DE FORMATIONE
FRACTIONVM CONTINVARVM.

Auctore

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§. 1.

Principium vniuerſale ad fractiones continuas perducens reperitur in ſerie infinita quantitatum A, B, C etc.; quarum ternae ſibi ſuccedentes ſecundum certam legem, ſiue conſtantem ſiue vtcunque variabilem ita a ſe inuicem pendent, vt ſit

$$fA = gB + hC,\ f'B = g'C + h'D,\ f''C = g'' + D + h''E,$$
$$f'''D = g'''E + h''F \text{ etc.}$$

Hinc enim deducuntur ſequentes aequalitates:

$$\frac{fA}{B} = g + \frac{hC}{B} = g + \frac{f'h}{f'B:C}$$
$$\frac{f'B}{C} = g' + \frac{h'D}{C} = g' + \frac{f''h'}{f''C:D}$$
$$\frac{f''C}{D} = g'' + \frac{h''E}{D} = g'' + \frac{f'''h''}{f'''D:E}$$
$$\frac{f'''D}{E} = g''' + \frac{h''F}{E} = g''' + \frac{f''''h'''}{f''''E:F}$$

etc. etc.

Quod si iam posteriores valores in prioribus continuo substituantur, sponte emerget sequens fractio continua:

$$\frac{fA}{B} = g + \cfrac{f^{I} h}{g^{I} + \cfrac{f^{II} h^{I}}{g^{II} + \cfrac{f^{III} h^{II}}{g^{III} + \cfrac{f^{IIII} h^{III}}{g^{IIII} + \text{etc.}}}}}$$

cuius ergo valor per solos duos primos terminos A & B seriei determinatur.

§. 2. Quoties igitur talis progressio quantitatum A, B, C, D, E etc. habetur, cuius lex ita fuerit comparata, vt terni quique eius termini sibi succedentes secundum legem quamcunque a se inuicem pendeant, toties inde deducitur fractio continua, cuius valor assignari potest. Quamobrem si formula quaecunque ita fuerit comparata, vt eius euolutio perducat ad huiusmodi seriem quantitatum A, B, C, D, E, etc. quarum quisque terminus per duos praecedentes determinatur, inde fractiones continuae deriuari poterunt, quod quomodo fiat, commodissime per aliquot exempla ostendi poterit.

I. Euolutio formulae.

$$s = x^{n}(\alpha - \beta x - \gamma x x).$$

§. 3. In hac formula exponens n indefinitus spectatur, successiue recipiens omnes valores 1, 2, 3, 4, 5, 6 etc., vnde, dummodo fuerit $n > 0$, haec formula euanescit, posito $x = 0$, tum vero etiam euanescit, sumto

$x = -$

$$x = -\frac{\beta \pm \sqrt{(\beta\beta + 4\alpha\gamma)}}{2\gamma}.$$

His notatis differentietur ista formula, vt fiat

$$ds = n\alpha x^{n-1} dx - (n+1)\beta x^n dx - (n+2)\gamma x^{n+1} dx,$$

vnde per partes integrando et integrationem tantum indicando fiet

$$n\alpha \int x^{n-1} dx = (n+1)\beta \int x^n dx + (n+2)\gamma \int x^{n+1} dx + s,$$

Hinc, si post quamque integrationem, ita peractam, vt integrale euanescat posito $x = 0$, statuatur

$$x = -\frac{\beta \pm \sqrt{(\beta\beta + 4\alpha\gamma)}}{2\gamma},$$

quippe quo casu fit $s = 0$, erit

$$n\alpha \int x^{n-1} dx = (n+1)\beta \int x^n dx + (n+2)\gamma \int x^{n+1} dx,$$

quae est eiusmodi relatio inter ternas formulas integrales sibi succedentes, qualem desideramus pro formatione fractionis continuae; quandoquidem hae formulae integrales, si loco n successiue scribantur numeri 1, 2, 3, 4, 5, 6 etc. nobis suppeditant quantitates A, B, C, D etc.

