NOVA METHODVS
QVANTITATES INTEGRALES DETERMINANDI.

Auctore

L. EVLERO.

§. 1.

Cum mihi saepius occurrissent formulae differentiales, quae per logarithmum quantitatis variabilis erant diuisae, veluti $\frac{P\,dz}{lz}$, nunquam perspicere potui, ad quodnam genus quantitatum earum integralia sint referenda, quin etiam maxime difficile videbatur eorum valores saltem vero proxime assignare. Quod quidem ad formulam integralem simplicissimam huius generis $\int \frac{dz}{lz}$ attinet, facile patet, si eam ita integrari concipiam, vt euanescat posito $z = 0$, tum vero statuatur $z = 1$, quantitatem infinite magnam esse prodituram; quod si enim variabilis z iam proxime ad vnitatem accesserit, vt sit $z = 1 - u$, existente ω quantitate infinite parua, tum ob $dz = -du$ et $lz = l(1-u) = -u$ haec formula erit $\int \frac{du}{u}$, cuius valor vtique fit infinitus. At vero dantur omnino huiusmodi formulae integrales $\int \frac{P\,dz}{lz}$, quae, etiamsi ponatur $z = 1$, tamen valores finitae magnitudinis sortiuntur: quod determinasse

nasse eo magis operae pretium videtur, quod nulla adhuc cognita est via istos valores inuestigandi.

§. 2. Consideremus exempli gratia hanc formulam satis simplicem $\int \frac{(z-1)dz}{lz}$, quae memorata lege integrata valorem finitum habere facile ostendi potest: Posito enim $\frac{z-1}{lz} = y$, vt formula nostra fiat $\int y\,dz$, ideoque exprimat aream curuae, pro abscissa z applicatam habentis $= y$, ista area a termino $z = 0$ vsque ad terminum $z = 1$ extensa vtique valorem finitum non multo maiorem quam $\frac{1}{2}$ repraesentabit; posita enim abscissa $z = 0$, fiet etiam applicata $y = 0$, at sumta $z = 1$ pro applicata $y = \frac{z-1}{lz}$ tam numerator quam denominator euanescit, ergo eorum loco substitutis suis differentialibus fiet $y = z = 1$. Pro abscissis autem mediis ponamus $z = e^{-n}$, existente e numero, cuius logarithmus hyperbolicus est vnitas, erit

$$y = \frac{e^{-n} - 1}{-n} = \frac{e^{n} - 1}{n e^{n}},$$

quae, si n fuerit numerus valde magnus, vt abscissa z fiat minima, applicata erit proxime $y = \frac{1}{n}$; qui ergo valor multo maior erit quam abscissa z; forma scilicet huius curuae similis erit figurae adiectae, vbi AP denotat abscissam z et PM applicatam y, abscissae vero $AB = 1$ respondet applicata $BC = 1$ qua curua descripta eius area AMCB non multum superabit aream trianguli ABC. quae est $= \frac{1}{2}$. Tab. I. Fig. 1.

 §. 3.

§. 3. Nuper autem, in aliis inueſtigationibus occupatus, praeter expectationem inueni, hanc aream aequalem eſſe logarithmo hyperbolico binarii, ita vt ea per fractiones decimales ſit $l\,2 = 0{,}6931471805$; ſequenti autem ratiocinio huc ſum perductus: Cum reuera ſit $l\,z = \frac{z^0 - 1}{0}$, quia differentiando vtrinque prodit $\frac{dz}{z} = \frac{dz}{z}$, et ſumto $z = 1$ vtraque expreſſio euaneſcit, loco 0 ſcribo $\frac{1}{i}$, denotante i numerum infinitum, eritque $l\,z = i(z^{\frac{1}{i}} - 1)$, hincque applicata

$$y = \frac{z - 1}{i(z^{\frac{1}{i}} - 1)} = \frac{1 - z}{i(1 - z^{\frac{1}{i}})},$$

et formula integralis $\int \frac{(1 - z)\,dz}{i(1 - z^{\frac{1}{i}})}$. Nunc igitur ſtatuo $z^{\frac{1}{i}} = x$, vt fiat $z = x^i$, vbi notetur pro vtroque integrationes termino $z = 0$ et $z = 1$ etiam fore $x = 0$ et $x = 1$; quia igitur hinc fit $dz = i x^{i-1} dx$, formula integralis euadit

$$\int \frac{x^{i-1}\,dx\,(1 - x^i)}{(1 - x)},$$

quam ergo integrari oportet a termino $x = 0$ vsque ad terminum $x = 1$.

§. 4. Spectemus nunc i vt numerum valde magnum, et fractio $\frac{1 - x^i}{1 - x}$ reſoluitur in hanc progreſſionem geometricam

$1 + x$

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \ldots\ldots x^{i-1}$$

cuius ſinguli termini in $x^{i-1}\,dx$ ducti et integrati praebent hanc ſeriem

$$\frac{x^i}{i} + \frac{x^{i+1}}{i+1} + \frac{x^{i+2}}{i+2} + \frac{x^{i+3}}{i+3} + \ldots\ldots\ldots + \frac{x^{2i-1}}{2i-1}$$

quae vtique euaneſcit facto $x = 0$. Nunc igitur ſumatur $x = 1$ et valor quaeſitus noſtrae formulae integralis erit

$$\frac{1}{i} + \frac{1}{i+1} + \frac{1}{i+2} + \frac{1}{i+3} + \ldots\ldots + \frac{1}{2i-1}$$

vbi quidem litera i denotat numerum infinite magnum, ita vt numerus horum terminorum ſit revera infinitus. Nihilo vero minus, quia ſinguli termini ſunt infinite parui, haec ſeries ſummam habebit finitam, quam ſequenti modo ad ſeriem ordinariam reducere licet.

§. 5. Series inuenta ſpectari poteſt tanquam differentia inter binas ſequentes progreſſiones harmonicas

$$A = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \ldots\ldots + \frac{1}{2i-1}$$

$$B = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \ldots\ldots + \frac{1}{i-1}$$

quandoquidem differentia $A - B$ ipſam ſeriem inventam exhibet; quia autem numerus terminorum ſeriei A eſt $2i - 1$, ſeriei vero $B = i - 1$, ille duplo maior eſt quam hic, quocirca, vt ſeriem regularem obtineamus, ſingulos terminos ſeriei B per ſaltum a ſeriei A termino ſecundo, quarto, ſexto, octauo etc. auferamus, quo pacto simul ad finem vtriusque peruenietur, eritque

$$A - B = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \text{etc.}$$

in infinitum, cuius ergo valor est $l\,2$, ita vt nunc quidem solide sit demonstratum, formulae integralis propositae $\int \frac{(z-1)\,dz}{l\,z}$ casu $z = 1$ valorem reuera esse $= l\,2$.

§. 6. Simile ratiocinium etiam ad formulam integralem generaliorem $\int \frac{(z^m - 1)\,dz}{l\,z}$ accommodari potest, ac tandem reperietur casu $z = 1$ eius valorem fore $l\,(m+1)$; quia igitur pari modo erit

$$\int \frac{(z^n - 1)\,dz}{l\,z} = l\,(n+1),$$

si hanc ab illa subtrahamus, prodit sequens integratio

$$\int \frac{(z^m - z^n)\,dz}{l\,z} = l\,\frac{m+1}{n+1}$$

si scilicet integratio a termino $z = 0$ vsque ad terminum $z = 1$ extendatur.

§. 7. Quia autem haec demonstratio per quantitates infinitas et infinite paruas procedit, merito aliam methodum planam et consuetam desideramus, quae ad easdem summas perducere valeat; quae quidem inuestigatio maxime ardua videbitur. Interim tamen, cum nuper consideratio functionum duas variabiles inuoluentium me ad integrationem formularum differentialium prorsus singularium perduxisset, quae aliis methodis frustra tentantur, ex eodem prin-

principio quoque integrationes hic exhibitas deriuandas esse intellexi. Hanc igitur methodum tanquam fontem prorsus nouum, ex quo integrationes, aliis methodis inaccessas, haurire liceat, clare et perspicue explicabo, cui negotio istam disquisitionem praecipue destinaui.

Lemma I.

§. 8. Si P fuerit functio quaecunque duarum variabilium z et u, ac ponatur $\int P\,dz = S$, vt etiam S sit functio binarum variabilium z et u, tum erit

$$\int dz \left(\frac{dP}{du}\right) = \left(\frac{dS}{dz}\right).$$

Demonstratio.

Cum in integratione formulae $\int P\,dz$ sola z vt variabilis spectetur, erit $\left(\frac{dS}{dz}\right) = P$, quae formula denuo differentiata sola u pro variabili habita praebet $\left(\frac{ddS}{du\,dz}\right) = \left(\frac{dP}{dv}\right)$, quae in dz ducta et integrata producit $\left(\frac{dS}{du}\right) = \int dz \left(\frac{dP}{du}\right)$ quandoquidem ex principiis calculi integralis est

$$\int dz \left(\frac{ddS}{dz\,du}\right) = \left(\frac{dS}{du}\right) \text{ q. e. d.}$$

Corollarium I.

§. 9. Eodem modo per huiusmodi differentialia vbi tantum u pro variabili spectatur vlterius progredi licet, vnde sequentes oriuntur integrationes

$$\left(\frac{ddS}{du^2}\right) = \int dz \left(\frac{ddP}{du^2}\right) \text{ et}$$

$$\left(\frac{d^3S}{du^3}\right) = \int dz \left(\frac{d^3P}{du^3}\right)$$

etc. etc.

