

DE

OSCILLATIONIBVS MINIMIS PENDVLI QVOTCVNQVE PON- DVSCVLIS ONVSTI.

Auctore

L. E V L E R O.

Problema.

Si filo tenuissimo sive grauitatis experti quotcumque ponduscula A, B, C, D in datis a se inuenientibus interuallis fuerint alligata, idque ex punto O suspensum et utcunque ad motum concitatum oscillationes minimas peragat, eius statum et motum ad quoduis tempus definire.

Solutio.

§. 1. Ex punto suspensionis O ducatur recta verticalis O V, et quicunque motus pendulo primum Tab. III Fig. 1. fuerit impressus elapsō tempore $= t$ pendulum tenet situm in figura expressum O A B C D etc. et ex singulis pondusculis ad verticalem O V agantur normales A P, B Q, C R, D S etc. Iam quia singula ponduscula dantur, eorum massae seu pondera designentur litteris A, B, C, D etc. et quia eorum interualla etiam dantur ponamus distancias

$$O A = a; A B = b; B C = c; C D = d \text{ etc.}$$

N. m. 3

Porro

Porro pro singulis pondusculis statuantur coordinatae

$$OP=x; OQ=x'; OR=x''; OS=x''' \text{ etc.}$$

$$PA=y; QB=y'; RC=y''; SD=y''' \text{ etc.}$$

Tum vero ductis verticalibus A q, B r, C s etc. vocentur anguli quibus singula interualla a situ verticali declinant

$$AOp=p; BAq=q; CBr=r; DCs=s \text{ etc.}$$

ex quibus illae coordinatae ita determinantur ut sit

$$\begin{array}{ll} x = a\cos.p & y = a\sin.p \\ x' = a\cos.p + b\cos.q & y' = a\sin.p + b\sin.q \\ x'' = a\cos.p + b\cos.q + c\cos.r & y'' = a\sin.p + b\sin.q + c\sin.r \\ x''' = a\cos.p + b\cos.q + c\cos.r + d\cos.s & y''' = a\sin.p + b\sin.q + c\sin.r + d\sin.s \\ \text{etc.} & \text{etc.} \end{array}$$

§. 2. His positis, pro motu determinando
vocetur

$$\text{tensio fili } OA = P$$

$$\text{tensio fili } AB = Q$$

$$\text{tensio fili } BC = R$$

$$\text{tensio fili } CD = S$$

atque hinc, si tempus t in minutis secundis exprimitur eiusque differentiale dt pro constante habeatur,
altitudo autem ex qua grauia uno minuto secundo
libere delabuntur notetur littera g , principia mecha-
nica sequentes suppeditant aequationes

$$\frac{ddx}{g dt^2}$$

$$\begin{array}{l}
 \frac{d d x}{z g d t^2} = I - \frac{P \cos. p}{A} + \frac{Q \cos. q}{A} \\
 \frac{d d x'}{z g d t^2} = I - \frac{Q \cos. q}{B} + \frac{R \cos. r}{B} \\
 \frac{d d x''}{z g d t^2} = I - \frac{R \cos. r}{C} + \frac{S \cos. s}{C} \\
 \frac{d d x'''}{z g d t^2} = I - \frac{S \cos. s}{D} \\
 \text{etc.} \quad \text{etc.}
 \end{array}
 \quad
 \begin{array}{l}
 \frac{d d y}{z g d t^2} = - \frac{P \sin. p}{A} + \frac{Q \sin. q}{A} \\
 \frac{d d y'}{z g d t^2} = - \frac{Q \sin. q}{B} + \frac{R \sin. r}{B} \\
 \frac{d d y''}{z g d t^2} = - \frac{R \sin. r}{C} + \frac{S \sin. s}{C} \\
 \frac{d d y'''}{z g d t^2} = - \frac{S \sin. s}{D} \\
 \text{etc.} \quad \text{etc.}
 \end{array}$$

harum aequationum numerus, qui duplo maior est quam numerus pendulorum, sufficit tam ad singulas tensiones P, Q, R, S etc. quam ad angulos p, q, r, s etc. determinandos pro quois tempore t.

§. 3. Haec ita se habent in genere quantaecumque etiam fuerint oscillationes, quo autem casu vii ulterius progredi licet, quam ob rem cogimur inuestigationes nostras tantum ad eos casus accommodare, quibus oscillationes sunt quam minimae, vti in problemate enunciatur. Quin igitur hoc casu omnes anguli p, q, r, s esse debent quam minimi, pro eorum cosinibus scribere licebit unitatem; pro sinibus autem ipsos angulos p, q, r, s etc. Hinc igitur singulae abscissae et applicatae fortientur valores

$$\begin{array}{ll}
 x = a & y = ap \\
 x' = a+b & y' = ap+bq \\
 x'' = a+b+c & y'' = ap+bq+cr \\
 x''' = a+b+c+d & y''' = ap+bq+cr+ds \\
 \text{etc.} & \text{etc.}
 \end{array}$$