§. 4. Scribamus igitur loco n ordine numeros naturales 1, 2, 3, 4, etc. vt prodeant istae relationes:

$$\alpha \int dx = 2\beta \int x\,dx + 3\gamma \int xx\,dx$$
$$2\alpha \int x\,dx = 3\beta \int xx\,dx + 4\gamma \int x^3 dx$$
$$3\alpha \int xx\,dx = 4\beta \int x^3 dx + 3\gamma \int x^4 dx$$
$$4\alpha \int x^3 dx = 5\beta \int x^4 dx + 6\gamma \int x^5 dx$$

etc. etc.

Hinc igitur habebimus

$$A = \int dx = x = -\frac{\beta \pm \sqrt{(\beta\beta + 4\alpha\gamma)}}{2\gamma},$$
$$B = \int x\,dx = \tfrac{1}{2}xx = \tfrac{1}{2}\left(\frac{-\beta \pm \sqrt{(\beta\beta + 4\alpha\gamma)}}{2\gamma}\right)^2,$$
$$C = \int xx\,dx = \tfrac{1}{3}x^3,\quad D = \int x^3 dx = \tfrac{1}{4}x^4$$

etc. etc.

 Tunc

Tunc vero pro literis f, g, h habebuntur isti valores:

$$f = \alpha, \; f' = 2\alpha, \; f'' = 3\alpha, \; f''' = 4\alpha \text{ etc.}$$
$$g = 2\beta, \; g' = 3\beta, \; g'' = 4\beta, \; g''' = 5\beta \text{ etc.}$$
$$h = 3\gamma, \; h' = 4\gamma, \; h'' = 5\gamma, \; h''' = 6\gamma \text{ etc.}$$

ex quibus valoribus resultat sequens fractio continua:

$$\frac{\alpha A}{B} = 2\beta + \cfrac{6\alpha\gamma}{3\beta + \cfrac{12\alpha\gamma}{4\beta + \cfrac{20\alpha\gamma}{5\beta + \cfrac{30\alpha\gamma}{6\beta + \text{etc.}}}}}$$

cuius ergo valor est

$$\frac{4\alpha\gamma}{-\beta + \sqrt{(\beta\beta + 4\alpha\gamma)}} = \beta + \sqrt{(\beta\beta + 4\alpha\gamma)}.$$

§. 5. Quo haec fractio continua concinnior reddatur, loco $\alpha\gamma$ scribamus $\frac{1}{2}\delta$, et prodibit

$$\beta + \sqrt{(\beta\beta + 2\delta)} = 2\beta + \cfrac{3\delta}{3\beta + \cfrac{6\delta}{4\beta + \cfrac{10\delta}{5\beta + \cfrac{15\delta}{6\beta + \text{etc.}}}}}$$

Quoniam autem haec expressio capite truncata videtur, adiecto hoc capite ponamus

$$s = \beta + \cfrac{\delta}{2\beta + \cfrac{3\delta}{3\beta + \cfrac{6\delta}{4\beta + \cfrac{10\delta}{5\beta + \text{etc.}}}}}, \text{ eritque}$$

$s = \beta$

$$s = \beta + \frac{\delta}{\beta + \sqrt{(\beta\beta + 2\delta)}}$$

quae expressio reducitur ad hanc:

$$s = \tfrac{1}{2}\beta + \tfrac{1}{2}\sqrt{(\beta\beta + 2\delta)}.$$

§. 6. Haec autem fractio continua adhuc ad maiorem simplicitatem reduci potest, si loco δ scribamus 2ε, vt sit

$$\tfrac{1}{2}\beta + \tfrac{1}{2}\sqrt{(\beta\beta + 4\varepsilon)} = \beta + \cfrac{2\varepsilon}{2\beta + \cfrac{6\varepsilon}{3\beta + \cfrac{12\varepsilon}{4\beta + \cfrac{20\varepsilon}{5\beta + 20 \text{ etc.}}}}}$$

Quod si iam prima fractio deprimatur per 2, secunda per 3, tertia per 4, quarta per 5 etc. prodibit sequens forma:

$$\tfrac{1}{2}\beta + \tfrac{1}{2}\sqrt{(\beta\beta + 4\varepsilon)} = \beta + \cfrac{\varepsilon}{\beta + \cfrac{\varepsilon}{\beta + \cfrac{\varepsilon}{\beta + \cfrac{\varepsilon}{\beta + \text{ etc.}}}}}$$

quae est simplicissima, cuius summa si tanquam incognita spectetur, ac vocetur $= z$, erit vtique $z = \beta + \frac{\varepsilon}{z}$, ideoque $zz = \beta z + \varepsilon$, vnde fit $z = \frac{\beta + \sqrt{(\beta\beta + 4\varepsilon)}}{2}$, quae est eadem.