Corol-

Corollarium 2.

§. 10. Quod si ergo formula $\int P\,dz$ fuerit integrabilis, ita vt eius integrale S exhiberi possit, tum etiam omnes istae formulae integrales

$$\int dz\left(\frac{dP}{du}\right),\ \int dz\left(\frac{ddP}{du^2}\right),\ \int dz\left(\frac{d^3P}{du^3}\right)\ \text{etc.}$$

integrationem admittent, atque adeo ipsa integralia exhiberi poterunt.

Scholion.

§. 11. Ex his quidem formulis si in genere tractentur, parum vtilitatis in calculum integralem redundat. At si functio P ita fuerit comparata, vt integrale $\int P\,dz$, casu saltem particulari, quo post integrationem variabili z certus quidam valor puta $z = a$ tribuitur, commode exhiberi potest, vt hoc casu quantitas S abeat in functionem solius variabilis u satis simplicem, tum integrationes memoratae perinde locum habebunt, si quidem post singulas integrationes ponatur $z = a$, atque hinc ad eiusmodi integrationes plerumque peruenitur, quas aliis methodis vix, ac ne vix quidem perficere liceat: atque hinc oritur

Primum principium integrationum.

§. 12. Si P eiusmodi fuerit functio binarum variabilium z et u, vt valor integralis $\int P\,dz$ saltem casu certo $z = a$ commode exprimi queat, qui valor sit $= S$, functio scilicet ipsius u tantum; tum etiam sequentia integralia si quidem post integrationem

tionem pariter ſtatuatur $z = a$ commode exhiberi poterunt ſcilicet

$$\int P\, dz = S$$
$$\int dz \left(\frac{dP}{du}\right) = \left(\frac{dS}{du}\right)$$
$$\int dz \left(\frac{ddP}{du^2}\right) = \left(\frac{ddS}{du^2}\right)$$
$$\int dz \left(\frac{d^3 P}{du^3}\right) = \left(\frac{d^3 S}{du^3}\right)$$
$$\int dz \left(\frac{d^4 P}{du^4}\right) = \left(\frac{d^4 S}{du^4}\right)$$
etc. etc.

Exemplum 1.

§. 13. Si fuerit $P = z^u$, erit quidem in genere

$$\int P\, dz = \frac{z^{u+1}}{u+1};$$

vnde caſu $z = 1$ hic valor ſatis ſimplex naſcitur $\frac{1}{u+1}$, ita vt ſit $S = \frac{1}{u+1}$, cum deinde per differentiationes continuas, dum ſola u pro variabili habetur, prodeat

$\left(\frac{dP}{du}\right) = z^u\, lz$, tum vero $\left(\frac{ddP}{du^2}\right) = z^u\, (lz^2)$, porro $\left(\frac{d^3 P}{du^3}\right) = z^u\, (lz)^3$, $\left(\frac{d^4 P}{du^4}\right) = z^u\, (lz)^4$, etc.

hinc ſequentes obtinentur valores integrales, ſi quidem poſt ſingulas integrationes ſtatuatur $z = 1$

$$\int z^u\, dz = \frac{1}{u+1}$$
$$\int z^u\, dz\, lz = -\frac{1}{(u+1)^2}$$
$$\int z^u\, dz\, (lz)^2 = +\frac{1 \cdot 2}{(u+1)^3}$$
$$\int z^u\, dz\, (lz)^3 = -\frac{1 \cdot 2 \cdot 3}{(u+1)^4}$$
$$\int z^u\, dz\, (lz)^4 = +\frac{1 \cdot 2 \cdot 3 \cdot 4}{(u+1)^5}$$
$$\int z^u\, dz\, (lz)^5 = -\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(u+1)^6}$$
$$\int z^u\, dz\, (lz)^6 = +\frac{1 \cdot \ldots \cdot 6}{(u+1)^6}$$
$$\int z^u\, dz\, (lz)^7 = -\frac{1 \cdot \ldots \cdot 7}{(u+1)^8}$$

 vnde

vnde concludimus generaliter fore

$$\int z^u\, dz\, (lz)^n = \pm \frac{1.\ 2.\ 3.\ 4 \ldots n}{(u+1)^{n+1}}$$

vbi signum + valet si n sit numerus par, alterum vero − si n sit numerus impar. Hae quidem integrationes iam aliunde satis sunt notae, id quod mirum non est, quoniam tam simplicem formulam pro P assumsimus: breuiter igitur repetamus eos casus, quos iam nuper expediui.

Exemplum 2.

§. 14. Si fuerit $P = \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}}$

iam dudum demonstraui, formulae $\int P\, dz$ valorem integralem casu quo post integrationem ponitur $z = 1$, esse $S = \frac{\pi}{2n \cos. \frac{\pi u}{2n}}$

hinc ergo cum sit $\left(\frac{dP}{du}\right) = - \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}}\, lz$

tum vero $\left(\frac{ddP}{du^2}\right) = + \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}}\, (lz)^2$

et $\left(\frac{d^3 P}{du^3}\right) = - \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}}\, (lz)^3$

etc. etc.

ex cognito valore S sequentes nacti sumus integrationes

$$\text{I. } \int \frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\, dz = S = \frac{\pi}{2n \operatorname{cof.} \frac{\pi u}{2n}}$$

$$\text{II. } \int \frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\, dz\, lz = \left(\frac{dS}{du}\right)$$

$$\text{III. } \int \frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\, dz\, (lz)^2 = \left(\frac{ddS}{du^2}\right)$$

$$\text{IV. } \int \frac{-z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\, dz\, (lz)^3 = \left(\frac{d^3S}{du^3}\right)$$

$$\text{V. } \int \frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\, dz\, (lz)^4 = \left(\frac{d^4S}{du^4}\right)$$

etc. etc.

Exemplum 3.

§. 15. Si fuerit $P = \frac{z^{n-u-1} - z^{n+u-1}}{1 - z^{2n}}$

simili modo demonstraui valorem formulae integralis $\int P dz$, casu quo post integrationem ponitur $z = 1$, fore

$$S = \frac{\pi}{2n} \operatorname{tang.} \frac{\pi u}{2n};$$

atque hinc sequentes integrationes pro eodem casu $z = 1$ fuerunt deductae

$$\text{I. } \int \frac{z^{n-u-1} - z^{n+u-1}}{1 - z^{2n}} dz = S = \frac{\pi}{2n} \text{ tang. } \frac{\pi u}{2n}$$

$$\text{II. } \int \frac{-z^{n-u-1} - z^{n+u-1}}{1 - z^{2n}} dz\, lz = \left(\frac{dS}{du}\right)$$

$$\text{III. } \int \frac{z^{n-u-1} + z^{n+u-1}}{1 - z^{2n}} dz\, (lz)^2 = \left(\frac{ddS}{du^2}\right)$$

$$\text{IV. } \int \frac{-z^{n-u-1} - z^{n+u-1}}{1 - z^{2n}} dz\, (lz)^3 = \left(\frac{d^3 S}{du^3}\right)$$

$$\text{V. } \int \frac{z^{n-u-1} - z^{n+u-1}}{1 - z^{2n}} dz\, (lz)^4 = \left(\frac{d^4 S}{du^4}\right)$$

etc. etc.

Scholion.

§. 16. Quo igitur vberiores fructus ex hoc principio expectare queamus, praecipuum negotium huc redit; vt eiusmodi functiones binarum variabilium z et u pro P inueftigemus, ita vt valor formulae integralis faltem certo quodam cafu puta $z = 1$ fuccincte affignari poffit, quemadmodum in allatis exemplis fieri licuit. Quemadmodum autem hoc principium ex continua differentiatione eft deductum, ita eodem modo continua integratio ad vfum noftrum accommodari poterit.

Lemma II.

§. 17. Si P fuerit functio duarum variabilium z et u, ac ponatur $\int P\, dz = S$ vt etiam S fit functio duarum variabilium z et u, tum erit $\int S\, du = \int d$

$=\int dz \int P\,du$, vbi in integralibus formulis $\int P\,du$ et $\int S\,du$ ſola u pro variabili habetur, in formula autem $\int dz \int P\,du$ ſola z.

Demonſtratio.

Ponatur $\int S\,du = V$, vt ſit $S = \left(\frac{dV}{du}\right)$, ideoque $\left(\frac{dV}{du}\right) = \int P\,dz$ eritque $\left(\frac{ddV}{dz\,du}\right) = P$; vnde, per du multiplicando et integrando, erit $\left(\frac{dV}{dz}\right) = \int P\,du$, ex quo ſequitur $V = \int dz \int P\,du = \int S\,du$. q. e. d.

Corollarium 1.

§. 18. Hoc modo etiam integratio repeti poteſt; vnde orietur talis aequatio $\int du \int S\,du = \int dz \int du \int P\,du$; hinc autem plerumque parum vtilitatis expectari poteſt, niſi forte iſtae integrationes commode ſuccedant.

Corollarium 2.

§. 19. Quod ſi ergo formula $\int P\,dz$ fuerit integrabilis, ſcilicet $=S$ altera hinc deducta $\int dz \int P\,du$ eatenus tantum integrari poterit, quatenus integrale $\int S\,du$ integrare licet.

