frac{n(n-1)}{2\cdot 3}\right) = \frac{2\cdot 4(n+2)(n+1)n(n-1)(n-2)(n-3)}{5\cdot 6\ 2\cdot 3\cdot 1\cdot 2\cdot 3\cdot 4}$$

et in genere:

$$N = \ldots\ldots\left(\frac{(n-\nu)(n-\nu-1)}{(\nu+1)(\nu+2)} - \frac{2(n-\nu)(n-1)}{(\nu+1)\ 2} + \frac{n(n-1)}{2\cdot 3}\right) \ldots\ldots \frac{\nu(\nu-1)}{(\nu+1)(\nu+2)}\,\frac{(n+2)(n+1)}{2\cdot 3}$$

III. Sit C = o et D = o, at pro denominatore erit [illegible]

$$\mathfrak{C} - \frac{(n-2)}{2}\alpha + \frac{(n-1)(n-2)}{2\cdot 3} = 0, \quad \text{hinc} \quad \alpha = \frac{n-3}{[illegible]}$$
$$\mathfrak{C} - \frac{(n-3)}{3}\alpha + \frac{(n-2)(n-3)}{3\cdot 4} = 0, \qquad \mathfrak{C} = \frac{(n-1)(n-2)}{3\cdot 4}$$

hincque pro numeratore:

$$A = \frac{n}{1} - \frac{(n-2)}{2} = \frac{n+2}{2}$$
$$B = \frac{n(n-1)}{1\cdot 2} - \frac{n(n-2)}{[illegible]} + \frac{(n-1)(n-2)}{[illegible]} = \frac{(n+2)(n+1)}{[illegible]}$$
[illegible]
$$E = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{(n-3)(n-4)}{4\cdot 5} - \frac{(n-3)(n-2)}{4\cdot 3} + \frac{(n-1)(n-2)}{3\cdot 4}\right) = \frac{(n+2)(n+1)n(n-1)(n-2)}{3\cdot 4\cdot 5\cdot 1\cdot 2\cdot 3}$$

Quia autem sufficit terminos, qui euanescentes antecedant, nosse, sequentes non determino, quia eorum lex deinceps patebit.

IV. Sit D = o et E = o, erit pro denominatore

$$\mathfrak{C} - \frac{(n-3)}{3}\alpha + \frac{(n-2)(n-3)}{3\cdot 4} = 0, \quad \text{hinc} \quad \alpha = \frac{2(n-3)}{4}$$
$$\mathfrak{C} - \frac{(n-4)}{4}\alpha + \frac{(n-3)(n-4)}{4\cdot 5} = 0, \qquad \mathfrak{C} = \frac{(n-2)(n-3)}{4\cdot 5}$$

at pro numeratore reperietur:

$$A = \frac{n}{1} - \frac{2(n-3)}{5} = \frac{3(n+2)}{5}$$

$$B = \frac{n(n-1)}{1.\,2} - \frac{2n(n-3)}{1.\,5} + \frac{(n-2)(n-3)}{4.\,5} = \frac{3(n+2)(n+1)}{5.\,4}$$

$$C = \frac{n}{1}\left(\frac{(n-1)(n-2)}{2.\,3} - \frac{2(n-1)(n-3)}{2.\,5} + \frac{(n-2)(n-3)}{4.\,5}\right) = \frac{(n+2)(n+1)n}{5.\,4.\,3}.$$

V. Sit $E = 0$ et $F = 0$, atque ex allatis facile concludimus fore primo

$$\alpha = \frac{2(n-4)}{6};\ \beta = \frac{(n-3)(n-4)}{5.\,6} \text{ tum vero}$$

$$A = \frac{4(n+2)}{6};\ B = \frac{6(n+2)(n+1)}{6.\,5};\ C = \frac{4(n+2)(n+1)(n)}{6.\,5.\,4} \text{ et}$$

$$D = \frac{1(n+2)(n+1)(n)(n-1)}{6.\,5.\,4.\,3}.$$

Generaliter ergo denique has eliciemus determinationes:

$$\alpha = \frac{2(n-\omega)}{\omega+2};\quad \beta = \frac{(n-\omega)(n-\omega+1)}{(\omega+2)(\omega+1)}$$

$$A = \frac{\omega}{1.} \cdot \frac{(n+2)}{\omega+2}$$

$$B = \frac{\omega(\omega-1)}{1.\,2.} \cdot \frac{(n+2)(n+1)}{(\omega+2)(\omega+1)}$$

$$C = \frac{\omega(\omega-1)(\omega-2)}{1.\,2.\,3.} \cdot \frac{(n+2)(n+1)n}{(\omega+2)(\omega+1)\omega}$$

$$D = \frac{\omega(\omega-1)(\omega-2)(\omega-3)}{1.\,2.\,3.\,4} \cdot \frac{(n+2)(n+1)n(n-1)}{(\omega+2)(\omega+1)\omega(\omega-1)}$$

$$E = \frac{\omega(\omega-1)(\omega-2)(\omega-3)(\omega-4)}{1.\,2.\,3.\,4.\,5} \cdot \frac{(n+2)(n+1)n(n-1)(n-2)}{(\omega+2)(\omega+1)\omega(\omega-1)(\omega-2)}$$

etc.

vnde etiam coefficientes terminorum post euanescentes sequentium facile formantur.

Coroll. I.

15. Quando pro denominatore in genere est:

$$\alpha = \frac{2(n-\omega)}{\omega+2} \quad \text{et} \quad \beta = \frac{(n-\omega)(n-\omega+1)}{(\omega+2)(\omega+1)}$$

pro numeratore habebimus:

$$A = \frac{\omega}{\omega + 2} \cdot \frac{n + 2}{1}$$

$$B = \frac{\omega(\omega - 1)}{(\omega + 2)(\omega + 1)} \cdot \frac{(n + 2)(n + 1)}{1 \cdot 2}$$

$$C = \frac{(\omega - 1)(\omega - 2)}{(\omega + 2)(\omega + 1)} \cdot \frac{(n + 2)(n + 1)n}{1 \cdot 2 \cdot 3}$$

$$D = \frac{(\omega - 2)(\omega - 3)}{(\omega + 2)(\omega + 1)} \cdot \frac{(n + 2)(n + 1)n(n - 1)}{1 \cdot 2 \cdot 3 \cdot 4}$$

$$E = \frac{(\omega - 3)(\omega - 4)}{(\omega + 2)(\omega + 1)} \cdot \frac{(n + 2)(n + 1)n(n - 1)(n - 2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

etc.

quorum valorum analogia ad eos, qui in primo problemate sunt inuenti, iam satis luculenter ordinem sequentium, vbi denominator pluribus constabit terminis, declarat.

Coroll. 2.

16. Neglectis terminis in numeratore post euanescentes sequentibus, habebimus approximationes sequentes:

$$\text{si } \omega = 0 \text{ erit } (1 + x)^n = \frac{1}{1 - nx + \frac{n(n + 1)}{1 \cdot 2} xx}$$

quae quidem in hoc genere plurimum a veritate discrepat.

Coroll. 3.

