# SOLVTIO GEMINA PROBLEMATIS, QVO MOTVS CORPORIS, FILO ALICVBI ALLIGATI, SVPER PLANO HORIZONTALI QVAERITVR.

Auctore
L. EVLERO.

§. 1.

Cum nuper hanc quaestionem tractassem, perueni ad aequationem differentialem primi gradus, inter binas variabiles, non nimis complexam, quae autem ita erat comparata, vt omnia Calculi artificia adhuc cognita ei resoluendae non sufficere viderentur. Postquam autem rem pluribus modis tentassem, tandem per longas ambages ad eius integrationem sum perductus, atque adeo summa admiratione deprehendi, eius integrale algebraice exprimi posse. Tum vero temporis contentus fui, ipsam tantum aequationem integralem cum publico communicare, simul pollicitus, totam Analysin, qua eo sum deductus, alia occasione ob oculos ponere, quod igitur nunc praestare constitui. Deinde vero aliam solutionem eiusdem quaestionis multo simpliciorum sum adeptus, quam hic etiam expositurus ero.

So-

## Solutio prior Problematis.

§. 2. Sit B C D corpus propositum, cuius massa sit $= M$, et centrum grauitatis in C, cuius respectu momentum inertiae sit $M\,k\,k$, hocque corpus in B alligatum sit filo $BA = a$, in puncto A fixo; puncti porro B distantia a centro grauitatis sit $BC = b$, quibus positis elapso tempore $= t$ referatur corpus ad axem fixum A E, ponaturque angulus $EAB = \Phi$, et, producta recta C B vsque ad hunc axem in L, vocetur angulus $ELB = \psi$, qui ergo superat angulum priorem $\Phi$ angulo $ABL = \omega$, ita vt sit $\omega = \psi - \Phi$. Tab. III. Fig. 2.

§. 3. Calculo igitur secundum principia motus ad haec elementa applicato, posui porro celeritatem angularem puncti B circa $A = u$, puncti vero C circa $B = v$, ita vt sit $u = \frac{d\Phi}{dt}$ et $v = \frac{d\psi}{dt}$. Deinde posui $u = pv$ et $b \cos.\omega = z$, ac breuitatis gratia feci $bb + kk = bc$, hincque aequatio differentialis prodiit ista:

$$a(1-p)(bc-zz)\,dp + (bc + aap^2 + apz(1+p))\,dz = 0$$

quae autem, posito $z = as$ et $\frac{bc}{aa} = nn$, ad hanc formam simpliciorem reducitur:

$$dp(1-p)(nn-ss) + ds(nn + p^2 + ps(1+p)) = 0$$

vbi ergo est $s = \frac{b.\cos.\omega}{a}$ et $nn = \frac{bb+kk}{aa}$; sicque in hanc aequationem inter binas variabiles $p$ et $s$ vnica quantitas constans, et quidem data, $nn$ ingreditur. In id igitur mihi erat incumbendum, vt istius aequationis integrale inuestigarem.

§. 4. Quoniam in hac aequatione quatuor terminorum species reperiuntur, vbi scilicet binae variabiles $p$

 et

et $s$ vel vnicam tantum dimenſionem, vel duas, vel tres, vel quatuor, obtinent; has quatuor partes ſeorſim repraeſento:

I. $nn(dp+ds)$ III. $psds-ssdp$

II. $-nnpdp$ IV. $pssdp+pp(p+s)ds$.

§. 5. Incipiamus nunc a parte vltima, quam autem, ponendo $s=pq$, primo in hanc formam transfundamus: $\text{IV}=p^3q(2q+1)dp+p^4(1+q)dq$. Iam ſi haec forma diuidatur per $p^4q(2q+1)$, fiet integrabilis; erit enim $\frac{\text{IV}}{p^4q(2q+1)}=\frac{dp}{p}+\frac{(1+q)dq}{q(2q+1)}$; vnde integrando colligitur $\int\frac{\text{IV}}{p^4q(2q+1)}=l\frac{pq}{\sqrt{(2q+1)}}$. Hinc, retrogrediendo ad differentialia, erit