Secundum principium integrationum.

§. 20. Si P eiusmodi fuerit functio duarum variabilium z et u, vt formulae integralis $\int P\,dz$ valor certo ſaltem caſu puta $z = a$ commode exhiberi queat, ita, vt hoc caſu quantitas S fiat functio ſolius variabilis u; tum etiam pro eodem caſu $z = a$

 huius

huius formulae integralis $\int dz \int P\,du$ valor assignari poterit, si modo formulam $\int S\,du$ integrare licuerit.

Exemplum I.

§. 21. Sumamus $P = z^u$, eritque $\int P\,dz = \frac{z^{u+1}}{u+1}$; quae formula casu $z = 1$ abit in $\frac{1}{u+1}$, quod ergo loco S scribatur. Tum vero quia est

$$\int P\,du = \int z^u\,du = \frac{z^u}{lz}, \text{ et quia}$$

$$\int S\,du = l(u+1), \text{ erit } \int \frac{z^u\,dz}{lz} = l(u+1);$$

si quidem post illam integrationem ponatur $z = 1$. Quia autem omnis integratio additionem constantis postulat, hic potius statui oportebit

$$\int \frac{z^u\,dz}{lz} = l(u+1) + C;$$

atque hic quidem facile intelligitur, hanc constantem C esse debere infinitam, quoniam in formula integrali fractio $\frac{z^u}{lz}$ posito $z = 1$ fit infinita, ita vt hinc parum pro instituto nostro sequi videatur.

Corollarium I.

§. 22. Quoniam autem haec constans C non a variabili u pendet, ea retinebit eundem valorem, quicunque numeri determinati pro u accipiantur. Sumamus

mamus igitur primo $u = m$, tum vero etiam $u = n$, vt habeamus iſtos valores

$$\text{I. } \int \frac{z^m\,dz}{lz} = l(m+1) + C \text{ et}$$

$$\text{II. } \int \frac{z^n\,dz}{lz} = l(n+1) + C$$

quarum altera ab altera ſubtracta relinquet iſtam integrationem notatu digniſſimam

$$\int \frac{(z^m - z^n)\,dz}{lz} = l\frac{m+1}{n+1}$$

quemadmodum iam ſupra ex longe aliis principiis demonſtrauimus.

Corollarium. 2.

§. 23. Si ad alteram integrationem aſcendamus, quia eſt $\int P\,du = \frac{z^u}{lz}$, erit $\int du \int P\,du = \frac{z^u}{(lz)^2}$; tum vero ob

$\int S\,du = l(u+1) + C$, erit $\int du \int S\,du = (u+1)(l(u+1)-1) + Cu + D$ ſicque habebimus

$$\int \frac{z^u\,dz}{(lz)^2} = (u+1)(l(u+1)-1) + Cu + D$$

vbi conſtantes C et D non ab u pendent; quare vt eas eliminemus tres caſus determinatos euoluamus

$$\text{I. } \int \frac{z^m\,dz}{(lz)^2} = (m+1)l(m+1) - m - 1 + Cm + D$$

II.

$$\text{II.}\ \int \frac{z^n\, dz}{(lz)^2} = (n+1)l(n+1) - n - 1 + Cn + D$$

$$\text{III.}\ \int \frac{z^k\, dz}{(lz)^2} = (k+1)l(k+1) - k - 1 + Ck + D$$

eritque $\text{I} - \text{III} = (m+1)l(m+1) - (k+1)l(k+1) + k - m + C(m-k)$

et $\text{II} - \text{III} = (n+1)l(n+1) - (k+1)l(k+1) + k - n + C(n-k)$

hincque deducimus

$$(\text{I}-\text{III})(n-k) - (\text{II}-\text{III})(m-k) = \begin{cases} (m+1)(n-k)l(m+1) \\ -(k+1)(n-k)l(k+1) + (k-m)(n-k) \\ -(n+1)(m-k)l(n+1) - (k-n)(m-k) \\ +(k+1)(m-k)lk+1) \end{cases}$$

atque hinc deducimus sequentem integrationem

$$\int \frac{dz\,((n-k)z^m - (m-k)z^n + (m-n)z^k)}{(lz)^2} =$$

$$+(m+1)(n-k)\,l(m+1)$$
$$-(n+1)(m-k)\,l(n+1)$$
$$+(k+1)(m-n)\,l(k+1).$$

Corollarium 3.

§. 14. Operae pretium erit aliquot casus euoluere, vbi quidem numeros m, n et k inter se inaequales accipi conuenit, quia aliter omnes termini se destruerent.

I. Sit igitur $m = 2$, $n = 1$ et $k = 0$ erit

$$\int \frac{(z-1)^2\, dz}{(lz)^2} = 3\,l\,3 - 4\,l\,2 = l\,\tfrac{27}{16}$$

II. Sit $m = B$, $n = 1$ et $k = 0$ eritque

$$\int \frac{(z^3 - 3z + 2)\, dz}{(lz)^2} = \int dz(z-1)^2(z+2 = 4l4 - 6l2 = 2l2 = l4$$

III.

III. Sit $m = 3$, $n = 2$ et $k = 0$ et erit

$$\int \frac{(2z^3 - 3zz + 1)\,dz}{(lz)^2} = \int \frac{dz(z-1)^2(2z+1)}{(lz)^2} = 8l4 - 9l3 = l\frac{4^8}{3^9}$$

IV. Sit $m = 3$, $n = 2$ et $k = 1$ et prodit

$$\int \frac{(z^3 - 2zz + z)\,dz}{(lz)^2} = \int \frac{z\,dz(z-1)^2}{(lz)^2} = 4l4 - 6l3 + 2l2 = l\frac{2^{10}}{3^6}.$$

Corollarium 4.

§. 25. In his casibus notatu dignum occurrit, quod numerator in formulis integralibus factorem habet $(z-1)^2$, quod ideo necessario vsu venit, ne valores integralium euadant infiniti. Quia enim denominator $(lz)^2$ euanescit casu $z = 1$, si ponamus $z = 1 - \omega$ existente ω infinite paruo, erit

$$lz = -\omega \text{ et } (lz)^2 = +\omega\omega.$$

Necesse ergo est vt in numeratore adsit factor, qui casu $z = 1 - \omega$ itidem praebeat $\omega\omega$, quod euenit si ibi factor fuerit $(z-1)^2$.

Scholion.

§. 26. Integratio, quam in corollario primo sumus nacti, ideo omni digna videtur attentione, quod valores integrales inde nati casu $z = 1$ nullo adhuc modo assignare potuerim, etiamsi tam simpliciter per logarithmos exprimantur. At vero integrationes, in Corollario secundo inuentae, etiamsi multo magis arduae videantur, tamen ex prioribus ope reductionum cognitarum non difficulter deriuari possunt; id quod pro vnico casu ostendisse sufficiet. Ponamus

$$\int \frac{dz(z-1)^2}{(lz)^2} = \frac{p}{lz} + \int \frac{q\,dz}{lz}$$ eritque differentiando

$$\frac{dz(z-1)^2}{(lz)^2} = \frac{dp}{lz} - \frac{p\,dz}{z(lz)^2} + \frac{q\,dz}{lz}$$

vnde aequatis terminis seorsim vel per $(lz)^2$ vel per lz diuisae habebimusque has duas aequalitates

$$(z-1)^2 = -\frac{p}{z} \text{ et } dp = -q\,dz$$

ex quarum priore oritur $p = -z(z-1)^2$, hincque

$$\frac{dp}{dz} = -3zz + 4z - 1$$ ideoque $q = 3zz - 4z + 1$

ita vt sit

$$\int \frac{dz(z-1)^2}{(lz)^2} = \frac{-z(z-1)^2}{lz} + \int \frac{(3zz-4z+1)dz}{lz}$$

hic autem prius membrum posito $z=1$ sponte euanescit; posito enim $z = 1-\omega$ vt sit $lz = -\omega$, erit $p = -\omega\omega(1-\omega)$ ideoque $\frac{p}{lz} = \omega(1-\omega) = 0$ ob $\omega = 0$, posterius vero membrum in has partes discerpi potest

$$3\int \frac{(zz-z)dz}{lz} - \int \frac{(z-1)dz}{lz}$$

prioris autem partis integrale est $3l\frac{3}{2}$, posterioris vero $-1\,l\,2$; sicque totum hoc integrale erit

$$3\,l\tfrac{3}{2} - l\,2 = 3\,l\,3 - 4\,l\,2 = l\tfrac{27}{16}$$

prorsus vti inuenimus. Hoc igitur modo si in genere statuamus

$$\int \frac{V\,dz}{(lz)^2} = \frac{p}{lz} + \int \frac{q\,dz}{lz}$$ erit differentiando

$$\frac{V\,dz}{(lz)^2} = \frac{dp}{lz} - \frac{p\,dz}{z(lz)^2} + \frac{q\,dz}{lz}$$

vnde istae duae fluunt aequalitates

$$p = -Vz \text{ et } q = -\frac{dp}{dz}.$$

Iam

Iam vt terminus $\frac{p}{lz}$ euanefcat pofito $z = 1$, numerator p factorem habere debet $(z-1)^2$; qui ergo etiam factor effe debet quantitatis V. Sit igitur

$V = \frac{U(z-1)^2}{z}$ eritque $p = -U(z-1)^2$, vnde fit

$dp = -dU(z-1)^2 - 2Udz(z-1) = (z-1)(dU(z-1) - 2Udz)$,

hincque fit

$q\,dz = (z-1)(2U\,dz - dU(z-1))$;

quia ergo q factorem habet $z-1$ formula $\int \frac{q\,dz}{lz}$ femper in partes refolui poteft, quarum integralia per corollarium primum affignare licet, fi modo U fuerit aggregatum ex quotcunque poteftatibus ipfius z; vnde fequens deducitur theorema.

