NOVA METHODVS
MOTVM PLANETARVM DETERMINANDI

Auctore

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§. 1.

Agitur hic de celeberrimo illo Problemate *Kepleriano*, cuius plurimae solutiones passim sunt traditae, quae omnes in hoc conueniunt, vt ex data anomalia vera Planetae eius anomalia media definiatur; cum tamen ad vsum astronomicum vicissim ex data anomalia media vera assignari deberet. Quanquam autem non difficile erat solutionem inuentam per conuersionem ad hunc scopum traducere, methodum directam, qua ex media anomalia inuestigari queat anomalia vera, vsu non esse cariturum arbitror. Quamobrem hic constitui eam methodum, qua in determinando motu Lunae feliciter sum vsus, ad Planetas primarios etiam adcommodare, quo clarius natura et vis huius methodi ob oculos exponatur; quandoquidem in illa eiusmodi artificia occurrunt, quae propter multitudinem elementorum, quibus motus Lunae implicatur, non satis dilucide perspiciuntur.

Tab. II. Fig. 18. §. 2. Referat igitur tabula planum, in quo Planeta moueatur, vbi punctum S ſit centrum Solis, axis vero S♈ ad principium arietis dirigatur. Planeta autem elapſo tempore τ, quod in diebus exprimi aſſumo, perpenerit in P, vnde ad axem demiſſo perpendiculo P Q vocentur binae coordinatae $SQ = x$ et $QP = y$; ipſa autem diſtantia Planetae a Sole ſit $SP = v$, ita vt $vv = xx + yy$. His poſitis conſtat principia motus ſequentes dare binas aequationes:

$$\frac{ddx}{\Delta d\tau^2} = -\frac{x}{v^3} \text{ et } \frac{ddy}{\Delta d\tau^2} = -\frac{y}{v^3},$$

vbi elementum $d\tau$ ſumitur conſtans, et littera Δ denotat certam quantitatem conſtantem, quam ex motu Terrae mox definiemus. Hic autem integrationi harum formularum non immoror, quam tum demum feliciori ſucceſſu ſum ſuſcepturus, cum has aequationes ad vſum commodiorem transformauero, vbi totum negotium multo facilius ſuccedet.

§. 3. Quoniam hic ambae coordinatae x et y, quarum valores ad quoduis tempus aſſignari oportet, maximis variationibus ſunt obnoxiae, dum per totam Planetae orbitam modo fieri poſſunt poſitiuae modo negatiuae, eas ante omnia ad alium axem transferri conueniet, vbi multo minores variationes ſint ſubiturae. Hunc in finem ſtatim motum medium eiusdem Planetae in calculum introduco, quo ſcilicet ſingulae reuolutiones motu aequabili circa ſolem in circulo peragantur. Ducatur igitur recta S M ad locum medium, quem Planeta eodem tempore eſt occupaturus; quippe qui locus ex tabulis mediorum motuum facillime innoteſcit. Vocemus igitur eius longitudinem,

nem, seu angulum ♈ $SM = \zeta$, qui ergo tempori est proportionalis, hancque rectam SM pro hoc tempore tanquam axem spectemus, ad quem locum Planetae P per coordinatas orthogonales referamus, quae sint

$Sq = X$, $qP = Y$, vnde iterum fit $vv = XX + YY$; ex his vero priores coordinatae ita definiuntur, vt sit

$$x = X \operatorname{cof.} \zeta - Y \operatorname{fin.} \zeta \text{ et } y = X \operatorname{fin.} \zeta + Y \operatorname{cof.} \zeta.$$

Atque nunc iam istud commodum sumus assecuti, vt, nisi Planeta enormem habuerit excentricitatem, hae quantitates X et Y exiguas tantum mutationes sint passurae, dum noua abscissa $Sq = X$ nunquam multum a distantia Planetae media a Sole est discrepatura; applicata autem $Pq = Y$ nunquam certos limites, non adeo remotos, est transgressura. Si enim Planeta excentricitate penitus careret, perpetuo foret X quantitas constans et $Y = 0$, quandoquidem hoc casu motus Planetae in motum medium recideret.

§. 4. Cum igitur sit $x = X \operatorname{cof.} \zeta - Y \operatorname{fin.} \zeta$, erit

$$dx = dX \operatorname{cof.} \zeta - dY \operatorname{fin.} \zeta - d\zeta (X \operatorname{fin.} \zeta + Y \operatorname{cof.} \zeta)$$

et denuo differentiando, ob $d\zeta$ constans, fiet

$$ddx = ddX \operatorname{cof.} \zeta - ddY \operatorname{fin.} \zeta - 2 d\zeta (dX \operatorname{fin.} \zeta + dY \operatorname{cof.} \zeta) - d\zeta^2 (X \operatorname{cof.} \zeta - Y \operatorname{fin.} \zeta).$$

Eodem modo, cum sit

$$y = X \operatorname{fin.} \zeta + Y \operatorname{cof.} \zeta, \text{ erit}$$

$$dy = dX \operatorname{fin.} \zeta + dY \operatorname{cof.} \zeta + d\zeta (X \operatorname{cof.} \zeta - Y \operatorname{fin.} \zeta) \text{ et}$$

$$ddy = ddX \operatorname{fin.} \zeta + ddY \operatorname{cof.} \zeta + 2 d\zeta (dX \operatorname{cof.} \zeta - dY \operatorname{fin.} \zeta) - d\zeta^2 (X \operatorname{fin.} \zeta + Y \operatorname{cof.} \zeta).$$

Hinc igitur colligimus sequentes valores:

$$dd x \cos. \zeta + dd y \sin. \zeta = dd X - 2 d\zeta\, d Y - X d\zeta^2$$
$$dd y \cos. \zeta - dd x \sin. \zeta = dd Y + 2 d\zeta\, d X - Y d\zeta^2$$

§. 5. At vero ex aequationibus fundamentalibus ſequitur fore

$$\frac{d d x \cos. \zeta + d d y \sin. \zeta}{\Delta d \tau^2} = \frac{- x \cos. \zeta - y \sin. \zeta}{v^3} = \frac{- X}{v^3},$$
$$\frac{d d y \cos. \zeta - d d x \sin. \zeta}{\Delta d \tau^2} = \frac{- y \cos. \zeta + x \sin. \zeta}{v^3} = \frac{- Y}{v^3},$$

quae ergo aequationes, ſubſtitutis valoribus modo inuentis, nobis praebebunt ſequentes formulas per X et Y expreſſas:

$$\frac{d d X - 2 d\zeta\, d Y - X d\zeta^2}{\Delta d \tau^2} = \frac{- X}{v^3}$$
$$\frac{d d Y + 2 d\zeta\, d X - Y d\zeta^2}{\Delta d \tau^2} = \frac{- Y}{v^3},$$

ex quibus ergo binas nouas coordinatas X et Y definiri conuenit.

§. 6. Antequam autem hoc negotium ſuſcipiamus, ex motu Terrae medio, quem pro penitus cognito aſſumere licet, quantitatem conſtantem Δ determinemus. Hunc in finem ſtatuamus diſtantiam mediam Terrae a Sole $= 1$, et iam loco Planetae P ſubſtituamus ipſam Terram, quaſi motu ſuo medio in circulo, cuius radius $= 1$, circa Solem moueretur, fietque hoc caſu $X = 1$ et $Y = 0$, hincque $v = 1$, vnde noſtrae aequationes euadent:

$$- \frac{d\zeta^2}{\Delta d \tau^2} = - 1 \text{ et } 0 = 0.$$

Hinc ergo fit

$$d\zeta^2 = \Delta d\tau^2, \text{ ideoque } \Delta = \frac{d\zeta^2}{d\tau^2},$$

vbi ζ denotat longitudinem Terrae mediam, quae ſi pro tempore τ dierum vocetur $= t$, erit $\Delta = \frac{d t^2}{d \tau^2}$; ſicque in noſtris formulis loco $\Delta d\tau^2$ ſcribi conueniet $d t^2$. Quamobrem,