$$\text{IV}=p^4q(2q+1)\,d.\,l\frac{pq}{\sqrt{(2q+1)}}.$$

Quare ſi breuitatis gratia ponatur $\frac{pq}{\sqrt{(2q+1)}}=r$, vt ſit

$$p=\frac{r\sqrt{(2q+1)}}{q} \text{ erit } \text{IV}=\frac{r^3(2q+1)^3}{q}dr.$$

Poſito porro $\frac{r(2q+1)}{q}=t$, ob

$$r=\frac{qt}{2q+1}, \text{ erit } \text{IV}=t^3dr=t^3d.\frac{qt}{2q+1}.$$

§. 6. Formula tertia $psds-ssdp$, ita repraeſentata: $pss\left(\frac{ds}{s}-\frac{dp}{p}\right)$, ſtatim dat hanc expreſſionem: $\text{III}=pss.\,d.\,l\frac{s}{p}$, ſiue, loco $s$ et $p$ ſubſtitutis valoribus,

$$\text{III}=p^3qq.\,d.\,l\,p=\frac{t^3q}{(2q+1)^{\frac{3}{2}}}dq\ \left(\text{ob } p=\frac{t}{\sqrt{(2q+1)}}\right).$$

Erit ergo

$$\frac{\text{III}}{t^3}=\frac{q\,dq}{(2q+1)^{\frac{3}{2}}}=d.\frac{q+1}{\sqrt{(2q+1)}}, \text{ ideoque } \text{III}=t^3.\,d.\frac{q+1}{\sqrt{(2q+1)}}.$$

§. 7.

§. 7. Colligamus iam tertiam et quartam partem, quarum summa erit $\mathrm{III} + \mathrm{IV} = t^3 . d . \left(\frac{qt}{2q+1} + \frac{q+1}{\sqrt{(2q+1)}}\right)$. Reliquae duae partes facili negotio eruentur. Cum enim sit $\mathrm{II} = -nnp\,dp = -\frac{nn}{2} d . pp$, loco $p$ scribendo valorem inuentum $\frac{t}{\sqrt{(2q+1)}}$, erit $\mathrm{II} = -\frac{1}{2} nn . d . \frac{tt}{2q+1}$. Prima denique, quae est $nn(dp + ds) = nn . d . (p + s)$, erit

$$\mathrm{I} = nn . d . p(1+q) = nn . d . \frac{(1+q)t}{\sqrt{(2q+1)}}.$$

Ergo colligendo fiet

$$\mathrm{I} + \mathrm{II} = \tfrac{1}{2} nn . d . \left(\frac{2(1+q)t}{\sqrt{(2q+1)}} - \frac{tt}{2q+1}\right).$$

§. 8. Ponatur nunc

$$\frac{qt}{2q+1} + \frac{q+1}{\sqrt{(2q+1)}} = \mathrm{P} \text{ et } \frac{2(1+q)}{\sqrt{(2q+1)}} - \frac{t}{2q+1} = \mathrm{Q},$$

eritque $\mathrm{I} + \mathrm{II} = \frac{1}{2} nnt\,d . \mathrm{Q}$ et $\mathrm{III} + \mathrm{IV} = t^3 . d . \mathrm{P}$, ideoque habebimus

$$\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} = 0 = t^3 d\mathrm{P} + \tfrac{1}{2} nn(t\,d\mathrm{Q} + \mathrm{Q}\,dt).$$

Cum autem sit

$$\mathrm{P} - \tfrac{1}{2}\mathrm{Q} = \frac{qt}{2q+1} + \frac{q+1}{\sqrt{2q+1}} - \frac{q+1}{\sqrt{2q+1}} + \frac{t}{2(2q+1)} = \frac{t}{2}$$

erit $\mathrm{Q} = 2\mathrm{P} + t$, quo substituto aequatio proposita tandem in hanc simpliciorem transmutatur:

$$t(nn + tt)\,d\mathrm{P} + nn\mathrm{P}\,dt - nnt\,dt = 0$$

siue in hanc:

$$d\mathrm{P} + \frac{nn\mathrm{P}\,dt}{t(nn+tt)} = \frac{nn\,dt}{nn+tt}.$$

§. 9. Consideretur iam coefficiens ipsius $\mathrm{P}$, qui est $\frac{nn\,dt}{t(nn+tt)} = \frac{dt}{t} - \frac{t\,dt}{nn+tt}$, cuius integrale est $l\frac{t}{\sqrt{(nn+tt)}}$. Quamobrem tota aequatio, si per fractionem $\frac{t}{\sqrt{(nn+tt)}}$ multiplicetur, fiet integrabilis. Prodit enim