Theorema.

§. 27. Si fuerit

$P = Az^{\alpha} + Bz^{\beta} + Cz^{\gamma} + Dz^{\delta} + \text{etc.}$

ita, vt fumma coefficientium

$A + B + C + D \text{ etc.} = 0$ tum erit

$\int \frac{P\,dz}{lz} = Al(\alpha+1) + Bl(\beta+1) + Cl(\gamma+1) + Dl(\delta+1)$

fi quidem poft integrationem ftatuatur $z = 1$.

Demonftratio.

Cum hoc ipfo cafu, quo poft integrationem ponitur $z = 1$ fit

$$\int \frac{z^n\,dz}{lz} = l(n+1) + \Delta$$

denotante Δ illam conftantem infinitam integratione ingreffam erit

$$A\int\frac{z^\alpha\, dz}{l z} = A\, l(\alpha+1) + A\,\Delta$$

eodemque modo

$$B\int\frac{z^\beta\, dz}{l z} = B\, l(\beta+1) + B\,\Delta$$

etc. etc.

si haec integralia omnia in vnam summam colligantur erit

$$(A + B + C + D + \text{etc.})\,\Delta = 0\,,$$

hincque erit integrale quaesitum

$$\int\frac{P\,dz}{l z} = A\,l(\alpha+1) + B\,l(\beta+1) + C\,l(\gamma+1) + D\,l(\delta+1)\ \text{etc.}$$

q. e. d.

Corollarium 1.

§. 28. Quia supponimus

$$A + B + C + D + \text{etc.} = 0$$

euidens est formulam

$$P = A\,z^\alpha + B\,z^\beta + C\,z^\gamma + D\,z^\delta + \text{etc.}$$

factorem habere $z - 1$ quemadmodum iam ante notauimus.

Corollarium 2.

§. 29. Quia est

$$(z-1)^n = z^n - \tfrac{n}{1} z^{n-1} + \tfrac{n(n-1)}{1.\ 2} z^{n-2} - \tfrac{n(n-1)(n-2)}{1.\ 2.\ 3} z^{n-3}$$

hoc valore loco P posito, erit $A = 1$ et $\alpha = n$; deinde $B = -\frac{n}{1}$ et $\beta = n-1$; porro $C = \frac{n(n-1)}{1.\ 2}$ et $\gamma = n-2$; etc.

hinc

hinc igitur erit

$$\int \frac{(z-1)^n\,dz}{lz} = l(n+1) - \tfrac{n}{1} ln + \tfrac{n.\,n-1}{1.\,2} l(n-1) - \tfrac{n.(n-1)(n-2)}{1.\,2.\,3} l(n-2)$$
$$+ \tfrac{n(n-1)(n-2)(n-3)}{1.\,2.\,3.\,4} l(n-3) + \text{etc.}$$

si modo exponens n fuerit nihilo maior, vel saltem vnitate non minor, quia alioquin casu $z=1$ fractio $\frac{(z-1)^n}{lz}$ fieret infinita; hoc autem non obstante area supra considerata fiet finita, ita vt sufficiat dummodo sit $n > 0$.

Exemplum 2.

§. 30. Sit $P = \frac{z^{n-u-1} + z^{n+u-1}}{1+z^{2n}}$ erit $\int P\,dz = \frac{\pi}{2n \operatorname{cos.} \frac{\pi u}{2n}}$;

si quidem post integrationem ponatur $z=1$ quem ergo valorem literae S tribuimus. Nunc spectata z vt constante erit

$$\int P\,du = \frac{1}{1+z^{2n}} \left(\int z^{n-u-1}\,du + \int z^{n+u-1}\,du\right)$$

ideoque

$$\int P\,du = -\frac{z^{n-u-1} + z^{n+u-1}}{(1+z^{2n})\,lz}$$ vnde fiet

$$\int S\,du = \int \frac{z^{n-u-1} + z^{n+u-1}}{1+z^{2n}} \cdot \frac{dz}{lz};$$

cum igitu sit $\operatorname{cos.} \frac{\pi u}{2n} = \operatorname{sin.} \frac{\pi(n-u)}{2n}$ erit

$$\int S\,du = \int \frac{\pi\,du}{2n \operatorname{sin.} \frac{\pi(n-u)}{2n}};$$

hinc si ponamus

$$\frac{\pi(n-u)}{2n} = \Phi, \text{ erit } d\Phi = -\frac{\pi\, du}{2n}, \text{ ideoque}$$

$$\int S\, du = -\int \frac{d\Phi}{\sin.\Phi} = -l\,\text{tang.}\tfrac{1}{2}\Phi,$$

quocirca habebimus

$$\int S\, du = -l\,\text{tang.}\frac{\pi(n-u)}{4n},$$

ita vt posito post integrationem $z = 1$ assecuti simus hanc integrationem

$$\int \frac{-z^{n-u-1} + z^{n+u-1}}{1+z^{2n}} \cdot \frac{dz}{lz} = -l\,\text{tang.}\frac{\pi(n-u)}{4n} = +l\,\text{tang.}\frac{\pi(n+u)}{4n}.$$

Exemplum 3.

§. 31. Sit $P = \frac{z^{n-u-1} - z^{n+u-1}}{1-z^{2n}}$ erit

$$\int P\, dz = \frac{\pi}{2n}\,\text{tang.}\frac{\pi u}{2n} = S$$

vnde fit

$$\int S\, du = -l\,\text{cos.}\frac{\pi u}{2n},$$

hinc cum sit

$$\int P\, du = -\frac{z^{n-u-1} - z^{n+u-1}}{(1-z^{2n})\, lz}$$

nanciscimur sequentem integrationem, si quidem integrale a termino $z = 0$ vsque ad terminum $z = 1$ extendatur

$$\int \frac{z^{n-u-1} + z^{n+u-1}}{1-z^{2n}} \cdot \frac{dz}{lz} = +l\,\text{cos.}\frac{\pi u}{2n}.$$

Haec quidem duo posteriora exempla iam ante fusius expediui; vnde iis magis euoluendis non immoror: sed ad sequens problema progredior.

Pro-

Problema.

Si proponantur hae duae feries infinitae

$P = z\cos. u + z^2 \cos. 2u + z^3 \cos. 3u + z^4 \cos. 4u + z^5 \cos. 5u + \text{etc.}$ et

$Q = z\sin. u + z^2 \sin. 2u + z^3 \sin. 3u + z^4 \sin. 4u + z^5 \sin. 5u + \text{etc.}$

quae binas variabiles z et u inuoluunt, inuenire relationes inter formulas integrales $\int \frac{P\,dz}{z}$, $\int P\,du$ et $\int \frac{Q\,dz}{z}$, $\int Q\,du$ aliasque formulas integrales per continuam integrationem inde natas.

Solutio.

§. 32. Cum vtraque feries fit recurrens, reperitur per formulas finitas

$$P = \frac{z\cos. u - zz}{1 - 2z\cos. u + zz} \text{ et } Q = \frac{z\sin. u}{1 - 2z\cos. u + zz}.$$

Vnde fit

$$\int \frac{P\,dz}{z} = \int \frac{dz\cos. u - z\,dz}{1 - 2z\cos. u + zz} = -l\sqrt{1 - 2z\cos. u + zz}$$

et $\int Q\,du = \int \frac{z\,du\sin. u}{1 - 2z\cos. u + zz} = +l\sqrt{1 - 2z\cos. u + zz}$

ita vt fit

$\int \frac{P\,dz}{z} = -\int Q\,du$; tum vero etiam erit

$$\int \frac{Q\,dz}{z} = \int \frac{dz\sin. u}{1 - 2z\cos. u + zz} = \text{A tang.} \frac{z\sin. u}{1 - z\cos. u};$$

at fi ifte arcus differentietur fumto folo angulo u variabili, erit

$$\frac{1}{du}\text{A tang.} \frac{z\sin. u}{1 - z\cos. u} = \frac{z\cos. u - zz}{1 - 2z\cos. u + zz}$$

ita vt fit $\int \frac{Q\,dz}{z} = \int P\,du$.