§. 10. Quatenus hic yy eſt valde paruum prae $(1+x)^2$, formula noſtra irrationalis ſequenti modo in ſeriem euoluetur:

$$((1+x)^2+yy)^{-\frac{3}{2}} = \frac{1}{(1+x)^3} - \frac{3\,yy}{2.\,(1+x)^5} + \frac{3.\,5.\,y^4}{2.\,4\,(1+x)^7} - \frac{3.\,5.\,7.\,y^6}{2.\,4.\,6\,(1+x)^9} + \text{etc.}$$

quo valore ſubſtituto noſtrae aequationes euadent:

$$\frac{ddx}{dt^2} - \frac{2\,n\,dy}{dt} - nn\,(1+x) = \frac{-nn}{(1+x)^2} + \frac{3.\,nn\,yy}{2.\,(1+x)^4} - \frac{3.\,5.\,nn\,y^4}{2.\,4\,(1+x)^6} + \frac{3.\,5.\,7.\,nn\,y^6}{2.\,4.\,6\,(1+x)^8}\ \text{etc.}$$

$$\frac{ddy}{dt^2} + \frac{2\,n\,dx}{dt} - nny = -\frac{nny}{(1+x)^3} + \frac{3\,nn\,y^3}{2.\,(1+x)^5} - \frac{3.\,5.\,nn\,y^5}{2.\,4\,(1+x)^7} + \frac{3.\,5.\,7.\,nn\,y^7}{2.\,4.\,6\,(1+x)^9}\ \text{etc.}$$

§. 11. Quatenus autem quoque x prae vnitate eſt valde paruum, denominatores noſtrarum aequationum denuo in ſeries conuertantur ope huius reductionis:

$$(1+x)^{-\lambda} = 1 - \lambda x + \frac{\lambda(\lambda+1)\,xx}{1.\,2} - \frac{\lambda(\lambda+1)(\lambda+2)\,x^3}{1.\,2.\,3} + \text{etc.}$$

vnde fit

$$\frac{1}{(1+x)^2} = 1 - 2x + 3xx - 4x^3 + 5x^4 - 6x^5 + 7x^6 - 8x^7 + \text{etc.}$$

$$\frac{1}{(1+x)^3} = 1 - 3x + 6xx - 10x^3 + 15x^4 - 21x^5 + 28x^6 - 36x^7 + \text{etc.}$$

$$\frac{1}{(1+x)^4} = 1 - 4x + 10xx - 20x^3 + 35x^4 - 56x^5 + 84x^6 - 120x^7 + \text{etc.}$$

$$\frac{1}{(1+x)^5} = 1 - 5x + 15xx - 35x^3 + 70x^4 - 126x^5 + 210x^6 - 330x^7 + \text{etc.}$$

$$\frac{1}{(1+x)^6} = 1 - 6x + 21xx - 56x^3 + 126x^4 - 252x^5 + 462x^6 - 792x^7 + \text{etc.}$$

etc. etc.

§. 12. Subſtituamus nunc iſtos valores in noſtris aequationibus, ac terminos ad partem dextram ſecundum dimenſiones, quas literae x et y in iis obtinent, diſponamus, quo clarius conuergentia terminorum ob oculos ponatur, siquidem literas x et y vt quantitates valde paruas respicere licet. Erit igitur

$$\frac{ddx}{dt^2} - \frac{2ndy}{dt} = 3nnx - 3nnxx + 4nnx^3 - 5nnx^4 + 6nnx^5 - 7nnx^6 \text{ etc.}$$
$$+\tfrac{3}{2}nnyy - 6nnxyy + 15nnxxyy - 30nnx^3yy + \tfrac{105}{2}nnx^4yy$$
$$-\tfrac{15}{8}nny^4 + \tfrac{45}{4}nnxy^4 - \tfrac{315}{8}nnxxy^4$$
$$+\tfrac{35}{16}nny^6$$

$$\frac{ddy}{dt^2} + \frac{2ndx}{dt} = +3nnxy - 6nnxxy + 10nnx^3y - 15nnx^4y + 21nnx^5y$$
$$+\tfrac{3}{2}nny^3 - \tfrac{15}{2}nnxy^3 + \tfrac{45}{2}nnxxy^3 - \tfrac{105}{2}nnx^3y^3$$
$$-\tfrac{15}{8}nny^5 + \tfrac{105}{8}nnxy^5.$$

§. 13. Diuidamus has aequationes per nn, et cum sit $nt = \zeta$, existente ζ longitudine media Planetae, siue angulo ♈SM, quem ex tabula motuum mediorum Planetae ad quoduis tempus depromere licet, erit $ndt = d\zeta$ et $nndt^2 = d\zeta^2$, vnde nostrae aequationes aliquanto fient simpliciores, scilicet:

$$\frac{ddx}{d\zeta^2} - \frac{2dy}{d\zeta} = 3x - 3xx + 4x^3 - 5x^4 + 6x^5 - 7x^6 \text{ etc.}$$
$$+\tfrac{3}{2}yy - 6xyy + 15xxyy - 30x^3yy + \tfrac{105}{2}x^4yy \text{ etc.}$$
$$-\tfrac{15}{8}y^4 + \tfrac{45}{4}xy^4 - \tfrac{315}{8}xxy^4 \text{ etc.}$$
$$+\tfrac{35}{16}y^6 \text{ etc.}$$

$$\frac{ddy}{d\zeta^2} + \frac{2dx}{d\zeta} = 3xy - 6xxy + 10x^3y - 15x^4y + 21x^5y \text{ etc.}$$
$$+\tfrac{3}{2}y^3 - \tfrac{15}{2}xy^3 + \tfrac{45}{2}xxy^3 - \tfrac{105}{2}x^3y^3 \text{ etc.}$$
$$-\tfrac{15}{8}y^5 + \tfrac{105}{8}xy^5 \text{ etc.}$$

§. 14. Iam obseruauimus, si orbita Planetae excentricitate careret, perpetuo fore tam $x = 0$ quam $y = 0$. Eatenus igitur hae quantitates non euanescent, quatenus adest

adeſt excentricitas. Statuamus igitur excentricitatem eſſe $= e$, et cum ea tanquam ſatis exigua ſpectetur, facile intelligitur, ambas literas x et y per huiusmodi ſeries convergentes exprimi poſſe:

$$x = eP + eeQ + e^3R + e^4S + e^5T \text{ etc.}$$
$$y = ep + eeq + e^3r + e^4s + e^5t + e^6u \text{ etc.}$$

ex quibus ſtatim colligitur

$$dx = edP + eedQ + e^3dR + e^4dS + e^5dT + e^6dU + \text{etc.}$$
$$dy = edp + eedq + e^3dr + e^4ds + e^5dt + e^6du + \text{etc.}$$
$$ddx = eddP + eeddQ + e^3ddR + e^4ddS + e^5ddT + e^6ddU + \text{etc.}$$
$$ddy = eddp + eeddq + e^3ddr + e^4dds + e^5ddt + e^6ddu + \text{etc.}$$

tum vero pro membris compoſitis habebimus has formulas vsque ad poteſtatem ſextam ipſius e continuatas:

$$\begin{aligned} xx = eePP &+ 2e^3PQ + 2e^4PR + 2e^5PS + 2e^6PT \\ &+ e^4QQ + 2e^5QR + 2e^6QS \\ &+ e^6RR \end{aligned}$$

$$\begin{aligned} xy = eepP &+ e^3pQ + e^4pR + e^5pS + e^6pT \\ &+ e^3qP + e^4qQ + e^5qR + e^6qS \\ &+ e^4rP + e^5rQ + e^6rR \\ &+ e^5sP + e^6sQ \\ &+ e^6tP \end{aligned}$$

$$\begin{aligned} yy + eepp &+ 2e^3pq + 2e^4pr + 2e^5ps + 2e^6pt \\ &+ e^4qq + 2e^5qr + 2e^6qs \\ &+ e^6rr \end{aligned}$$

$$\begin{aligned} x^3 = e^3P^3 &+ 3e^4PPQ + 3e^5PPR + 3e^6PPS \\ &+ 3e^5PQQ + 6e^6PQR \\ &+ e^6Q^3 \end{aligned}$$