17. Ponamus $\omega = 1$, eritque proxime:

$$(1 + x)^n = \frac{1 + \frac{n + 2}{3} x}{1 - \frac{2(n - 1)}{3} x + \frac{n(n - 1)}{2 \cdot 3} xx}$$

 sit

fin autem $\omega = 2$ erit adhuc propius:

$$(1+x)^n = \frac{1 + \frac{2(n+2)}{4}x + \frac{(n+2)(n+1)}{4\cdot 3}xx}{1 - \frac{2(n-2)}{4}x + \frac{(n-2)(n-1)}{4\cdot 3}xx}$$

et fi $\omega = 3$ erit

$$(1+x)^n = \frac{1 + \frac{3(n+2)}{5}x + \frac{3(n+2)(n+1)}{5\cdot 4}xx + \frac{(n+2)(n+1)n}{5\cdot 4\cdot 3}x^3}{1 - \frac{2(n-3)}{5}x + \frac{(n-3)(n-2)}{5\cdot 4}xx}$$

fi $\omega = 4$ erit

$$(1+x)^n = \frac{1 + \frac{4(n+2)}{6}x + \frac{6(n+2)(n+1)}{6\cdot 5}xx + \frac{4(n+2)(n+1)n}{6\cdot 5\cdot 4}x^3 + \frac{(n+2)(n+1)n(n-1)}{6\cdot 5\cdot 4\cdot 3}x^4}{1 - \frac{2(n-4)}{6}x + \frac{(n-4)(n-3)}{6\cdot 5}xx}$$

fi $\omega = 5$ erit $(1+x)^n =$

$$\frac{1 + \frac{5(n+2)}{7}x + \frac{10(n+2)(n+1)}{7\cdot 6}x^2 + \frac{10(n+2)(n+1)n}{7\cdot 6\cdot 5}x^3 + \frac{5(n+2)(n+1)n(n-1)}{7\cdot 6\cdot 5\cdot 4}x^4 + \frac{(n+2)(n+1)n(n-1)(n-2)}{7\cdot 6\cdot 5\cdot 4\cdot 3}x^5}{1 - \frac{2(n-5)}{7}x + \frac{(n-5)(n-4)}{7\cdot 6}xx}$$

Quae expreffiones ex coëfficientibus poteftatum binomii expedite vlterius continuantur. Quo longius vero continuantur, eo minus a veritate aberrabunt.

Coroll. 4.

18. Generaliter autem hanc formulae $(1+x)^n$ transformationem commodius exhibere non licet, quam vt dicamus effe

$$(1+x)^n = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + etc.}{1 - \alpha x + \beta xx}$$

exiftentibus coefficientium valoribus:

$$A = \frac{(\omega+1)\omega}{(\omega+2)(\omega+1)} \cdot \frac{n+2}{1} \qquad \alpha = \frac{2(n-\omega)}{\omega+2}$$

$$B = \frac{\omega(\omega-1)}{(\omega+2)(\omega+1)} \cdot \frac{(n+2)(n+1)}{1 \cdot 2} \qquad \beta = \frac{(n-\omega)(n-\omega-1)}{(\omega+2)(\omega+1)}$$

$$C = \frac{(\omega-1)(\omega-2)}{(\omega+2)(\omega+1)} \cdot \frac{(n+2)(n+1)n}{1 \cdot 2 \cdot 3}$$

$$D = \frac{(\omega-2)(\omega-1)}{(\omega+2)(\omega+1)} \cdot \frac{(n+2)(n+1)n(n-1)}{1 \cdot 2 \cdot 3 \cdot 4}$$

$$E = \frac{(\omega-3)(\omega-4)}{(\omega+2)(\omega+1)} \cdot \frac{(n+2)(n+1)n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

etc.

Coroll. 5.

19. Hic iterum patet, cum quantitas ω b arbitrio nostro pendeat, si capiatur $\omega = n$, prodi e $\alpha = 0$, $\beta = 0$ et

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \text{etc.}$$

Sin autem fit $\omega = \infty$ erit $\alpha = -2$, et $\beta = 1$; vn e

$$(1+x)^n = \frac{1 + \frac{n+2}{1}x + \frac{(n+2)(n+1)}{1 \cdot 2}x^2 + \frac{(n+2)(n+1)n}{1 \cdot 2 \cdot 3}x^3 + \text{etc.}}{1 + 2x + xx}$$

seu $(1+x)^n = \dfrac{(1+x)^{n+2}}{(1+x)^2}$, cuius ratio est manifesta

Problema III.

20. *Binomii potestatem* $(1+x)^n$ *transformare* *i* *huiusmodi seriem maxime conuergentem:*

$$(1+x)^n = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Cx^6 + \text{etc}}{1 - \alpha x + \beta x^2 - \gamma x^3}$$

denominatore existente quadrinomio.

Soluti

Solutio.

Sequens ergo aequatio conftrui debet:

$$0 = 1 + \frac{n}{1}x + \frac{n(n-1)}{1.\,2}x^2 + \frac{n(n-1)(n-2)}{1.\,2.\,3}x^3 + \frac{n(n-1)(n-2)(n-3)}{1.\,2.\,3.\,4}x^4 + \text{etc.}$$
$$-\alpha - \frac{n}{1}\alpha - \frac{n(n-1)}{1.\,2}\alpha - \frac{n(n-1)(n-2)}{1.\,2.\,3}\alpha - \text{etc.}$$
$$+ \beta + \frac{n}{1}\beta + \frac{n(n-1)}{1.\,2}\beta + \text{etc.}$$
$$- \gamma - \frac{n}{1}\gamma - \text{etc.}$$
$$-1 - A - B - C - D - \text{etc.}$$

Hic iam effici poteft, vt in ferie coefficientium A, B, C, D etc. terni fucceffiui euanefcant: Sumantur ergo terni quicunque fucceffiue euanefcentes, et obtinebuntur tres huiusmodi aequationes:

$$\gamma - \frac{(n-\omega+2)}{\omega-1}\beta + \frac{(n-\omega+2)(n-\omega+1)}{(\omega-1)(\omega+0)}\alpha - \frac{(n-\omega+2)(n-\omega+1)(n-\omega)}{(\omega-1)(\omega-0)(\omega+1)} = 0$$
$$\gamma - \frac{(n-\omega+1)}{\omega}\beta + \frac{(n-\omega+1)(n-\omega)}{\omega(\omega+1)}\alpha - \frac{(n-\omega+1)(n-\omega)(n-\omega-1)}{\omega(\omega+1)(\omega+2)} = 0$$
$$\gamma - \frac{(n-\omega)}{\omega+1}\beta + \frac{(n-\omega)(n-\omega-1)}{(\omega+1)(\omega+2)}\alpha - \frac{(n-\omega)(n-\omega-1)(n-\omega-2)}{(\omega+1)(\omega+2)(\omega+3)} = 0.$$

Hinc differentiis fumendis habebitur:

$$\frac{(n+1)}{(\omega-1)\omega}\beta - \frac{2(n+1)(n-\omega+1)}{(\omega-1)\omega(\omega+1)}\alpha + \frac{3(n+1)(n-\omega+1)(n-\omega)}{(\omega-1)\omega(\omega+1)(\omega+2)} = 0$$
$$\frac{(n+1)}{\omega(\omega+1)}\beta - \frac{2(n+1)(n-\omega)}{\omega(\omega+1)(\omega+2)}\alpha + \frac{3(n+1)(n-\omega)(n-\omega-1)}{\omega(\omega+1)(\omega+2)(\omega+3)} = 0$$

fiue:

$$\beta - \frac{2(n-\omega+1)}{\omega+1}\alpha + \frac{3(n-\omega+1)(n-\omega)}{(\omega+1)(\omega+2)} = 0$$
$$\beta - \frac{2(n-\omega)}{\omega+2}\alpha + \frac{3(n-\omega)(n-\omega-1)}{(\omega+2)(\omega+3)} = 0$$

quarum aequationum differentia dat:

$$\frac{2(n+1)}{(\omega+1)(\omega+2)}\alpha - \frac{2.\,3\,(n+2)(n-\omega)}{(\omega+1)(\omega+2)(\omega+3)} = 0$$

hincque fit:

$$\alpha = \frac{3(n-\omega)}{\omega+3};\quad \beta = \frac{3(n-\omega)(n-\omega+1)}{(\omega+3)(\omega+2)};\ \text{et}\ \gamma = \frac{(n-\omega)(n-\omega+1)(n-\omega+2)}{(\omega+3)(\omega+2)(\omega+1)}$$

