

SPECULATIONES  
ANALYTICAE

Auctore

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**C**um super inuenissem integrale huius formulae differentialis  $\frac{(x^\alpha - x^\beta) dx}{\ln x}$ , si ita capiatur ut euanscat posito  $x = 0$ , tum vero statuarum  $x = 1$ , aequari huic valori:  $\frac{\alpha - \beta}{\beta + 1}$ : haec integratio eo magis attentione digna mihi videbatur, quod ejus veritas per nullas methodos hactenus visitatas ostendi posset. Quamobrem nullum plane est dubium, quin ea plurimum in recessu habeat, et ad multa alia praeclara inuenta in Analysis perducere queat. Haud igitur ingratum Geometris fore arbitror, si nonnullas speculationes quae super hac formula se mihi obtulerunt exposuero.

§ 1. Quoniam ista integratio se ad omnes plane exponentes pro literis  $\alpha$  et  $\beta$  assumtos extendit, atque adeo valores imaginarii non excluduntur, ponamus  $\alpha = n\sqrt{-1}$ , et  $\beta = -n\sqrt{-1}$ , eritque  $x^\alpha - x^\beta = x^{n\sqrt{-1}} - x^{-n\sqrt{-1}}$ , quae formula cum reducatur ad hanc:  $e^{n\sqrt{-1}x\sqrt{-1}} - e^{-n\sqrt{-1}x\sqrt{-1}}$ , notum est eius valorem esse  $= 2\sqrt{-1} \sin(n\sqrt{-1}x)$ ; quo valore substituto prodit

$$2\sqrt{-1} \int \frac{x^{\alpha} \sin(n\sqrt{-1}x)}{\ln x} dx = \frac{1}{1 - n^2}.$$

H 2

Constat

Constat autem huius formulæ  $L \frac{1+n\sqrt{-1}}{1-n\sqrt{-1}}$  valorem esse  $x\sqrt{-1} A \tan g. n$ , quandoquidem sumto  $n$  variabili eius differentiatio dat

$$d. L \frac{1+n\sqrt{-1}}{1-n\sqrt{-1}} = \frac{2dn\sqrt{-1}}{1+n^2},$$

euia integrale manifesto est  $x\sqrt{-1} A \tan g. n$ ; hinc igitur adipiscitur sequens Theorema:

*Theorema.* — Ista formula integralis  $\int_{-Lx}^{dx \sin. nLx}$  a termino  $x=0$  usque ad terminum  $x=1$  extensa exprimit arcum circuli cuius tangens  $= n$ ; unde sumto  $n=1$ , erit  $\int_{-Lx}^{dx \sin. x} = \pi$ , denotante  $\pi$  semiperipheriam circuli, cuius radius  $= 1$ .

*§. 2.* Quamvis autem haec integratio ex nostra forma generali, quæ aliis methodis inaccessa videtur, sit deducta: tamen eius veritas per resolutiones consuetas sequenti modo ostendi potest, sicutque ex hoc casu integratio generalis eam maius firmamentum accipiet; Cum enim per seriem infinitam sit

$$\sin. nLx = \frac{nLx}{1} - \frac{n^3(Lx)^3}{1 \cdot 2 \cdot 3} + \frac{n^5(Lx)^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \text{etc.}$$

erit

$$\int_{-Lx}^{dx \sin. nLx} = \int_{-Lx}^{dx} \left( n - \frac{n^3(Lx)^2}{1 \cdot 2 \cdot 3} + \frac{n^5(Lx)^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \text{ etc.} \right);$$

constat autem esse

$$\int dx(Lx)^2 = x(Lx)^2 - 2 \int dxLx = x(Lx)^2 - 2xLx + 2 \cdot 1 \cdot x;$$

quæ expressio posito  $x=1$  reducitur ad 2. 1; simili modo fieri

$$\int dx(Lx)^4 = x(Lx)^4 - 4 \int dx(Lx)^2 = x(Lx)^4 - 4x(Lx)^2 + 4 \cdot 3 \int dx(Lx)^2,$$

quac

quae positi  $x = 1$  ob  $Ix = 0$  praebet  $4 \cdot 3 \cdot 2 \cdot 1^5$   
eodemque modo erit  $\int dx(Ix)^5 = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1^5$ .  
His igitur singulis valoribus integralibus introductis  
proueniet

$$\int \frac{dx \sin. n Ix}{Ix} = n - \frac{x + n^3}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{4 \cdot 3 \cdot 2 \cdot n^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{6 \dots x n^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \dots + \frac{n^9}{n^5} - \frac{n^{11}}{x^7} + \text{etc.}$$

cuius serici summa manifesto est A tang.  $n$ .

§. 3. Hic casus nobis ansam praebet etiam hanc formulam integralem inuestigandi  $\int \frac{dx \cos. n Ix}{Ix}$  ; quae quidem non immediate in nostra forma generali continetur ; et quia est

$$\cos nIx = 1 - \frac{n^2 (Ix)^2}{1 \cdot 2} + \frac{n^4 (Ix)^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{n^6 (Ix)^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.}$$

ex primo termino oritur  $\int \frac{dx}{Ix}$ , cuius quidem valorem ostendi esse infinitum. Pro sequentibus autem terminis erit

$$\int dxIx = xIx - x = -x \quad \text{et} \quad \int dx(Ix)^3 = -1 \cdot 2 \cdot 3 \cdot \text{et} \\ \int dx(Ix)^5 = -1 \cdot \dots \cdot 5 \text{ etc.}$$

quibus valoribus substitutis obtinebimus

$$\int \frac{dx \cos. n Ix}{Ix} = \int \frac{dx}{Ix} + \frac{n^2}{2} - \frac{n^4}{4} + \frac{n^6}{6} - \frac{n^8}{8},$$

quae expressio manifesto reducitur ad hanc :

$$\int \frac{dx}{Ix} + \frac{1}{2}I(x + n^2).$$

Quia autem primus terminus hanc summam reddit infinitam, hinc subtrahamus aliam similem.

$$\int \frac{dx \cos. m Ix}{Ix} = \int \frac{dx}{Ix} + \frac{1}{2}I(x + m^2)$$

et habebimus

$$\int \frac{dx (\cos. n Ix - \cos. m Ix)}{Ix^2} = \frac{1}{2}I \frac{x + n^2}{x + m^2}$$

atque haec integratio non minus notatu digna videtur quam praecedens.