$$xxy = e^3 pPP + 2e^4 pPQ + 2e^5 pPR + 2e^6 pPS$$
$$+ e^5 pQQ + 2e^6 pQR$$
$$+ e^4 qPP + 2e^5 qPQ + 2e^6 qPR$$
$$+ e^6 qQQ$$
$$+ e^5 rPP + 2e^6 rPQ$$
$$+ e^6 sPP$$

$$xyy = e^3 ppP + 2e^4 pqP + 2e^5 prP + 2e^6 psP$$
$$+ e^5 qqP + 2e^6 qrP$$
$$+ e^4 ppQ + 2e^5 pqQ + 2e^6 prQ$$
$$+ e^6 qqQ$$
$$+ e^5 ppR + 2e^6 pqR$$
$$+ e^6 ppS$$

$$y^3 = e^3 p^3 + 3e^4 ppq + 3e^5 ppr + 3e^6 pps$$
$$+ 3e^5 pqq + 6e^6 pqr$$
$$+ e^6 q^3$$

$$x^4 = e^4 P^4 + 4e^5 P^3 Q + 4e^6 P^3 R$$
$$+ 6e^6 PPQQ$$

$$x^3 y = e^4 pP^3 + 3e^5 pPPQ + 3e^6 pPPR$$
$$+ 3e^6 pPQQ$$
$$+ e^5 qP^3 + 3e^6 qPPQ$$
$$+ e^6 rP^3$$

$$xxyy = e^4 ppPP + 2e^5 ppPQ + 2e^6 ppPR$$
$$+ e^6 ppQQ$$
$$+ 2e^5 pqPP + 4e^6 pqPQ$$
$$+ 2e^6 prPP$$
$$+ e^6 qqPP$$

$$xy^3 = e^4 p^3 P + 3e^5 ppqP + 3e^6 pprP$$
$$+ 3e^6 pqqP$$
$$+ e^5 p^3 Q + 3e^6 ppqQ$$
$$+ e^6 p^3 R$$

$y^4 = e^4$

$$y^4 = e^4 p^4 + 4 e^5 p^3 q + 4 e^6 p^3 r + 6 e^6 p p q q$$

$$x^5 = e^5 P^5 + 5 e^6 P^4 Q$$

$$x^4 y = e^5 p P^4 + 4 e^6 p P^3 Q + e^6 q P^4$$

$$x^3 y y = e^5 p p P^3 + 3 e^6 p p P P Q + 2 e^6 p q P^3$$

$$x x y^3 = e^5 p^3 P P + 3 e^6 p p q P P + 2 e^6 p^3 P Q$$

$$x y^4 = e^5 p^4 P + 4 e^6 p^3 q P + e^6 p^4 Q$$

$$y^5 = e^5 p^5 + 5 e^6 p^4 q$$

$$x^6 = e^6 P^6$$

$$x^5 y = e^6 p P^5$$

$$x^4 y y = e^6 p p P^4$$

$$x^3 y^3 = e^6 p^3 P^3$$

$$x x y^4 = e^6 p^4 P P$$

$$x y^5 = e^6 p^5 P$$

$$y^6 = e^6 p^6$$

§. 15. Concipiamus nunc omnes hos valores in noſtris aequationibus ſubſtitui, et quia quantitates P, Q, R et p, q, r ab excentricitate e immunes eſſe debent, neceſſe eſt, vt in his aequationibus omnes termini, paribus poteſtatibus ipſius e affecti, ſeorſim inter ſi aequentur, vnde vtraque aequatio in plures diſcerpetur, dum ſcilicet omnes termini ſimpliciter per e multiplicati inter ſe aequales ſtatuuntur, deinde vero illi termini, qui per $e\,e$ ſunt

 affecti,

affecti, tum ii, qui per e^3 sunt affecti etc. Hoc igitur modo plures adipiscimur ordines aequationum, ex quibus quantitates incognitas P, Q, R, S et p, q, r, s etc. determinari oportebit. Hos igitur ordines aequationum hic ob oculos ponamus.

Ordo primus,

continens partes sola litera e affectas.

I. $\frac{ddP}{d\zeta^2} - \frac{2dp}{d\zeta} = 3P$

II. $\frac{ddp}{d\zeta^2} + \frac{2dP}{d\zeta} = 0.$

Ordo secundus,

continens partes per ee affectas.

I. $\frac{ddQ}{d\zeta^2} - \frac{2dq}{d\zeta} = 3Q - 3PP + \frac{3}{2}pp$

II. $\frac{ddq}{d\zeta^2} + \frac{2dQ}{d\zeta} = 3Pp$

Ordo tertius,

continens partes per e^3 affectas.

I. $\frac{ddR}{d\zeta^2} - \frac{2dr}{d\zeta} = 3R - 6PQ + 4P^3 + 3pq - 6ppP$

II. $\frac{ddr}{d\zeta^2} + \frac{2dR}{d\zeta} = 3pQ + 3qP - 6pPP + \frac{3}{2}p^3.$

Ordo quartus,

continens partes per e^4 affectas.

I. $\frac{ddS}{d\zeta^2} - \frac{2ds}{d\zeta} = 3S - 6PR - 3QQ + 3pr + \frac{3}{2}qq - 6ppQ$
$+ 12PPQ - 12pqP - 5P^4 + 15ppPP - \frac{15}{8}p^4$

II. $\frac{dds}{d\zeta^2} + \frac{2dS}{d\zeta} = 3pR + 3qQ + 3rP - 12pPQ - 6qPP + \frac{9}{2}ppq$
$+ 10pP^3 - \frac{15}{2}p^3P$

Ordo

Ordo quintus,

continens partes per e^5 affectas.

$$\text{I. } \frac{ddT}{d\zeta^2} - \frac{2dt}{d\zeta} = 3T - 6PS - 6QR + 3ps + 3qr$$
$$+ 12PPR + 12PQQ - 12prP - 6qqP$$
$$- 20P^3Q + 30ppPQ + 30pqPP - \tfrac{15}{2}p^3q$$
$$+ 6P^5 - 30ppP^3 + \tfrac{45}{4}p^4P.$$

$$\text{II. } \frac{ddt}{d\zeta^2} + \frac{2dT}{d\zeta} = 3pS + 3qR + 3rQ + 3sP$$
$$- 12pPR - 6pQQ - 12qPQ - 6rPP$$
$$+ \tfrac{9}{2}ppr + \tfrac{9}{2}pqq + 30pPPQ + 10qP^3$$
$$- \tfrac{45}{2}ppqP - \tfrac{15}{2}p^3Q - 15pP^4 + \tfrac{45}{2}p^3PP$$
$$- \tfrac{15}{8}p^5$$

Ordo sextus,

continens partes per e^6 affectas.

$$\text{I. } \frac{ddU}{d\zeta^2} - \frac{2du}{d\zeta} = 3U - 6PT - 6QS - 3RR + 3pt$$
$$+ 3qs + \tfrac{3}{2}rr + 12PPS + 24PQR + 4Q^3$$
$$- 12psP - 12qrP - 12prQ - 6qqQ$$
$$- 12pqR - 6ppS - 20P^3R - 30PPQQ$$
$$+ 30ppPR + 15ppQQ + 60pqPQ$$
$$+ 30prPP + 15qqPP - \tfrac{15}{2}p^3r - \tfrac{45}{4}p^2q^2$$
$$+ 30P^4Q - 90p^2P^2Q - 60pqP^3 + 45p^3qP$$
$$+ \tfrac{45}{4}p^4Q - 7P^6 + \tfrac{105}{2}ppP^4 - \tfrac{315}{8}p^4PP + \tfrac{35}{16}p^6.$$

$$\text{II. } \frac{ddu}{d\zeta^2} + \frac{2dU}{d\zeta} = 3pT + 3qS + 3rR + 3sQ + 3tP$$
$$- 12pPS - 12pQR - 12qPR - 6qQQ$$
$$- 12rPQ - 6sPP + \tfrac{9}{2}ppS + 9pqr + \tfrac{3}{2}q^3$$
$$+ 30pPPR + 30pPQQ + 30qPPQ$$
$$+ 10rP^3 - \tfrac{45}{2}pprP - \tfrac{45}{2}pqqP - \tfrac{45}{2}ppqQ$$
$$- \tfrac{15}{2}p^3R - 60pP^3Q - 15qP^4 + \tfrac{135}{2}ppqPP$$
$$+ 45p^3PQ - \tfrac{75}{8}p^4q + 21pP^5 - \tfrac{105}{2}p^3P^3 + \tfrac{105}{8}p^5P.$$

§. 16.