$$\frac{t\,dP}{\surd(nn+tt)} + \frac{nnP\,dt}{(nn+tt)^{\frac{3}{2}}} = \frac{nnt\,dt}{(nn+tt)^{\frac{3}{2}}},$$

cuius integrale eſt

$$\frac{Pt}{\surd(nn+tt)} = -\frac{nn}{\surd(nn+tt)} + C,$$

vnde colligitur $Pt = C\surd(nn+tt) - nn$. Cum igitur poſuerimus

$$P = \frac{qt}{2q+1} + \frac{q+1}{\surd(2q+1)},$$ erit etiam

$$Pt = \frac{qtt}{2q+1} + \frac{t(q+1)}{\surd(2q+1)},$$

vnde naſcitur haec aequatio algebraica:

$$C\surd(nn+tt) - nn = \frac{qtt}{2q+1} + \frac{t(q+1)}{\surd 2q+1},$$

quae, ſubſtituendo loco $t$ valorem $p\surd(2q+1) = \surd p(2s+p)$, ad hanc reducitur:

$$C\surd(nn+2ps+pp) - nn = p+s+ps,$$

ſiue ad hanc:

$$C = \frac{nn+p+s+ps}{\surd(nn+2ps+pp)},$$

quod eſt integrale completum aequationis differentialis propoſitae:

$$dp(1-p)(nn-ss) + ds(nn+p^2+ps(1+p)) = 0$$

## Altera ſolutio multo ſimplicior et elegantior.

§. 10. Hanc ſolutionem mihi immediate ex primis formulis differentio-differentialibus deriuare licuit, quae, poſitis coordinatis $AX = x$ et $XC = y$ et tenſione fili $AB = T$, ſunt:

$$\text{I. } \frac{ddx}{\alpha\,dt^2} = -\frac{T\cos.\Phi}{M};\quad \text{II. } \frac{ddy}{\alpha\,dt^2} = -\frac{T\sin.\Phi}{M};$$

$$\text{III. } \frac{dd\psi}{\alpha\,dt^2} = -\frac{Tb\sin.\omega}{M};$$

vbi

vbi $\alpha$ denotat celeritatem, lapsu granium libero vno minuto secundo acquisitam. Coordinatae autem $x$ et $y$ ita per angulos $\phi$ et $\psi$ definiuntur, vt sit

$$x = a \operatorname{cos.} \phi + b \operatorname{cos.} \psi \text{ et } y = a \operatorname{sin.} \phi + b \operatorname{sin.} \psi,$$

vnde differentiando fit

$$dx = -a\,d\phi \operatorname{sin.} \phi - b\,d\psi \operatorname{sin.} \psi$$
$$dy = +a\,d\phi \operatorname{cos.} \phi + b\,d\psi \operatorname{cos.} \psi$$
$$ddx = -a\,dd\phi \operatorname{sin.} \phi - a\,d\phi^2 \operatorname{cos.} \phi - b\,dd\psi \operatorname{sin.} \psi - b\,d\psi^2 \operatorname{cos.} \psi$$
$$ddy = +a\,dd\phi \operatorname{cos.} \phi - a\,d\phi^2 \operatorname{sin.} \phi + b\,dd\psi \operatorname{cos.} \psi - b\,d\psi^2 \operatorname{sin.} \psi$$

§. 11. Iam formulae ita combinentur; primo scilicet I sin. $\phi$ — II cos. $\phi = 0$, quae ergo, facta substitutione, dabit hanc aequationem:

$$a\,dd\phi + b\,dd\psi \operatorname{cos.} \omega - b\,d\psi^2 \operatorname{sin.} \omega = 0.$$

Deinde fiat ista combinatio: I. cos. $\phi$ + II sin. $\phi = -\frac{T}{M}$, vnde ergo nascitur haec aequatio:

$$a\,d\phi^2 + b\,dd\psi \operatorname{sin.} \omega + b\,d\psi^2 \operatorname{cos.} \omega = \frac{T}{M}.$$

At vero ex tertia formula est $\frac{T}{M} = \frac{kk\,dd\psi}{b \operatorname{sin.} \omega}$, vnde substituto hoc valore in superiore aequatione erit

$$kk\,dd\psi + ab\,d\phi^2 \operatorname{sin.} \omega + bb\,dd\psi \operatorname{sin.} \omega^2 + bb\,d\psi^2 \operatorname{sin.} \omega \operatorname{cos.} \omega = 0.$$

