# DE MOTV OSCILLATORIO DVORVM CORPORVM EX FILO SVPER TROCHLEAS TRADVCTO SVSPENSORVM.

Auctore

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§. 1.

Filo AMNB, super duas trochleas M et N traducto, appensa sint duo corpora A et B. Per puncta M et N ducantur rectae verticales MP et NQ, ad easque horizontales AP et BQ, et elapso tempore $t$ corpora teneant situm in figura repraesentatum. Tum pro situ corporum ponantur coordinatae $MP = x$ et $PA = y$, $NQ = x'$ et $QB = y'$, et quia longitudo fili manet inuariata, statuamus $MN = M$, $MA = a + z$ et $NB = b - z$, vt tota fili longitudo sit $= a + b + M$; tum vero ponamus angulos $AMP = \eta$ et $BNQ = \theta$ eritque Tab. III. Fig. 1.

$$x = (a + z) \operatorname{cof.} \eta = \text{ et } y = (a + z) \operatorname{fin.} \eta;$$

eodemque modo

$$x' = (b - z) \operatorname{cof.} \theta \text{ et } y' = (b - z) \operatorname{fin.} \theta.$$

§. 2. Ponatur nunc tensio fili $= T$, a qua quia ambo corpora sursum trahuntur, dum propria grauitate deorſum nituntur, principia motus nobis ſuppeditant quatuor ſequentes aequationes:

$$\text{I. } \frac{ddx}{2g\,dt^2} = \frac{A - T\cos.\eta}{A}; \quad \text{III. } \frac{ddx'}{2g\,dt^2} = \frac{B - T\sin.\theta}{B}$$

$$\text{II. } \frac{ddy}{2g\,dt^2} = -\frac{T\sin.\eta}{A}; \quad \text{IV. } \frac{ddy'}{2g\,dt^2} = -\frac{T\sin.\theta}{B},$$

ex quibus quatuor aequationibus 1°. tenſionem fili T: 2° quantitatem $z$; 3° et 4° angulos $\eta$ et $\theta$ definiri oportet.

§. 3. At vero differentiando habebimus

$$dx = dz\cos.\eta - (a+z)\,d\eta\,\sin.\eta \text{ et } ddx = (ddz - (a+z)\,d\eta^2)\cos.\eta - (2dzd\eta + (a+z)dd\eta)\sin.\eta$$

$$dy = dz\sin.\eta + (a+z)\,d\eta\cos.\eta \text{ et } ddy = (ddz - (a+z)\,d\eta^2)\sin.\eta + (2dzd\eta + (a+z)dd\eta)\cos.\eta$$

Eodem modo reperietur

$$ddx' = -(ddz + (b-z)\,d\theta^2)\cos.\theta + (2\,dz\,d\theta - (b-z)\,dd\theta)\sin.\theta$$

$$ddy' = -(ddz + (b-z)\,d\theta^2)\sin.\theta - (2\,dz\,d\theta - (b-z)\,dd\theta)\cos.\theta$$

quibus valoribus ſubſtitutis, noſtrae aequationes erunt:

$$\text{I. } \frac{(ddz - (a+z)\,d\eta^2)\cos.\eta - (2\,dz\,d\eta + (a+z)\,dd\eta)\sin.\eta}{2g\,dt^2} = \frac{A - T\cos.\eta}{A}.$$

$$\text{II. } \frac{(ddz - (a+z)\,d\eta^2)\sin.\eta + (2\,dz\,d\eta + (a+z)\,dd\eta)\cos.\eta}{2g\,dt^2} = -\frac{T\sin.\eta}{A}.$$

$$\text{III. } -\frac{(ddz + (b-z)\,d\theta^2)\cos.\theta + (2\,dz\,d\theta - (b-z)\,dd\theta\sin.\theta}{2g\,dt^2} = \frac{B - T\cos.\theta}{B}.$$

$$\text{IV. } -\frac{(ddz + (b-z)\,d\theta^2)\sin.\theta - (2\,dz\,d\theta - (b-z)\,dd\theta)\cos.\theta}{2g\,dt^2} = -\frac{T\sin.\theta}{B}.$$