§. 16. In constitutione horum ordinum tota vis istius methodi potissimum continetur, quae adeo multo latius patet, cum etiam eius ope motus lunares expediri queant. Cum enim hactenus ne vllam quidem integrationem tentauerimus, nunc binae aequationes cuiusque ordinis facili negotio integrari poterunt, namque in ordine primo tantum occurrunt binae incognitae P et p, quarum valores per integrationem ad functiones temporis siue anguli ζ reducuntur, quibus inuentis secundus ordo binas tantum incognitas Q et q complectitur, quas pari modo per tempus exprimere licebit; tum vero simili modo ex tertio ordine definientur literae incognitae R et r, et ita porro, vnde tandem veri valores pro binis incognitis principalibus x et y colligentur. Totum autem hoc integrationis negotium in sequenti Problemate generali ostendamus.

Problema generale.

§. 17. Propositis duabus aequationibus differentialibus secundi gradus:

$$\text{I. } \frac{ddZ}{d\zeta^2} - \frac{2dz}{d\zeta} = 3Z + M \text{ et}$$

$$\text{II. } \frac{ddz}{d\zeta^2} + \frac{2dZ}{d\zeta} = N,$$

vbi M et N denotant functiones quascunque temporis siue anguli ζ, inuestigare valores binarum quantitatum incognitarum Z et z.

Solutio.

Incipiamus ab aequatione posteriore, quae ducta in $d\zeta$ et integrata dat $\frac{dz}{d\zeta} + 2Z = \int N d\zeta$, vnde fit

$\frac{dz}{d\zeta}$

$$\frac{dz}{d\zeta} = \int N\, d\zeta - 2\, Z,$$

qui valor in priore aequatione fubftitutus praebet $\frac{ddZ}{d\zeta^2} - 2\int N\, d\zeta + Z = M$, fiue $\frac{ddZ}{d\zeta^2} + Z = M + 2\int N\, d\zeta = L$, ponendo breuitatis ergo $L = M + 2\int N\, d\zeta$. Conftat autem, fi effet $L = 0$, tum fore $Z = \alpha \operatorname{cof.} \zeta$, qui valor, etiamfi tantum eft integrale particulare, tamen fufficit ad integrale completum inueftigandum. Statuamus igitur pro noftro cafu effe $Z = v \operatorname{cof.} \zeta$, eritque

$$\frac{dZ}{d\zeta} = -v \operatorname{fin.} \zeta + \frac{dv}{d\zeta} \operatorname{cof.} \zeta \text{ et}$$

$$\frac{ddZ}{d\zeta^2} = -v \operatorname{cof.} \zeta - \frac{2\, dv}{d\zeta} \operatorname{fin.} \zeta + \frac{ddv}{d\zeta^2} \operatorname{cof.} \zeta,$$

ficque noftra aequatio euadet

$$\frac{ddv}{d\zeta^2} \operatorname{cof.} \zeta - \frac{2\, dv}{d\zeta} \operatorname{fin.} \zeta = L$$

quae per $d\zeta . \operatorname{cof.} \zeta$ multiplicata praebet

$$\frac{ddv}{d\zeta} \operatorname{cof.} \zeta^2 - 2\, dv \operatorname{fin.} \zeta \operatorname{cof.} \zeta = L\, d\zeta \operatorname{cof.} \zeta,$$

cuius integrale eft

$$\frac{dv}{d\zeta} \operatorname{cof.} \zeta^2 = \int L\, d\zeta \operatorname{cof.} \zeta,$$

vnde colligitur

$$dv = \frac{d\zeta}{\operatorname{cof.} \zeta^2} \int L\, d\zeta \operatorname{cof.} \zeta,$$

quae aequatio denuo integrata dat

$$v = \int \frac{d\zeta}{\operatorname{cof.} \zeta^2} \int L\, d\zeta \operatorname{cof.} \zeta = \operatorname{tang.} \zeta \int L\, d\zeta \operatorname{cof.} \zeta - \int L\, d\zeta \operatorname{fin.} \zeta,$$

quo valore inuento erit

$$Z = \operatorname{fin.} \zeta \int L\, d\zeta \operatorname{cof.} \zeta - \operatorname{cof.} \zeta \int L\, d\zeta \operatorname{fin.} \zeta,$$

vbi binae formulae integrales iam binas conftantes per integrationes ingreffas inuoluunt. Cum deinde fit

$$\frac{dz}{d\zeta} = \int N\, d\zeta - 2\, Z, \text{ erit}$$

$$\frac{dz}{d\zeta} = \int N\, d\zeta - 2 \operatorname{fin.} \zeta \int L\, d\zeta \operatorname{cof.} \zeta + 2 \operatorname{cof.} \zeta \int L\, d\zeta \operatorname{fin.} \zeta,$$

hincque integrando deducitur

$$z = \int d\zeta \int N\, d\zeta - 2\int d\zeta\, \text{fin.}\,\zeta \int L\, d\zeta\, \text{cof.}\,\zeta$$
$$+ 2\int d\zeta\, \text{cof.}\,\zeta \int L\, d\zeta\, \text{fin.}\,\zeta$$

quae aequatio, fi pofteriores formulae integrales duplicatae reducantur, fequentem induet formam:

$$z = \int d\zeta \int N\, d\zeta + 2\,\text{cof.}\,\zeta \int L\, d\zeta\, \text{cof.}\,\zeta + 2\,\text{fin.}\,\zeta \int L\, d\zeta\, \text{fin.}\,\zeta$$
$$- 2\int L\, d\zeta.$$

§. 18. Reftituamus nunc loco L valorem affumtum $M + 2\int N\, d\zeta$, ac pro valore ipfius Z reperiemus

$$\int L\, d\zeta\, \text{cof.}\,\zeta = \int M\, d\zeta\, \text{cof.}\,\zeta + 2\,\text{fin.}\,\zeta \int N\, d\zeta$$
$$- 2\int N\, d\zeta\, \text{fin.}\,\zeta \quad \text{et}$$

$$\int L\, d\zeta\, \text{fin.}\,\zeta = \int M\, d\zeta\, \text{fin.}\,\zeta - 2\,\text{cof.}\,\zeta \int N\, d\zeta$$
$$+ 2\int N\, d\zeta\, \text{cof.}\,\zeta,$$

vnde fit

$$Z = \text{fin.}\,\zeta \int M\, d\zeta\, \text{cof.}\,\zeta - \text{cof.}\,\zeta \int M\, d\zeta\, \text{fin.}\,\zeta$$
$$+ 2\int N\, d\zeta - 2\,\text{fin.}\,\zeta \int N\, d\zeta\, \text{fin.}\,\zeta - 2\,\text{cof.}\,\zeta \int N\, d\zeta\, \text{cof.}\,\zeta.$$

Deinde vero pro altero valore z, vbi eaedem formulae iam reductae occurrunt, habebimus

$$z = 2\,\text{cof.}\,\zeta \int M\, d\zeta\, \text{cof.}\,\zeta + 2\,\text{fin.}\,\zeta \int M\, d\zeta\, \text{fin.}\,\zeta$$
$$- 2\int M\, d\zeta - 4\,\text{cof.}\,\zeta \int N\, d\zeta\, \text{fin.}\,\zeta$$
$$+ 4\,\text{fin.}\,\zeta \int N\, d\zeta\, \text{cof.}\,\zeta - 3\int d\zeta \int N\, d\zeta.$$

§. 19. Quod fi ergo M et N fuerint functiones quaecunque ipfius ζ, valores integrales pro Z et z inuenti non parum euadunt perplexi: Verum in refolutione noftrorum ordinum commode vfu venit, vt quantitates M et N perpetuo per finus et cofinus huiusmodi angulorum: $n\zeta + \alpha$, exprimantur, quemadmodum mox videbimus. Quando autem literae M et N huiusmodi induunt valores, tum integralia

tegralia quaesita satis succincte et concinne assignare licebit, id quod in sequenti Problemate speciali clarius ob oculos ponamus.