§. 7. Verum ista summa simplicissima immediate deduci potest ex ipsa formula initio assumta

$$s = x^{n}(\alpha - \beta x - \gamma x x),$$

quam

quam quoniam nihilo aequalem posuimus, erit vtique $\alpha = \beta x + \gamma x x$, eodemque modo

$$\alpha x = \beta x x + \gamma x^3, \ \alpha x x = \beta x^3 + \gamma x^4, \text{ etc.}$$

ita vt pro ferie A, B, C, D, etc. habeamus hanc fimplicem feriem poteftatum: $1, x, x^2, x^3, x^4$ etc., tum vero omnes literae, f, g, h etc. fiunt α, β, γ etc. vnde oritur ifta fractio continua:

$$\frac{\alpha}{x} = \beta + \cfrac{\alpha\gamma}{\beta + \cfrac{\alpha\gamma}{\beta + \cfrac{\alpha\gamma}{\beta + \text{etc.}}}}$$

vbi eft $\frac{1}{x} = \frac{\beta + \sqrt{(\beta\beta + 4\alpha\gamma)}}{2\alpha}$. Huius ergo fractionis valor eft $\frac{1}{2}\beta + \frac{1}{2}\sqrt{(\beta\beta + 4\alpha\gamma)}$, vt ante, ob $\alpha\gamma = \varepsilon$.

II. Euolutio formulae.

$$s = x^n (a - x).$$

§. 8. Haec igitur formula euanefcit, ponendo $x = a$; hinc autem fit $ds = n a x^{n-1} dx - (n+1) x^n dx$, quae expreffio cum duobus tantum conftet terminis, reducatur ad fractionem, cuius denominator fit $\alpha + \beta x$, ita vt fiat

$$ds = \frac{n a \alpha x^{n-1} dx + (\beta n a - \alpha(n+1)) x^n dx - \beta(n+1) x^{n+1} dx}{\alpha + \beta x}.$$

His igitur membris feorfim integratis fiet

$$s = n a \alpha \int \frac{x^{n-1} dx}{\alpha + \beta x} + (n \beta a - (n+1)\alpha) \int \frac{x^n dx}{\alpha + \beta x} - \beta(n+1) \int \frac{x^{n+1} dx}{\alpha + \beta x}$$

quae

quare ſi poſt ſingulas integrationes ſtatuamus $x = a$, vt fiat $s = 0$, habebimus hanc reductionem:

$$n a \alpha \int \frac{x^{n-1} d x}{\alpha + \beta x} = ((n+1)\alpha - n \beta a) \int \frac{x^n d x}{\alpha + \beta x} + (n+1) \beta \int \frac{x^{n+1} d x}{\alpha + \beta x}.$$

§. 9. Loco n ſubſtituamus nunc ſucceſſiue numeros 1, 2, 3, 4 etc. atque comparatione cum formulis generalibus inſtituta habebimus

$$A = \int \frac{d x}{\alpha + \beta x}, \quad B = \int \frac{x d x}{\alpha + \beta x}, \quad C = \int \frac{x x d x}{\alpha + \beta x} \text{ etc.}$$

vbi quidem poſt integrationem fieri debet $x = a$. Praeterea vero habebimus

$$f = a \alpha, \ f' = 2 a \alpha, \ f'' \, 3 a \alpha, \ f''' \, 4 a \alpha, \text{ etc.}$$

$$g = 2\alpha - \beta a, \ g' = 3\alpha - 2\beta a, \ g'' = 4\alpha - 3\beta a, \text{ etc.}$$

$$h = 2\beta, \ h' = 3\beta, \ h'' = 4\beta, \ h''' = 5\beta, \text{ etc.}$$

atque ex his oritur ſequens fractio continua:

$$\frac{\alpha a A}{B} = (2\alpha - \beta a) + \cfrac{4 a \alpha \beta}{(3\alpha - 2\beta a) + \cfrac{9 a \alpha \beta}{(4\alpha - 3\beta a) + \cfrac{16 a \alpha \beta}{(5\alpha - 4\beta a) + \text{etc.}}}}$$