4. Cum autem in genere sit

$$\cos. a - \cos. b = 2 \sin. \frac{a+b}{2} \sin. \frac{b-a}{2} \text{ erit}$$

$$\cos. n l x - \cos. m l x = 2 \sin. \frac{m+n}{2} l x \sin. \frac{m-n}{2} l x,$$

ita ut sit

$$\int d x \frac{\sin. \frac{m+n}{2} l x \sin. \frac{m-n}{2} l x}{l x} = \frac{1}{l} \frac{1}{\frac{1}{2} + \frac{m^2 - n^2}{4 l^2}},$$

quod si ergo ponamus  $m = p + q$  et  $n = p - q$  sequens adipiscemur theorema maxime notatu dignum:

*Theorema 2.* Ista forma integralis

$$\int \frac{d x}{l x} \sin. p l x \sin. q l x \text{ est } \frac{1}{l} \frac{1}{\frac{1}{2} + \frac{(p+q)^2}{4 l^2}},$$

si scilicet integratio a termino  $x = 0$  usque ad terminum  $x = l$  extenditur; quae integratio eo magis est notatu digna, quia in ea nullus arcus circularis occurrit, etiam si priorem in se complecti videatur, quod autem secus se habet, quia  $\sin. q l x$  ad unitatem reduci nequit quin simul quantitas  $q$  reddatur variabilis.

§. 5. Operae igitur pretium erit inuestigare, quomodo etiam integrale huius theorematis ex forma nostra generali deriuari queat. Hunc in finem consideremus istam formam integralē

$$\int \frac{d x}{l x} (x^\alpha - x^\beta)(x^\gamma - x^\delta),$$

quae in has duas resoluitur

$$\int \frac{d x}{l x} (x^\alpha + \gamma - x^\beta + \gamma) - \int \frac{d x}{l x} (x^\alpha + \delta - x^\beta + \delta),$$

cuius

cuius prioris valor est  $I\left(\frac{\alpha+\gamma+i}{\beta+\gamma+i}\right)$ , posterioris vero  $I\left(\frac{\alpha+\delta+i}{\beta+\delta+i}\right)$ , ita ut habeamus

$$\int \frac{dx}{Lx} (x^\alpha - x^\beta) (x^\gamma + x^\delta) = I\left(\frac{(\alpha+\gamma+i)(\beta+\delta+i)}{(\beta+\gamma+i)(\beta+\delta+i)}\right).$$

Nunc igitur statuamus  $\alpha = pV - i$  et  $\beta = -pV - i$ , deinde  $\gamma = qV - i$  et  $\delta = -qV - i$ , ut fiat

$$x^\alpha - x^\beta = 2V - i \sin. pLx \text{ et } x^\gamma - x^\delta = 2V - i \sin. qLx;$$

sic enim nostra forma integralis induet hanc formam  $= 4 \int \frac{dx}{Lx} \sin. pLx \sin. qLx$ . Pro eius antem valore reperimus

$$\begin{aligned} \alpha + \gamma + i &= i + (p - q)V - i; \quad \beta + \delta + i = i - (p + q)V - i; \\ \beta + \gamma + i &= i + (q - p)V - i; \quad \text{et } \alpha + \delta + i = i + (p - q)V - i, \end{aligned}$$

quibus valoribus substitutis valor integralis prodit

$$I\left(\frac{i+(p+q)^2}{i+(p-q)^2}\right) = -I\left(\frac{i+(p-q)^2}{i+(p+q)^2}\right),$$

unde manifesto sequitur integratio postremi theoremati:

$$\int \frac{dx}{Lx} (\sin. pLx \sin. qLx) = \pm I\left(\frac{i+(p-q)^2}{i+(p+q)^2}\right).$$

§. 6. Hinc occasionem arripimus etiam hanc formam generalem euoluendii

$$\int \frac{dx}{Lx} (x^\alpha - x^\beta) (x^\gamma + x^\delta),$$

cuius valor erit

$$= I\left(\frac{\alpha+\gamma+i}{\beta+\gamma+i}\right) + I\left(\frac{\alpha+\delta+i}{\beta+\delta+i}\right) = I\left(\frac{(\alpha+\gamma+i)(\alpha+\delta+i)}{(\beta+\gamma+i)(\beta+\delta+i)}\right).$$

Nunc iterum faciamus

$$\alpha = pV - i \text{ et } \beta = -pV - i;$$

tum vero

$$\gamma = qV - i \text{ et } \delta = -qV - i,$$

fiet-

sicutque

$$x^p - x^q = 2V - i \sin. p l x \text{ et } x^q + x^p = 2 \cos. q l x,$$

ita ut ipsa formula integralis oriatur

$$+ V - i \int \frac{dx}{l x} \sin. p l x \cos. q l x.$$

Pro valore autem integrali habebimus

$$\alpha + \gamma + i = i + (p+q)V - i; \beta + \gamma + i = i + (q-p)V - i;$$

$$\alpha + \delta + i = i + (p-q)V - i; \gamma + \delta + i = i - (p+q)V - i,$$

vide valor integralis prodit

$$I\left(\frac{i+(p+q)V-i}{i-(p+q)V-i}\right) \cdot \left(\frac{i+(p-q)V-i}{i-(p-q)V-i}\right) = I\left(\frac{i+(p-q)V-i}{i-(p+q)V-i}\right) + I\left(\frac{i+(p-q)V-i}{i-(p-q)V-i}\right).$$