Problema speciale.

§. 20. Si binae aequationes differentiales secundi gradus huiusmodi habeant formam:

$$\text{I. } \frac{ddZ}{d\zeta^2} - \frac{2dz}{d\zeta} - 3Z = C \operatorname{cos.} (n\zeta + \alpha),$$

$$\text{II. } \frac{ddz}{d\zeta^2} + \frac{2dZ}{d\zeta} = c \operatorname{sin.} (n\zeta + \alpha),$$

inuenire valores integrales pro binis quantitatibus incognitis Z et z.

Solutio.

Haud difficulter quidem Solutio huius Problematis ex praecedenti deduci posset, siquidem valores integrales completi desiderarentur; verum quia pro nostro instituto integralia particularia sufficere possunt, quemadmodum in sequentibus manifesto patebit, ista integralia multo facilius inuestigare poterimus, quandoquidem forma integralium haud difficulter perspicitur. Hunc in finem statuamus $Z = F \operatorname{cos.} (n\zeta + \alpha)$ et $z = f \operatorname{sin.} (n\zeta + \alpha)$, vbi tantum coefficientes F et f definiri oportet. Substituamus igitur istos valores assumtos in binis nostris aequationibus propositis, et cum sit

$$\frac{dZ}{d\zeta} = -nF \operatorname{sin.} (n\zeta + \alpha) \text{ et } \frac{ddZ}{d\zeta^2} = -nnF \operatorname{cos.} (n\zeta + \alpha)$$

$$\frac{dz}{d\zeta} = +nf \operatorname{cos.} (n\zeta + \alpha) \text{ et } \frac{ddz}{d\zeta^2} = -nnf \operatorname{sin.} (n\zeta + \alpha)$$

aequatio prior in omnibus terminis continebit $\operatorname{cos.} (n\zeta + \alpha)$ posterior vero $\operatorname{sin.} (n\zeta + \alpha)$, quibus factoribus omissis eae induent has formas:

$$-nnF - 2nf - 3F = C \quad \text{et} \quad -nnf - 2nF = c$$

ex quibus inueftigari oportet literas F et f. Ex pofteriore quidem ftatim colligitur $f = -\frac{2F}{n} - \frac{c}{nn}$, qui valor in priore fubftitutus praebet $-nnF + F + \frac{2c}{n} = C$, vnde fit

$$F = \frac{2c - nC}{n(nn-1)} \quad \text{et} \quad f = \frac{2nC - c(3+nn)}{nn(nn-1)}.$$

§. 21. Pro hoc ergo cafu propofito valores integrales quaefiti erunt

$$Z = \frac{2c - nC}{n(nn-1)} \operatorname{cof.}(n\zeta + \alpha) \quad \text{et} \quad z = \frac{2nC - (3+nn)c}{nn(nn-1)} \operatorname{fin.}(n\zeta + \alpha)$$

atque hinc intelligitur, fi quantitates M et N plures contineant huiusmodi terminos, veluti fi aequationes noftrae effent

$$\frac{ddZ}{d\zeta^2} - \frac{2dz}{d\zeta} - 3Z = C \operatorname{cof.}(n\zeta + \alpha) + C' \operatorname{cof.}(n'\zeta + \alpha') + C'' \operatorname{cof.}(n''\zeta + \alpha'') + \text{etc.}$$

$$\frac{ddz}{d\zeta^2} + \frac{2dZ}{d\zeta} = c \operatorname{fin.}(n\zeta + \alpha) + c' \operatorname{fin.}(n'\zeta + \alpha') + c'' \operatorname{fin.}(n''\zeta + \alpha'') + \text{etc.}$$

tum, quia Z et z vbique vnicam tantum habent dimenfionem, fore

$$Z = \frac{2c - nC}{n(nn-1)} \operatorname{cof.}(n\zeta + \alpha) + \frac{2c' - n'C'}{n'(n'n'-1)} \operatorname{cof.}(n'\zeta + \alpha') + \frac{2c'' - n''C''}{n''(n''n''-1)} \operatorname{cof.}(n''\zeta + \alpha'') + \text{etc.}$$

$$z = \frac{2nC - (3+nn)c}{nn(nn-1)} \operatorname{fin.}(n\zeta + \alpha) + \frac{2n'C' - (3+n'n')c'}{n'n'(n'n'-1)} \operatorname{fin.}(n'\zeta + \alpha') + \text{etc.}$$

His igitur praemiffis fingulos noftros ordines percurramus.

Refolutio aequationum primi ordinis.

§. 22. Quoniam hic binae aequationes propofitae funt:

ddP

$$\frac{ddP}{d\zeta^2} - \frac{2dp}{d\zeta} - 3P = 0,$$
$$\frac{ddp}{d\zeta^2} + \frac{2dP}{d\zeta} = 0$$

earum integratio ita in genere instituatur. Primo posterior integrata dat $\frac{dp}{d\zeta} + 2P = \alpha$, vnde fit $\frac{dp}{d\zeta} = \alpha - 2P$, qui valor in prima substitutus praebet: $\frac{ddP}{d\zeta^2} + P = 2\alpha$. Ponatur hic, vt in solutione generali, $P = v \text{ cof. } \zeta$, fietque

$$\frac{ddv}{d\zeta^2} \text{cof.} \zeta - \frac{2dv}{d\zeta} \text{fin.} \zeta = 2\alpha$$

quae aequatio, in $d\zeta \text{ cof.} \zeta$ ducta et integrata, praebet

$$\frac{dv}{d\zeta} \text{cof.} \zeta^2 = 2\alpha \text{ fin.} \zeta + \beta,$$

hincque fit

$$dv = \frac{2\alpha d\zeta \text{ fin.} \zeta}{\text{cof.} \zeta^2} + \frac{\beta d\zeta}{\text{cof.} \zeta^2},$$

vnde integrando oritur

$$v = \frac{2\alpha}{\text{cof.} \zeta} + \beta \text{ tang.} \zeta + \gamma,$$

quocirca habebimus

$$P = 2\alpha + \beta \text{ fin.} \zeta + \gamma \text{ cof.} \zeta.$$

Tum vero porro erit

$$\frac{dp}{d\zeta} = -3\alpha - 2\beta \text{ fin.} \zeta - 2\gamma \text{ cof.} \zeta, \text{ vnde fit}$$
$$p = -3\alpha\zeta + 2\beta \text{ cof.} \zeta - 2\gamma \text{ fin.} \zeta + \delta$$

sicque quatuor in calculum ingressae sunt quantitates constantes α, β, γ, δ, quemadmodum integratio binarum aequationum differentialium secundi gradus postulat, vt integralia completa reperiantur.

§. 23. Quoniam nostrum Problema vtique est determinatum, ex conditionibus quas innoluit valores singularum harum constantium determinari oportet. Ac primo quidem quia angulus ζ denotat longitudinem mediam Pla-

 netae

netae, a qua locus verus nunquam vltra datos limites di- ſcrepare poteſt, manifeſtum eſt eſſe debere $\alpha = 0$. Niſi enim eſſet $\alpha = 0$, quantitas p, ideoque et x, creſcente ζ, tandem in infinitum excreſcere poſſet, id quod indicium eſſet, motum medium non rite eſſe conſtitutum, ex quo neceſſe eſt fieri $\alpha = 0$. Secundo ob eandem rationem quoque quarta conſtans δ nihilo aequari debet, vt locus medius cum vero in certis orbitae locis conueniat. Tertio pro binis reliquis conſtantibus faciamus $\beta = k$ ſin. ε et $\gamma = k$ coſ. ε, vt obtineamus

$$P = k \text{ ſin. } \varepsilon \text{ ſin. } \zeta + k \text{ coſ. } \varepsilon \text{ coſ. } \zeta = k \text{ coſ. } (\zeta - \varepsilon) \text{ et}$$

$$p = 2\, k \text{ ſin. } \varepsilon \text{ coſ. } \zeta - 2\, k \text{ coſ. } \varepsilon \text{ ſin. } \zeta = -2\, k \text{ ſin. } (\zeta - \varepsilon)$$