§. 12. Duas igitur nacti sumus aequationes differentiales secundi gradus, quas hoc modo per litteras A et B indicemus:

$$\mathrm{A} = a\,dd\phi + b\,dd\psi \operatorname{cos.} \omega - b\,d\psi^2 \operatorname{sin.} \omega = 0,$$

$$\mathrm{B} = kk\,dd\psi + ab\,d\phi^2 \operatorname{sin.} \omega + bb\,dd\psi \operatorname{sin.} \omega^2 + bb\,d\psi^2 \operatorname{sin.} \omega \operatorname{cos.} \omega = 0,$$

in quas tantum bini anguli $\phi$ et $\psi$, vna cum $\omega = \psi - \phi$ ingrediuntur, et nunc totum negotium eo redit, vt eiusmodi di combinatio harum aequationum instituatur, quae ad formulam integrabilem perducat. Hoc autem praestabit ista combinatio: $B + A\,(a + b\cos.\omega) = 0$, vnde oritur ista aequatio

$$\left.\begin{array}{l} kkdd\psi + abd\,\Phi^2 \sin.\omega + bbdd\psi \sin.\omega^2 + bbd\,\psi^2 \sin.\omega \cos.\omega \\ + aadd\Phi - abd\,\psi^2 \sin.\omega + bbdd\psi \cos.\omega^2 - bbd\,\psi^2 \sin.\omega \cos.\omega \\ \qquad + abdd\psi \cos.\omega \\ \qquad + abdd\Phi \cos.\omega \end{array}\right\} = 0.$$

§. 13. In hac aequatione occurrit terminus $ab\sin.\omega$, qui ob $d\psi - d\Phi = d\omega$ praebet hoc membrum:

$$-\,abd\omega\,(d\Phi + d\psi)\sin.\omega,$$

sicque nunc nostra aequatio, vlterius reducta, erit:

$$\left.\begin{array}{l} kkdd\psi + aadd\Phi - abd\omega \sin.\omega\,(d\Phi + d\psi) \\ + bbdd\psi + abdd\Phi \cos.\omega \\ + abdd\psi \cos.\omega\,. \end{array}\right\} = 0$$

siue

$$aadd\Phi + ab\,(dd\Phi + dd\psi)\cos.\omega + (bb + kk)\,dd\psi$$
$$-\,abd\omega\,(d\Phi + d\psi)\sin.\omega = 0.$$

Vbi iam manifestum est, huius aequationis integrale esse

$$ab\,(d\Phi + d\psi)\cos.\omega + aad\Phi + (bb + kk)\,d\psi = C\,dt.$$

Quia enim elementum $dt$ constans est assumtum, id propter homogeneitatem constanti est adiungendum.

§. 14. Egregie autem haec aequatio integralis conuenit cum ea quam methodo priore inuenimus; ad quod ostendendum introducamus celeritates angulares $u$ et $v$, et cum sit $\frac{d\Phi}{dt} = u$ et $\frac{d\psi}{dt} = v$, habebitur ista aequatio:

Iam

$$a b \cos. \omega (u + v) + a a u + (b b + k k) v = C.$$

Iam haec aequatio, per $a a$ diuisa, et posito vt supra fecimus

$\frac{b \cos. \omega}{a} = s$ et $\frac{b b + k k}{a a} = n n$, hanc induet formam:

$$s (u + v) + u + n n v = C.$$

Nunc fiat $u p = v$, eritque $v s (p + 1) + p v + n n v = C$. At vero principium conseruationis virium vivarum in praecedente dissertatione perduxit ad hanc aequationem:

$$f f = a a v v (n n + p p + 2 p s),$$

vnde deducitur $v = \frac{f}{a \sqrt{(n n + p p + 2 p s)}}$, qui valor in nostra aequatione substitutus suppeditat sequentem:

$$C = \frac{f}{a} \frac{n n + p + s + p s}{\sqrt{(n n + p p + 2 p s)}} \text{ siue } \frac{n n + p + s + p s}{\sqrt{(n n + p p + 2 p s)}} = \frac{C a}{f},$$

quae est ea ipsa aequatio, ad quam nos praecedens integratio perduxit. Manifestum igitur est, hanc posteriorem integrationem priore multo esse simpliciorem et elegatiorem.