Est vero

$$I\left(\frac{i+(p+q)V-i}{i-(p+q)V-i}\right) = 2V - i A \tan. (p+q)$$

codemque modo

$$I\left(\frac{i+(p-q)V-i}{i-(p-q)V-i}\right) = 2V - i A \tan. (p-q);$$

quibus valoribus substitutis resultat ista integratio:

$$\int \frac{dx}{l x} \sin. p l x \cos. q l x = \frac{1}{2} A \tan. (p+q) + \frac{1}{2} A \tan. (p-q).$$

Cum igitur sit ingenere

$$A \tan. a + A \tan. b = A \tan. \frac{a+b}{1-ab}$$

erit summa arcuum modo inuenta  $= A \tan. \frac{ip}{1-pp+qq}$

et valor integralis  $\frac{1}{2} A \tan. \frac{ip}{1-pp+qq}$ : hinc sequens

*Theorema 3.* Ista formula integralis

$$\int \frac{dx}{l x} \sin. p l x \cos. q l x$$

a termino  $x=0$  usque ad  $x=i$  extensa aequalis  
est huic valori:

$$\frac{1}{2} A \tan. \frac{ip}{1-pp+qq}$$

§. 7. Quod si ergo sumamus  $q = p$ , ob  
 $\sin. p \ln x \cos. q \ln x = \frac{1}{2} \sin. 2p \ln x$ ,  
prodibit ista integratio:

$$\frac{1}{2} \int \frac{dx}{\ln x} \sin. 2p \ln x = \frac{1}{2} A \operatorname{tang.} 2p,$$

id quod prorsus conuenit cum theoremate primo;  
at vero etiam ingenere ad Theorema primum re-  
duci potest. Cum enim sit

$$\sin. a \cos. b = \frac{1}{2} \sin. (a + b) + \frac{1}{2} \sin. (a - b),$$

formula nostra in has partes diuiditur:

$$\frac{1}{2} \int \frac{dx}{\ln x} \sin. (p + q) \ln x + \frac{1}{2} \int \frac{dx}{\ln x} \sin. (p - q) \ln x.$$

Prioris igitur partis valor erit ex theoremate

$$\frac{1}{2} A \operatorname{tang.} (p + q),$$

posterioris vero partis

$$= \frac{1}{2} A \operatorname{tang.} (p - q),$$

quae forma utique reducitur ad eam quam hic dedimus.

§. 8. Nunc autem integrationem nostram ge-  
neralem

$$\int \frac{dx}{\ln x} (x^a - x^b) = \frac{x^a + x^b}{a + b},$$

aliquanto generalius ad angulos redicamus, ponendo

$$a = m + n\sqrt{-1}, \quad b = m - n\sqrt{-1},$$

vt fiat

$$x^a - x^b = x^m (x^{n\sqrt{-1}} - x^{-n\sqrt{-1}}) = 2\sqrt{-1} x^m \sin. nx;$$

tum vero erit

$$\frac{a + b}{a - b} = \frac{m + n\sqrt{-1}}{m - n\sqrt{-1}},$$

quae fractio posito  $n = k(m + 1)$  reducitur ad hanc

$$\frac{m + k\sqrt{-1}}{m - k\sqrt{-1}}. \quad \text{Est vero}$$

$$\frac{m + k\sqrt{-1}}{m - k\sqrt{-1}} = 2\sqrt{-1} A \operatorname{tang.} k = 2\sqrt{-1} A \operatorname{tang.} \frac{\pi}{m+1},$$

¶ Tom. XX. Nov. Comin. I sicque

sicque impetramus sequens theorema :

*Theorema 4.* Ista formula integralis

$$\int \frac{dx}{lx} x^m \sin. n l x$$

a termino  $x = 0$  usque ad terminum  $x = 1$  extensa semper aequalis erit huic valori: A tang.  $\frac{n}{m+1}$ , quod theorema sumto  $m = 0$  ad primum reducitur; vbi in primis notatu dignum occurrit, quod, quoties  $\frac{n}{m+1}$  eundem habet valorem, toties etiam formae integrales aequales inter se euadunt.

§. 9. Verum etiam hoc theorema in genere ad primum reduci potest. Si enim ponatur  $x^{m+1} = y$  erit

$$x^m dx = \frac{dy}{m+1} \text{ et } lx = \frac{ly}{m+1}$$

his valoribus substitutis fiet -

$$\int \frac{dy}{ly} \sin. \frac{n}{m+1} ly,$$

quae cum similis sit primo theoremati, eius valor manifesto est  $= A \tan. \frac{n}{m+1}$ ;

quoniam autem hic posuimus  $x^{m+1} = y$ , ambo termini integrationis hic etiam erunt  $y = 0$  et  $y = 1$ .

§. 10. Per hoc ergo theorema, cum sit

$$A \tan. \frac{1}{2} + A \tan. \frac{1}{2} = A \tan. 1 = \frac{\pi}{4},$$

pro priore erit  $n = 1$  et  $m = 1$ , pro posteriore vero  $n = 1$  et  $m = 2$ , hinc igitur habebitur ista integratio :

$$\int \frac{dx}{lx} (x + x^2) \sin. lx = \frac{\pi}{4}.$$

Deinde

Deinde cum per seriem infinitam sit

$$\frac{1}{1-x} = A \tan g. \frac{1}{x} + A \tan g. \frac{1}{x^2} + A \tan g. \frac{1}{x^3} + A \tan g. \frac{1}{x^4} + A \tan g. \frac{1}{x^5} + \text{etc.}$$

cuius seriei terminus generalis est  $A \tan g. \frac{1}{x^n}$ , habebimus hanc integrationem satis memorabilem:

$$\int \frac{dx}{x^2(1-x)} (x^2 + x^4 + x^8 + x^{16} + \text{etc.}) \sin. I x = \frac{\pi}{4},$$

quod eo magis est notatum dignum, quod series infinita  $x^2 + x^4 + x^8 + x^{16}$  nullo modo ad summam finitam reduci potest.