§. 4. Hinc iam per idoneas combinationes formentur aequationes ſequentes ſimpliciores:

I. cos. $\eta$ + II. ſin. $\eta$ dat:

$$\text{I}^{\circ}.\ \frac{ddz - (a+z)\,d\eta^2}{2g\,dt^2} = \frac{A\cos.\eta - T}{A}; \text{ porro}$$

— I. ſin. $\eta$ + II. cof. $\eta$ dat:

II°.

II°. $\frac{2\,dz\,d\eta + (a+z)\,dd\eta}{2\,g\,d\,t^2} = -\,\text{fin.}\,\eta.$

$-$ III. cof. $\theta$ $-$ IV. fin. $\theta$ praebet:

III°. $\frac{ddz + (b-z)\,d\theta^2}{2\,g\,a\,t^2} = \frac{T - B\,\text{cof.}\,\theta}{B}.$

$+$ III. fin. $\theta$ $-$ IV. cof. $\theta$ producit:

IV°. $\frac{2\,dz\,d\theta - (b-z)\,dd\theta}{2\,g\,a\,t^2} = \text{fin.}\,\theta.$

Sicque tantum in I$^a$ et III$^a$. tenfio T occurrit, fecunda autem et quarta tenfionis funt immunes.

§. 5. Ad tenfionem igitur eliminandam vtamur hac noua combinatione: I$^a$. A $+$ III$^a$. B, quae praebet

$$\frac{(A+B)\,ddz - A\,(a+z)\,d\eta^2 + B\,(b-z)\,d\theta^2}{2\,g\,d\,t^2} = A\,\text{cof.}\,\eta - B\,\text{cof.}\,\theta$$

quae ergo aequatio cum fuperiorum fecunda et quarta totam folutionem problematis continet, vnde tam quantitatem $z$ quam angulos $\eta$ et $\theta$ definiri oportet.

§. 6. Antequam refolutionem harum aequationum aggrediamur, quatuor aequationes primum inuentas alio modo tractemus, et quia eft

$$dx = dz\,\text{cof.}\,\eta - (a+z)\,d\eta\,\text{fin.}\,\eta \text{ et } dy = dz\,\text{fin.}\,\eta + (a+z)\,d\eta\,\text{cof.}\,\eta$$

vtamur fequentibus combinationibus:

I. $2\,dx$ $+$ II. $2\,dy$, vnde fit

$$\frac{2\,dx\,ddx + 2\,dy\,ddy}{2\,g\,d\,t^2} = 2\,dx - \frac{2\,T\,dz}{A},$$

quae aequatio integrata dat

$$\frac{dx^2 + dy^2}{2\,g\,d\,t^2} = 2\,x - 2\int\frac{T\,dz}{A}.$$

Nunc vero haec combinatio: I. $x$ $+$ II. $y$ praebet

$$\frac{x\,ddx + y\,ddy}{2\,g\,d\,t^2} = x - \frac{T\,(a+z)}{A},$$

Haec aequatio addatur ad priorem et prodibit

$$\frac{dx^2 + dy^2 + xddx + yddy}{2g\,dt^2} = 3x - 2\int\frac{T\,dz}{A} - \frac{T}{A}(a+z),$$

vbi quidem partis ad finistram integrale est $\frac{x\,dx + y\,dy}{2g\,dt^2}$; at ex membro ad dextram nihil concludi posset. Pari modo non succederet haec combinatio: I. $y$ — II. $x$, quae dat $\frac{yddx - xddy}{2g\,dt^2} = y$, vbi etiam membri finistri integrale est $\frac{y\,dx - x\,dy}{2g\,dt^2}$, sed iterum membrum alterum nullam reductionem patitur.