ex quo poſteriore valore patet, ſi fuerit $\zeta = \varepsilon$, tum fore $p = 0$, ideoque etiam $y = 0$, quatenus ſcilicet a p pendet, ita vt hoc caſu locus verus cum medio conueniat. Quamobrem ſi aſſumamus cum Aſtronomis, hoc euenire in ipſo Aphelio, conſtans ε exhibebit longitudinem Aphelii. Tum vero pro quouis alio ſitu angulus $\zeta - \varepsilon$ exhibet anomaliam mediam, quae cum ſit praecipuum elementum in motu Planetarum, ſtatuamus breuitatis gratia $\zeta - \varepsilon = \theta$, eritque $P = k$ coſ. θ et $p = -2\, k$ ſin. θ; vnde ex hoc ſaltem ordine erit $x = e\, k$ coſ. θ, vbi $e\, k$ iam denotat excentricitatem, quandoquidem, ſumta anomalia media $\theta = 0$, foret diſtantia Planetae a Sole $a\,(1 + e\, k)$; at vero ſumto $\theta = 180°$ prodiret diſtantia Perihelii $= a\,(1 - e\, k)$. Quare cum excentricitas ſupponatur $= e$, neceſſe eſt vt fiat $k = 1$, conſequenter conſtantibus noſtris rite determinatis reſolutio ordinis primi ita ſe habebit: $P =$ coſ. θ et $p = -2$ ſin. θ; vbi θ exprimit anomaliam mediam, et cum ſit $\theta = \zeta - \varepsilon$, erit vtique $d\,\theta = d\,\zeta$, ideoque in ordinibus ſequentibus loeo $d\zeta$ ſcribere licebit $d\theta$, ita vt ſola anomalia media in calculum ſit ingreſſura.

Reſolutio aequationum ſecundi ordinis.

§. 24. In hoc ordine continentur quantitates incognitae Q et q, his aequationibus expreſſae:

$$\text{I. } \frac{d\,d\,Q}{d\theta^2} - \frac{2\,d\,q}{d\theta} - 3\,Q = -3\,P\,P + \tfrac{3}{2}\,p\,p,$$

$$\text{II. } \frac{d\,d\,q}{d\theta^2} + \frac{2\,d\,Q}{d\theta} = 3\,P\,p;$$

vbi cum inuenerimus $P = \text{coſ.}\,\theta$ et $p = -2\,\text{ſin.}\,\theta$, erit

$$P^2 = \tfrac{1}{2} + \tfrac{1}{2}\,\text{coſ.}\,2\,\theta,\ P\,p = -\text{ſin.}\,2\,\theta \text{ et } p\,p = 2 - 2\,\text{coſ.}\,2\,\theta,$$

hincque noſtrae aequationes erunt

$$\frac{d\,d\,Q}{d\theta^2} - \frac{2\,d\,q}{d\theta} - 3\,Q = +\tfrac{3}{2} - \tfrac{9}{2}\,\text{coſ.}\,2\,\theta = M,$$

$$\frac{d\,d\,q}{d\theta^2} + \frac{2\,d\,Q}{d\theta} = -3\,\text{ſin.}\,2\,\theta = N$$

Quo nunc hic ſolutione ſpeciali (§. 20.) vti queamus, ambae literae M et N in duas partes diſcerptae concipiantur, quarum priores contineant angulum $0\,\theta$, alterae vero angulum $2\,\theta$, quandoquidem has literas ita repraeſentari licet: $M = \tfrac{3}{2}\,\text{coſ.}\,0\,\theta - \tfrac{9}{2}\,\text{coſ.}\,2\,\theta$ et $N = \text{ſin.}\,0\,\theta - 3\,\text{ſin.}\,2\,\theta$. Quia autem pro prioribus partibus fieret $n = 0$, formulae ſupra inuentae euadent incongruae, vnde hunc caſum ſeorſim euolui conueniet. Sit ergo in genere

$$\frac{d\,d\,Z}{d\theta^2} - \frac{2\,d\,z}{d\theta} - 3\,Z = C \text{ et } \frac{d\,d\,z}{d\theta^2} + \frac{2\,d\,Z}{d\theta} = 0,$$

vbi eſt $C = \tfrac{3}{2}$. Ex poſteriore aequatione fit $\frac{d\,z}{d\theta} = \alpha - 2\,Z$, vnde prior euadit $\frac{d\,d\,Z}{d\theta^2} + Z = 2\,\alpha + C = D$. Poſito nunc $Z = v\,\text{coſ.}\,\theta$ erit $\frac{d\,d\,v}{d\theta^2}\,\text{coſ.}\,\theta - \frac{2\,d\,v}{d\theta}\,\text{ſin.}\,\theta = D$, quae aequatio ducta in $d\theta\,\text{coſ.}\,\theta$ praebet integrale

$$\frac{d\,v}{d\theta}\,\text{coſ.}\,\theta^2 = D\,\text{ſin.}\,\theta + \beta, \text{ hincque}$$

$dv =$

$$d v = \frac{D\, d\, \theta\, \text{fin.}\, \theta}{\text{cof.}\, \theta^2} + \frac{\beta\, d\, \theta}{\text{cof.}\, \theta^2}$$

vnde integrando colligitur

$$v = \frac{D}{\text{cof.}\, \theta} + \beta\, \text{tang.}\, \theta + \gamma; \text{ erit ergo}$$

$$Z = D + \beta\, \text{fin.}\, \theta + \gamma\, \text{cof.}\, \theta, \text{ confequenter}$$

$$\frac{d z}{d \theta} = \alpha - 2 D - 2 \beta\, \text{fin.}\, \theta - 2 \gamma\, \text{cof.}\, \theta$$

et integrando

$$z = \alpha \theta - 2 D \theta + 2 \beta\, \text{cof.}\, \theta - 2 \gamma\, \text{fin.}\, \theta + \delta,$$

vbi ante omnia obferuandum eft, conftantem α ita accipi debere, vt fiat $\alpha - 2 D = 0$, quia alioquin motus medius non rite effet conftitutus. Erit ergo

$$\alpha = 2 D = 4 \alpha + 2 C, \text{ vnde fit } \alpha = -\tfrac{2}{3} C;$$

tum vero ob rationes fupra allegatas etiam effe oportet $\delta = 0$. Quod autem ad conftantes β et γ attinet, quia angulus θ in primo ordine iam rite conftitutus fupponitur, euidens eft poni debere $\beta = 0$ et $\gamma = 0$; ficque habebimus $Z = -\frac{1}{3} C$ et $z = 0$, qui valores adeo ex ipfis aequationibus differentialibus concludi potuiffent, dum fcilicet ambae vt conftantes effent fpectatae.

§. 25. Priores igitur partes pro litteris noftris Q et q praebent $Q = -\frac{1}{3} C = -\frac{1}{2}$ et $q = 0$; pro partibus autem pofterioribus, quia eft $n = 2$, formulae fupra (§. 20) exhibitae nulla laborant ambiguitate, et cum pro hoc cafu fit $C = -\frac{9}{2}$ et $c = -3$, colligitur $F = \frac{1}{2}$, et $f = \frac{1}{4}$, ita vt nunc fit $Z = \frac{1}{2}\, \text{cof.}\, 2\, \theta$ et $z = \frac{1}{4}\, \text{fin.}\, 2\, \theta$. Vtrosque ergo valores coniungendo habebimus pro ordine fecundo has determinationes:

$$Q = -\tfrac{1}{2} + \tfrac{1}{2}\, \text{cof.}\, 2\, \theta \text{ et } q = \tfrac{1}{4}\, \text{fin.}\, 2\, \theta.$$

Refolu-

Resolutio aequationum tertii ordinis.