§. 11. Sed reuertamur ad nostram integrationem principalem, qua est

$$\int \frac{dx}{1-x} (x^\alpha - x^6) = I \frac{\alpha+1}{6+1},$$

cuius veritatem etiam hoc modo ostendere licet; cum sit  $x^\alpha = e^{\alpha \ln x}$  denotante  $e$  numerum cuius logarithmus hyperbolicus  $= \gamma$ , erit per seriem infinitam

$$x^\alpha = 1 + \frac{\alpha \ln x}{1} + \frac{\alpha \alpha (\ln x)^2}{1 \cdot 2} + \frac{\alpha^3 (\ln x)^3}{1 \cdot 2 \cdot 3} + \frac{\alpha^4 (\ln x)^4}{1 \cdot 2 \cdot 3 \cdot 4}$$

hincque colligitur fore

$$x^\alpha - x^6 = (\alpha - 6) \frac{1}{1} \ln x + (\alpha \alpha - 6 \cdot 6) \frac{1}{1 \cdot 2} (\ln x)^2 + (\alpha^3 - 6^3) \frac{1}{1 \cdot 2 \cdot 3} (\ln x)^3 + \text{etc.}$$

quae series per  $\frac{dx}{1-x}$  multiplicata et integrata, ob

$$\int dx (1-x)^n = 1 + 2 + 3 + \dots + n$$

(vbi signum superius valet, si  $n$  est numerus par, inferius si impar); praebet positio  $x=1$  sequentem seriem:

$$\alpha - 6 - \frac{(\alpha^2 - 6^2)}{2} + \frac{(\alpha^3 - 6^3)}{3} - \frac{(\alpha^4 - 6^4)}{4} + \text{etc.}$$

quae series manifesto praebet

$$I(1+\alpha) - I(1+6) = I \frac{\alpha+1}{6+1}.$$

§. 12. Quo valor huius formulae succinctius exprimatur, loco  $\alpha$  et  $\beta$  scribamus  $\alpha = 1$  et  $\beta = n$ , ut habeamus

$$\int \frac{dx}{x^{\alpha}} (x^{\alpha-1} - x^{\beta-1}) = \int \frac{dx}{x^{\alpha}} (x^{\alpha} - x^{\beta}) = l_{\beta}^{\alpha}$$

Quod si ergo sumamus  $\alpha = e^m$  et  $\beta = e^n$ , nanciscemur sequentem integrationem fatis concinnam:

$$\int \frac{dx}{x^{\alpha}} (x^{\alpha} - x^{\beta}) = m - n.$$

§. 13. Inuestigemus nunc integrale huius formula differentialis

$$\frac{dx}{x^{\alpha}} \left( \frac{x^{\alpha} - x^{\beta}}{1 + x^n} \right), \text{ et cum sit}$$

$$\frac{1}{1 + x^n} = 1 - x^n + x^{2n} - x^{3n} + x^{4n} \text{ etc.}$$

colligitur hinc integrale quaesitum

$$l_{\beta}^{\alpha} = l_{\beta+n}^{\alpha} + l_{\beta+2n}^{\alpha} - l_{\beta+3n}^{\alpha},$$

vnde nanciscimus sequens theorema:

*Theorema 5.* Ista formula integralis

$$\int \frac{dx}{x^{\alpha}} \left( \frac{x^{\alpha} - x^{\beta}}{1 + x^n} \right)$$

a termino  $x = 0$  vsque ad terminum  $x = 1$  extensa semper aequatur huic formulae logarithmicae:

$$l_{\beta}^{\alpha} \cdot \frac{\beta+n}{\alpha+n} \cdot \frac{\alpha+2n}{\beta+2n} \cdot \frac{\beta+3n}{\alpha+3n} \cdot \frac{\alpha+4n}{\beta+4n} \text{ etc.}$$

§. 14. Cum igitur alibi demonstrauerim, huius producti in infinitum continuati

$$\frac{a}{b} \cdot \frac{(c+b)}{c+a} \cdot \frac{a+k}{b+k} \cdot \frac{c+b+k}{c+a+k} \cdot \frac{a+2k}{b+2k} \cdot \frac{c+b+2k}{c+a+2k}$$

valo-

valorem aequari huic expressioni

$$\frac{\int z^{e-1} dz (1-z^k)^{\frac{b-k}{k}}}{\int z^{e-1} dz (1-z^k)^{\frac{a-k}{k}}},$$

$$\int z^{e-1} dz (1-z^k)^{\frac{b-k}{k}}$$

applicatione ad nostrum casum facta erit

$$a = \alpha, b = \beta, k = n, k = 2n$$

hincque valor nostri producti infiniti

$$\frac{\int z^{n-1} dz (1-z^{2n})^{\frac{\beta-n}{2n}}}{\int z^{n-1} dz (1-z^{2n})^{\frac{\alpha-n}{2n}}}$$

$$\int z^{n-1} dz (1-z^{2n})^{\frac{\beta-n}{2n}}$$

quae ambae formulae integrales a termino  $z=0$  vsque ad terminum  $z=1$  sunt extendendae, atque hinc colligimus sequens theorema.