§. 10. Integratione autem inſtituta fit

$$\int \frac{d x}{\alpha + \beta x} = \frac{1}{\beta} l \frac{\alpha + \beta x}{\alpha},$$

quandoquidem integralia euaneſcere debent facto $x = 0$. Nunc igitur fiat $x = a$, eritque $A = \frac{1}{\beta} l \frac{\alpha + \beta a}{\alpha}$. Porro

$$\int \frac{x d x}{\alpha + \beta x} = \frac{1}{\beta} \left(x - \frac{\alpha}{\beta} l \frac{\alpha + \beta x}{\alpha}\right), \text{ factoque } x = a \text{ fiet}$$

$$B = \frac{a}{\beta} - \frac{\alpha}{\beta \beta} l \frac{\alpha + \beta a}{\alpha},$$

quamobrem valor noſtrae fractionis continuae erit

$$\frac{\alpha\, a\, \beta\, l\, \frac{\alpha + \beta\, a}{\alpha}}{a\, \beta - \alpha\, l\, \frac{\alpha + \beta\, a}{\alpha}}$$

euidens autem eſt, nihil de vniuerſalitate perire, etiamſi ſumatur $a = 1$; tum enim erit

$$\frac{\alpha\, \beta\, l\, \frac{\alpha + \beta}{\alpha}}{\beta - \alpha\, l\, \frac{\alpha + \beta}{\alpha}} = (2\alpha - \beta) + \cfrac{4\alpha\beta}{(3\alpha - 2\beta) + \cfrac{9\alpha\beta}{(4\alpha - 3\beta) + \text{etc.}}}$$

§. 11. Tota autem haec expreſſio manifeſto vnice pendet a ratione numerorum α et β; vnde ſumamus $\alpha = 1$ et $\beta = n$, atque orietur haec fractio continua:

$$\frac{n\, l(1+n)}{n - l(1+n)} = (2 - n) + \cfrac{4n}{(3 - 2n) + \cfrac{9n}{(4 - 3n) + \cfrac{16n}{(5 - 4n) + \text{etc.}}}}$$

cui ſi praefigamus ſecundum ordinis legem $1 + n$ et ſummam ſtatuamus $= s$, vt ſit

$$s = 1 + \cfrac{n}{(2 - n) + \cfrac{4n}{(3 - 2n) + \cfrac{9n}{(4 - 3n) + \cfrac{16n}{(5 - 4n) + \text{etc.}}}}}$$

erit

$$s = 1 + \frac{n(n - l(1+n))}{n\, l(1+n)} = \frac{1 + n - l(1+n)}{l(1+n)} = \frac{n}{l(1+n)}.$$

§. 12.

§. 12. Exempla aliquot percurramus, sitque primo $n = 1$, erit

$$\frac{1}{l\,2} = 1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \text{etc.}}}}}$$

Posito autem $n = 2$ erit

$$\frac{2}{l\,3} = 1 + \cfrac{2}{0 + \cfrac{8}{-1 + \cfrac{18}{-2 + \cfrac{32}{-3 + \cfrac{50}{-4 + \text{etc.}}}}}}$$

quae autem expressio, ob quantitates negatiuas, non satis est commoda; quod cum eueniat quando $n > 1$, operae pretium erit eos casus euoluere, quibus n vnitate minor accipitur.

§. 13. Quo hoc facilius fieri possit, reuertamur ad expressionem literas α et β continentes, atque capite, quod deerat suppleto, prodit ista forma:

$$\frac{\beta}{l\,\frac{\alpha+\beta}{\alpha}} = \alpha + \cfrac{\alpha\beta}{(2\alpha-\beta) + \cfrac{4\alpha\beta}{(3\alpha-2\beta) + \cfrac{9\alpha\beta}{(4\alpha-3\beta) + \text{etc.}}}}$$