§. 7. Mirum autem non est, hunc motum, qualem in genere contemplamur, prorsus esse inextricabilem, quoniam ambo corpora A et B etiam inaequalia esse possent: hoc autem casu grauius inter oscillandum descenderet, leuius vero ascenderet, sicque motus prodiret nimis complicatus, quam vt per calculum determinari posset. Quamobrem necesse est nostram inuestigationem tantum ad corpora aequalia restringere, quia alioquin status aequilibrii locum habere non posset. Praeterea vero etiam necesse est diuagationes, seu angulos $\eta$ et $\theta$ quam minimos assumere; vnde facile intelligitur, tensionem fili hoc casu ponderi cuiusque corporis fore aequalem, ita vt sit $T = A = B$. Denique patet, nisi corporibus initio motus verticalis fuerit impressus, ambo corpora durante motu vix esse vel ascensura vel descensura, sicque etiam quantitas $z$ quasi vt infinite parua tractari poterit.

§. 8. Ponamus igitur ambo corpora A et B inter se aequalia, ac primo quidem remoueamus vtrumque motum oscillatorium, ita vt sit $\eta = 0$ et $\theta = 0$, ac remanebunt

bunt tantum aequationes prima et tertia

$$\frac{ddz}{2g\,dt^2} = 1 - \frac{T}{A};$$

$$\frac{d\,d\,z}{2g\,dt^2} = \frac{T}{A} - 1;$$

quae inuicem additae dant $\frac{2\,ddz}{2g\,dt^2} = 0$, et a se inuicem subtractae relinquunt $0 = -\frac{2T}{A} + 2$. Ex priore ergo sequitur $\frac{dz}{dt} = \alpha$, ideoque $z = \alpha\,t$; vnde patet, corpus A motu vniformi descendere celeritate $= \alpha$, alterum vero corpus B eadem celeritate ascendere. Ex posteriore vero fit $T = A$; tensio scilicet fili perpetuo erit eadem et aequalis ponderi vnius corporis.

§. 9. Nunc igitur tribuamus vtrique corpori quandam inclinationem $\eta$ et $\theta$, quasi infinite exiguam, et manifestum est, priorem motum inde non sensibiliter turbari, ita vt adhuc sit $z = \alpha\,t$ et $T = A$, nisi quatenus ob motum minimum corporum tensio aliquantillum immutetur; vnde literam B in calculo retineamus. Quatuor ergo aequationes nostrae erunt:

$$\text{I} - \frac{(a+\alpha t)\,d\eta^2}{2g\,dt^2} = 1 - \frac{T}{A};\quad \text{II}\ \frac{2\alpha\,dt\,d\eta + (a+\alpha t)\,dd\eta}{2g\,dt^2} = -\eta$$

$$\text{III}\ \frac{(b-\alpha t)\,d\theta^2}{2g\,dt^2} = \frac{T}{A} - 1;\quad \text{IV}\ \frac{2\alpha\,dt\,d\theta - (b-\alpha t)\,dd\theta}{2g\,dt^2} = \theta,$$

vbi ex secunda et quarta elici opportet ambos angulos $\eta$ et $\theta$. Prima vero ac tertia, quia inuoluunt quasi infinite-parua secundi ordinis, tantum inseruient correctionibus minimis, tam tensionis T quam quantitatis $z$, accuratius determinandis, quas igitur hic praetermittere licebit.