His autem valoribus pro denominatore inuentis pro numeratore reperientur:

$$A = \frac{\omega}{\omega+3}\cdot\frac{n+3}{1}$$

$$B = \frac{\omega(\omega-1)}{(\omega+3)(\omega+2)}\cdot\frac{(n+3)(n+2)}{1\cdot 2}$$

$$C = \frac{\omega(\omega-1)(\omega-2)}{(\omega+3)(\omega+2)(\omega+1)}\cdot\frac{(n+3)(n+2)(n+1)}{1\cdot 2\cdot 3}$$

$$D = \frac{(\omega-1)(\omega-2)(\omega-3)}{(\omega+3)(\omega+2)(\omega+1)}\cdot\frac{(n+3)(n+2)(n+1)n}{1\cdot 2\cdot 3\cdot 4}$$

$$E = \frac{(\omega-2)(\omega-3)(\omega-4)}{(\omega+3)(\omega+2)(\omega+1)}\cdot\frac{(n+3)(n+2)(n+1)n(n-1)}{1\cdot 2\cdot 3\cdot 4\cdot 5}$$

$$F = \frac{(\omega-3)(\omega-4)(\omega-5)}{(\omega+3)(\omega+2)(\omega+1)}\cdot\frac{(n+3)(n+2)(n+1)n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$$

etc.

ac denominator formabitur ex his valoribus:

$$\alpha = \frac{3(n-\omega)}{\omega+3}$$

$$\beta = \frac{3(n-\omega)(n-\omega+1)}{(\omega+3)(\omega+2)}$$

$$\gamma = \frac{(n-\omega)(n-\omega+1)(n-\omega+2)}{(\omega+3)(\omega+2)(\omega+1)}$$

quibus substitutis erit

$$(1+x)^n = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \text{etc.}}{1 - \alpha x + \beta x^2 - \gamma x^3}$$

Coroll. 1.

21. Manifestum hic est, quicunque numeri integer positiuus pro ω assumatur, in numerator semper terminos ternos successiuos in nihilum abire Ita si sit $\omega = 0$ erit

(1+x)

$$(1+x)^n = \frac{1 + * + * + * - \frac{(n+3)(n+2)(n+1)n}{1\cdot 2\cdot 3\cdot 4}x^4 - \frac{4(n+3)(n+2)(n+1)n(n-1)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5 - \text{etc.}}{1 - \frac{n}{1}x + \frac{n(n+1)}{1\cdot 2}x^2 - \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}x^3}$$

Unde reiectis in numeratore terminis, qui post euanescentes sequuntur, erit proxime:

$$(1+x)^n = \frac{1}{1 - \frac{n}{1}x + \frac{n(n+1)}{1\cdot 2}x^2 - \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}x^3}.$$

Coroll. 2.

22. Simili modo ponendo $\omega = 1$ erit proxime:

$$(1+x)^n = \frac{1 + \frac{n+3}{4}x}{1 - \frac{3(n-1)}{4}x + \frac{3(n-1)n}{4\cdot 3}x^2 - \frac{(n-1)n(n+1)}{4\cdot 3\cdot 2}x^3}$$

at si sumatur $\omega = 2$ erit

$$(1+x)^n = \frac{1 + \frac{2(n+3)}{5}x + \frac{(n+3)(n+2)}{5\cdot 4}x^2}{1 - \frac{3(n-1)}{5}x + \frac{3(n-2)(n-1)}{5\cdot 4}x^2 - \frac{(n-2)(n-1)n}{5\cdot 4\cdot 3}x^3}$$

posito vero $\omega = 3$ erit

$$(1+x)^n = \frac{1 + \frac{3(n+3)}{6}x + \frac{3(n+3)(n+2)}{6\cdot 5}x^2 + \frac{(n+3)(n+2)(n+1)}{6\cdot 5\cdot 4}x^3}{1 - \frac{3(n-1)}{6}x + \frac{3(n-1)(n-2)}{6\cdot 5}x^2 - \frac{(n-3)(n-2)(n-1)}{6\cdot 5\cdot 4}x^3}.$$

Coroll. 3.

23. Postrema haec formula ideo est notatu digna, quod numerator et denominator pari terminorum numero constat, et quod alter in alterum abit, si exponens n sumatur negatiue. Haec ergo expressio conferenda est cum similibus ex problematibus superioribus ortis:

$$(1+x)^n = \frac{1 + \frac{(n+1)}{2}x}{1 - \frac{(n-1)}{2}x} \ldots (\S.\ 7.)$$

(1+x:

$$(1+x)^n = \frac{1 + \frac{2(n+2)}{4}x + \frac{(n+2)(n+1)}{4 \cdot 3}x^2}{1 - \frac{2(n-2)}{4}x + \frac{(n-2)(n-1)}{4 \cdot 3}x^2} \ldots (\S.\ 17.)$$

vnde ſimul ordo huiusmodi formularum facile colligitur.

Problema IV.

24. *Binomii poteſtatem $(1+x)^n$ transformare in huiusmodi progreſſionem maxime conuergentem:*

$$(1+x)^n = \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + etc.}{1 - \alpha x + \beta x^2 - \gamma x^3 + \delta x^4 - \varepsilon x^5 + \zeta x^6 - etc.}$$

denominatore exiſtente multinomio quocunque.

Solutio.

Si ſolutiones praecedentium problematum conſulamus, leui attentione adhibita inde ſequentem ſolutionem generalem colligimus:

$$A = \frac{\omega}{\omega+\Phi} \cdot \frac{n+\Phi}{1}$$

$$B = \frac{\omega(\omega-1)}{(\omega+\Phi)(\omega+\Phi-1)} \cdot \frac{(n+\Phi)(n+\Phi-1)}{1 \cdot 2}$$

$$C = \frac{\omega(\omega-1)(\omega-2)}{(\omega+\Phi)(\omega+\Phi-1)(\omega+\Phi-2)} \cdot \frac{(n+\Phi)(n+\Phi-1)(n+\Phi-2)}{1 \cdot 2 \cdot 3}$$

$$D = \frac{\omega(\omega-1)(\omega-2)(\omega-3)}{(\omega+\Phi)(\omega+\Phi-1)(\omega+\Phi-2)(\omega+\Phi-3)} \cdot \frac{(n+\Phi)(n+\Phi-1)(n+\Phi-2)(n+\Phi-3)}{1 \cdot 2 \cdot 3 \cdot 4}$$

etc.

deinde vero pro denominatore:

$$\alpha = \frac{\Phi(n-\omega)}{1(\omega+\Phi)}$$

$$\beta = \frac{\Phi(\Phi-1)}{1 \cdot 2} \cdot \frac{(n-\omega)(n-\omega+1)}{(\omega+\Phi)(\Phi+\Phi-1)}$$

$$\gamma = \frac{\Phi(\Phi-1)(\Phi-2)}{1 \cdot 2 \cdot 3} \cdot \frac{(n-\omega)(n-\omega+1)(n-\omega+2)}{(\omega+\Phi)(\omega+\Phi-1)(\omega+\Phi-2)}$$

$$\delta = \frac{\Phi(\Phi-1)(\Phi-2)(\Phi-3)}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{(n-\omega)(n-\omega+1)(n-\omega+2)(n-\omega+3)}{(\omega+\Phi)(\omega+\Phi-1)(\omega+\Phi-2)(\omega+\Phi-3)}$$

etc.

qui

qui valores ad praecedentium formam propius reducuntur vt fit:

$$\alpha = \frac{\Phi}{\Phi + \omega} \cdot \frac{n - \omega}{1}$$

$$\beta = \frac{\Phi(\Phi - 1)}{(\Phi + \omega)(\Phi + \omega - 1)} \cdot \frac{(n - \omega)(n - \omega + 1)}{1. \quad 2}$$

$$\gamma = \frac{\Phi(\Phi - 1)(\Phi - 2)}{(\Phi + \omega)(\Phi + \omega - 1)(\Phi + \omega - 2)} \cdot \frac{(n - \omega)(n - \omega + 1)(n - \omega + 2)}{1. \quad 2. \quad 3.}$$

$$\delta = \frac{\Phi(\Phi - 1)(\Phi - 2)(\Phi - 3)}{(\Phi + \omega)(\Phi + \omega - 1)(\Phi + \omega - 2)(\Phi + \omega - 3)} \cdot \frac{(n - \omega)(n - \omega + 1)(n - \omega + 2)(n - \omega + 3)}{1. \quad 2. \quad 3. \quad 4.}$$

etc.