Quia igitur abscissae hoc casu fiunt constantes, earum differentialia euanescent; ex quibus nascentur sequentes aequationes:

$$o = A$$

$$\circ = A - P + Q; \quad \circ = B - Q + R; \quad \circ = C - R + S; \\ \circ = D - S$$

ex quibus statim singulae tensiones facillime definiuntur, scilicet

$S = D$; $R = C + D$; $Q = B + C + D$ et $P = A + B + C + D$; etc.
hinc ad calculum contrahendum ponamus breuitatis gratia

$$\frac{P}{A} = 1 + \frac{B+C+D}{A} = \alpha \text{ hinc erit } \frac{Q}{A} = \frac{B+C+D}{A} = \alpha - 1 \\ \frac{Q}{B} = 1 + \frac{C+D}{B} = \beta \quad \frac{R}{B} = \frac{C+D}{B} = \beta - 1 \\ \frac{R}{C} = 1 + \frac{D}{C} = \gamma \quad \frac{S}{C} = \frac{D}{C} = \gamma - 1 \\ \frac{S}{D} = 1 \quad = \delta \quad \text{etc.} \quad \text{etc.} \quad \text{etc.}$$

§. 4. Quod si iam pro applicatis y , y' , y'' itemque pro tensionibus P , Q , R , S etc. suos scribamus valores, adipiscemur sequentes aequationes differentiales secundi gradus:

$$\text{I. } \frac{addp}{2gd^2} = ap + (\alpha - 1)q$$

$$\text{II. } \frac{addp + bddq}{2gd^2} = bq + (\beta - 1)r$$

$$\text{III. } \frac{addp + bddq + cddr}{2gd^2} = cr + (\gamma - 1)s$$

$$\text{IV. } \frac{addp + bddq + cddr + ddds}{2gd^2} = ds = -s.$$

Sicque totum negotium ad resolutionem harum aequationum differentio-differentialium reducitur, quoque artificia prorsus singularia postulat.

§. 5. Quia in omnibus his aequationibus variabiles p , q , r , s etc tantum unicam tenent dimensionem, euidens est, his aequationibus satisficeri posse,

si inter quantitates p, q, r, s certae rationes constantes statuantur. Sit igitur

$$p = \mathfrak{A}z; q = \mathfrak{B}z; r = \mathfrak{C}z; s = \mathfrak{D}z$$

sic enim illae aequationes sequentes induent formas:

$$\text{I. } \frac{\mathfrak{A}a ddz}{z g dt^2} = -\alpha \mathfrak{A}z + (\alpha - 1) \mathfrak{B}z$$

$$\text{II. } \frac{(\mathfrak{A}a + \mathfrak{B}b) ddz}{z g dt^2} = -\mathfrak{C}\mathfrak{B}z + (\mathfrak{C} - 1) \mathfrak{C}z$$

$$\text{III. } \frac{(\mathfrak{A}a + \mathfrak{B}b + \mathfrak{C}c) ddz}{z g dt^2} = -\gamma \mathfrak{C}z + (\gamma - 1) \mathfrak{D}z$$

$$\text{IV. } \frac{(\mathfrak{A}a + \mathfrak{B}b + \mathfrak{C}c + \mathfrak{D}d) ddz}{z g dt^2} = -\delta \mathfrak{D}z + -\mathfrak{D}z$$

quae aequationes cum omnes inter se conuenire debant, singulas ad hanc formam reuocemus:

$$\frac{ddz}{z g dt^2} = -\frac{z}{k}$$

quo valore in singulis substituto nanciscemur sequentes quatuor aequationes inter meras quantitates constantes, scilicet

$$\text{I. } -\frac{\mathfrak{A}a}{k} = -\alpha \mathfrak{A} + (\alpha - 1) \mathfrak{B}$$

$$\text{II. } -\frac{\mathfrak{A}a - \mathfrak{B}b}{k} = -\mathfrak{C}\mathfrak{B} + (\mathfrak{C} - 1) \mathfrak{C}$$

$$\text{III. } -\frac{\mathfrak{A}a - \mathfrak{B}b - \mathfrak{C}c}{k} = -\gamma \mathfrak{C} + (\gamma - 1) \mathfrak{D}$$

$$\text{IV. } -\frac{\mathfrak{A}a - \mathfrak{B}b - \mathfrak{C}c - \mathfrak{D}d}{k} = -\mathfrak{D}.$$

§. 6. Ex his iam aequationibus determinare licebit coëfficientes assumtos $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. Ex prima enim erit

$$\mathfrak{B} = \frac{\mathfrak{A}}{\alpha - 1} (\alpha - \frac{a}{k}); \text{ ex secunda erit}$$

$$\mathfrak{C} = \frac{1}{\mathfrak{C} - 1} (\mathfrak{C}\mathfrak{B} - \frac{\mathfrak{A}a - \mathfrak{B}b}{k}) \text{ siue } \mathfrak{C} = \frac{\mathfrak{B}}{\mathfrak{C} - 1} (\mathfrak{C} - \frac{b}{k}) - \frac{\mathfrak{A}}{\mathfrak{C} - 1} \frac{a}{k}$$

eodem modo ex tertia elicimus

$$\mathfrak{D} = \frac{\mathfrak{C}}{\gamma - 1} (\gamma - \frac{c}{k}) - \frac{\mathfrak{B}}{\gamma - 1} \frac{b}{k} - \frac{\mathfrak{A}}{\gamma - 1} \frac{a}{k}$$