§. 10. Pro angulo igitur $\eta$ inueniendo habemus hanc aequationem:

$$2\alpha\, dt\, d\eta + (a + \alpha t)\, dd\eta + 2g\eta\, dt^2 = 0,$$

quam quidem facile esset ad differentialia primi gradus reducere, quod autem nobis parum lucri esset allaturum. Ad ipsam aequationem autem commodius referendam ponamus $a = i\alpha$ et $\frac{2g}{\alpha} = n$, vt habeamus hanc aequationem:

$$2\, dt\, d\eta + (i + t)\, dd\eta + n\eta\, dt^2 = 0,$$

pro cuius integrali inueniendo fingamus hanc seriem:

$$\eta = A + Bt + Ctt + Dt^3 + Et^4 + Ft^5 + \text{etc.}$$

eritque

$$\frac{d\eta}{dt} = B + 2Ct + 3Dtt + 4Et^3 + 5Ft^4 + \text{etc. et}$$

$$\frac{dd\eta}{dt^2} = 1.2C + 2.3Dt + 3.4Ett + 4.5Ft^3\ \text{etc.}$$

qui valores substituantur vt sequitur:

$$\left.\begin{array}{lllll}
\frac{i.\, dd\eta}{dt^2} = 1.2iC & + 2.3iDt & + 3.4iFtt & + 4.5iFt^3 & + \text{etc.} \\
\frac{t.\, dd\eta}{dt^2} = & + 1.2C & + 2.3D & + 3.4E & + \text{etc.} \\
\frac{2\, d\eta}{dt} = 2B & + 4C & + 6D & + 8E & + \text{etc.} \\
n\eta = nA & + nB & + nC & + nD & + \text{etc.}
\end{array}\right\} = 0$$

vnde deducuntur hae determinationes:

$$C = \frac{-2B - nA}{1.2.i};\ D = \frac{-6C - nB}{2.3i};\ E = \frac{-12D - nC}{3.4i};\ \text{etc.}$$

§. 11. Quia autem hic singuli coefficientes a binis praecedentibus pendent, huic incommodo medelam afferemus, ponendo $i + t = s$, vt habeamus hanc aequationem:

$$2\, ds\, d\eta + s\, dd\eta + n\eta\, ds^2 = 0,\ \text{et nunc ponamus}$$

$$\eta = A + Bs + Css + Ds^3 + \text{etc.}$$

qua

qua ſerie ſubſtituta fiet

$$\left.\begin{array}{l} \frac{s\,dd\eta}{ds^2} = 1.2\,C\,s + 2.3\,D\,ss + 3.4\,E\,s^3 + 4.5\,F\,s^4 + \text{etc.} \\ \frac{2\,d\eta}{ds} = 2B + 4C\,s + 6\,D\,ss + 8\,E\,s^3 + 10\,F\,s^4 + \text{etc.} \\ n\,\eta = nA + nB\,s + nC\,ss + nD\,s^3 + nE\,s^4 + \text{etc.} \end{array}\right\} = 0$$

Hinc fit

$$B = \frac{-nA}{2};\ C = \frac{-nB}{6};\ D = \frac{-nC}{12};\ E = \frac{-nD}{20};\ \text{etc.}$$

vnde ſeries aſſumta ita prodit expreſſa:

$$\frac{\eta}{A} = 1 - \frac{ns}{2} + \frac{nnss}{2.6} - \frac{n^3 s^3}{2.6.12} + \frac{n^4 s^4}{2.6.12.20} - \text{etc.}$$

quae, quia nullo caſu abrumpitur, nihil prodeſt, niſi forte quamdiu tempus, ideoque et $s$, eſt quantitas valde parua.

§. 12. Notatu etiam digna eſt transformatio iſtius aequationis, ſtatuendo $\eta = \frac{y}{s}$; hinc enim erit

$$d\eta = \frac{dy}{s} - \frac{y\,ds}{ss} \text{ et } dd\eta = \frac{ddy}{s} - \frac{2\,dy\,ds}{ss} + \frac{2\,y\,ds^2}{s^3};$$

qui valores ſubſtituti producent hanc aequationem:

$$ddy + \frac{n\,y\,ds^2}{s} = 0.$$

Quod ſi hic ponamus

$y = e^{\int u\,ds}$, vt ſit $dy = u\,ds\,e^{\int u\,ds}$ et $ddy = (du\,ds + u^2\,ds^2)\,e^{\int u\,ds}$, fietque $du + uu\,ds + \frac{n\,ds}{s} = 0$, quae aequatio quia eſt formae Riccatianae, quam nullo adhuc modo tractare licuit, omnis opera in ea euoluenda fruſtra conſumetur; ita vt determinationem huius motus oſcillatorii, quo corpora A et B ciebuntur, dum filum ſuper trochleas vniformiter promouetur, pro caſu deſperato declarare ſimus coacti.