§. 33. Verum eaedem relationes facilius ex ipfis feriebus deriuantur. Cum enim fit

$P = z$

$P = z\cos. u + z^2\cos. 2u + z^3\cos. 3u + z^4\cos. 4u$ etc. erit

$\int \frac{P\,dz}{z} = \frac{z\cos. u}{1} + \frac{zz\cos. 2u}{2} + \frac{z^3\cos. 3u}{3}$ etc.

et $\int P\,du = \frac{z\sin. u}{1} + \frac{zz\sin. 2u}{2} + \frac{z^3\sin. 3u}{3}$ etc.

et quia eſt

$Q = z\sin. u + zz\sin. 2u + z^3\sin. 3u +$ etc. erit

$\int \frac{Q\,dz}{z} = \frac{z\sin. u}{1} + \frac{zz\sin. 2u}{2} + \frac{z^3\sin. 3u}{3}$ etc.

et $\int Q\,du = -\frac{z\cos. u}{1} - \frac{z^2\cos. 2u}{2} - \frac{z^3\cos. 3u}{3}$ etc.

vnde manifeſtum eſt fore

$\int \frac{P\,dz}{z} = -\int Q\,du$ et $\int \frac{Q\,dz}{z} = \int P\,du.$

§. 34. Quo hoc modo vlterius progredi liceat ſtatuamus breuitatis gratia

$P' = \frac{z\cos. u}{1} + \frac{z^2\cos. 2u}{2} + \frac{z^3\cos. 3u}{3}$ etc. et $Q' = \frac{z\sin. u}{1} + \frac{z^2\sin. 2u}{2} + \frac{z^3\sin. 3u}{3}$ etc.

$P'' = \frac{z\cos. u}{1^2} + \frac{z^2\cos. 2u}{2^2} + \frac{z^3\cos. 3u}{3^2}$ etc. et $Q'' = \frac{z\sin. u}{1^2} + \frac{zz\sin. 2u}{2^2} + \frac{z^3\sin. 3u}{3^2}$ etc.

$P''' = \frac{z\cos. u}{1^3} + \frac{zz\cos. 2u}{2^3} + \frac{z^3\cos. 3u}{3^3}$ etc. et $Q''' = \frac{z\sin. u}{1^3} + \frac{zz\sin. 2u}{2^3} + \frac{z^3\sin. 3u}{3^3}$ etc.

$P'''' = \frac{z\cos. u}{1^4} + \frac{zz\cos. 2u}{2^4} + \frac{z^3\cos. 3u}{3^4}$ etc. et $Q'''' = \frac{z\sin. u}{1^4} + \frac{zz\sin. 2u}{2^4} + \frac{z^3\sin. 3u}{3^4}$ etc.

etc. etc. etc. etc.

et hinc comparationes ante inuentae continuabuntur

$P' = \int \frac{P\,dz}{z} = -\int Q\,du, \quad Q' = \int \frac{Q\,dz}{z} = \int P\,du.$

$P'' = \int \frac{P'\,dz}{z} = -\int Q'\,du, \quad Q'' = \int \frac{Q'\,dz}{z} = \int P'\,du.$

$P''' = \int \frac{P''\,dz}{z} = -\int Q''\,du, \quad Q''' = \int \frac{Q''\,dz}{z} = \int P''\,du.$

$P'''' = \int \frac{P'''\,dz}{z} = -\int Q'''\,du, \quad Q'''' = \int \frac{Q'''\,dz}{z} = \int P'''\,du.$

etc. etc. etc. etc. etc. etc.

vnde

vnde plures infignes relationes deduci poffunt.

§. 35. Maxime autem notatu dignae et ad noftrum inftitutum accommodatae funt eae relationes, vbi formulae integrales, in quibus fola z eft variabilis, reducuntur ad alias formulas integrales, in quibus fola u eft variabilis, cuiusmodi funt, quae fequuntur

$$P' = \int \frac{P\,dz}{z} = -\int Q\,du$$

$$P'' = \int \frac{dz}{z} \int \frac{P\,dz}{z} = -\int du \int P\,du$$

$$P''' = \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{P\,dz}{z} = +\int du \int du \int Q\,du$$

$$P'''' = \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{P\,dz}{z} = +\int du \int du \int du \int P\,du$$

$$P^{V} = \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{P\,dz}{z} = -\int du \int du \int du \int du \int Q\,du$$

etc.

Similique modo pro altero genere

$$Q' = \int \frac{Q\,dz}{z} = +\int P\,du$$

$$Q'' = \int \frac{dz}{z} \int \frac{Q\,dz}{z} = -\int du \int Q\,du$$

$$Q''' = \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{Q\,dz}{z} = -\int du \int du \int P\,du$$

$$Q'''' = \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{Q\,dz}{z} = +\int du \int du \int du \int Q\,du$$

$$Q^{V} = \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{Q\,dz}{z} = +\int du \int du \int du \int du \int P\,du$$

etc.

§. 36. Quod fi iam noftrarum ferierum, fiue, quod eodem redit, quantitatum

P, P', P'', P''', P'''' etc. et Q', Q'', Q''', Q'''' etc.

eos tantum valores defideremus, quos adipifcuntur

 pofito

posito $z = 1$, hoc commodi assequimur, vt in formulis integralibus vbi solus angulus u pro variabili habetur, statim ante integrationes ponere liceat $z = 1$, hoc autem facto erit

$$P = \frac{\cos. u - 1}{2 - 2\cos. u} = -\tfrac{1}{2} \quad \text{et} \quad Q = \frac{\sin. u}{2 - 2\cos. u} = \tfrac{1}{2}\cot.\tfrac{1}{2}u$$

tum vero porro

$$\int P\,du = A - \tfrac{1}{2}u$$

$$\int du \int P\,du = B + Au - \tfrac{1}{4}uu$$

$$\int du \int du \int P\,du = C + Bu + \tfrac{1}{2}Auu - \tfrac{1}{12}u^3$$

$$\int du \int du \int du \int P\,du = D + Cu + \tfrac{1}{2}Buu + \tfrac{1}{6}Au^3 - \tfrac{1}{48}u^4$$

at pro formulis, vbi est q, calculus non tam concinne succedit; erit enim

$$Q = \tfrac{1}{2}\cot.\tfrac{1}{2}u$$

$$\int Q\,du = l\sin.\tfrac{1}{2}u$$

$$\int du \int Q\,du = \int du\, l\sin.\tfrac{1}{2}u$$

quae formula cum omnem integrationem respuat, vix vlterius progredi licet; interim tamen erit

$$\int du \int du \int Q\,du = \int du \int du\, l\sin.\tfrac{1}{2}u$$

$$\int du \int du \int du \int Q\,du = \int du \int du \int du\, l\sin.\tfrac{1}{2}u.$$

§. 37. Quod ad priores formulas variabilem z inuoluentes attinet, per notas reductiones elicitur

$$\int \frac{P'\,dz}{z} = \int \frac{dz}{z} \int \frac{P\,dz}{z} = lz \int \frac{P\,dz}{z} - \int \frac{P\,dz}{z}\, lz$$

vbi prius membrum $lz\int P\,dz$ euanescit posito $z = 1$, tum vero

$$\int \frac{dz}{z} \int \frac{P'\,dz}{z} = \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{P\,dz}{z} = + \int \frac{P\,dz}{z} \frac{(lz)^2}{2}$$

quibus expressionibus vlterius exhibitis colligimus fore

P' =

$P' = \int \frac{P\,dz}{z}$	$Q' = \int \frac{Q\,dz}{z}$
$P'' = -\int \frac{P\,dz}{z}\, lz$	$Q'' = -\int \frac{Q\,dz}{z}\, lz$
$P''' = +\int \frac{P\,dz}{z}\, \frac{(lz)^2}{1.2}$	$Q''' = +\int \frac{Q\,dz}{z}\, \frac{(lz)^2}{1.2}$
$P^{IV} = -\int \frac{P\,dz}{z}\, \frac{(lz)^3}{1.2.3}$	$Q^{IV} = -\int \frac{Q\,dz}{z}\, \frac{(lz)^3}{1.2.3}$.

§. 38. Ex his igitur ſequentium formularum integralium valores aſſignare poſſumus caſu quo $z = 1$

$$P = -\tfrac{1}{2}$$
$$P' = \int \frac{P\,dz}{z} = -\,l\,\text{ſin.}\,\tfrac{1}{2}u$$
$$P'' = -\int \frac{P\,dz}{z}\, lz = -B - Au + \tfrac{1}{4}uu$$
$$P''' = +\int \frac{P\,dz}{z}\, \frac{(lz)^2}{1.2} = \int du \int du\; l\,\text{ſin.}\,\tfrac{1}{2}u$$
$$P^{IV} = -\int \frac{P\,dz}{z}\, \frac{(lz)^3}{1.2.3} = D + Cu + \tfrac{1}{2}Buu + \tfrac{1}{6}Au^3 - \tfrac{1}{48}u^4$$
$$P^{V} = +\int \frac{P\,dz}{z}\, \frac{(lz)^4}{1.2.3.4} = \int du \int du \int du \int du\; l\,\text{ſin.}\,\tfrac{1}{2}u$$

etc.

Eodemque modo

$$Q = \tfrac{1}{2}\cot.\,\tfrac{1}{2}u$$
$$Q' = \int \frac{Q\,dz}{z} = A - \tfrac{1}{2}u$$
$$Q'' = -\int \frac{Q\,dz}{z}\, \frac{lz}{1} = -\int du\; l\,\text{ſin.}\,\tfrac{1}{2}u$$
$$Q''' = +\int \frac{Q\,dz}{z}\, \frac{(lz)^2}{2} = -C - Bu - \tfrac{1}{2}Auu + \tfrac{1}{12}u^3$$
$$Q^{IV} = -\int \frac{Q\,dz}{z}\, \frac{(lz)^3}{6} = \int du \int du \int du\; l\,\text{ſin.}\,\tfrac{1}{2}u$$
$$Q^{V} = +\int \frac{Q\,dz}{z}\, \frac{(lz)^4}{24} = E + Du + \tfrac{1}{2}Cuu + \tfrac{1}{6}Bu^3 + \tfrac{1}{24}Au^4 - \tfrac{1}{240}u^5$$

etc.