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Qui

Qui valores in quarta substituti producent aequationem algebraicam, ex qua quantitatem incognitam k determinari opportebit; ubi aequatio tot inuoluat radices, quot dantur ponduscula: ita ut pro k totidem diuersi valores sint prodituri. Quarta autem aequatio quae hic est ultima hac forma reprezentatur:

$$\mathfrak{A}a + \mathfrak{B}b + \mathfrak{C}c + \mathfrak{D}d - \mathfrak{D}k = 0.$$

§ 7. Substituamus nunc successiue valores ex prioribus aequationibus inuenitos in posterioribus; et quia erat:

$$\mathfrak{B} = \frac{\mathfrak{W}}{a-1} (\alpha - \frac{a}{k}) \text{ fiet:}$$

$$\mathfrak{C} = \frac{\mathfrak{W}}{(\alpha-1)(\beta-1)} (\alpha - \frac{a}{k})(\beta - \frac{b}{k}) - \frac{\mathfrak{W}}{(\alpha-1)(\gamma-1)} \frac{a}{k}$$

$$\mathfrak{D} = \frac{\mathfrak{W}}{(\alpha-1)(\beta-1)(\gamma-1)} (\alpha - \frac{a}{k})(\beta - \frac{b}{k})(\gamma - \frac{c}{k}) - \frac{\mathfrak{W}}{(\alpha-1)(\gamma-1)} (\alpha - \frac{a}{k}) \frac{b}{k} - \frac{\mathfrak{W}}{\gamma-1} \frac{a}{k} - \frac{\mathfrak{W}}{(\beta-1)(\gamma-1)} (\gamma - \frac{c}{k}) \frac{a}{k}$$

qui valores ad sequentes formas reducuntur:

$$(\alpha-1) \frac{\mathfrak{B}}{\mathfrak{W}} = \alpha - \frac{a}{k}$$

$$(\alpha-1)(\beta-1) \frac{\mathfrak{C}}{\mathfrak{W}} = \alpha \beta - \frac{a(\alpha+\beta-1)}{k} + \frac{ab}{kk}$$

$$(\alpha-1)(\beta-1)(\gamma-1) \frac{\mathfrak{D}}{\mathfrak{W}} = \alpha \beta \gamma - \frac{a(\alpha \beta + \alpha \gamma + \beta \gamma - \alpha - \beta - \gamma + 1) - ab(\beta + \gamma - 1)}{k} - \frac{abc}{k^2} \\ + \frac{ab(\beta + \gamma - 1) + a(\alpha + \beta - 1) + b(c - \alpha)}{kk}$$

§ 8. Quod si iam istos valores in aequatione inuenientur substituamus, pro determinatione quantitatis k prodibit aequatio quarti gradus, ad quam coenodius inueniendam illam aequationem multiplicemus per:

$$\frac{(\alpha-1)(\beta-1)(\gamma-1)}{k} \mathfrak{B} = \frac{(\alpha-1)(\beta-1)(\gamma-1)}{k} \mathfrak{B} + \frac{(\alpha-1)(\beta-1)(\gamma-1)}{k} \mathfrak{C} \beta$$

$$\frac{(\alpha-1)(\beta-1)(\gamma-1)}{k} \mathfrak{D} = \frac{(\alpha-1)(\beta-1)(\gamma-1)}{k} \mathfrak{D} = 0 \quad \text{facto:}$$

facto autem calculo aequatio ista biquadratica ita reperietur expressa :

$$\begin{aligned}
 & + ab\cdot \mathfrak{C}\gamma \\
 a\mathfrak{C}\gamma k^4 - a\mathfrak{C}\gamma(a+b+c+d)k^3 + bc\cdot a\gamma & \\
 & + cd\cdot a\mathfrak{C} \\
 & + ac\cdot \gamma(a+\mathfrak{C}-1) \\
 & + bd\cdot a(\mathfrak{C}+\gamma-1) \\
 & + ad(a\mathfrak{C}+a\gamma+\mathfrak{C}\gamma-a-\mathfrak{C}-\gamma+1) \} k \\
 \\
 & - bcd\cdot a \\
 & - ab\cdot \gamma \\
 & - abd(\mathfrak{C}+\gamma-1) \\
 & - acd(a+\mathfrak{C}-1) \} k + abcd = 0
 \end{aligned}$$

vbi obseruasse iuuabit, primo litteras a , b , c , d semper denotare distantias positivas; tum vero litteras α , \mathfrak{C} , γ , esse numeros positivos atque adeo unitate maiores. Hinc enim ratio intelligi poterit, cur omnes quatuor radices huius aequationis proditurae sint reales: eas autem omnes esse positivas permutatio signorum declarat.

§. 9. Ipsi resolutioni huius aequationis hic non immoramus, quandoquidem si numerus pondiscularum esset maior a tali inuestigatione prorsus abstinere cogeremur: designemus igitur quatuor huius aequationis radices litteris k , k' , k'' et k''' ex quarum singulis peculiares valores pro litteris A , B , C et D colligemus, quos pariter hoc modo designemus A' , B' , C' , D' ; A'' , B'' , C'' , D'' et A''' , B''' , C''' , D''' ; vbi quidem patet, litteras A , A' , A'' , A''' arbitrio nostro

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penitus relinqu; ita vt hoc modo quatuor habeamus quantitates pro lubitu accipiendas.