§. 13.

§. 13. Interim tamen, quia longitudo fili MA continuo crescit, ita vt pendulum, corpus A sustinens, continuo crescat, evidens est, oscillationes continuo tardiores fieri debere; vnde si pro tempore praesente quantitas $s$ tanquam constans spectaretur, omisso primo termino, vt haberemus:

$$d\,d\,y + \frac{n\,y\,d\,s^2}{s} = 0, \text{ integrale foret}$$

$$y = \mathfrak{A} \text{ fin. } (\lambda + s \sqrt{\tfrac{n}{s}}) \text{ fiue}$$

$$\eta\, s = \mathfrak{A} \text{ fin. } (\lambda + \sqrt{n\,s}) \text{ fiue}$$

$$\eta = \frac{\mathfrak{A}\,\alpha}{a + \alpha\,t} \text{ fin. } (\lambda + \frac{\sqrt{2\,g\,(a + \alpha\,t)}}{\alpha});$$

quae expressio non multum videtur a scopo aberrare.

§. 14. Vt autem appareat, quantum ille valor

$$y = \mathfrak{A} \text{ fin. } (\lambda + s \sqrt{\tfrac{n}{s}}) = \mathfrak{A} \text{ fin. } (\lambda + \sqrt{n\,s})$$

a veritate discrepet, eum in aequatione differentiali $s\,d\,d\,y + n\,y\,d\,s^2 = 0$ substituamus. Quia ergo est

$$\frac{d\,y}{d\,s} = \frac{\mathfrak{A}\sqrt{n}}{2\sqrt{s}} \text{ cof. } (\lambda + \sqrt{n\,s}) \text{ et}$$

$$\frac{d\,d\,y}{d\,s^2} = -\frac{n\,\mathfrak{A}}{4\,s} \text{ fin. } (\lambda + \sqrt{n\,s}) - \frac{n\,\mathfrak{A}}{4\,s\sqrt{n\,s}} \text{ cof. } (\lambda + \sqrt{n\,s})$$

habebimus hanc aequationem:

$$-\frac{n\,\mathfrak{A}}{4} \text{ fin. } (\lambda + \sqrt{n\,s}) - \frac{n\,\mathfrak{A}}{4\sqrt{n\,s}} \text{ cof. } (\lambda + \sqrt{n\,s}) + n\,\mathfrak{A} \text{ fin. } (\lambda + \sqrt{n\,s}) = 0$$

fiue

$$\frac{3\,n\,\mathfrak{A}}{4} \text{ fin. } (\lambda + \sqrt{n\,s}) - \frac{\mathfrak{A}\sqrt{n}}{4\sqrt{s}} \text{ cof. } (\lambda + \sqrt{n\,s}) = 0$$

vnde aberratio diiudicari debet.

§. 15. Quanquam igitur casus, quo B = A, facillimus videbatur, tamen statim ac filum promouetur nihil plane

plane de motu corporum definire licet; quando autem filum quiescit, ita vt sit $\alpha = 0$, tum vtrumque corpus perinde oscillationes suas peraget, ac si firmiter esset suspensum. Tanto igitur minus erit sperandum, si corpora inter se inaequalia statuere velimus. Interim tamen occurrent certi quidam casus, quibus praeter omnem expectationem motum definire licebit, quos ergo vtique operae pretium erit accuratius euoluisse.