§. 39. Cum igitur ſit

$$P = \frac{z\,\text{coſ.}\,u - zz}{1 - 2z\,\text{coſ.}\,u + zz} \quad \text{et} \quad Q = \frac{z\,\text{ſin.}\,u}{1 - 2z\,\text{coſ.}\,u + zz}$$

 hacte-

hactenus id ſumus aſſecuti, vt harum duarum formularum integralium

$$\int \frac{dz\,(\text{coſ.}\,u - z)}{1 - 2z\,\text{coſ.}\,u + zz}(lz)^n \text{ et } \int \frac{dz\,\text{ſin.}\,u}{1 - 2z\,\text{coſ.}\,u + zz}(lz)^n$$

valores caſu $z = 1$ commode per angulum u aſſignare valeamus, ſi modo conſtaret, quo facto quantitates A, B, C, D determinari oporteat, id quod vix alio modo niſi per ipſas ſeries, vnde hae quantitates ſunt natae, fieri poſſe videtur.

§. 40. Omiſſis igitur formulis integralibus, quae quantitatem Q inuoluunt, quippe quarum integratio minus ſuccedit, alteras tantum conſideremus, et poſito ſtatim $z = 1$ vbi fit $P = -\frac{1}{2}$ ita vt fit

$$\text{coſ.}\,u + \text{coſ.}\,2u + \text{coſ.}\,3u + \text{coſ.}\,4u = -\tfrac{1}{2}$$

ſi per du multiplicemus et integremus, habebimus

$$Q' = \frac{\text{ſin.}\,u}{1} + \frac{\text{ſin.}\,2u}{2} + \frac{\text{ſin.}\,3u}{3} + \frac{\text{ſin.}\,4u}{4} + \frac{\text{ſin.}\,5u}{5}\,\text{etc.} = A - \tfrac{1}{2}u$$

quae conſtans nihilo aequalis videri poteſt, quia poſito $u = 0$ ſumma ſeriei euaneſcere videtur; at ſumto angulo u infinite paruo ſeries praebebit

$$u + u + u + u + u + u + \text{etc. in infinitum}$$

notum autem eſt, talem ſeriem ſummam finitam habere poſſe, vnde hoc caſu omiſſo ſtatuamus $u = \pi$, ſeu potius $u = \pi + \omega$ prodibitque haec ſeries exiſtente ω angulo infinite paruo

$$-\omega + \omega - \omega + \omega - \omega + \omega - \omega + \text{etc.}$$

vbi quia ſigna alternantur, nullum eſt dubium, quin ſumma ſeriei euaneſcat, quae cum eſſe debeat $A - \frac{\pi}{2}$ euidens

euidens eſt, fieri conſtantem $A = \frac{1}{2}\pi$, ita, vt iam habeamus

$$Q' = \frac{\sin. u}{1} + \frac{\sin. 2u}{2} + \frac{\sin. 3u}{3} + \frac{\sin. 4u}{4} + \frac{\sin. 5u}{5} \text{ etc.} = \frac{\pi - u}{2}.$$

Hoc modo conſtantem determinandi Illuſtr. *Daniel Bernoulli* primus eſt vſus, qui praeterea multa praeclara circa indolem harum ſerierum annotauit.

§. 41. Multiplicemus porro hanc vltimam ſeriem per $-du$, et integratio dabit

$$P'' = \frac{\cos. u}{1^2} + \frac{\cos. 2u}{2^2} + \frac{\cos. 3u}{3^2} + \frac{\cos. 4u}{4^2} + \text{etc.} = B - \frac{\pi u}{2} + \frac{uu}{4}$$

ad quam conſtantem inueniendam ponamus primo $u = 0$ fietque

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} \text{ etc.} = B$$

Cuius ſeriei ſummam iam pridem primus demonſtraui eſſe $= \frac{\pi\pi}{6}$; verum ſi haec veritas nobis eſſet ignota, egregia illa methodo a magno *Bernoullio* adhibita vtamur ac ponamus $u = \pi$ eritque

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} \text{etc.} = B - \frac{\pi\pi}{2} + \frac{\pi\pi}{4} = B - \frac{\pi\pi}{4}$$

ambae hae ſeries additae dabunt

$$\frac{2}{2^2} - \frac{2}{4^2} + \frac{2}{6^2} - \frac{2}{8^2} \text{ etc.} = 2B - \frac{\pi\pi}{4}$$

cuius duplum praebet

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} = 4B - \frac{\pi\pi}{2} = B$$

vnde colligitur $B = \frac{\pi\pi}{6}$ ita vt ſit

$$P'' = \frac{\cos. u}{1^2} + \frac{\cos. 2u}{2^2} + \frac{\cos. 3u}{3^2} + \frac{\cos. 4u}{4^2} = \frac{\pi\pi}{6} - \frac{\pi u}{2} + \frac{uu}{4}.$$

§. 42. Eodem modo vlterius progrediamur, et denuo per du multiplicando et integrando adipiſcimur

$$Q^{III} = \frac{\sin. u}{1^3} + \frac{\sin. 2u}{2^3} + \frac{\sin. 3u}{3^3} + \frac{\sin. 4u}{4^3} + \text{etc.} = C + \frac{\pi\pi u}{6} - \frac{\pi uu}{4} + \frac{u^3}{12}$$

vbi si statuatur $u = 0$, summa seriei manifesto euanescit, prodiret enim posito $u = \omega$

$$\frac{\omega}{1^2} + \frac{\omega}{2^2} + \frac{\omega}{3^2} + \frac{\omega}{4^2} + \text{etc.} = \frac{\omega\pi\pi}{6}$$

quae ob $\omega = 0$ fit $= 0$ sicque erit $C = 0$ ideoque

$$Q^{III} = \frac{\sin. u}{1^3} + \frac{\sin. 2u}{2^3} + \frac{\sin. 3u}{3^3} + \frac{\sin. 4u}{4^3} \text{ etc.} = \frac{\pi\pi u}{6} - \frac{\pi uu}{4} + \frac{u^3}{12}.$$

§. 43. Ducatur haec series in $-du$ et integratio praebebit

$$P^{IV} = \frac{\cos. u}{1^4} + \frac{\cos. 2u}{2^4} + \frac{\cos. 3u}{3^4} + \frac{\cos. 4u}{4^4} \text{ etc.} = D - \frac{\pi\pi uu}{12} + \frac{\pi u^3}{12} - \frac{u^4}{48}$$

hinc sumto $u = 0$ fiet

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} \text{ etc.} = D$$

nunc vero fiat etiam $u = \pi$ fietque

$$-\frac{1}{1^4} + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} \text{ etc.} = D - \frac{\pi^4}{48}$$

hae autem ambae series additae dant

$$\frac{2}{2^4} + \frac{2}{4^4} - \frac{2}{6^4} - \frac{2}{8^4} + \text{etc.} = 2D - \frac{\pi^4}{48}$$

quae octies sumta vt numeratores fiant $= 2^4$ praebebit

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc.} = 16D - \frac{\pi^4}{6}$$

vnde oritur $D = \frac{\pi^4}{90}$ quae est eadem summa seriei

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} \text{ etc.}$$

quam iam dudum inueneram, habebimus iam

$$P^{IIII} = \frac{\cos. u}{1^4} + \frac{\cos. 2u}{2^4} + \frac{\cos. 3u}{3^4} + \frac{\cos. 4u}{4^4} = \frac{\pi^4}{90} - \frac{\pi^2 u^2}{12} + \frac{\pi u^3}{12} - \frac{u^4}{48}.$$

§. 44. Multiplicando iterum per du et integrando consequimur

$Q^V =$

$$Q^{V} = \frac{\sin. u}{1^5} + \frac{\sin. 2u}{2^5} + \frac{\sin. 3u}{3^5} + \frac{\sin. 4u}{4^5} + \text{etc.} = E + \frac{\pi^4 u}{90} - \frac{\pi^2 u^3}{36} + \frac{\pi u^4}{48} - \frac{u^5}{240}$$

vbi vti in casu penultimo constans E iterum fit $= 0$ ita vt habeamus

$$Q^{V} = \frac{\sin. u}{1^5} + \frac{\sin. 2u}{2^5} + \frac{\sin. 3u}{3^5} + \frac{\sin. 4u}{4^5} \text{ etc.} = \frac{\pi^4 u}{90} - \frac{\pi^2 u^3}{36} + \frac{\pi u^4}{48} - \frac{u^5}{240}.$$