§. 10. Prosequamur igitur nostrum calculum pro sola radice k , cui respondent coëfficientes A, B, C, D , quandoquidem quod pro hac radice fuerit compertum facillime quoque ad reliquas radices applicatur. Cum igitur statuissimus hanc aequationem differentialem secundi gradus $\frac{ddz}{dt^2} = \frac{z}{k}$, ita vt sit $\frac{ddz}{dt^2} + \frac{2g}{k}z = 0$ si ponamus $\frac{z}{k} = \lambda \lambda$, vt sit $\lambda = \sqrt{\frac{g}{k}}$ si quidem k semper est quantitas realis positiva, notum est post duplicom integrationem prodire $z = f(\lambda t + \vartheta)$: vbi ϑ est angulus ab arbitrio nostro pendens. altera autem constans arbitraria f sine restrictione unitati aequalis ponи potest: propterea quod coëfficiens A iam est arbitrarius. Hinc igitur ad quodvis tempus t singuli anguli p, q, r, s ita determinabuntur, vt sit

$$\text{I. } p = A \sin(\lambda t + \vartheta); q = B \sin(\lambda t + \vartheta); r = C \sin(\lambda t + \vartheta) \\ \text{et } s = D \sin(\lambda t + \vartheta)$$

similique modo si ex reliquis radicibus, k', k'', k''' ponamus

$$\lambda' = \sqrt{\frac{g}{k'}}, \lambda'' = \sqrt{\frac{g}{k''}}, \lambda''' = \sqrt{\frac{g}{k'''}}$$

praeter illam solutionem adhuc habebimus tres sequentes

$$\text{II. } p = A' \sin(\lambda' t + \vartheta'); q = B' \sin(\lambda' t + \vartheta'); r = C' \sin(\lambda' t + \vartheta'); \\ s = D' \sin(\lambda' t + \vartheta')$$

$$\text{III. } p = A'' \sin(\lambda'' t + \vartheta''); q = B'' \sin(\lambda'' t + \vartheta''); r = C'' \sin(\lambda'' t + \vartheta''); \\ s = D'' \sin(\lambda'' t + \vartheta'')$$

$$\text{IV. } p = \mathfrak{A}^{\text{III}} \sin.(\lambda^{\text{III}} t + \vartheta^{\text{III}}); q = \mathfrak{B}^{\text{III}} \sin.(\lambda^{\text{III}} t + \vartheta^{\text{III}}); r = \mathfrak{C}^{\text{III}} \sin.(\lambda^{\text{III}} t + \vartheta^{\text{III}}); \\ s = \mathfrak{D}^{\text{III}} \sin.(\lambda^{\text{III}} t + \vartheta^{\text{III}}).$$

§. 11. Singulae autem hae quatuor solutiones maxime sunt particulares: propterea quod duas tantum constantes arbitrarias inuoluunt, scilicet \mathfrak{A} et ϑ , dum solutio generalis ob quatuor aequationes differentio-differentiales octo constantes arbitrarias complecti deberet. Qualis igitur motus singulis respondeat operae pretium erit accuratius inuestigare; ac primo quidem quoniam pro qualibet casu quatuor anguli p, q, r, s eandem perpetuo inter se seruant rationem, motus erit maxime regularis et pendulum per iode oscillations suas pereget ac si esset simplex; atque quia elapso tempore $t = \frac{2\pi}{\lambda}$ si loco t scribamus $t + \frac{2\pi}{\lambda}$ singuli anguli in eundem statum reuertuntur, ideoque pendulum interea duos oscillationes absoluisse censetur, sicque tempus vnius oscillationis erit $= \frac{\pi}{\lambda} = \frac{\pi \sqrt{k}}{\sqrt{2g}}$, quod adeo tempus in minutis secundis exprimitur. Eodem modo pro secunda radice k' erit tempus vnius cuiusque oscillationis $= \frac{\pi \sqrt{k'}}{\sqrt{2g}}$; pro tercia radice $= \frac{\pi \sqrt{k''}}{\sqrt{2g}}$ et pro quarta $\frac{\pi \sqrt{k'''}}{\sqrt{2g}}$.

§. 12. Cum igitur hae quatuor solutiones simplices problemati nostro satisfaciant, quoniam in aequationibus differentio - differentialibus ad quas nos solutio perduxit singulae quantitates p, q, r, s vbiique unam tantum dimensionem tenent, solutiones illae particulares quomodo cunque inter se combinentur pro-

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blemati pariter satisfacent, vnde sequens solutio generalis conficitur:

$$\begin{aligned} p &= \mathfrak{A} \sin.(\lambda t + \vartheta) + \mathfrak{A}' \sin.(\lambda' t + \vartheta') + \mathfrak{A}'' \sin.(\lambda'' t + \vartheta'') + \mathfrak{A}''' \sin.(\lambda''' t + \vartheta''') \\ q &= \mathfrak{B} \sin.(\lambda t + \vartheta) + \mathfrak{B}' \sin.(\lambda' t + \vartheta') + \mathfrak{B}'' \sin.(\lambda'' t + \vartheta'') + \mathfrak{B}''' \sin.(\lambda''' t + \vartheta''') \\ r &= \mathfrak{C} \sin.(\lambda t + \vartheta) + \mathfrak{C}' \sin.(\lambda' t + \vartheta') + \mathfrak{C}'' \sin.(\lambda'' t + \vartheta'') + \mathfrak{C}''' \sin.(\lambda''' t + \vartheta''') \\ s &= \mathfrak{D} \sin.(\lambda t + \vartheta) + \mathfrak{D}' \sin.(\lambda' t + \vartheta') + \mathfrak{D}'' \sin.(\lambda'' t + \vartheta'') + \mathfrak{D}''' \sin.(\lambda''' t + \vartheta''') \end{aligned}$$