Ponamus nunc $n = n - m$ et $\beta = 2m$, vt obtineamus se-

 quen-

quentem formam:

$$\frac{2m}{l\frac{n+m}{n-m}} = n-m+\cfrac{2m(n-m)}{2n-4m+\cfrac{8m(n-m)}{3n-7m+\cfrac{18m(n-m)}{4n-10m+\text{etc.}}}}$$

vnde sequentes casus speciales deducuntur.
Si $m=1$ et $n=3$ erit

$$\frac{2}{l\,2} = 2+\cfrac{4}{2+\cfrac{16}{2+\cfrac{36}{2+\cfrac{64}{2+\text{etc.}}}}}$$

quae fractio per 2 diuisa et reducta praebet istam:

$$\frac{1}{l\,2} = 1+\cfrac{1}{1+\cfrac{4}{1+\cfrac{9}{1+\cfrac{16}{1+\text{etc.}}}}}$$

quae iam supra est inuenta.
Sit $m=1$ et $n=4$ erit

$$\frac{2}{l\frac{5}{3}} = 3+\cfrac{6}{4+\cfrac{24}{5+\cfrac{54}{6+\cfrac{96}{7+\text{etc.}}}}}$$

$$= 3+\cfrac{6.1}{4+\cfrac{6.4}{5+\cfrac{6.9}{6+\cfrac{6.16}{7+\text{etc.}}}}}$$

Sit

Sit $m = 1$ et $n = 5$, erit

$$\frac{2}{l\frac{3}{2}} = 4 + \cfrac{8}{6 + \cfrac{32}{8 + \cfrac{72}{10 + \cfrac{128}{12 + \text{etc.}}}}}$$

siue

$$\frac{1}{l\frac{3}{2}} = 2 + \cfrac{2}{3 + \cfrac{8}{4 + \cfrac{18}{5 + \cfrac{32}{6 + \text{etc.}}}}}$$

$$= 2 + \cfrac{2.1}{3 + \cfrac{2.4}{4 + \cfrac{2.9}{5 + \cfrac{2.16}{6 + \text{etc.}}}}}$$

III. Euolutio formulae.

$$s = x^n (1 - x^2)$$

§. 14. Haec ergo formula euanescit casibus $x = 0$ et $x = 1$. Quoniam vero hinc fit

$$ds = n x^{n-1} dx - (n+2) x^{n+1} dx,$$

reducatur hoc differentiale ad denominatorem $\alpha + \beta x x$, fietque

$$ds = \frac{n \alpha x^{n-1} dx + (n\beta - (n+2)\alpha) x^{n+1} dx - (n+2) \beta x^{n+3} dx.}{\alpha + \beta x x}$$

Hinc

Hinc iam iterum integrando fit

$$s = n\alpha\int\frac{x^{n-1}dx}{\alpha+\beta xx} + (n\beta-(n+2)\alpha)\int\frac{x^{n+1}dx}{\alpha+\beta xx} - (n+2)\beta\int\frac{x^{n+3}dx}{\alpha+\beta xx}.$$

Quod si iam post integrationes statuatur $x = 1$, prodibit haec integralium reductio:

$$n\alpha\int\frac{x^{n-1}dx}{\alpha+\beta xx} = ((n+2)\alpha - n\beta)\int\frac{x^{n+1}dx}{\alpha+\beta xx} + (n+2)\beta\int\frac{x^{n+3}dx}{\alpha+\beta xx}.$$

§. 15. Quoniam hic potestates ipsius x binario augentur, exponenti n successiue tribuamus valores 1, 3, 5, 7, 9 etc. ac statuatur:

$$A = \int\frac{dx}{\alpha+\beta xx},\quad B = \int\frac{xx\,dx}{\alpha+\beta xx},\quad C = \int\frac{x^4\,dx}{\alpha+\beta xx}\ \text{etc.}$$

Deinde vero literae f, g, h cum suis deriuatis erunt:

$$f = \alpha,\ f' = 3\alpha,\ f'' = 5\alpha,\ f''' = 7\alpha,\ \text{etc.}$$
$$g = 3\alpha-\beta,\ g' = 5\alpha-3\beta,\ g'' = 7\alpha-5\beta,\ \text{etc.}$$
$$h = 3\beta,\ h' = 5\beta,\ h'' = 7\beta,\ h''' = 9\beta,\ \text{etc.}$$

vnde nascitur sequens fractio continua:

$$\frac{\alpha A}{B} = 3\alpha-\beta + \cfrac{9\alpha\beta}{5\alpha-3\beta + \cfrac{25\alpha\beta}{7\alpha-5\beta + \cfrac{49\alpha\beta}{9\alpha-7\beta + \text{etc.}}}}$$