*Theorema 6.* Ifsa formula integralis

$\int \frac{dx}{x^l x} \left( \frac{x^\alpha - x^\beta}{1+x^n} \right)$ , a termino  $x=0$  vsque ad terminum  $x=1$  extensa aequalis est huic valori  $\frac{l!}{Q}$ , existente

$$P = \int z^{n-1} dz (1-z^{2n})^{\frac{\beta-n}{2n}} \text{ et}$$

$$Q = \int z^{n-1} dz (1-z^{2n})^{\frac{\alpha-n}{2n}}.$$

dum scilicet etiam haec formulae integrales posteriores a termino  $z=0$  vsque ad terminum  $z=1$  extenduntur.

*§. 15.* Sumamus igitur  $n=1$ , vt formula nostra integralis fiat  $\frac{\int dx}{x^l x} \frac{x^\alpha - x^\beta}{1+x}$ ,

ac tum erit

$$P = \int dz (1 - zz)^{\frac{e-z}{z}} \text{ et } Q = \int dz (1 - zz)^{\frac{a-z}{z}},$$

vnde pro  $\alpha$  et  $\beta$  sequentes casus euoluamus. Sit primo  $\alpha = 2$  et  $\beta = 1$ , erit  $P = A \sin z = \frac{\pi}{2}$  et  $Q = z = 1$ , ideoque  $\frac{P}{Q} = \frac{\pi}{2}$  vnde colligimus fore  $\int \frac{dx}{1+x} \cdot \frac{x-1}{x+1} = l \frac{\pi}{2}$ .

§. 16. Sumamus nunc  $\alpha = 3$  et  $\beta = 1$ , vt fiat  $\frac{x^\alpha - x^6}{1+x} = x(x-1)$  hincque formula nostra integralis erit  $\int \frac{dx}{1+x}(x-1)$ , cuius valorem nouimus esse  $= l \frac{\pi}{2}$  at vero ex formula nostra generali erit

$$P = \frac{\pi}{2} \text{ et } Q = \int dz \sqrt{1-zz} = \int \frac{dz}{\sqrt{1-zz}} - \int \frac{zz dz}{\sqrt{1-zz}}$$

At verō per reductiones notas est

$$\int \frac{zz dz}{\sqrt{1-zz}} = \frac{1}{2} \int \frac{dz}{\sqrt{1-zz}},$$

sicque erit

$$Q = \frac{1}{2} \int \frac{dz}{\sqrt{1-zz}} = \frac{1}{2} \frac{\pi}{2},$$

vnde fit  $\frac{P}{Q} = Q$ , qui valor perfecte congruit cum ante assignato.

§. 17. Quoniam in quantitate P non occurrit exponens  $\alpha$ , in altero vero Q tantum  $\beta$  occurrat, superius theorema ita in duas partes distribuere licet, vt sit

$$\int \frac{dx}{1+x} \cdot \frac{x^{\alpha-1}}{1+x^n} = C - l/z^n - dz(1-z^{2n})^{\frac{\alpha-2n}{2n}} \text{ et}$$

$$\int \frac{dx}{1+x} \cdot \frac{x^{\beta-1}}{1+x^n} = C - l/z^n - dz(1-z^{2n})^{\frac{\beta-2n}{2n}}$$

vbi

vbi  $C$  denotant certam constantem, quae autem in differentia duarum huiusmodi formularum integrarium e calculo egreditur.

§. 18. Possimus etiam nostram formulam integraliem principalem

$$\int \frac{dx}{x^{\alpha} - x^{\beta}} (x^{\alpha} - x^{\beta}) = I \frac{x}{\beta}$$

ita transformare, ut in ea exponentes infiniti occur-  
rant, quae ob hoc ipsum attentione non indigna vi-  
detur. Denotet igitur  $i$  numerum infinite magnum,  
et quia  $Ix$  ita exprimere licet, vt sit  $Ix = i(x^{\frac{1}{i}} - 1)$ ,  
formula nostra hanc induet formam :

$$\int \frac{dx}{i x (x^{\frac{1}{i}} - 1)} (x^{\alpha} - x^{\beta}) = I \frac{x}{\beta}.$$

Nunc igitur ad exponentem fractum tollendum sta-  
tuamus  $x^{\frac{1}{i}} = z$ , vt sit  $x = z^i$ , hincque  $\frac{dx}{x} = \frac{i dz}{z}$ ;  
tum vero  $x^{\alpha} = z^{\alpha i}$  et  $x^{\beta} = z^{\beta i}$ , et quia adhuc idem  
termini integrationis habentur  $z = 0$  et  $z = 1$ , hinc  
sequens theorema resultat :

*Theorema 7.* Denotante  $i$  numerum infinite  
magnum ista formula integralis

$$\int \frac{dz (z^{\alpha i} - z^{\beta i})}{z(z-1)}$$

a termino  $z=0$  vsque ad terminum  $z=1$  extensa  
semper æqualis est huic valori  $I \frac{x}{\beta}$ .

§. 19. Cum sit

$$\frac{z^{\alpha i}}{z-1} = z^{\alpha i-1} + z^{\alpha i-2} + z^{\alpha i-3} + z^{\alpha i-4} \text{ etc.}$$

erit

crit

$$\int \frac{z^{ei} dz}{z(z-i)} = \frac{1}{ai-1} + \frac{1}{ai-2} + \frac{1}{ai-3} + \frac{1}{ai-4} \text{ etc.} + C$$

eodemque modo crit

$$\int \frac{z^{ei} dz}{z(z-i)} = \frac{1}{6i-1} + \frac{1}{6i-2} + \frac{1}{6i-3} + \frac{1}{6i-4} \text{ etc.} + C$$

vnde patet, differentiam hanc duarum serierum esse  
 $\frac{dz}{z}$ , ita si fuerit  $a \neq z$  et  $b = i$  prodibit ista in-  
 tegratio

$$\int \frac{dz(z^r - z^i)}{z(z-i)} = \frac{1}{2i-1} + \frac{1}{2i-2} + \frac{1}{2i-3} + \frac{1}{2i-4} + \text{etc.} \dots + \frac{1}{i}$$

quoniam sequentes termini per seriem posteriorem tolluntur. Constat autem huius seriei summam esse  $I_2$ .