in his enim formulis octo occurruunt constantes arbitriae, scilicet quatuor coëfficientes $\mathfrak{A}, \mathfrak{A}', \mathfrak{A}''$, \mathfrak{A}''' quippe per quos reliqui determinantur; tum quatuor anguli $\vartheta, \vartheta', \vartheta'', \vartheta''''$, quemadmodum gemina integratio quatuor illarum aequationum postulat. Hinc igitur patet, principium illustriss. D. Bernoulli, quo omnes huiusmodi oscillationes ex duobus vel pluribus motibus oscillatoriis simplicibus et regularibus componi statuit, omnino in primis notus principiis esse fundatum atque adeo ex iis immediate deduci posse.

§. 13. Ope harum igitur formularum ad quodvis tempus t singuli illi anguli p, q, r et s assignari possuntque status penduli definiri poterit. Quin etiam horum angulorum variationes momentaneae celeritates praebebunt, quibus status penduli quovis temporis momento dt immutatur. Cum enim formulae

$$\frac{dp}{dt}, \frac{dq}{dt}, \frac{dr}{dt} \text{ et } \frac{ds}{dt}$$

exprimant celeritates angulares, quibus isti anguli tempore dt augentur, hae celeritates ita se habebunt

$$\frac{dp}{dt}$$

$$\frac{dt}{dt} = \lambda \mathfrak{A} \cos(\lambda t + \vartheta) + \lambda' \mathfrak{A}' \cos(\lambda' t + \vartheta') + \lambda'' \mathfrak{A}'' \cos(\lambda'' t + \vartheta'') \\ + \lambda''' \mathfrak{A}''' \cos(\lambda''' t + \vartheta''')$$

$$\frac{dq}{dt} = \lambda \mathfrak{B} \cos(\lambda t + \vartheta) + \lambda' \mathfrak{B}' \cos(\lambda' t + \vartheta') + \lambda'' \mathfrak{B}'' \cos(\lambda'' t + \vartheta'') \\ + \lambda''' \mathfrak{B}''' \cos(\lambda''' t + \vartheta''')$$

$$\frac{dr}{dt} = \lambda \mathfrak{C} \cos(\lambda t + \vartheta) + \lambda' \mathfrak{C}' \cos(\lambda' t + \vartheta') + \lambda'' \mathfrak{C}'' \cos(\lambda'' t + \vartheta'') \\ + \lambda''' \mathfrak{C}''' \cos(\lambda''' t + \vartheta''')$$

$$\frac{ds}{dt} = \lambda \mathfrak{D} \cos(\lambda t + \vartheta) + \lambda' \mathfrak{D}' \cos(\lambda' t + \vartheta') + \lambda'' \mathfrak{D}'' \cos(\lambda'' t + \vartheta'') \\ + \lambda''' \mathfrak{D}''' \cos(\lambda''' t + \vartheta''')$$

sicque omnia sumus adepti, quae circa solutionem huius problematis desiderari possunt.

Corollarium.

§ 14. Maxima igitur difficultas in resolutione aequationis algebraicae ex qua omnes valores litterae k determinari oportet, occurrit; praecipue si pendulum pluribus pondusculis fuerit oneratum. Tum vero etiam quemadmodum pro singulis valoribus ipsius k coëfficientes \mathfrak{B} , \mathfrak{C} et \mathfrak{D} definiri commode queant nondum satis liquet pro pluribus quam quatuor pondusculis. Quo igitur hanc investigationem faciliorem reddamus, differentias inter binas aequationes se in sequentes (§. 5.) exhibitas consideremus.

$$\text{I. } -\frac{\mathfrak{A}^a}{k} = -\alpha \mathfrak{A} + (\alpha - 1) \mathfrak{B} \text{ siue } 0 = \mathfrak{A} \left(\frac{a}{k} - \alpha \right) + (\alpha - 1) \mathfrak{B}$$

$$\text{I-II. } \frac{\mathfrak{B}^b}{k} = -\alpha \mathfrak{A}' + \mathfrak{B}(\alpha + \beta - 1) - \mathfrak{C}(\beta - 1)$$

$$\text{II-III. } \frac{\mathfrak{C}^c}{k} = -\gamma \mathfrak{B} + \mathfrak{C}(\beta + \gamma - 1) - \mathfrak{D}(\gamma - 1)$$

$$\text{III-IV. } \frac{\mathfrak{D}^d}{k} = -\gamma \mathfrak{C} + \gamma \mathfrak{D}$$

vnde

vnde facile patet quomodo hae aequalitates sint
continuandae, si pondusculorum numerus fuerit
maior.