Etsi autem ex hac lege etiam denominator in infinitum continuari possit; tamen ex principio, vnde eum deduximus, patet eum non vltra terminos euanescentes produci debere, siquidem pro Φ sumatur numerus positiuus integer.

Coroll. 1.

25. Denominator ergo ex numeratore formari potest, si numeri Φ et ω inter se permutentur, et loco n scribatur $-n$. At posito $-n$ pro $+n$ formula $(1 + x)^n$ abit in $(1 + x)^{-n}$, vnde si fuerit $(1 + x)^n = \frac{P}{Q}$ erit $(1 + x)^{-n} = \frac{Q}{P}$, ex quo ratio huius conuersionis eo clarius perspicitur.

Coroll. 2.

26. Cum igitur numerator et denominator inter se permutari possint, etiam numeratorem apud terminos euanescentes abrumpere licet; tum vero denominatorem in infinitum continuari oportet, vt fractio obtineatur potestati $(1 + x)^n$ aequalis.

Coroll. 3.

27. Si ſumatur $\Phi = \omega$, numerator et denominator multo magis inter ſe aſſimilantur, ac tantum ratione ſigni exponentis n a ſe inuicem diſcrepabunt. Erit autem tunc:

$$A = \frac{\omega}{2\omega} \cdot \frac{n+\omega}{1}$$

$$B = \frac{\omega(\omega-1)}{2\omega(2\omega-1)} \cdot \frac{(n+\omega)(n+\omega-1)}{1 \cdot 2}$$

$$C = \frac{\omega(\omega-1)(\omega-2)}{2\omega(2\omega-1)(2\omega-2)} \cdot \frac{(n+\omega)(n+\omega-1)(n+\omega-2)}{1 \cdot 2 \cdot 3}$$

$$D = \frac{\omega(\omega-1)(\omega-2)(\omega-3)}{2\omega(2\omega-1)(2\omega-2)(2\omega-3)} \cdot \frac{(n+\omega)(n+\omega-1)(n+\omega-2)(n+\omega-3)}{1 \cdot 2 \cdot 3 \cdot 4}$$

etc.

$$\alpha = \frac{\omega}{2\omega} \cdot \frac{n-\omega}{1}$$

$$\beta = \frac{\omega(\omega-1)}{2\omega(2\omega-1)} \cdot \frac{(n-\omega)(n-\omega+1)}{1 \cdot 2}$$

$$\gamma = \frac{\omega(\omega-1)(\omega-2)}{2\omega(2\omega-1)(2\omega-2)} \cdot \frac{(n-\omega)(n-\omega+1)(n-\omega+2)}{1 \cdot 2 \cdot 3}$$

$$\delta = \frac{\omega(\omega-1)(\omega-2)(\omega-3)}{2\omega(2\omega-1)(2\omega-2)(2\omega-3)} \cdot \frac{(n-\omega)(n-\omega+1)(n-\omega+2)(n-\omega+3)}{1 \cdot 2 \cdot 3 \cdot 4}$$

etc.

Coroll. 4.

28. Hinc formulae ſuperiores (23) ad approximandum perquam idoneae deriuantur:

$$(1+x)^n = \frac{1 + \frac{1}{2} \cdot \frac{n+1}{1} x}{1 - \frac{1}{2} \cdot \frac{n-1}{1} x}$$

$$(1+x)^n = \frac{1 + \frac{2}{4} \cdot \frac{n+2}{1} x + \frac{2 \cdot 1}{4 \cdot 3} \cdot \frac{(n+2)(n+1)}{1 \cdot 2} x^2}{1 - \frac{2}{4} \cdot \frac{n-2}{1} x + \frac{2 \cdot 1}{4 \cdot 3} \cdot \frac{(n-2)(n-1)}{1 \cdot 2} x^2}$$

$(1+x)$

$$(1+x)^n = \frac{1 + \frac{3}{6}\cdot\frac{(n+3)}{1}x + \frac{3\cdot 2}{6\cdot 5}\cdot\frac{(n+3)(n+2)}{1\cdot\ \ 2}x^2 + \frac{3\cdot 2\cdot 1}{6\cdot 5\cdot 4}\cdot\frac{(n+3)(n+2)(n+1)}{1\cdot\ \ 2\cdot\ \ 3}x^3}{1 - \frac{3}{6}\cdot\frac{(n-3)}{1}x + \frac{3\cdot 2}{6\cdot 5}\cdot\frac{(n-3)(n-2)}{1\cdot\ \ 2}x^2 - \frac{3\cdot 2\cdot 1}{6\cdot 5\cdot 4}\cdot\frac{(n-3)(n-2)(n-1)}{1\cdot\ \ 2\cdot\ \ 3}x^3}$$

quae quomodo vlterius continuari debeant sponte patet.

Scholion 1.

29. Hae formulae eo magis sunt notatu dignae, quo minus earum ratio patet; nam etsi tam in numeratore quam denominatore lex progressionis est perspicua, secundum quam vterque in infinitum continuatur, tamen iam animaduertimus, alterutrum tantum in infinitum produci oportere, altero ex finito terminorum numero constante, ibi scilicet quouis casu terminari debet, vbi termini aliquot euanescere incipiunt; etiamsi deinceps iterum termini finitae magnitudinis occurrant. Haec autem ita sunt interpretanda, si in valoribus litterarum A, B, C, etc. α, β, γ etc. factor numeratoris euanescens a factore denominatoris euanescente tolli censeatur, ita vt fractio $\frac{\omega - m}{2\omega - 2m}$ casu $\omega = m$ vnitati aequalis statuatur. Sin autem, vti calculi ratio exigit, haec fractio tantum semissi vnitatis aequalis capiatur, tum continui ratio non amplius infringitur; ac si hac lege reperta tam numerator quam denominator etiam vltra terminos euanescentes in infinitum continuatur, fractio resultans formulae $(1+x)^n$ perfecte erit aequalis. Quod idem in genere est tenendum, dummodo inter numeros Φ et ω certa ratio statuatur, ita vt si $\Phi = \lambda\omega$ fractionis $\frac{\omega - m}{(\lambda+1)\omega - (\lambda+1)m}$

etiam casu $\omega = m$ sumatur $= \frac{1}{\lambda+1}$ ex quo haec cautio neutiquam principio continuitatis aduersari est putanda.

Scholion 2.