§. 26. Binae aequationes huius ordinis ita se habent:

$$\text{I. } \frac{d\,d\,R}{d\,\theta^2} - \frac{2\,d\,r}{d\,\theta} - 3\,R = -6\,P\,Q + 4\,P^3 + 3\,p\,q - 6\,p\,p\,P = M$$

$$\text{II. } \frac{d\,d\,r}{d\,\theta^2} + \frac{2\,d\,R}{d\,\theta} = 3\,p\,Q + 3\,q\,P - 6\,p\,P\,P + \tfrac{3}{2}\,p^3 = N.$$

Cum igitur iam inuenerimus

$$P = \text{cos.}\,\theta,\; p = -2\,\text{sin.}\,\theta,\; Q = -\tfrac{1}{2} + \tfrac{1}{2}\,\text{cos.}\,2\,\theta \text{ et}$$
$$q = \tfrac{1}{4}\,\text{sin.}\,2\,\theta,$$

per notas reductiones angulorum, quibus est

$$\text{cos.}\,\alpha\,\text{cos.}\,\beta = \tfrac{1}{2}\,\text{cos.}\,(\alpha - \beta) + \tfrac{1}{2}\,\text{cos.}\,(\alpha + \beta)$$
$$\text{sin.}\,\alpha\,\text{cos.}\,\beta = \tfrac{1}{2}\,\text{sin}\,(\alpha - \beta) + \tfrac{1}{2}\,\text{sin.}\,(\alpha + \beta)$$
$$\text{sin.}\,\alpha\,\text{sin.}\,\beta = \tfrac{1}{2}\,\text{cos.}\,(\alpha - \beta) - \tfrac{1}{2}\,\text{cos.}\,(\alpha + \beta)$$
$$\text{cos.}\,\alpha\,\text{sin.}\,\beta = \tfrac{1}{2}\,\text{sin.}\,(\alpha + \beta) - \tfrac{1}{2}\,\text{sin.}\,(\alpha - \beta)$$

colligimus pro priore aequatione

$$P\,Q = -\tfrac{3}{4}\,\text{cos.}\,\theta + \tfrac{1}{4}\,\text{cos.}\,3\,\theta,\; p\,q = -\tfrac{1}{4}\,\text{cos.}\,\theta + \tfrac{1}{4}\,\text{cos.}\,3\,\theta,$$
$$P^3 = \tfrac{3}{4}\,\text{cos.}\,\theta + \tfrac{1}{4}\,\text{cos.}\,3\,\theta,\; p\,p\,P = \text{cos.}\,\theta - \text{cos.}\,3\,\theta,$$

hincque colligitur

$$M = -\tfrac{9}{4}\,\text{cos.}\,\theta + \tfrac{25}{4}\,\text{cos.}\,3\,\theta.$$

Simili modo pro altera aequatione erit

$$p\,Q = \tfrac{3}{2}\,\text{sin.}\,\theta - \tfrac{1}{2}\,\text{sin.}\,3\,\theta,\quad q\,P = +\tfrac{1}{8}\,\text{sin.}\,\theta + \tfrac{1}{8}\,\text{sin.}\,3\,\theta,$$
$$p\,P\,P = -\tfrac{1}{2}\,\text{sin.}\,\theta - \tfrac{1}{2}\,\text{sin.}\,3\,\theta,\; p^3 = -6\,\text{sin.}\,\theta + 2\,\text{sin.}\,3\,\theta,$$

vnde fit

$$N = -\tfrac{9}{8}\,\text{sin.}\,\theta + \tfrac{39}{8}\,\text{sin.}\,3\,\theta.$$

§. 27. Hic literae M et N iterum ex duabus constant partibus, pro quarum prioribus est $n = 1$, pro posterioribus autem $n = 3$. Priore autem casu formulae supra datae fiunt incongruae, vnde hunc casum seorsim euolui conueniet. Sit igitur

$$\frac{ddZ}{d\theta^2} - \frac{2dz}{d\theta} - 3Z = C \operatorname{cof.} \theta \quad \text{et} \quad \frac{ddz}{d\theta^2} + \frac{2dZ}{d\theta} = c \operatorname{fin.} \theta,$$

ita vt fit $C = -\frac{9}{4}$ et $c = -\frac{9}{8}$. Iam ex pofteriore fit,

$$\frac{dz}{d\theta} = -c \operatorname{cof.} \theta - 2Z + \alpha,$$

hinque prior aequatio prodibit

$$\frac{ddZ}{d\theta^2} + Z = (C - 2c) \operatorname{cof.} \theta + 2\alpha.$$

Cum autem noftro cafu fit $C - 2c = 0$, habebimus

$$\frac{ddZ}{d\theta^2} + Z = 2\alpha.$$

Hinc fi vt hactenus ponatur $Z = v \operatorname{cof.} \theta$, erit

$$\frac{ddv}{d\theta^2} \operatorname{cof.} \theta - \frac{2dv}{d\theta} \operatorname{fin.} \theta = 2\alpha,$$

et integrando

$$\frac{dv}{d\theta} \operatorname{cof.} \theta^2 = 2\alpha \operatorname{fin.} \theta + \beta, \text{ vnde colligitur}$$

$$dv = \frac{2\alpha\, d\theta \operatorname{fin.} \theta}{\operatorname{cof.} \theta^2} + \frac{\beta\, d\theta}{\operatorname{cof.} \theta^2},$$

hincque porro

$$v = \frac{2\alpha}{\operatorname{cof.} \theta} + \beta \operatorname{tang.} \theta + \gamma,$$

confequenter

$$Z = 2\alpha + \beta \operatorname{fin.} \theta + \gamma \operatorname{cof.} \theta, \text{ ex quo porro fit}$$

$$\frac{dz}{d\theta} = -c \operatorname{cof.} \theta - 3\alpha - 2\beta \operatorname{fin.} \theta + 2\gamma \operatorname{cof.} \theta,$$

vnde reperitur

$$z = -c \operatorname{fin.} \theta - 3\alpha\theta + 2\beta \operatorname{cof.} \theta - 2\gamma \operatorname{fin.} \theta + \delta.$$

§. 28. Cum hic motum medium rite conftitutum effe affumamus, neceffe eft vt fit $\alpha = 0$, tum vero etiam debet effe $\delta = 0$. Quod autem ad conftantes β et γ attinet, quia etiam Aphelium rite conftitutum affumimus, debet etiam

etiam esse $\beta = 0$ et $\gamma = 0$, sicque valores nostri quaesiti erunt

$$Z = 0 \text{ et } z = -c \sin.\theta = +\tfrac{3}{8}\sin.\theta.$$

Hic autem probe obseruasse iuuabit, si non fuisset $C - 2c = 0$, ad terminos fuisse peruentum, qui ipsum angulum θ continuissent, quos nullo modo per constantes tollere licuisset. Hoc scilicet casu motus Planetae verus non ad medium reuocari potuisset; vnde ista conditio $C - 2c = 0$ necessario in natura rei est fundata, atque etiam in sequentibus ordinibus, quoties Sinus et Cosinus anguli simplicis θ occurrunt, semper necessario euadere debet $C = 2c$, indeque semper erit $Z = 0$ et $z = -c\sin.\theta$.

§. 29. Pro terminis autem posterioribus, angulum 3θ inuoluentibus, formulae supra datae tuto adhiberi possunt: erit enim

$$n = 3,\ C = \tfrac{75}{4} \text{ et } c = \tfrac{39}{8}, \text{ vnde fit } F = -\tfrac{3}{8} \text{ et } f = -\tfrac{7}{24},$$

quocirca ex hoc ordine nanciscimur sequentes determinationes:

$$R = -\tfrac{3}{8}\cos.3\theta \text{ et } r = +\tfrac{9}{8}\sin.\theta - \tfrac{7}{24}\sin.3\theta.$$

Resolutio aequationum quarti ordinis,

§. 30. Binae huius ordinis aequationes ita se habent:

$$\text{I. } \frac{ddS}{d\theta^2} - \frac{2dS}{d\theta} - 3S = \left.\begin{aligned} &-6PR - 3QQ + 3pr + \tfrac{3}{4}qq \\ &+6ppQ + 12PPQ - 12pqP \\ &-5P^4 + 15ppPP - \tfrac{15}{8}p^4 \end{aligned}\right\} = M$$

 II.

$$\text{II.}\ \frac{dds}{d\theta^2}+\frac{2\,dS}{d\theta}=\left.\begin{array}{l}3pR+3qQ+3rP\\ -12pPQ-6qPP+\frac{9}{2}ppq\\ +10pP^3-\frac{15}{2}p^3P\end{array}\right\}=N.$$