§. 16. Quia est $B = \int\frac{xx\,dx}{\alpha+\beta xx}$, erit

$$B = \frac{1}{\beta}\int dx - \frac{\alpha}{\beta}\int\frac{dx}{\alpha+\beta xx},\ \text{ideoque}\ B = \frac{1}{\beta} - \frac{\alpha}{\beta}A,$$

quo valore substituto habebimus

$\alpha\beta A$

$$\frac{\alpha\beta A}{1-\alpha A} = 3\alpha - \beta + \cfrac{9\alpha\beta}{5\alpha - 3\beta + \cfrac{25\alpha\beta}{7\alpha - 5\beta + \text{etc.}}}$$

cui, quia caput deeſt, praefigamus $\alpha + \beta + \alpha\beta$; tum autem erit ſumma $\beta + \frac{1}{A}$, ita vt habeamus

$$\beta + \frac{1}{A} = \alpha + \beta + \cfrac{\alpha\beta}{3\alpha - \beta + \cfrac{9\alpha\beta}{5\alpha - 3\beta + \cfrac{25\alpha\beta}{7\alpha - 5\beta + \text{etc.}}}}$$

exiſtente $A = \int \frac{dx}{\alpha + \beta xx}$, integrali ita ſumto, vt euaneſcat poſito $x = 0$, tum vero facto $x = 1$.

§. 17. Euoluamus primo caſum ſimpliciſſimum, quo $\alpha = 1$ et $\beta = 1$, vbi erit $A = \frac{\pi}{4}$, vnde habebimus

$$1 + \frac{4}{\pi} = 2 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \text{etc.}}}}}$$

ſiue erit

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \text{etc.}}}}$$

quae eſt ipſa fractio continua olim a *Brounkero* primum producta, cuius inueſtigatio, cum a *Walliſio* per calculos valde taedioſos ſit eruta, hic quaſi ſponte ex noſtra formula ſeſe prodidit.

§. 18.

§. 18. Noſtra autem forma generalis infinitas alias ſimiles expreſſiones ſuppeditat, prouti literae α et β vario modo accipiuntur. Ac primo quidem, ſi α et β fuerint numeri poſitiui, valor literae A ſemper per arcum circularem exprimetur, contra vero per logarithmos. Sit igitur primo $\beta = 1$, eritque

$$A = \int \frac{dx}{\alpha + xx} = \frac{1}{\sqrt{\alpha}} \text{ A tang. } \frac{x}{\sqrt{\alpha}} = \frac{1}{\sqrt{\alpha}} \text{ A tang. } \frac{1}{\sqrt{\alpha}},$$

vnde naſcitur haec fractio continua:

$$1 + \frac{\sqrt{\alpha}}{\text{A tang. } \frac{1}{\sqrt{\alpha}}} = \alpha + 1 + \cfrac{\alpha}{3\alpha - 1 + \cfrac{9\alpha}{5\alpha - 3 + \cfrac{25\alpha}{7\alpha - 5 + \text{etc.}}}}$$

Hinc igitur ſi ſumatur $\alpha = 3$, quia A tang. $\frac{1}{\sqrt{3}} = \frac{\pi}{6}$, habebimus

$$1 + \frac{6\sqrt{3}}{\pi} = 4 + \cfrac{3}{8 + \cfrac{27}{12 + \cfrac{75}{16 + \cfrac{147}{20 + \text{etc.}}}}}$$

ſiue

$$1 + \frac{6\sqrt{3}}{\pi} = 4 + \cfrac{3 \cdot 1}{8 + \cfrac{3 \cdot 9}{12 + \cfrac{3 \cdot 25}{16 + \cfrac{3 \cdot 49}{20 + \text{etc.}}}}}$$

§. 19. Sit nunc B numerus positiuus quicunque, et quia est

$$A = \int \frac{dx}{\alpha + \beta xx} = \frac{1}{\beta} \int \frac{dx}{\frac{\alpha}{\beta} + xx},$$

integrando fit $A = \frac{1}{\sqrt{\alpha\beta}} A \text{ tang. } \sqrt{\frac{\beta}{\alpha}}$. Hinc igitur habebimus

$$\beta + \frac{\sqrt{\alpha\beta}}{A \text{ tang. } \sqrt{\frac{\beta}{\alpha}}} = \alpha + \beta + \cfrac{\alpha\beta}{3\alpha - \beta + \cfrac{9\alpha\beta}{5\alpha - 3\beta + \text{etc.}}}$$