30. Quo haec clarius perspiciantur, consideremus casum $\omega = 0$, et ob $\frac{\omega}{2\omega} = \frac{1}{2}$, erit numerator nostrae fractionis:

$$1 + \tfrac{1}{2}\cdot\tfrac{n}{1}x + \tfrac{1}{2}\cdot\tfrac{n(n-1)}{1\cdot 2}x^2 + \tfrac{1}{2}\cdot\tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \tfrac{1}{2}\cdot\tfrac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^4 + \text{etc.}$$

et denominator:

$$1 - \tfrac{1}{2}\cdot\tfrac{n}{1}x + \tfrac{1}{2}\cdot\tfrac{n(n+1)}{1\cdot 2}x^2 - \tfrac{1}{2}\cdot\tfrac{n(n+1)(n+2)}{1\cdot 2\cdot 3}x^3 + \tfrac{1}{2}\cdot\tfrac{n(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3\cdot 4}x^4 - \text{etc.}$$

manifestum autem est numeratoris valorem esse $= \frac{1}{2} + \frac{1}{2}(1+x)^n$ denominatoris vero $= \frac{1}{2} + \frac{1}{2}(1+x)^{-n}$, illumque ergo per hunc diuisum praebere $(1+x)^n$. Simili modo si ponatur $\omega = 1$, erit

pro numeratore	pro denominatore
$A = \frac{1}{2}\cdot\frac{n+1}{1}$	$\alpha = \frac{1}{2}\cdot\frac{n-1}{1}$
$B = 0$	$\beta = 0$
$C = -\frac{1}{4}\cdot\frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}$	$\gamma = -\frac{1}{4}\cdot\frac{(n-1)n(n+1)}{1\cdot 2\cdot 3}$
$D = -\frac{1}{4}\cdot\frac{(n+1)n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4}$	$\delta = -\frac{1}{4}\cdot\frac{(n-1)n(n+1)(n+2)}{1\cdot 2\cdot 3\cdot 4}$
$E = -\frac{1}{4}\cdot\frac{(n+1)n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4\cdot 5}$	$\varepsilon = -\frac{1}{4}\cdot\frac{(n-1)n(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3\cdot 4\cdot 5}$
etc.	etc.

atque hinc colligitur fore

numeratorem $= \frac{1}{2} + \frac{1}{4}(n+1)x + (\frac{1}{2} - \frac{1}{4}(n-1)x)(1+x)^n$

denominatorem $= \frac{1}{2} - \frac{1}{4}(n-1)x + (\frac{1}{2} + \frac{1}{4}(n+1)x)(1+x)^{-n}$

quorum ille per hunc diuisus manifesto praebet formulam propositam $(1+x)^n$. Sin autem in denominatore termini litteris γ, δ, ε etc. affecti omitterentur

terentur, tum in numeratore loco fractionis $\frac{\omega - 1}{2\omega}$ [illegible] vnitas statui deberet ob legem supra stabilitam, vnde valores C, D, E etc. duplo prodirent maiores; foretque numeratoris valor $= (1 - \frac{1}{2}(n-1)x)(1+x)^n$, denominator vero $= 1 - \frac{1}{2}(n-1)x$ qua fractione iterum veritas obtinetur. Videamus ergo, quomodo per huiusmodi formulas tam quantitates radicales, quam exponentiales et logarithmi commode vero proxime exhiberi queant; quando quidem constat tam logarithmos quam exponentiales quantitates ad formam $(1+x)^n$ reuocari posse.

Problema V.

31. Radicem quadratam ex quouis numero non-quadrato proposito per formulas ante exhibitas proxime assignare.

Solutio.

Sit numerus propositus non quadratus $= aa + b$, et ponatur $\frac{b}{aa} = x$ erit $aa + b = aa(1+x)$ ideoque $\sqrt{(aa+b)} = a(1+x)^{\frac{1}{2}}$. Habebimus ergo $n = \frac{1}{2}$, et ex praecedente problemate formulae continuo magis ad $\sqrt{(aa+b)}$ appropinquantes erunt:

$$\sqrt{(aa+b)} = \frac{1 + [illegible] \frac{b}{a^2}}{1 + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{b}{a^2}}\,a$$

$$\sqrt{(aa+b)} = \frac{1 + \frac{2}{4}\cdot\frac{5}{2}\cdot\frac{b}{a^2} + \frac{2\cdot1}{4\cdot3}\cdot\frac{5\cdot3}{2\cdot4}\cdot\frac{b^2}{a^4}}{1 + \frac{2}{4}\cdot\frac{3}{2}\cdot\frac{b}{a^2} + \frac{2\cdot1}{4\cdot3}\cdot\frac{3\cdot1}{2\cdot4}\cdot\frac{b^2}{a^4}}\,a$$

$$\sqrt{(aa+b)} = \frac{1 + \frac{3}{6}\cdot\frac{7}{2}\cdot\frac{b}{a^2} + \frac{3\cdot 2}{6\cdot 5}\cdot\frac{7\cdot 5}{2\cdot 4}\cdot\frac{b^2}{a^4} + \frac{3\cdot 2\cdot 1}{6\cdot 5\cdot 4}\cdot\frac{7\cdot 5\cdot 3}{2\cdot 4\cdot 6}\cdot\frac{b^3}{a^6}}{1 + \frac{3}{6}\cdot\frac{5}{2}\cdot\frac{b}{a^2} + \frac{3\cdot 2}{6\cdot 5}\cdot\frac{5\cdot 3}{2\cdot 4}\cdot\frac{b^2}{a^4} + \frac{3\cdot 2\cdot 1}{6\cdot 5\cdot 4}\cdot\frac{5\cdot 3\cdot 1}{2\cdot 4\cdot 6}\cdot\frac{b^3}{a^6}}\, a$$

etc.

Euolutis autem his factoribus et posito breuitatis ergo $\frac{b}{aa} = x$ consequemur formas sequentes:

$$\sqrt{(aa+b)} = \frac{1+\frac{3}{4}x}{1+\frac{1}{4}x}\, a$$

$$\sqrt{(aa+b)} = \frac{1+\frac{5}{4}x+\frac{5}{16}xx}{1+\frac{3}{4}x+\frac{1}{16}xx}\, a$$

$$\sqrt{(aa+b)} = \frac{1+\frac{7}{4}x+\frac{7}{8}xx+\frac{7}{64}x^3}{1+\frac{5}{4}x+\frac{3}{8}xx+\frac{1}{64}x^3}\, a$$

$$\sqrt{(aa+b)} = \frac{1+\frac{9}{4}x+\frac{27}{16}xx+\frac{15}{32}x^3+\frac{9}{256}x^4}{1+\frac{7}{4}x+\frac{15}{16}xx+\frac{5}{32}x^3+\frac{1}{256}x^4}\, a$$

$$\sqrt{(aa+b)} = \frac{1+\frac{11}{4}x+\frac{11}{4}xx+\frac{77}{64}x^3+\frac{55}{256}x^4+\frac{11}{1024}x^5}{1+\frac{9}{4}x+\frac{7}{4}xx+\frac{35}{64}x^3+\frac{15}{256}x^4+\frac{1}{1024}x^5}\, a$$

etc.

Sin autem ponamus $\frac{x}{4} = y$ seu $y = \frac{b}{4aa}$ erit

$$\sqrt{(aa+b)} = \frac{1+3y}{1+y}\, a$$

$$\sqrt{(aa+b)} = \frac{1+5y+5yy}{1+3y+yy}\, a$$

$$\sqrt{(aa+b)} = \frac{1+7y+14yy+7y^3}{1+5y+6yy+y^3}\, a$$

$$\sqrt{(aa+b)} = \frac{1+9y+27yy+30y^3+9y^4}{1+7y+15yy+10y^3+y^4}\, a$$

$$\sqrt{(aa+b)} = \frac{1+11y+44yy+77y^3+55y^4+11y^5}{1+9y+28yy+35y^3+15y^4+y^5}\, a$$

etc.