§. 20. Plurima adhuc alia conjectaria ex ista integratione memorabili deduci possent, quibus autem hic non immorabor: sed potius ipsam Analysin quae ad hanc integrationem perduxit accuratius perpendam. Considerati scilicet potentiam  $x^n$ , cuius exponentis  $n$  pro Iubitu siue ut constans siue ut variabilis spectari queat, et cum sit

$$\int \frac{x^n dx}{x} = \frac{x^n}{n},$$

ideoque si post integrationem sumatur  $x = i$ , crit  
 $\int \frac{dx}{x} x^n = \frac{1}{n}$ ; quae formula ergo fundamentum con-  
 stituit vnde sequentia deducemus.

§. 21. Hanc iam formulam per  $du$  multiplicatam integremus, spectata  $x$  ut constante, et quia

summa

summa  $\int x^u du = \frac{x^{u+1}}{1+x}$ , tum vero constat hanc integrationem ab altera vbi  $x$  erat variabilis non turbari; habebimus nunc istam integrationem  $\int \frac{dx}{x(1+x)} x^u = lu + A$ , vbi  $A$  denotat constantem per integrationem ingressam, quae igitur e medio tolletur, si duas huiusmodi formas a se inuicem subtrahamus, vnde si primo sumamus  $u = \alpha$ , tum vero  $u = \beta$ , et posterius integrale a priori subtrahamus, prodibit nostra forma principialis initio commemorata

$$\int \frac{dx}{x(1+x)} (x^\alpha - x^\beta) = l \frac{\alpha}{\beta}.$$

§. 22. Simili autem modo a formula integraли  $\int \frac{dx}{x(1+x)} x^u = lu + A$  vterius progrediamur, qua per  $du$  multiplicata et ex sola variabilitate ipsius  $u$  integrata ob  $\int x^u du = \frac{x^{u+1}}{1+x}$  vt ante perueniemus ad istam integrationem:

$\int \frac{dx}{x(1+x)^2} x^u = \int du lu + Au + B = ulu - u + Au + B$   
vbi ergo ternas formulas particulares inter se combinari oportet, vt ambae constantes  $A$  et  $B$  ex calculo deturbentur; quia autem loco  $A$  scribere licet  $A + 1$  erit

$$\int \frac{dx}{x(1+x)^2} x^u = u lu + Au + B.$$

§. 23. Quod si iam denuo per  $du$  multiplicemus et integremus, mutatis literis constantibus, quo formula concinnior reddatur, reperiemus

$$\int \frac{dx}{x(1+x)^3} x^u = \frac{1}{2} u u lu + Au u + Bu + C$$

eodemque modo vterius

$$\int \frac{dx}{x(1-x)^2} x^u = \frac{1}{2} u^2 \ln x + A u^2 + B u + C u + D$$

$$\int \frac{dx}{x(1-x)^2} x^v = \frac{1}{2} v^2 \ln x + A v^2 + B v + C v + D v + E$$

etc.

etc.

Vnde intelligitur, continuo plures casus particulares inuicem coniungi debere, vt omnes quantitates constantes A, B, C, D, etc. ex calculo expellantur.

§. 24. Hoc igitur modo euoluamus formulam §. 22. inuentam, et exponenti  $u$  tribuamus hos tres valores:  $\alpha$ ,  $\beta$  et  $\gamma$ , vt obtineamus istas tres formulas:

$$I. \int \frac{dx}{x(1-x)^2} x^\alpha = \alpha \ln \alpha + A \alpha + B$$

$$II. \int \frac{dx}{x(1-x)^2} x^\beta = \beta \ln \beta + A \beta + B$$

$$III. \int \frac{dx}{x(1-x)^2} x^\gamma = \gamma \ln \gamma + A \gamma + B$$

vnde eliminando B duas hanc nancimur aequationes:

$$I - II = \alpha \ln \alpha - \beta \ln \beta + A(\alpha - \beta) \text{ et}$$

$$II - III = \beta \ln \beta - \gamma \ln \gamma + A(\beta - \gamma)$$

$$(I - II)(\beta - \gamma) - (II - III)(\alpha - \beta) = (\beta - \gamma)\alpha \ln \alpha - (\beta - \gamma)\beta \ln \beta - (\alpha - \beta)\beta \ln \beta + (\alpha - \beta)\gamma \ln \gamma$$

quae reducitur ad hanc

$$I(\beta - \gamma) + II(\gamma - \alpha) + III(\alpha - \beta) = (\beta - \gamma)\alpha \ln \alpha + (\gamma - \alpha)\beta \ln \beta + (\alpha - \beta)\gamma \ln \gamma.$$

§. 25. Hinc igitur pro formulis ad istud genus referendis constituere poterimus sequens theorema fundamentale:

Theore-

*Theorema 8.* Ista formula integralis

$$\int \frac{dx}{x^{\beta} (x^{\gamma})^{\alpha}} ((\beta - \gamma) x^{\alpha} + (\gamma - \alpha) x^{\beta} + (\alpha - \beta) x^{\gamma})$$

a termino  $x = 0$  usque ad terminum  $x = 1$  extensa semper aequalis est huic valori:

$$(\beta - \gamma) \alpha / \alpha + (\gamma - \alpha) \beta / \beta + (\alpha - \beta) \gamma / \gamma.$$

§. 26. Circa hanc formam imprimis notasse iuvabit, formulam

$$(\beta - \gamma) x^{\alpha} + (\gamma - \alpha) x^{\beta} + (\alpha - \beta) x^{\gamma}$$

non solum per  $x = 1$  esse diuisibilem, sed etiam per  $(x - 1)^2$ ; prius inde patet, quod posito  $x = 1$  fit  $\beta - \gamma + \gamma - \alpha + \alpha - \beta = 0$

posteriorius vero, quod eius etiam differentiale posito  $x = 1$  fit  $\alpha \beta - \gamma + \beta \gamma - \alpha + \gamma \alpha - \beta = 0$ , id quod natura rei postulat, quia in denominatore  $(1/x)^2$  posito  $x = 1$  continetur quadratum quantitatis euaneſcentis.