§. 16. Primum igitur iterum faciamus $\eta = 0$ et $\theta = 0$, et aequationes nostrae erunt:

I. $\frac{ddz}{2g\,dt^2} = 1 - \frac{T}{A}$.

III. $\frac{ddz}{2g\,dt^2} = \frac{T}{B} - 1$; vnde fit $T = \frac{2AB}{A+B}$, hincque

$$\frac{ddz}{2g\,dt^2} = \frac{A-B}{A+B} = \frac{1}{n}, \text{ vt fit } n = \frac{A+B}{A-B}.$$

Hinc iam erit

$$\frac{dz}{2g\,dt} = \frac{t}{n}, \text{ ideoque } dz = \frac{2gt\,dt}{n} \text{ et } z = \frac{gtt}{n},$$

vbi constantes non addimus, quia hinc multo magis quam supra in aequationes inextricabiles illaberemus; sic igitur assecuti sumus has duas aequationes: $T = \frac{2AB}{A+B}$ et $z = \frac{gtt}{n}$, existente $n = \frac{A+B}{A-B}$, ita vt $n$ sit numerus positiuus, si $A > B$, contra vero negatiuus.

§. 17. Nunc etiam vtrique corpori minimas tribuamus inclinationes, a quibus cum praecedentes valores non immutari sint censendi, tantum secunda et quarta aequationum nostrarum in computum erunt ducendae, quae ob $dz = \frac{2gt\,dt}{n}$ erunt:

$$\frac{4gt\,dt\,d\eta + (an + gtt)\,dd\eta}{2gn\,dt^2} = -\eta \text{ et}$$

$$\frac{4gt\,dt\,d\theta - (bn - gtt)\,dd\theta}{2gn\,dt^2} = +\theta;$$

quae cum inter se sint similes, tractasse solam primam sufficiet, quae quo commodior reddatur, faciamus $n\,a = i\,g$, vt sit $i = \frac{n\,a}{g}$, et aequatio resoluenda erit:

$$4\,t\,dt\,d\eta + (i + tt)\,dd\eta + 2\,n\,\eta\,dt^2 = 0$$

quam etiam vt supra per series integrare tentemus.

§. 18. Fingamus igitur sequentem seriem:

$\eta = A + Bt + Ctt + Dt^3 + Et^4 + Ft^5 + Gt^6 +$ etc. erit

$\frac{d\eta}{dt} = B + 2Ct + 3Dtt + 4Et^3 + 5Ft^4 + 6Gt^5 +$ etc. et

$\frac{dd\eta}{dt^2} = 1.2C + 2.3Dt + 3.4Ett + 4.5Ft^3 + 5.6Gt^4 +$ etc.

quibus substitutis fiet:

$$\left.\begin{array}{lllllll}
\frac{i\,dd\eta}{dt^2} = 1.2\,iC & + 2.3\,iDt & + 3.4\,iEtt & + 4.5\,iFt^3 & + 5.6\,iGt^4 & + \text{etc.} \\
\frac{tt\,dd\eta}{dt^2} = & & + 1.2\,C & + 2.3\,D & + 3.4\,E & + \text{etc.} \\
\frac{4t\,d\eta}{dt} = & 4B & + 8C & + 12D & + 16E & + \text{etc.} \\
2n\eta = 2An + & 2Bn & + 2Cn & + 2Dn & + 2En & + \text{etc.}
\end{array}\right\} = 0$$

vnde sequuntur sequentes denominationes:

$$C = -\frac{2An}{1.2\,i};\; D = -\frac{B(2n+4)}{2.3\,i};\; E = -\frac{C(2n+10)}{3.4\,i};$$

$$F = -\frac{D(2n+18)}{4.5\,i};\; G = -\frac{E(2n+28)}{5.6\,i};\; H = -\frac{F(2n+40)}{5.6\,i}\ \text{etc.}$$

sicque bini primi coefficientes A et B manent indeterminati.