Coroll.

Coroll. 1.

32. Si formularum harum numeratores et denominatores attentius contemplemur, non difficulter obſeruabimus, vtrosque conſtituere progreſſionem recurrentem ſecundi ordinis, et quemlibet terminum ita dependere a binis praecedentibus, vt ſi terni termini ordine ſint P, Q, R, ſemper ſit $R = (1 + 2y)Q - yyP$ ſeu ſcala relationis habeatur $1 + 2y, - yy$.

Coroll. 2.

33. Si pro y ſtatuamus valorem $\frac{b}{4aa}$, et numeratorem denominatoremque a fractionibus liberemus, habebimus ſequentes formulas:

$$\surd(aa+b) = \frac{4a^2+3b}{4a^2+b}a$$

$$\surd(aa+b) = \frac{16a^4+20a^2b+5bb}{16a^4+12a^2b+bb}a$$

$$\surd(aa+b) = \frac{64a^6+112a^4b+56a^2b^2+7b^3}{64a^6+80a^4b+24a^2b^2+b^3}a$$

$$\surd(aa+b) = \frac{256a^8+576a^6b+432a^4b^2+120a^2b^3+9b^4}{256a^8+448a^6b+240a^4b^2+40a^2b^3+b^4}$$

etc.

Coroll. 3.

34. In his formulis iterum tam numeratores quam denominatores ſeriem conſtituunt recurrentem, cuius ſcala relationis eſt $2(2aa+b), -bb$, ita vt, ſi P, Q, R denotent tres terminos ſe inuicem excipientes, futurum ſit

$$R = 2(2aa+b)Q - bbP.$$

At

At feriei numeratorum duo termini initiales funt 1, et $4aa+3b$ denominatorum vero 1 et $4aa+b$, vnde reliqui facile reperiuntur.

Coroll. 4.

35. Si fractio $\frac{b}{4aa}$ ad minores terminos reduci poteft, his potius loco ipforum b et $4aa$ vti conueniet. Ponatur ergo in minimis terminis: $\frac{b}{4aa}=\frac{y}{z}$, atque habebimus:

$$\sqrt{(aa+b)}=\frac{z+3y}{z+y}a$$

$$\sqrt{(aa+b)}=\frac{zz+5yz+5yy}{zz+2yz+yy}a$$

$$\sqrt{(aa+b)}=\frac{z^3+7yz^2+14y^2z+7y^3}{z^3+5yz^2+6y^2z+y^3}a$$

hincque erit $R=(z+2y)Q-yyP$.

Coroll. 5.

36. Hae fractiones adhuc commodius exprimi poffunt hoc modo:

$$\sqrt{(aa+b)}=\frac{z+2y+y\cdot 1}{z+2y-y\cdot 1}a$$

$$\sqrt{(aa+b)}=\frac{z^2+4yz+3yy+y(z+2y)}{z^2+4yz+3yy-y(z+2y)}a$$

$$\sqrt{(aa+b)}=\frac{z^3+6yz^2+10y^2z+4y^3+y(z^2+4yz+3y^2)}{z^3+6yz^2+10y^2z+4y^3-y(z^2+4yz+3y^2)}a$$

$$\sqrt{(aa+b)}=\frac{z^4+8yz^3+21y^2z^2+20y^3z+5y^4+y(z^3+6yz^2+10y^2z+4y^3)}{z^4+8yz^3+21y^2z^2+20y^3z+5y^4-y(z^3+6yz^2+10y^2z+4y^3)}a$$

etc.

Coroll. 6.

37. Pro his fractionibus formandis fufficit vnicam hanc feriem conftituiffe:

$1,\ z+2y;\ z^2+4yz+3y^2;\ z^3+6yz^2+10y^2z+4y^3 \ldots P;\ Q;\ R$

quae

quae pariter est recurrens ad legem $R = (z + 2y)Q - yyP$. Formata autem hac serie erit proxime $\sqrt{(aa + b)} = \frac{Q + yP}{Q - yP}\,a$: quae scilicet fractio ex binis terminis se immediate sequentibus illius seriei facillime formatur.

Exemplum I.

38. *Radicem quadratam ex 2 proxime exhibere.*

Cum sit $aa + b = 2$ erit $a = 1$ et $b = 1$, vnde $\frac{b}{4aa} = \frac{1}{4} = \frac{y}{z}$; ergo $y = 1$ et $z = 4$, atque $z + 2y = 6$. Quare ex scala relationis $R = 6Q - P$ formetur haec series recurrens:

$1;\ 6;\ 35;\ 204;\ 1189;\ 6930;\ 40391 \ldots P, Q, R$

et fractiones $\frac{Q + P}{Q - P}$ ad $\sqrt{2}$ continuo magis appropinquantes sunt:

$\sqrt{2} = \frac{7}{5};\ \frac{41}{29};\ \frac{239}{169};\ \frac{1393}{985};\ \frac{8119}{5741};\ \frac{47321}{33461}.$

Exemplum 2.

39. *Radicem quadratam ex 3 proxime exhibere.*

Cum sit $aa + b = 3$, statuatur $a = 1$, erit $b = 2$; et $\frac{b}{4aa} = \frac{2}{4} = \frac{1}{2}$ vnde fit $y = 1$ et $z = 2$; ergo $z + 2y = 4$. Quare ex scala relationis $R = 4Q - P$ formetur haec series recurrens:

$1;\ 4;\ 15;\ 56;\ 209;\ 780;\ 2911;\ 10864 \ldots P, Q, R$

eritque proxime $\sqrt{3} = \frac{Q + P}{Q - P}$, siue

$\sqrt{3} = \frac{5}{3};\ \frac{19}{11};\ \frac{71}{41};\ \frac{265}{153};\ \frac{989}{571};\ \frac{3691}{2131};\ \frac{13775}{7953}$ etc.

Aliter. Vel statuamus $a = 2$; vt sit $b = -1$; erit $\frac{b}{4aa} = -\frac{1}{16} = \frac{y}{z}$ vnde $y = -1$; $z = 16$ et $z + 2y = 14$. Quare

Quare ex ſcala relationis; $R = 14Q - P$ formetur ſeries recurrens:

$$1;\ 14;\ 195;\ 2716;\ 37829;\ 526890 \ldots P,\ Q,\ R$$

eritque proxime $\sqrt{3} = \frac{Q-P}{Q+P}\,2$ ſiue

$$\sqrt{3} = \tfrac{13}{15}.\,2;\ \tfrac{181}{209}.\,2;\ \tfrac{2521}{2911}.\,2;\ \tfrac{35113}{40545}.\,2;\ \text{etc. vel}$$

$$\sqrt{3} = \tfrac{26}{15};\ \tfrac{362}{209};\ \tfrac{5042}{2911};\ \tfrac{70226}{40545};\ \text{etc.}$$

Problema VI.

40. *Radicem cubicam ex quouis numero non-cubo propoſito per formulas ante exhibitas proxime aſſignare.*

Solutio.