§. 45. Multiplicemus denuo per $-du$ prodibitque integrando

$$P^{VI} = \frac{\cos. u}{1^6} + \frac{\cos. 2u}{2^6} + \frac{\cos. 3u}{3^6} + \frac{\cos. 4u}{4^6} \text{ etc.} = F - \frac{\pi^4}{90} \cdot \frac{uu}{2} + \frac{\pi\pi}{6} \cdot \frac{u^4}{24} - \frac{\pi}{2} \cdot \frac{u^5}{120} + \frac{1}{2} \cdot \frac{u^6}{720}$$

vbi ad constantem determinandam ponatur $u = 0$ eritque

$$\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \text{etc.} = F$$

tum vero sumatur $u = \pi$ et fiet

$$-\frac{1}{1^6} + \frac{1}{2^6} - \frac{1}{3^6} + \frac{1}{4^6} - \text{etc.} = F - \frac{\pi^6}{480}.$$

quae additae dant

$$\frac{2}{2^6} + \frac{2}{4^6} + \frac{2}{6^6} + \frac{2}{8^6} + \text{etc.} = 2F - \frac{\pi^6}{480}.$$

quae multiplicetur per 32 vt omnes numeratores fiant $64 = 2^6$ et orietur

$$\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \text{etc.} = 64F - \frac{\pi^6}{15} = F$$

vnde colligitur $F = \frac{\pi^6}{945}$ ita vt sit

$$P^{VI} = \frac{\cos. u}{1^6} + \frac{\cos. 2u}{2^6} + \frac{\cos. 3u}{3^6} + \frac{\cos. 4u}{4^6} + \text{etc.} = \frac{\pi^6}{945} - \frac{\pi^4}{90} \cdot \frac{u^2}{2} + \frac{\pi^2}{6} \cdot \frac{u^4}{24} - \frac{\pi}{2} \cdot \frac{u^5}{120} + \frac{1}{2} \cdot \frac{u^6}{720}.$$

§. 46. Has series vlterius continuare superfluum foret, cum lex progressionis iam satis sit manifesta, praecipue si in subsidium vocentur summationes potestatum reciprocarum parium, quas olim vsque ad potestatem trigesimam supputatas dedi. Quod, quo clarius perspiciatur istas summas sequenti modo repraesentemus

$\frac{1}{1^2} +$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}\text{ etc.}=\alpha\pi\pi \quad \text{vt fit } \alpha=\tfrac{1}{6}$$

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\text{ etc.}=\beta\pi^4 \quad \text{vt fit } \beta=\tfrac{1}{90}$$

$$\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\frac{1}{6^6}\text{ etc.}=\gamma\pi^6 \quad \text{vt fit } \gamma=\tfrac{1}{945}$$

$$\frac{1}{1^8}+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\frac{1}{6^8}\text{ etc.}=\delta\pi^8 \quad \text{vt fit } \delta=\tfrac{1}{9450}$$

etc. etc. etc.

atque his positis sequentes habebimus integrationes pro casu scilicet $z=1$

$$Q^{I}=\int\frac{dz\,\sin.u}{1-2z\cos.u+zz}=\tfrac{1}{2}\pi-\tfrac{1}{2}u=\text{A tang.}\frac{\sin.u}{1-\cos.u}$$

$$P^{II}=-\int\frac{dz(\cos.u-z)}{1-2z\cos.u+zz}\cdot\frac{lz}{1}=\alpha\pi\pi-\tfrac{1}{2}\pi u+\tfrac{1}{2}\cdot\frac{uu}{2}$$

$$Q^{III}=+\int\frac{dz\,\sin.u}{1-2z\cos.u+zz}\cdot\frac{(lz)^2}{2}=\alpha\pi\pi\frac{u}{1}-\tfrac{1}{2}\pi\cdot\frac{uu}{2}+\tfrac{1}{2}\cdot\frac{u^3}{6}$$

$$P^{IV}=-\int\frac{dz(\cos.u-z)}{1-2z\cos.u+zz}\cdot\frac{(lz)^3}{6}=\beta\pi^4-\alpha\pi\pi.\frac{uu}{2}+\tfrac{1}{2}\pi.\frac{u^3}{6}-\tfrac{1}{2}.\frac{u^4}{24}$$

$$Q^{V}=+\int\frac{dz\,\sin.u}{1-2z\cos.u+zz}\cdot\frac{(lz)^4}{24}=\beta\pi^4.\frac{u}{1}-\alpha\pi\pi.\frac{u^3}{6}+\tfrac{1}{2}\pi.\frac{u^4}{24}-\tfrac{1}{2}.\frac{u^5}{120}$$

$$P^{VI}=-\int\frac{dz(\cos.u-z)}{1-2z\cos.u+zz}\cdot\frac{(lz)^5}{120}=\gamma\pi^6-\beta\pi^4.\frac{uu}{2}+\alpha\pi\pi.\frac{u^4}{24}-\tfrac{1}{2}\pi.\frac{u^5}{120}+\tfrac{1}{2}.\frac{u^6}{720}$$

$$Q^{VII}=+\int\frac{dz\,\sin.u}{1-2z\cos.u+zz}\cdot\frac{(lz)^6}{720}=\gamma\pi^6.\frac{u}{1}-\beta\pi^4\frac{u^3}{6}+\alpha\pi\pi\frac{u^5}{120}-\tfrac{1}{2}\pi.\frac{u^6}{720}+\tfrac{1}{2}.\frac{u^7}{5040}$$

etc. etc. etc.

§. 47. Operae pretium erit, aliquos casus, quibus angulo u datus valor tribuitur, ob oculos exponere. Ponamus igitur $u=0$ quo casu formulae nostrae alternatim euanescunt, reliquae vero praebebunt

$$-\int\frac{dz}{1-z}\,lz=\alpha\pi\pi=\frac{\pi\pi}{6}$$

$$-\int\frac{dz}{1-z}\,\frac{(lz)^3}{6}=\beta\pi^4=\frac{\pi^4}{90}$$

$$-\int\frac{dz}{1-z}\,\frac{(lz)^5}{120}=\gamma\pi^6=\frac{\pi^6}{945}$$

his affines sunt formulae, quae oriuntur ex positione

$u=\pi$

$u = \pi$ vbi iterum abeunt alternae finum u inuolventes et remanebunt fequentes

$$\int \frac{dz}{1+z}\, lz = -\frac{\pi\pi}{12} = -\tfrac{1}{2}\,\alpha\,\pi\pi$$

$$\int \frac{dz}{1+z}\,\frac{(lz)^3}{6} = -\frac{7\pi^4}{720} = -\tfrac{7}{8}\,\beta\,\pi^6$$

$$\int \frac{dz}{1+z}\,\frac{(lz)^5}{120} = -\tfrac{31}{32}\,\gamma\,\pi^6$$

$$\int \frac{dz}{1+z}\,\frac{(lz)^7}{720} = -\tfrac{127}{128}\,\delta\,\pi^8.$$

§. 48. Hic notatu dignum occurrit, quod valores alterni, quos hic omifimus, etiam euanefcant pofito $u = \pi$; deinde non minus notatu dignum eft, easdem formulas quoque euanefcere pofito $u = 2\pi$, fola prima excepta, quippe quae etiam non euanefcit pofito $u = 0$; reliquae vero, fcilicet tertia, quinta, feptima etc. certe euanefcunt cafibus $u = 0$ et $u = \pi$, quin etiam $u = 2\pi$. Quod quo clarius appareat, has formulas per factores repraefentemus eritque tertiae valor

$$= \tfrac{1}{12}\, u(\pi - u)(2\pi - u),$$

quintae vero valor reperitur

$$\tfrac{u}{720}(\pi - u)(2\pi - u)(4\pi\pi + 6\pi u - 3uu),$$

quod etiam in fequentibus vfu venit. Ingenere autem obferuari meretur, omnes noftras formulas fola prima excerpta eosdem fortiri valores, fiue ponatur $u = 0$ fiue $u = 2\pi$, quippe quibus tam idem finus quam cofinus refpondet. Videtur quidem eundem confenfum locum habere debere fi ponatur $u = 4\pi$ et $u = 6\pi$, verum Illuftr. *Bernoullius* iam luculenter oftendit, angulum u in his valoribus non vltra quatuor rectos augeri posse. Huiusmodi autem anomalia etiam in omnibus vulgaribus seriebus quibus arcus exprimuntur occurrit, atque adeo in *Leibniziana*, inqua est

$$u = \frac{tang.\, u}{1} - \frac{(tang.\, u)^3}{3} + \frac{(tang.\, u)^5}{5} - \frac{(tang.\, u)^7}{7} + \frac{(tang.\, u)^9}{9} - \text{etc.}$$

angulum u non vltra 180 gr. augere licet. Si enim poneremus $u = 180^\circ + u$ foret vtrique tang. u = tang. u neque tamen series illa exprimeret arcum $\pi + u$ sed tantum arcum u, cuiusmodi phaenomena etiam in aliis similibus seriebus locum habent. Quod autem prima series hinc plerumque excipi debeat, ratio in eo est sita, quod in formula integrali posito $u = 0$ denominator fiat $(1 - z)$ qui casu $z = 1$ euanescit ideoque formula in infinitum excrescit, id quod in sequentibus, quae per lz sunt multiplicatae, non amplius euenit, quia $\frac{lz}{1-z}$ casu $z = 1$ non amplius fit infinitus sed tantum $= -1$ et si maior potestas logarithmi adsit fit adeo $= 0$.

§. 49. Ponamus nunc etiam $n = 90^\circ$, seu $u = \frac{\pi}{2}$, vt sit cos. $u = 0$ et sin. $u = 1$ hocque casu omnes formulae generales sequentes obtinebunt valores

$$\int \frac{dz}{1+zz} = \frac{\pi}{4}$$

$$\int \frac{z\,dz}{1+zz}\, lz = -\frac{\pi\pi}{48}$$

$$\int \frac{dz}{1+zz}\, \frac{(lz)^2}{2} = \frac{\pi^3}{32}$$

$$\int \frac{z\,dz}{1+zz}\, \frac{(lz)^3}{6} = -\frac{7\pi^4}{11520}.$$

§. 50.