§. 15. Supra litterae B , C et D ex pri-
ma A determinauimus; nunc autem a postrema inci-
pientes singulas ex vltima D deriuemus, vnde fit vt
sequitur

$$\gamma C = D \left(\gamma - \frac{d}{k} \right)$$

$$C B = C \left(\gamma + \gamma - 1 - \frac{c}{k} \right) - D \left(\gamma - 1 \right)$$

$$\alpha A = B \left(\alpha + C - 1 - \frac{b}{k} \right) - C \left(\gamma - 1 \right)$$

vnde reperimus

$$\frac{C}{D} = \frac{1}{\gamma} \left(\gamma - \frac{d}{k} \right)$$

$$\frac{B}{D} = \frac{1}{\alpha \gamma} \left(\gamma - \frac{d}{k} \right) \left(\gamma + \gamma - 1 - \frac{c}{k} \right) - \left(\frac{\gamma - 1}{\alpha} \right)$$

$$\frac{A}{D} = \frac{1}{\alpha \beta \gamma} \left(\gamma - \frac{d}{k} \right) \left(\gamma + \gamma - 1 - \frac{c}{k} \right) \left(\alpha + C - 1 - \frac{b}{k} \right) - \frac{\gamma - 1}{\alpha \beta} \\ \left(\alpha + C - 1 - \frac{b}{k} \right) - \frac{\alpha - 1}{\alpha \gamma} \left(\gamma - \frac{d}{k} \right)$$

qui valores in prima aequatione substituti producent
istam aequationem :

$$0 = - \frac{1}{\alpha \beta \gamma} \left(\alpha - \frac{a}{k} \right) \left(\alpha + C - 1 - \frac{b}{k} \right) \left(\gamma + \gamma - 1 - \frac{c}{k} \right) \left(\gamma - \frac{d}{k} \right) \\ + \frac{\gamma - 1}{\alpha \beta} \left(\alpha - \frac{a}{k} \right) \left(\alpha + C - 1 - \frac{b}{k} \right) + \frac{\alpha - 1}{\alpha \gamma} \left(\alpha - \frac{a}{k} \right) \left(\gamma - \frac{d}{k} \right) \\ + \frac{\alpha - 1}{\alpha \gamma} \left(\gamma + \gamma - 1 - \frac{c}{k} \right) \left(\gamma - \frac{d}{k} \right) - \frac{(\alpha - 1)(\gamma - 1)}{\alpha}$$

quae aequatio manifesto ascendit ad quartum ordinem,
ex qua incognitae k quatuor valores inuestigari op-
portet: hocque modo operatio institui facile poterit,
si pondusculorum numerus fuerit maior.

Scho-

Scholion.

§ 16. Quamuis autem haec solutio sit maxime elegans, et problemati perfectissime satisfaciat, tamen maximae occurunt difficultates, si eam ad casum determinatum applicare voluerimus. Quod si enim pro statu initiali vbi $t = 0$ singulis angulis p, q, r, s datus valores tribuere velimus, simulque singulis pondusculis datas celeritates angularares, ad octo aequationes perueniemus, quae similes erunt duabus aequationibus ex angulo p natis: si enim requiratur ut initio fuerit angulus $p = f$ eiusque celeritas angularis $= i$ hae duea obtinentur aequationes:

$$f = \mathfrak{A} \sin \vartheta + \mathfrak{A}' \sin \vartheta' + \mathfrak{A}'' \sin \vartheta'' + \mathfrak{A}''' \sin \vartheta''' \text{ et}$$

$$i = \lambda \mathfrak{A} \cos \vartheta + \lambda' \mathfrak{A}' \cos \vartheta' + \lambda'' \mathfrak{A}'' \cos \vartheta'' + \lambda''' \mathfrak{A}''' \cos \vartheta'''$$

similesque binac aequationes obtinebuntur pro reliquis pondusculis. Nunc igitur requiritur ut ex his octo aequationibus octo illae constantes arbitariae

$$\mathfrak{A}, \mathfrak{A}', \mathfrak{A}'', \mathfrak{A}''' \text{ et } \vartheta, \vartheta', \vartheta'', \vartheta'''$$

definiantur, quem sane laborem vix quisquam exsequitur: si modo corpusculorum numerus ternarium superauerit; quamobrem iam dudum non dubitaui asseuerare, solutionem hanc quantumuis elegantem et perfectam plane esse ineptam, ut ad casus determinatos, quibus penduli status initialis praescribitur ad aptari possit. Ex quo manifesto sequitur, si problema ita proponatur, ut si pendulo datus status et motus initio imprimatur, motus deinceps secutures definiri debeat, longe aliam solutionem requiri, quae proinde ab hac maxime discrepare debet.

DE OSCILLATIONIBVS

EVOLVTIO CASVS

quo omnia ponduscula sunt aequalia eorumque interualla etiam aequalia $\equiv a$; ponatur autem breuitatis gratia $\frac{a}{k} \equiv u$.