Sit numerus propoſitus non cubus $= a^3 + b$, et ponatur $\frac{b}{a^3} = x$ erit $a^3 + b = a^3(1 + x)$ ideoque $\sqrt[3]{(a^3 + b)} = a(1 + x)^{\frac{1}{3}}$. Habemus ergo $n = \frac{1}{3}$; vnde ex §. 28. nanciſcemur has approximationes:

$$\sqrt[3]{(a^3+b)} = a.\,\frac{1 + \frac{1}{2}.\frac{4}{3}x}{1 + \frac{1}{2}.\frac{2}{3}x}$$

$$\sqrt[3]{(a^3+b)} = a.\,\frac{1 + \frac{2}{4}.\frac{7}{3}x + \frac{2.1}{4.3}.\frac{7.4}{3.3}xx}{1 + \frac{2}{4}.\frac{5}{3}x + \frac{2.1}{4.3}.\frac{5.2}{3.3}xx}$$

$$\sqrt[3]{(a^3+b)} = a.\,\frac{1 + \frac{3}{6}.\frac{10}{3}x + \frac{3.2}{6.5}.\frac{10.7}{3.3}xx + \frac{3.2.1}{6.5.4}.\frac{10.7.4}{3.3.3}x^3}{1 + \frac{3}{6}.\frac{8}{3}x + \frac{3.2}{6.5}.\frac{8.5}{3.3}xx + \frac{3.2.1}{6.5.4}.\frac{8.5.2}{3.3.3}x^3}$$

etc.

Enofli-

Euolutis autem his coefficientibus habebimus:

$$\sqrt[3]{(a^3+b)} = a.\,\frac{1+\frac{2}{3}x}{1+\frac{1}{3}x}$$

$$\sqrt[3]{(a^3+b)} = a.\,\frac{1+\frac{7}{6}x+\frac{7}{27}xx}{1+\frac{5}{6}x+\frac{5}{54}xx}$$

$$\sqrt[3]{(a^3+b)} = a.\,\frac{1+\frac{5}{3}x+\frac{7}{9}xx+\frac{7}{81}x^3}{1+\frac{4}{3}x+\frac{4}{9}xx+\frac{2}{81}x^3}$$

etc.

Coroll. 1.

41. Si loco x valorem $\frac{b}{a^3}$ substituamus, et fractiones implicatas tollamus, obtinebimus formulas sequentes:

$$\sqrt[3]{(a^3+b)} = \frac{3a^3+2b}{3a^3+\;b}\,a$$

$$\sqrt[3]{(a^3+b)} = \frac{54a^6+63a^3b+14bb}{54a^6+45a^3b+\;5bb}\,a$$

$$\sqrt[3]{(a^3+b)} = \frac{81a^9+135a^6b+63a^3b^2+7b^3}{81a^9+108a^6b+36a^3b^2+2b^3}\,a$$

etc.

vbi autem commodam progressionis legem definire non licet.

Coroll. 2.

42. Sufficit autem, forma vti priori; inde enim cubus ad numerum propositum propius accedens colligitur, cuius radix pro a posita nouum

dabit valorem pro b et x. Sic si radix cubica ex 2 quaeratur, erit statim $a = 1$, et proxime $\sqrt[3]{2} = \frac{5}{4}$. Sit iam $a = \frac{5}{4}$; et fit $b = 2 - a^3 = \frac{3}{64}$; et $x = \frac{3}{125}$; vnde erit denuo per formam priorem:

$$\sqrt[3]{2} = \frac{125+2}{125+1} \cdot \frac{5}{4} = \frac{127}{126} \cdot \frac{635}{504} \ldots$$

cuius fractionis cubus est $2 - $ [illegible], qui ergo a veritate tantum parte $\frac{1}{1000000}$ deficit.

Coroll. 3.

43. Simili modo formulae pro extractione radicum altiorum potestatum formari possunt. Ita si quaeratur $\sqrt[m]{(a^m + b)}$, ponatur $x = \frac{b}{a^m}$ et $n = \frac{1}{m}$, hincque habebitur:

$$\sqrt[m]{(a^m + b)} = \frac{2ma^m + (m+1)b}{2ma^m + (m-1)b} a$$

quae etiam sufficere potest ad radices quantumuis exacte definiendas.

Problema VII.

44. *Per formulas supra inuentas proxime exprimere logarithmum cuiusque numeri propositi.*

Solutio.

Sit $1 + x$ numerus propositus, et constat eius logarithmum hyperbolicum esse $l(1+x) = \frac{(1+x)^n - 1}{n}$

existen-

existente $n = 0$. Quodsi iam in formulis supra inventis n spectemus vt numerum infinite paruum; habebimus:

$$(1+x)^n = \frac{1+\frac{1}{2}(1+n)x}{1+\frac{1}{2}(1-n)x}$$

$$(1+x)^n = \frac{1+\frac{2}{4}(2+n)x+\frac{2\cdot 1}{4\cdot 3}(1+\frac{3}{2}n)xx}{1+\frac{2}{4}(2-n)x+\frac{2\cdot 1}{4\cdot 3}(1-\frac{3}{2}n)xx}$$

$$(1+x)^n = \frac{1+\frac{3}{6}(3+n)x+\frac{3\cdot 2}{6\cdot 5}(3+\frac{5}{2}n)x^2+\frac{3\cdot 2\cdot 1}{6\cdot 5\cdot 4}(1+\frac{11}{6}n)x^3}{1+\frac{3}{6}(3-n)x+\frac{3\cdot 2}{6\cdot 5}(3-\frac{5}{2}n)x^2+\frac{3\cdot 2\cdot 1}{6\cdot 5\cdot 4}(1-\frac{11}{6}n)x^3}$$

$$(1+x)^n = \frac{1+\frac{4}{8}(4+n)x+\frac{4\cdot 3}{8\cdot 7}(6+\frac{7}{2}n)x^2+\frac{4\cdot 3\cdot 2}{8\cdot 7\cdot 6}(4+\frac{13}{3}n)x^3+\frac{4\cdot 3\cdot 2\cdot 1}{8\cdot 7\cdot 6\cdot 5}(1+\frac{25}{12}n)x^4}{1+\frac{4}{8}(4-n)x+\frac{4\cdot 3}{8\cdot 7}(6-\frac{7}{2}n)x^2+\frac{4\cdot 3\cdot 2}{8\cdot 7\cdot 6}(4-\frac{13}{3}n)x^3+\frac{4\cdot 3\cdot 2\cdot 1}{8\cdot 7\cdot 6\cdot 5}(1-\frac{25}{12}n)x^4}$$

etc.

Quodsi iam hic ponatur $n = 0$, habebimus pro $l(1+x)$ sequentes approximationes:

$$l(1+x) = \frac{x}{1+\frac{1}{2}x}$$

$$l(1+x) = \frac{x+\frac{1}{2}xx}{1+x+\frac{1}{6}xx}$$

$$l(1+x) = \frac{x+xx+\frac{11}{60}x^3}{1+\frac{3}{2}x+\frac{3}{5}x^2+\frac{1}{20}x^3}$$

$$l(1+x) = \frac{x+\frac{3}{2}xx+\frac{13}{21}x^3+\frac{5}{84}x^4}{1+2x+\frac{9}{7}xx+\frac{2}{7}x^3+\frac{1}{70}x^4}$$

etc.

Vel si ponatur $x = \frac{m}{n}$, quoniam fractionum logarithmos potissimum indagare conuenit, et fractiones partiales tollantur, fiet

$$l(1+\tfrac{m}{n}) = \frac{2m}{2n+m}$$

$$l(1+\tfrac{m}{n}) = \frac{6mn+3mm}{6nn+6mn+mm}$$

$$l(1+\tfrac{m}{n}) = \frac{60mn^2+60m^2n+11m^3}{60n^3+90mn^2+36m^2n+3m^3}$$

$$l(1+\tfrac{m}{n}) = \frac{420mn^3+630m^2n^2+260m^3n+25m^4}{420n^4+840mn^3+540m^2n^2+120m^3n+6m^4}$$

etc.

haeque fractiones tam prope accedunt ad verum valorem $l(1+\frac{m}{n})$, vt seriei vulgaris

$$l(1+\tfrac{m}{n}) = \frac{m}{n} - \frac{m^2}{2n^2} + \frac{m^3}{3n^3} - \frac{m^4}{4n^4} \text{ etc.}$$

ingens terminorum numerus capi deberet ad parem approximationem obtinendam.