§. 19. Hinc igitur perspicitur, hanc seriem abrumpi sequentibus casibus: scil. $n = 0$; $n = -2$; $n = -5$; $n = -9$; $n = -14$; $n = -20$, hincque in genere si $n = -\frac{i(i+3)}{2}$; vbi quidem alternatim vel A vel B nihilo aequale sumi debet, ita vt his casibus motum desideratum assignare valeamus. Praecipuos igitur euoluamus:

I. Si

I. Si $n = -2$ erit $\eta = Bt$

II. Si $n = -5$ erit $\eta = A + \frac{10\,Att}{1.2\,i}$;

III. Si $n = -9$ erit $\eta = Bt + \frac{14\,Bt^3}{2.3\,i}$;

IV. Si $n = -14$ erit $\eta = A + \frac{28\,Att}{1.2\,i} + \frac{28.18\,A}{1.2.3.4\,ii}$;

V. Si $n = -20$ erit $\eta = Bt + \frac{36\,Bt^3}{2.3\,i} + \frac{36.\ 2\,Bt^5}{1.2.3.4.5\,ii}$;

VI. Si $n = -27$ erit $\eta = A + \frac{54\,Att}{1.2\,i} + \frac{54.44\,At^4}{1.2.3.4\,ii} + \frac{54.44.26\,At^6}{1.2.3.4.5.6\,i^3}$;

etc. etc. etc. etc.

§. 20. Euoluamus igitur casum $n = -2$, vnde pro nostris corporibus prodit $B = 3A$, ita vt corpus A sit ascensurum; tum igitur erit $\eta = Bt$, quod autem est integrale particulare, vnde ante omnia integrale completum inuestigari debet. Hunc in finem ponamus $\eta = tv$, ita vt sit $d\eta = tdv + vdt$ et $dd\eta = tddv + 2\,dtdv$, et prodibit

$$4\,tt\,dt\,dv + (i+tt)\,t\,ddv + 2\,(i+tt)\,dt\,dv = 0, \text{ siue}$$

$$(i+tt)\,t\,ddv + 2\,(i+3\,tt)\,dt\,dv = 0, \text{ vnde fit}$$

$$ddv = -\frac{2(i+3tt)\,dt\,dv}{t(i+tt)}, \text{ hinc}$$

$$\frac{ddv}{dv} = -\frac{2(i+3tt)\,dt}{t(i+tt)} = -\frac{2\,dt}{t} - \frac{4\,t\,dt}{i+tt},$$

vnde fit integrando

$$l\frac{dv}{dt} = -2\,lt - 2\,l(i+tt) + l\,C, \text{ consequenter}$$

$$\frac{dv}{dt} = \frac{C}{tt(i+tt)^2}, \quad \text{ideoque} \quad dv = \frac{C\,dt}{tt(i+tt)^2},$$

quae ita resoluitur:

$$dv = \frac{C\,dt}{ii\,tt} - \frac{C\,dt}{i(i+tt)^2} - \frac{C\,dt}{ii(i+tt)}, \text{ vnde fit}$$

$$v = -\frac{C}{iit} - \frac{C}{i}\int\frac{dt}{(i+tt)^2} - \frac{C}{ii}\int\frac{dt}{i+tt}.$$

§. 21. Erat autem $i = \frac{n\alpha}{g} = -\frac{2\alpha}{g}$, ideoque $i$ numerus negatiuus. Ponamus igitur $i = -cc$ et habebimus

$$v = -\frac{C}{c^4 t} + \frac{C}{cc}\int\frac{dt}{(tt-cc)^2} - \frac{C}{c^4}\int\frac{dt}{tt-cc}.$$

Ponamus porro $\int\frac{dt}{(tt-cc)^2} = \frac{\alpha t}{tt-cc} + \int\frac{\beta dt}{tt-cc}$, eritque differentiando et per $dt$ diuidendo

$$\frac{1}{(tt-cc)^2} = \frac{\alpha}{tt-cc} - \frac{2\alpha tt}{(tt-cc)^2} + \frac{\beta}{tt-cc},$$

vnde colligimus $\alpha = \beta = -\frac{1}{2cc}$, ita vt nunc fit

$$v = -\frac{C}{c^4 t} - \frac{Ct}{2c^4(tt-cc)} - \frac{3C}{2c^4}\int\frac{dt}{tt-cc}.$$