§. 17. Sit primo pondusculorum numerus $\equiv 2$ erit $a \equiv 2$ et $\mathfrak{C} \equiv 1$, vnde aequationes ex §. 13. erunt

$$\circ = -\mathfrak{A}(2-u) + \mathfrak{B} \text{ et } \mathfrak{B}u = -2\mathfrak{A} + 2\mathfrak{B};$$

ex posteriore scquitur

$$\mathfrak{A} = \frac{\mathfrak{B}(2-u)}{2}$$

qui valor in priore substitutus dat

$$\circ = -\frac{\mathfrak{B}(2-u)^2}{2} + \mathfrak{B} \text{ siue } (2-u)^2 - 2 = \circ$$

vnde statim deducitur

$$2-u = \pm \sqrt{2} \text{ ideoque } u = 2 \mp \sqrt{2} = \frac{a}{k},$$

tum vero \mathfrak{A} per \mathfrak{B} ita exprimitur vt sit $\mathfrak{A} = \frac{\mathfrak{B}(2-u)}{2}$
vbi \mathfrak{B} pro libitu accipi potest. Quare si bini va-
lores ipsius k sint k et k' , hisque respondeant litterae \mathfrak{A}' et \mathfrak{B}' solutio in his aequationibus continebi-
tur posito $\lambda = \sqrt{\frac{a}{k}}$ et $\lambda' = \sqrt{\frac{a}{k'}}$

$$p = \mathfrak{A} \sin.(\lambda t + \vartheta) + \mathfrak{A}' \sin.(\lambda' t + \vartheta') \text{ et}$$

$$q = \mathfrak{B} \sin.(\lambda t + \vartheta) + \mathfrak{B}' \sin.(\lambda' t + \vartheta').$$

§. 18. Sit pondusculorum numerus $\equiv 3$, erit
que $a \equiv 3$, $\mathfrak{C} \equiv 2$ et $\gamma \equiv 1$, vnde aequationes nostrae
erint

$$\circ = -\mathfrak{A}(3-u) + 2\mathfrak{B} \text{ siue } \circ = \mathfrak{A}(3-u) - 2\mathfrak{B}$$

$$\mathfrak{B}u = -3\mathfrak{A} + 4\mathfrak{B} - \mathfrak{C}$$

$$\mathfrak{C}u = -2\mathfrak{B} + 2\mathfrak{C}$$

ex

PENDULI MVLTIMEMBRIS.

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Ex hac fit $\mathfrak{B} = \frac{\mathfrak{C}(z-u)}{z}$ tum vero experiore;

$$\mathfrak{A} = \frac{\mathfrak{C}(z-u)(4-u)}{6} - \frac{\mathfrak{C}}{z} = \frac{\mathfrak{C}}{6}(6-6u+uu)$$

vnde aequatio prodit

$$\frac{(z-u)(z-u)}{z} - u - \frac{(z-u)}{z} - (z-u) = 0 \text{ siue}$$

$$6 - 18u + 9uu - u^2 = 0$$

cuius ergo dabuntur tres radices, ideoque valores pro k, k', k'' , ex quibus tota solutio facile conficitur.

§. 19. Sit pondusculorum numerus = 4, erit
 $\alpha = 4, \beta = 3, \gamma = 2, \delta = 1$, vnde nostrae aequationes erunt

$$0 = \mathfrak{A}(4-u) - 3\mathfrak{B}$$

$$\mathfrak{B}u = -4\mathfrak{A} + 6\mathfrak{B} - 2\mathfrak{C}$$

$$\mathfrak{C}u = -3\mathfrak{B} + 4\mathfrak{C} - \mathfrak{D}$$

$$\mathfrak{D}u = -2\mathfrak{C} + 2\mathfrak{D}$$

Ex ultima fit

$$\mathfrak{C} = \frac{\mathfrak{D}(z-u)}{z}$$

$$\mathfrak{B} = \frac{\mathfrak{D}(z-u)(4-u)}{6} - \frac{\mathfrak{D}}{z} = \frac{\mathfrak{D}}{6}(6-6u+uu)$$

$$\mathfrak{A} = \frac{\mathfrak{D}}{6}(24 - 36u + 12uu - u^2)$$

qui valores substituti hanc praebent aequationem

$$u^4 - 16u^3 + 72uu - 96u + 24 = 0$$

cuius quatuor radices quaeri oportet.

§. 20. Sit numerus pondusculorum = 5, erit
 $\alpha = 5, \beta = 4, \gamma = 3, \delta = 2, \epsilon = 1$, et aequationes nostrae erunt

$$\begin{aligned} \mathfrak{A} &= \mathfrak{A}(5-u) - 4\mathfrak{B} \\ \mathfrak{B}u &= -5\mathfrak{A} + 8\mathfrak{B} - 3\mathfrak{C} \\ \mathfrak{C}u &= -4\mathfrak{B} + 6\mathfrak{C} - 2\mathfrak{D} \\ \mathfrak{D}u &= -3\mathfrak{C} + 4\mathfrak{D} - \mathfrak{E} \\ \mathfrak{E}u &= -2\mathfrak{D} + 2\mathfrak{E} \end{aligned}$$

hic ex tribus posterioribus colligimus

$$\mathfrak{D} = \frac{\mathfrak{E}}{2}(2-u)$$

$$\mathfrak{C} = \frac{\mathfrak{E}}{6}(6-6u+uu)$$

$$\mathfrak{B} = \frac{\mathfrak{E}}{24}(24-36u+12uu-u^2)$$

tum vero inuenitur

$$\mathfrak{A} = \frac{\mathfrak{E}}{120}(120-240u+120uu-20u^2+u^4)$$

quibus valoribus substitutis aquatio sequens resultat

$$u^6 - 25u^4 + 200u^3 - 600uu + 600u - 120 = 0.$$

§. 21. Hinc generaliter si numerus pondusculorum fuerit $= n$, aquatio ex qua u definiri debet per legitimam inductionem colligitur fore