Coroll. 1.

45. Ita si logarithmum hyperbolicum binarii desideremus, ob $m = 1$ et $n = 1$, sequentes prodibunt approximationes:

$$l\,2 = \tfrac{2}{3};\ \tfrac{9}{13};\ \tfrac{131}{189};\ \tfrac{1335}{1926};\ [\tfrac{445}{642}]$$

quibus fractionibus in decimales conuersis, cum sit

$$l\,2 = 0,693147180559945 3$$

erit proxime

$l\,2 = 0,666666$

$l\,2 = 0,692307$

$l\,2 = 0,693121$

$l\,2 = 0,69314635$

vere $l\,2 = 0,69314718$

sicque

sicque quarta fractio a veritate tantum parte $\frac{93}{100000000}$ deficit.

Coroll. 2.

46. Numerorum autem binario minorum logarithmi multo adhuc exactius reperiuntur. Ita cum sit $l\frac{3}{2} = 0,4054651081081 64$ ponamus $m = 1$ et $n = 2$, nostraeque formulae dabunt proxime

$$l\frac{3}{2} = \frac{2}{5} = 0,40000000$$
$$l\frac{3}{2} = \frac{15}{37} = 0,40540540 5$$
$$l\frac{3}{2} = \frac{371}{915} = 0,40546448 1$$
$$l\frac{3}{2} = \frac{6425}{15847} = 0,40546510 16$$

error scilicet huius vltimae fractionis est $\frac{65}{10000000000}$ ideoque plus quam centies minor quam casu praecedente.

Coroll. 3.

47. Quando ergo fractio $\frac{m}{n}$ adeo semisse est minor, tum erit tam exacte

$$l\left(1+\frac{m}{n}\right) = \frac{420\, m n^3 + 630\, m^2 n^2 + 260\, m^3 n + 25\, m^4}{420\, n^4 + 840\, m n^3 + 540\, m^2 n + 120\, m^3 n + 6\, m^4}$$

vt error in fractione decimali post decimam demum notam percipiatur. Aliis autem methodis vix tam facile ad veritatem appropinquare licet.

Coroll. 4.

48. Si fractio $\frac{m}{n}$ fuerit valde parua, tum sufficiet vti prima vel secunda formula, ita si $\frac{m}{n} = \frac{1}{8}$; prima formula dat $l\frac{9}{8} = \frac{2}{17} = 0,11764$, et secunda;

$$l\frac{9}{8} = \frac{51}{433} = 0,11778291$$ at reuera est

$$l\frac{9}{8} = 0,11778303$$ vnde

vnde ſecunda formula circiter $\frac{1}{10000000}$ a veritate deficit.

Problema VIII.

49. *Quantitatem exponentialem e^x per formulas inuentas proxime exprimere, exiſtente e numero, cuius logarithmus hyperbolicus aequatur vnitati.*

Solutio.

Notum eſt eſſe $e^x = (1 + \frac{x}{n})^n$, ſi pro n ſumatur numerus infinitus. Scribamus ergo in formulis §. 28, $\frac{x}{n}$ loco x et ſimul ponamus $n = \infty$; atque obtinebimus ſequentes approximationes

$$e^x = \frac{1 + \frac{1}{2}x}{1 - \frac{1}{2}x}$$

$$e^x = \frac{1 + \frac{2}{4}x + \frac{1}{4\cdot 3}xx}{1 - \frac{2}{4}x + \frac{1}{4\cdot 3}xx}$$

$$e^x = \frac{1 + \frac{3}{6}x + \frac{3}{6\cdot 5}xx + \frac{1}{6\cdot 5\cdot 4}x^3}{1 - \frac{3}{6}x + \frac{3}{6\cdot 5}xx - \frac{1}{6\cdot 5\cdot 4}x^3}$$

$$e^x = \frac{1 + \frac{4}{8}x + \frac{6}{8\cdot 7}xx + \frac{4}{8\cdot 7\cdot 6}x^3 + \frac{1}{8\cdot 7\cdot 6\cdot 5}x^4}{1 - \frac{4}{8}x + \frac{6}{8\cdot 7}xx + \frac{4}{8\cdot 7\cdot 6}x^3 + \frac{1}{8\cdot 7\cdot 6\cdot 5}x^4}$$

etc.

vnde lex, qua ſequentes hujusmodi formulae confici debent, eſt manifeſta. Si fractiones partiales tollere velimus, habebimus:

$e^x =$

$$e^x = \frac{2+x}{2-x}$$

$$e^x = \frac{12+6x+xx}{12-6x+xx}$$

$$e^x = \frac{120+60x+12xx+x^3}{120-60x+12xx-x^3}$$

$$e^x = \frac{1680+840x+180xx+20x^3+x^4}{1680-840x+180xx-20x^3+x^4}$$

Coroll. 1.

50. Hinc ergo erit ipse numerus e in fractionibus proximis:

$e = \frac{3}{1};\ \frac{19}{7};\ \frac{193}{71};\ \frac{2721}{1001};$ etc.

quarum fractionum hanc legem obseruari conuenit, vt si ponatur:

$e = \frac{A}{\mathfrak{A}};\ \frac{B}{\mathfrak{B}};\ \frac{C}{\mathfrak{C}};\ \frac{D}{\mathfrak{D}};\ \frac{E}{\mathfrak{E}}$ etc. sit

$A = 3;\ B = 6A+1;\ C = 10B+A;\ D = 14C+B;\ E = 18D+C;$ etc.
$\mathfrak{A} = 1;\ \mathfrak{B} = 6\mathfrak{A}+1;\ \mathfrak{C} = 10\mathfrak{B}+\mathfrak{A};\ \mathfrak{D} = 14\mathfrak{C}+\mathfrak{B};\ \mathfrak{E} = 18\mathfrak{D}+\mathfrak{C};$ etc.
vbi multiplicatores 6, 10, 14, 18, etc. sunt numeri impariter pares.

Coroll. 2.

51. Cum igitur sit $e = 2,718281828459045235 36$ videamus quam prope fractiones inuentae accedant ad veritatem:

$e = \frac{3}{1} = 3,0000$
$e = \frac{19}{7} = 2,714285714$
$e = \frac{193}{71} = 2,718309859$
$e = \frac{2721}{1001} = 2,718281718$
etc.

vbi primi in partibus decimis, ſecunda in milleſimis, tertia in centies milleſimis, et quarta in centies centenis milleſimis aberrat.

Coroll. 3.

52. Talis lex progreſſionis etiam in formulis generalibus pro e^x deprehenditur. Si enim noſtras fractiones ponamus:

$$e^x = \frac{A}{\mathfrak{A}};\ \frac{B}{\mathfrak{B}};\ \frac{C}{\mathfrak{C}};\ \frac{D}{\mathfrak{D}};\ \frac{E}{\mathfrak{E}} \text{ etc. ſumtis } A=1 \text{ et } \mathfrak{A}=1, \text{ erit:}$$

$$B=2+x;\ C=6B+Axx;\ D=10C+Bxx;\ E=14D+Cxx; \text{ etc.}$$
$$\mathfrak{B}=2-x;\ \mathfrak{C}=6\mathfrak{B}+\mathfrak{A}xx;\ \mathfrak{D}=10\mathfrak{C}+\mathfrak{B}xx;\ \mathfrak{E}=14\mathfrak{D}+\mathfrak{C}xx; \text{ etc.}$$

vnde ſeries tam numeratorum, quam denominatorum facile continuatur.

SOLV-