§. 50. Consideremus etiam casum $u = 60$ gr. siue $u = \frac{\pi}{3}$ vt sit cos. $u = \frac{1}{2}$ et sin. $u = \frac{\sqrt{3}}{2}$ et formulae generales perducent ad sequentia integralia

$$\frac{\sqrt{3}}{2}\int\frac{dz}{1 - z + zz} = \frac{\pi}{3}$$

$$-\frac{1}{2}\int\frac{dz(1 - 2z)}{1 - z + zz}\, lz = \frac{\pi\pi}{36}$$

$$\frac{\sqrt{3}}{2}\int\frac{dz}{1 - z + zz}\frac{(lz)^2}{2} = \frac{5\pi^3}{162}.$$

Simili modo si ponamus $u = 120^\circ = \frac{2\pi}{3}$ vt sit cos. $u = -\frac{1}{2}$ et sin. $u = \frac{\sqrt{3}}{2}$ sequentes integrationes istis affines prodibunt

$$\frac{\sqrt{3}}{2}\int\frac{dz}{1 + z + zz} = \frac{\pi}{6}$$

$$\frac{1}{2}\int\frac{dz(1 + 2z)}{1 + z + zz}\, lz = -\frac{\pi\pi}{18}$$

$$\frac{\sqrt{3}}{2}\int\frac{dz}{1 + z + zz}\frac{(lz)^2}{2} = \frac{2\pi^3}{81}$$

sicque pro lubitu numerus huiusmodi integrationum specialium augeri poterit.

§. 51. Quemadmodum istae integrationes memorabiles ex priore serie nostra P posito $z = 1$ sunt deductae, ita eodem modo alteram seriem Q pertractemus. Cum igitur sit

$$Q = \text{sin.}\, u + \text{sin.}\, 2u + \text{sin.}\, 3u + \text{sin.}\, 4u \text{ etc.} = \tfrac{1}{2}\text{cot.}\, \tfrac{1}{2}u$$

si per $-du$ multiplicemus et integremus, reperitur series

$$P' = \frac{\text{cos.}\, u}{1} + \frac{\text{cos.}\, 2u}{2} + \frac{\text{cos.}\, 3u}{3} + \frac{\text{cos.}\, 4u}{4} + \text{etc.} = \tfrac{1}{2} - l\,\text{sin.}\, \tfrac{1}{2}u + A$$

pro qua constante determinanda ponatur $u = \pi$ vt sit

$$-1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \tfrac{1}{6} - \text{etc.} = A$$

quocirca sit $A = -l\,2$ ita, vt habeamus

$$P' = \frac{\text{cos.}\, u}{1} + \frac{\text{cos.}\, 2u}{2} + \frac{\text{cos.}\, 3u}{3} + \frac{\text{cos.}\, 4u}{4} + \text{etc.} = -l\,2\,\text{sin.}\, \tfrac{1}{2}u$$

 pro

pro quo valore scribamus breuitatis gratia $u\,\Delta : u$ si quidem eum spectamus tanquam certam ipsius u functionem ita vt sit $P' = \Delta : u$.

§. 52. Multiplicando porro per $d\,u$ et integrando nanciscimur hanc seriem

$$Q'' = \frac{\sin. u}{1^2} + \frac{\sin. 2u}{2^2} + \frac{\sin. 3u}{3^2} + \frac{\sin. 4u}{4^2} \text{ etc.} = \int du\,\Delta : u = \Delta' : u$$

vbi haec formula integralis inuoluet certam constantem quam facile definire licet ex casu $u = 0$ quia enim series euanescit, fieri debet $\Delta' : 0 = 0$ sicque integratio plene determinatur.

§. 53. Si eodem modo vlterius progrediamur multiplicando per $-d\,u$, prodibit haec series

$$P''' = \frac{\cos. u}{1^3} + \frac{\cos. 2u}{2^3} + \frac{\cos. 3u}{5^3} + \frac{\cos. 4u}{4^3} = -\int du\,\Delta' : = \Delta'' : u.$$

Iam ad constantem, quae in hac expressione continetur, definiendam sit 1° $u = 0$ eritque

$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3}$ etc. $= \Delta'' : 0$ 1°. sit $u = \pi$ et fiet

$-\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3}$ etc. $= \Delta'' : \pi$ quibus additis prodit

$\frac{2}{2^3} + \frac{2}{4^3} + \frac{2}{6^3} + \frac{2}{8^3} +$ etc. $= \Delta'' 0 + \Delta'' : \pi$ hacque quatuor sumta erit

$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} +$ etc. $= 4\,\Delta'' : 0 + 4\,\Delta'' : \pi = \Delta'' : 0$ vnde oritur

$$3\,\Delta'' : 0 + 4\,\Delta'' : \pi = 0$$

ex qua constans in formulam nostram integralem

$$\Delta'' : u = -\int d\,u\,\Delta'\,u$$

ingressa determinari debet.

§. 54. Multiplicemus denuo per $d\,u$ et integremus prodibitque

$$Q^{IV} = \frac{\sin. u}{1^4} + \frac{\sin. 2u}{2^4} + \frac{\sin. 3u}{3^4} + \frac{\sin. 4u}{4^4} + \text{ etc.} = \int du\,\Delta'' : u = \Delta''' : u$$

atque

atque haec functio $\Delta^{III}:u$ ita debet determinari, vt euanescat sumto $u=0$ siue vt fiat $\Delta^{III}:0=0$. Eodem modo vlterius progrediendo fiet

$$P^{V}=\frac{\cos. u}{1^5}+\frac{\cos. 2u}{2^5}+\frac{\cos. 3u}{3^5}+\frac{\cos. 4u}{4^5}=-\int du\,\Delta^{III}:u=\Delta^{IIII}:u$$

huiusque functionis indoles sequenti modo determinabitur: ponatur scilicet vt hactenus $u=0$ et $u=\pi$ eritque

$\frac{1}{1^5}+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\text{etc.}=\Delta^{IV}:0$ et

$-\frac{1}{1^5}+\frac{1}{2^5}-\frac{1}{3^5}+\frac{1}{4^5}-\frac{1}{5^5}+\text{etc.}=\Delta^{IV}:\pi$ hinc addendo

$\frac{2}{2^5}+\frac{2}{4^5}+\frac{2}{6^5}+\frac{2}{8^5}+\text{etc.}=\Delta^{IV}:0+\Delta^{IV}:\pi$ et multiplicando per 16

$\frac{1}{1^5}+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\text{etc.}=16\,\Delta^{IV}:0+16\,\Delta^{IV}:\pi=\Delta^{IV}:0$

sicque fieri debebit $15\,\Delta^{IV}:0+16\,\Delta^{IV}\pi=0$ etc.

§. 55. Hinc igitur sequentes adipiscemur integrationes pro casu $z=1$

I. $-\int\frac{dz(\cos. u-z)}{1-2z\cos. u+zz}=-l\,2\,\text{fin.}\,\frac{1}{2}u=\Delta:u$

II. $\int\frac{dz\,\text{fin.}\,u}{1-2z\cos. u+zz}\,lz=\int du\,\Delta\,u=\Delta^{I}:u$

III. $-\int\frac{dz(\cos. u-z)}{1-2z\cos. u+zz}\,\frac{(lz)^2}{2}=-\int du\,\Delta^{I}\,u=\Delta^{II}:u$

IV. $\int\frac{dz\,\text{fin.}\,u}{1-2z\cos. u+zz}\,\frac{(lz)^3}{6}=\int du\,\Delta^{II}\,u=\Delta^{III}\,u$

V. $-\int\frac{dz(\cos. u-z)}{1-2z\cos. u+zz}\,\frac{(lz)^4}{24}=-\int du\,\Delta^{III}\,u=\Delta^{IV}\,u$

VI. $\int\frac{dz\,\text{fin.}\,u}{1-2z\cos. u+zz}\,\frac{(lz)^5}{120}=\int du\,\Delta^{IV}\,u=\Delta^{V}\,u$

etc. etc. etc. etc.

Has autem expressiones facile quousque libuerit continuare licet, si modo integratio cuiusque integralis

 rite

rite instituatur; conditiones autem, quas impleri oportet, sequenti modo referri possunt

$\Delta' : 0 = 0$	$3\Delta'' : 0 + 4\Delta'' : \pi = 0$
$\Delta''' : 0 = 0$	$15\Delta^{IV} : + 16\Delta^{IV} : \pi = 0$
$\Delta^{V} : 0 = 0$	$63\Delta^{VI} : 0 + 64\Delta^{VI} : \pi = 0$
$\Delta^{VII} : 0 = 0$	$255\Delta^{VIII} : 0 + 255\Delta^{VIII} : \pi = 0$
etc. etc.	etc. etc.

caeterum quia posteriores integrationes absoluere non licet, hinc parum vtilitatis exspectare possumus.

§. 56. Caeterum methodus, qua hic sumus vsi, ad constantes per quamque integrationem ingressas determinandas, a celeberrimo *Bernoullio* primum est adhibita atque eo maiori attentione digna est aestimanda, quod eius ope summationes meae serierum reciprocarum potestatum obtineri possunt, quandoquidem credideram, eas non aliter nisi ex consideratione infinitorum arcuum qui vel eodem sinu vel cosinu gaudent, demonstrari posse.

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