$$\mathfrak{O} = I - \frac{n_u}{1^2} + \frac{n(n-1)uu}{1^2 \cdot 2^2} - \frac{n(n-1)(n-2)uu^2}{1^2 \cdot 2^2 \cdot 3^2} + \frac{n(n-1)(n-2)(n-3)uu^3}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2} \text{ etc.}$$

deinde vero si coëfficientium $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ ultimus fuerit \mathfrak{N} singuli sequenti modo determinabuntur

$$\begin{aligned} \mathfrak{E} &= I - \frac{n-1}{1^2 \cdot 2} u + \frac{(n-1)(n-2)}{1^2 \cdot 2^2 \cdot 3} u^2 - \frac{(n-1)(n-2)(n-3)}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4} u^3 + \frac{(n-1)(n-2)(n-3)(n-4)}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2 \cdot 5} u^4 \text{ etc.} \\ \mathfrak{B} &= I - \frac{n-2}{1^2 \cdot 2} u + \frac{(n-2)(n-3)}{1^2 \cdot 2^2 \cdot 3} uu - \frac{(n-2)(n-3)(n-4)}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4} u^2 + \frac{(n-2)(n-3)(n-4)(n-5)}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2 \cdot 5} u^3 \text{ etc.} \\ \mathfrak{C} &= I - \frac{n-3}{1^2 \cdot 2} u + \frac{(n-3)(n-4)}{1^2 \cdot 2^2 \cdot 3} uu - \frac{(n-3)(n-4)(n-5)}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4} u^2 + \frac{(n-3)(n-4)(n-5)(n-6)}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2 \cdot 5} u^3 \text{ etc.} \\ \mathfrak{D} &= I - \frac{n-4}{1^2 \cdot 2} u + \frac{(n-4)(n-5)}{1^2 \cdot 2^2 \cdot 3} uu - \frac{(n-4)(n-5)(n-6)}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4} u^2 + \frac{(n-4)(n-5)(n-6)(n-7)}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2 \cdot 5} u^3 \text{ etc.} \end{aligned}$$

etc.

etc.

etc.

Tantum

Tantum igitur restat methodus, cuius ope illius aequationis n radices elici possunt, quippe quibus inuen-tis solutio completa huius casus habebitur.

§. 22. Aequatio illa ordinis n , ex qua valores litterae u definire oportet etiam hoc modo concin-nius referri potest.

$$0 = u^n - \frac{n^2}{1} u^{n-1} + \frac{n^2(n-1)^2}{1 \cdot 2} u^{n-2} - \frac{n^2(n-1)^2(n-2)^2}{1 \cdot 2 \cdot 3} u^{n-3} + \frac{n^2(n-1)^2(n-2)^2(n-3)^2}{1 \cdot 2 \cdot 3 \cdot 4} u^{n-4} \text{ etc.}$$

Hinc autem coëfficientes \mathfrak{A} , \mathfrak{B} , \mathfrak{C} , \mathfrak{D} etc. etiam hoc modo exhiberi possunt

$$\pm 1 \cdot 2 \cdot 3 \dots n \frac{\mathfrak{A}}{n!} = u^{n-1} - \frac{n(n-1)}{1} u^{n-2} + \frac{n(n-1)^2(n-2)}{1 \cdot 2} u^{n-3} - \frac{n(n-1)^2(n-2)^2(n-3)}{1 \cdot 2 \cdot 3} u^{n-4} \text{ etc.}$$

$$\mp 1 \cdot 2 \cdot 3 \dots (n-1) \frac{\mathfrak{B}}{n!} = u^{n-2} - \frac{n(n-1)(n-2)}{1} u^{n-3} + \frac{(n-1)(n-2)^2(n-3)}{1 \cdot 2} u^{n-4} - \frac{(n-1)(n-2)^2(n-3)^2(n-4)}{1 \cdot 2 \cdot 3} u^{n-5} \text{ etc.}$$

$$\pm 1 \cdot 2 \cdot 3 \dots (n-2) \frac{\mathfrak{C}}{n!} = u^{n-3} - \frac{(n-2)(n-3)}{1} u^{n-4} + \frac{(n-2)(n-3)^2(n-4)}{1 \cdot 2} u^{n-5} - \frac{(n-2)(n-3)^2(n-4)^2(n-5)}{1 \cdot 2 \cdot 3} u^{n-6} \text{ etc.}$$

$$\mp 1 \cdot 2 \cdot 3 \dots (n-3) \frac{\mathfrak{D}}{n!} = u^{n-4} - \frac{(n-3)(n-4)}{1} u^{n-5} + \frac{(n-3)(n-4)^2(n-5)}{1 \cdot 2} u^{n-6} - \frac{(n-3)(n-4)^2(n-5)^2(n-6)}{1 \cdot 2 \cdot 3} u^{n-7} \text{ etc.}$$

etc.

etc.

vbi signorum ambigitorum superiora sunt accipienda
si n fuerit numerus impar, inferiora autem si par.