Est vero

$$\int\frac{dt}{tt-cc} = -\int\frac{dt}{cc-tt} = -\frac{1}{2c}\, l\frac{c+t}{c-t}$$

consequenter

$$v = -\frac{C}{c^4 t} - \frac{Ct}{2c^4(cc-tt)} + \frac{3C}{4c^5}\, l\frac{c+t}{c-t},$$

vel, posito breuitatis gratia $C = -Dc^5$, fiet

$$v = \frac{Dc}{t} + \frac{Dct}{2(cc-tt)} - \frac{3D}{4}\, l\frac{c+t}{c-t} + E.$$

§. 23. Inuento hoc valore erit angulus noster

$$\eta = Dc + \frac{Dctt}{2(cc-tt)} - \frac{3}{4}Dt\, l\frac{c+t}{c-t} + Et,$$

vbi notetur esse $cc = -i = +\frac{2\alpha}{g}$. Posito igitur $t = 0$ fiet $\eta = Dc$; sicque $Dc$ exprimit inclinationem initialem. Hinc crescente $t$ hoc pendulum MA ascendet, et angulus etiam crescit, nisi forte constans D fuerit negatiua; verum tempus $t$ non vltra $c$ augeri potest, quia alioquin expressio pro $\eta$ adeo in infinitum excresceret. Quod quo clarius appareat, consideremus etiam celeritatem angularem $\frac{d\eta}{dt} = \frac{Dc^3 t}{(cc-tt)^2} - \frac{3}{4}D\, l\frac{c+t}{c-t} - \frac{3}{2}\cdot\frac{Dct}{cc-tt} + E$. Nunc igitur ponamus initio, quo $t = 0$, fuisse $\eta = \alpha$ et $\frac{d\eta}{dt} = 0$, erit-

eritque $\alpha = Dc$ et $E = 0$, ideoque et $D = \frac{\alpha}{c}$, ficque erit

$$\eta = \alpha + \frac{\alpha tt}{2(cc - tt)} - \frac{3\alpha t}{4c} l \frac{c+t}{c-t} \text{ et}$$

$$\frac{d\eta}{dt} = \frac{\alpha cct}{(cc-tt)^2} - \frac{3\alpha}{4c} l \frac{c+t}{c-t} - \frac{3\alpha t}{2(cc-tt)}, \text{ fiue}$$

$$\frac{d\eta}{dt} = \frac{\alpha t(3tt - cc)}{2(cc-tt)^2} - \frac{3\alpha}{4c} l \frac{c+t}{c-t},$$

ficque hinc ad quoduis tempus $t$ tam angulum $\eta$ quam celeritatem angularem $\frac{d\eta}{dt}$ definire licet.

§. 24. Ex indole harum formularum perfpicuum eft, tempus $t$ non vltra terminum C augeri poffe, quippe quo tempore longitudo fili $MA = a + z$ ad nihilum redigitur, et tam angulus $\eta$, quam celeritas $\frac{d\eta}{dt}$ in infinitum excrefcunt, quod quidem cum pendulo infinite breui facile conciliari poteft. Verum iam multo ante, quam hoc euenit, angulus $\eta$ tam fit magnus, vt non amplius pro tam paruo haberi poffit, qualem natura noftri calculi fupponit; ficque etiam iftius motus determinatio mox erronea euadet. Quod autem ad ofcillationes alterius corporis maioris B attinet, earum motus ob defectum Analyfeos nullo plane cafu definire licet, quoniam omnes valores numeri $n$, quibus integratio fuccedit, funt negatiui, ideoque tantum in pendulo afcendente locum habere poffunt.

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