§. 27. Quo vis huius integrationis generalis clarius perspiciatur, casum enoluuisse operae pretium erit quo ponitur  $\alpha = n + 2$ ,  $\beta = n + 1$  et  $\gamma = n$  quandoquidem obtinebitur ista integratio:

$$\begin{aligned} \int \frac{x^{n+1} dx (x-1)^2}{(1/x)^2} &= (n+2)/(n-2) - 2(n+1)/(n+1) + nh \\ &= 1 \frac{(n+2)^n + n^n}{(n+1)^2 (n+1)}. \end{aligned}$$

§. 28. Tractemus eodem modo formulam integralē gradus tertii; in qua occurrat  $(1/x)^3$ , tribuendo exponenti  $\alpha$  quatuor valores  $\alpha, \beta, \gamma, \delta$ , unde oriuntur sequentes aequationes.

$$\text{I. } \int \frac{dx}{x(1/x)^3} x^\alpha = \frac{1}{2} \alpha \alpha / \alpha + A \alpha \alpha + B \alpha + C$$

$$\text{II. } \int \frac{dx}{x(1/x)^3} x^\beta = \frac{1}{2} \beta \beta / \beta + A \beta \beta + B \beta + C$$

$$\text{III. } \int \frac{dx}{x(1/x)^3} x^\gamma = \frac{1}{2} \gamma \gamma / \gamma + A \gamma \gamma + B \gamma + C$$

$$\text{IV. } \int \frac{dx}{x(1/x)^3} x^\delta = \frac{1}{2} \delta \delta / \delta + A \delta \delta + B \delta + C$$

Atque hinc erit primo

$$\text{I - II} = \frac{1}{2} \alpha \alpha / \alpha - \frac{1}{2} \beta \beta / \beta + A(\alpha \alpha - \beta \beta) + B(\alpha - \beta)$$

unde fit

$$\frac{1 - \alpha}{\alpha - \beta} = \frac{\alpha \alpha / \alpha - \beta \beta / \beta}{2(\alpha - \beta)} + A(\alpha + \beta) + B.$$

Eodemque modo erit

$$\frac{\text{II} - \text{III}}{\beta - \gamma} = \frac{\beta \beta / \beta - \gamma \gamma / \gamma}{2(\beta - \gamma)} + A(\beta + \gamma) + B$$

quarum formularum differentia dat

$$\frac{(1 - \alpha)}{\alpha - \beta} - \frac{(\text{II} - \text{III})}{\beta - \gamma} = \frac{\alpha \alpha / \alpha - \beta \beta / \beta}{2(\alpha - \beta)} - \frac{(\beta \beta / \beta - \gamma \gamma / \gamma)}{2(\beta - \gamma)} + A(\alpha - \gamma)$$

qua per  $\alpha - \gamma$  diuisa prodit

$$\frac{1 - \alpha}{(\alpha - \beta)(\alpha - \gamma)} - \frac{(\text{II} - \text{III})}{(\beta - \gamma)(\alpha - \gamma)} = \frac{\alpha \alpha / \alpha - \beta \beta / \beta}{2(\alpha - \beta)(\alpha - \gamma)} - \frac{(\beta \beta / \beta - \gamma \gamma / \gamma)}{2(\beta - \gamma)(\alpha - \gamma)} + A$$

eodemque modo erit

$$\frac{(\text{II} - \text{III})}{\beta - \gamma(\beta - \delta)} - \frac{\text{III} - \text{IV}}{(\gamma - \delta)(\beta - \delta)} = \frac{(\beta \beta / \beta - \gamma \gamma / \gamma)}{2(\beta - \gamma)(\beta - \delta)} - \frac{(\gamma \gamma / \gamma - \delta \delta / \delta)}{2(\gamma - \delta)(\beta - \delta)} + A$$

quae postrema a superiori sublata relinquit

$$\frac{1 - \alpha}{(\alpha - \beta)(\alpha - \gamma)} - \frac{(\text{II} - \text{III})}{(\beta - \gamma)(\alpha - \gamma)} - \frac{(\text{II} - \text{III})}{(\beta - \gamma)(\beta - \delta)} + \frac{(\text{III} - \text{IV})}{((\gamma - \delta)(\beta - \delta))} =$$

$$\left( \frac{\alpha \alpha / \alpha - \beta \beta / \beta}{2(\alpha - \beta)(\alpha - \gamma)} \right) - \left( \frac{\beta \beta / \beta - \gamma \gamma / \gamma}{2(\beta - \gamma)(\alpha - \gamma)} \right) - \left( \frac{\beta \beta / \beta - \gamma \gamma / \gamma}{2(\beta - \gamma)(\beta - \delta)} \right) + \left( \frac{\gamma \gamma / \gamma - \delta \delta / \delta}{2(\gamma - \delta)(\beta - \delta)} \right)$$

sicque iam omnes tres constantes  $A, B, C$  sunt elisae.