Hic ſinguli termini, vt hactenus factum eſt, euoluantur, ac pro priore quidem aequatione reperietur:

$$PR=-\tfrac{3}{16}\,\text{cof.}\,2\theta-\tfrac{3}{16}\,\text{cof.}\,4\theta,\qquad QQ=\tfrac{3}{8}-\tfrac{1}{2}\,\text{cof.}\,2\theta+\tfrac{1}{8}\,\text{cof.}\,4\theta,$$
$$pr=-\tfrac{9}{8}+\tfrac{17}{12}\,\text{cof.}\,2\theta-\tfrac{7}{24}\,\text{cof.}\,4\theta,\qquad qq=\tfrac{1}{32}-\tfrac{1}{32}\,\text{cof.}\,4\theta,$$
$$ppQ=-\tfrac{3}{2}+2\,\text{cof.}\,2\theta-\tfrac{1}{2}\,\text{cof.}\,4\theta,\qquad PPQ=-\tfrac{1}{8}+\tfrac{1}{8}\,\text{cof.}\,4\theta,$$
$$pqP=-\tfrac{1}{8}+\tfrac{1}{8}\,\text{cof.}\,4\theta,\qquad P^4=\tfrac{3}{8}+\tfrac{1}{2}\,\text{cof.}\,2\theta+\tfrac{1}{8}\,\text{cof.}\,4\theta,$$
$$ppPP=\tfrac{1}{2}-\tfrac{1}{2}\,\text{cof.}\,4\theta,\qquad p^4=6-8\,\text{cof.}\,2\theta+2\,\text{cof.}\,4\theta,$$

vnde colligitur:

$$M=-\tfrac{1221}{8}+\tfrac{251}{8}\,\text{cof.}\,2\theta-\tfrac{965}{64}\,\text{cof.}\,4\theta.$$

Pro altera vero aequatione erit

$$pR=-\tfrac{3}{8}\,\text{ſin.}\,2\theta+\tfrac{3}{8}\,\text{ſin.}\,4\theta,\qquad qQ=-\tfrac{1}{8}\,\text{ſin.}\,2\theta+\tfrac{1}{16}\,\text{ſin.}\,4\theta,$$
$$rP=+\tfrac{5}{12}\,\text{ſin.}\,2\theta-\tfrac{7}{48}\,\text{ſin.}\,4\theta,\qquad pPQ=\tfrac{1}{2}\,\text{ſin.}\,2\theta-\tfrac{1}{4}\,\text{ſin.}\,4\theta,$$
$$qPP=\tfrac{1}{8}\,\text{ſin.}\,2\theta+\tfrac{1}{16}\,\text{ſin.}\,4\theta,\qquad ppq=\tfrac{1}{2}\,\text{ſin.}\,2\theta-\tfrac{1}{4}\,\text{ſin.}\,4\theta,$$
$$pP^3=-\tfrac{1}{2}\,\text{ſin.}\,2\theta-\tfrac{1}{4}\,\text{ſin.}\,4\theta,\qquad p^3P=-2\,\text{ſin.}\,2\theta+\text{ſin.}\,4\theta,$$

vnde colligitur

$$N=\tfrac{21}{4}\,\text{ſin.}\,2\theta-\tfrac{61}{8}\,\text{ſin.}\,4\theta.$$

§. 30. Hic igitur occurrunt tres partes, quarum prima eſt conſtans, ſecunda vero angulo 2θ et tertia angulo 4θ affecta. Pro prima igitur parte erit

$$C=-\tfrac{1221}{8},\quad c=0\ \text{et}\ n=0,$$

vnde, quemadmodum in ordine ſecundo iam vidimus, colligitur

$$Z=-\tfrac{1}{3}C=+\tfrac{407}{8}\ \text{et}\ z=0.$$

Porro

Porro pro ſecunda parte eſt

$$n = 2,\ C = \tfrac{251}{8} \text{ et } c = \tfrac{21}{4}, \text{ vnde fit}$$

$$F = \frac{2c - 2C}{6} = \frac{c - C}{3} = -\tfrac{209}{24} \text{ et } f = -F - \tfrac{c}{4} = \tfrac{355}{48}.$$

Pro tertia tandem parte, vbi

$$n = 4,\ C = -\tfrac{963}{64} \text{ et } c = -\tfrac{61}{8}, \text{ erit}$$

$$F = \frac{2c - 4C}{60} = \frac{c - 2C}{30} = +\tfrac{719}{960}, \text{ hincque}$$

$$f = -\tfrac{F}{2} - \tfrac{c}{16} = \tfrac{49}{480},$$

quibus valoribus ſubſtitutis habebimus

$$S = \tfrac{407}{8} - \tfrac{209}{24} \operatorname{cof.} 2\theta + \tfrac{719}{960} \operatorname{cof.} 4\theta,$$

$$s = \tfrac{355}{48} \operatorname{fin.} 2\theta + \tfrac{49}{480} \operatorname{fin.} 4\theta.$$

§. 31. Nimis longum foret huiusmodi calculos pro ordinibus ſuperioribus exſequi, praecipue cum motus Planetarum primariorum iam aliunde ſatis ſit cognitus, atque hic inſtitutum noſtrum in eo tantum verſetur, vt ſpecimen huius nouae methodi tradamus. Omiſſis igitur ſequentibus ordinibus pro binis noſtris incognitis x et y ſequentes conſecuti ſumus valores:

$$x = e \operatorname{cof} \theta + ee\left(\tfrac{1}{2} - \tfrac{1}{2} \operatorname{cof.} 2\theta\right) - \tfrac{3}{8} e^3 \operatorname{cof.} 3\theta$$
$$+ e^4\left(\tfrac{407}{8} - \tfrac{209}{24} \operatorname{cof.} 2\theta + \tfrac{719}{960} \operatorname{cof.} 4\theta\right)$$

$$y = -2e \operatorname{fin.} \theta + \tfrac{1}{4} ee \operatorname{fin.} 2\theta + e^3\left(\tfrac{9}{8} \operatorname{fin.} \theta - \tfrac{7}{24} \operatorname{fin.} 3\theta\right)$$
$$+ e^4\left(\tfrac{355}{48} \operatorname{fin.} 2\theta + \tfrac{49}{480} \operatorname{fin.} 4\theta\right),$$

§. 32. Inuentis autem his valoribus x et y innoteſcet angulus M S P, qui vocatur aequatio centri Planetae, quae ſi ponatur $= \omega$, erit longitudo Planetae vera

$$= \Upsilon S P = \zeta + \omega;$$

vbi ζ denotat longitudinem Planetae mediam. Pro aequatioquatione autem centri ω, ob

$$Sq = a(1+x) \text{ et } Pq = ay, \text{ erit tang. } \omega = \frac{y}{1+x};$$

denique ex ipſa hac aequatione ω colligetur diſtantia Planetae a Solo

$$SP = \frac{a(1+x)}{\cos.\ \omega} = a(1+x) \text{ ſec. } \omega,$$

vbi a deſignat diſtantiam Planetae mediam a Sole.

§. 32. Pro Planetis quidem primariis neutiquam conſultum foret eiusmodi tabulas conſtruere, quae pro ſingulis anomaliis mediis exhiberent valores litterarum x et y, cum tabularum conſuetarum ope totum negotium multo facilius expediri queat. Verum ſi perturbationes, quas Planetae ob actionem mutuam ſibi inferunt, inueſtigare voluerimus, tum iſtam methodum, etſi per ſe non parum moleſtam, pari ſucceſſu applicare licebit, dum contra aliae methodi ob ſummam integrationum difficultatem vix in vſum vocari poſſunt. Huius igitur methodi vis potiſſimum in eo conſiſtit, quod in ſingulis ordinibus, quos conſtituimus, negotium integrationis mira facilitate expediri poteſt, id quod in hac diſſertatione inprimis oſtendere mihi erat propoſitum, quo facilius noua theoria mea motuum lunarium diiudicari poſſit, quandoquidem ea tota iſti artificio innititur.

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