§. 29. Quod si iam singula huius aequationis membra euoluantur et tam secundum numeros I, II, III, IV quam secundum formulas  $\alpha\alpha/\alpha$ ,  $\beta\beta/\beta$ ,  $\gamma\gamma/\gamma$ ,  $\delta\delta/\delta$  in ordinem disponantur, obtinebitur sequens aequatio:

$$\frac{I}{(\alpha-\beta)(\alpha-\gamma)} + \frac{II(\alpha-\delta)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{III(\alpha-\delta)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{IV(\alpha-\delta)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

$$= \frac{\alpha\alpha/\alpha}{2(\alpha-\beta)(\alpha-\gamma)} + \frac{(\alpha-\delta)\beta\beta/\beta}{2(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{(\alpha-\delta)\gamma\gamma/\gamma}{2(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{(\alpha-\delta)\delta\delta/\delta}{2(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

quae aequatio per  $\alpha-\delta$  diuisa ad pulcherrimam uniformitatem reducitur; quo facto sequens nanciscimur theorema ad hunc casum adcommodatum:

*Theorema 9.* Ita formula integralis

$$\int \frac{dx}{x(lx)} \left( \frac{x^\alpha}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{x^\beta}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} \right. \\ \left. + \frac{x^\gamma}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{x^\delta}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \right)$$

a termino  $x=0$  vsque ad terminum  $x=r$  extensa aequatur sequenti formulae

$$\frac{\alpha\alpha/\alpha}{2(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta\beta/\beta}{2(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma\gamma/\gamma}{2(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta\delta/\delta}{2(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

ex qua forma perspicuit quo modo ad casus magis compositos facile progredi licet.

§. 30. Ad hunc modum etiam praecedentes casus representare operae pretium erit. Ita pro divisori  $lx$  habebimus sequentem formam integralem:

$$\int \frac{dx}{x(lx)} \frac{x^\alpha}{(\alpha-\beta)} + \frac{x^\beta}{\beta-\alpha} = \frac{1}{\alpha-\beta} + \frac{1}{\beta-\alpha}$$

Deinde theorema §. 24. allatum ita referetur:

## SPECULATIONES

$$\int \frac{dx}{x(\ln x)^2} \left( \frac{x^\alpha}{(\alpha-\beta)(\alpha-\gamma)} + \frac{x^\beta}{(\beta-\alpha)(\beta-\gamma)} + \frac{x^\gamma}{(\gamma-\alpha)(\gamma-\beta)} \right) =$$

$$\frac{\alpha^{n-1}/\alpha}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n-1}/\beta}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n-1}/\gamma}{(\gamma-\alpha)(\gamma-\beta)}$$

atque istam formam sequitur illa quam in theore-  
mate ultimo retulimus.

§. 31. Nunc igitur hoc negotium in genere  
expedire poterimus pro quacunque potestate ipsius.  
1.  $x^n$  qui in denominatore formulae integralis oc-  
currat cuius exponens sit  $= n - 1$ , ut numerus  
membrorum fiat  $= n$ ; tum igitur accipientur pro  
libitu numeri  $\alpha, \beta, \gamma, \delta$  etc. quorum numerus sit  
 $= n$ , et quaerantur hinc sequentes valores:

$$\mathfrak{A} = (\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)(\alpha - \varepsilon) \text{ (etc.)}$$

$$\mathfrak{B} = (\beta - \alpha)(\beta - \gamma)(\beta - \delta)(\beta - \varepsilon) \text{ (etc.)}$$

$$\mathfrak{C} = (\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)(\gamma - \varepsilon) \text{ (etc.)}$$

$$\mathfrak{D} = (\delta - \alpha)(\delta - \beta)(\delta - \gamma)(\delta - \varepsilon) \text{ (etc.)}$$

etc. etc.

tum vero ponatur etiam breuitatis gratia hoc pro-  
ductum

$$1. 2. 3. 4. 5. \dots (n-2) = N$$

atque obtinebitur sequens forma integralis generalis-  
sima

$$\int \frac{dx}{x(\ln x)^{n-1}} \left( \frac{x^\alpha}{\mathfrak{A}} + \frac{x^\beta}{\mathfrak{B}} + \frac{x^\gamma}{\mathfrak{C}} + \frac{x^\delta}{\mathfrak{D}} + \text{etc.} \right) =$$

$$\frac{\alpha^{n-2}/\alpha}{N\mathfrak{A}} + \frac{\beta^{n-2}/\beta}{N\mathfrak{B}} + \frac{\gamma^{n-2}/\gamma}{N\mathfrak{C}} + \frac{\delta^{n-2}/\delta}{N\mathfrak{D}} + \text{etc.}$$

ubi notandum, casu  $n = 2$  fore  $N = 1$ .

§. 32.

§. 32. Ad haec uberiorius illustranda meminisse iubabit, me iam pridem insigne theorema arithmeticum demonstrasse circa huiusmodi fractiones  $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \text{etc.}$  quorum numerus sit ut ante  $= n$  vbi ostendi, omnes sequentes formulas nihilo aequari

$$\text{I. } \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} + \text{etc.} = 0$$

$$\text{II. } \frac{\alpha}{\alpha} + \frac{\beta}{\beta} + \frac{\gamma}{\gamma} + \frac{\delta}{\delta} + \text{etc.} = 0$$

$$\text{III. } \frac{\alpha\alpha}{\alpha} + \frac{\beta\beta}{\beta} + \frac{\gamma\gamma}{\gamma} + \frac{\delta\delta}{\delta} \text{ etc.} = 0$$

$$\text{IV. } \frac{\alpha^3}{\alpha} + \frac{\beta^3}{\beta} + \frac{\gamma^3}{\gamma} + \frac{\delta^3}{\delta} \text{ etc.} = 0$$

donec perueniatur ad potestatem exponentis  $n - 2$ ; at vero sumto exponente  $= n - 1$  semper fore demonstrauimus

$$\frac{\alpha^{n-1}}{\alpha} + \frac{\beta^{n-1}}{\beta} + \frac{\gamma^{n-1}}{\gamma} + \frac{\delta^{n-1}}{\delta} + \text{etc.} = 1$$

DE