Faciamus igitur $\alpha + \beta = 2n$ et $\alpha - \beta = 2m$, vt fit $\alpha = m + n$ et $\beta = n - m$, quibus valoribus positis erit

$$n - m + \frac{\sqrt{(nn - mm)}}{A \text{ tang. } \sqrt{\frac{n-m}{n+m}}} = 2n + \cfrac{nn - mm}{2n + 4m + \cfrac{9(nn - mm)}{2n + 8m + \text{etc.}}}$$

§. 20. Consideremus etiam casum, quo β est numerus negatiuus, et ponendo $\beta = -\gamma$, erit

$$A = \int \frac{dx}{\alpha - \gamma xx} = \frac{1}{\gamma} \int \frac{dx}{\frac{\alpha}{\gamma} - xx},$$

cuius integrale est

$$A = \frac{1}{2\sqrt{\alpha\gamma}}\, l \frac{\sqrt{\frac{\alpha}{\gamma}} + x}{\sqrt{\frac{\alpha}{\gamma}} - x};$$

facto ergo $x = 1$ erit

$$A = \frac{1}{2\sqrt{\alpha\gamma}}\, l \frac{\sqrt{\alpha} + \sqrt{\gamma}}{\sqrt{\alpha} - \sqrt{\gamma}},$$

 vnde

vnde nafcitur ifta fractio continua:

$$-\gamma+\frac{2\sqrt{\alpha\gamma}}{l\frac{\sqrt{\alpha}+\sqrt{\gamma}}{\sqrt{\alpha}-\sqrt{\gamma}}}=\alpha-\gamma-\cfrac{\alpha\gamma}{3\alpha+\gamma-\cfrac{9\alpha\gamma}{5\alpha+3\gamma-\cfrac{25\alpha\gamma}{7\alpha+5\gamma-\text{etc.}}}}$$

hocque modo nacti fumus nouas fractiones continuas, quarum valores etiam per logarithmos exhibere licet, et quae prorfus difcrepant ab illis, quas ante inuenimus.

§. 21. Hic cafus prae reliquis notatu dignus fe offert, quando $\gamma=\alpha$. Siue, quod eodem redit, $\alpha=1$ et $\gamma=1$; quia enim tum eft $l\frac{\sqrt{\alpha}+\sqrt{\gamma}}{\sqrt{\alpha}-\sqrt{\gamma}}=l\infty=\infty$, habebimus

$$-1=0-\cfrac{1}{4-\cfrac{9}{8-\cfrac{25}{12-\text{etc.}}}}$$

fiue mutatis fignis

$$1=\cfrac{1}{4-\cfrac{9}{8-\cfrac{25}{12-\text{etc.}}}}$$

hinc primus denominator

$$4-\cfrac{9}{8-\cfrac{25}{12-\text{etc.}}}$$ debet effe $=1$.

Erit ergo $0=3-\cfrac{9}{8-\cfrac{25}{12-\text{etc.}}}$

fiue

siue $1 = \cfrac{3}{8 - \cfrac{25}{12 - \text{etc.}}}$

vbi denominator debet esse $= 3$, vnde fit

$$0 = 5 - \cfrac{25}{12 - \text{etc.}}$$

cuius denominator debet esse $= 5$, vnde fit

$$0 = 7 - \cfrac{49}{16 - \cfrac{81}{20 - \text{etc.}}}$$

ex quo ordine facile veritas perspicitur.

§. 22. Sumamus $\alpha = 4$ et $\gamma = 1$ et nanciscemur hanc fractionem:

$$-1 + \frac{4}{l\,3} = 3 - \cfrac{4.1}{13 - \cfrac{4.9}{23 - \cfrac{4.25}{33 - \cfrac{4.49}{43 - \text{etc.}}}}}$$

Sin autem accipiamus $\alpha = 9$ et $\gamma = 1$ erit

$$-1 + \frac{6}{l\,2} = 8 - \cfrac{9.1}{28 - \cfrac{9.9}{48 - \cfrac{9.25}{68 - \cfrac{9.49}{88 - \text{etc.}}}}}$$

IV. Euolutio formulae.

$$s = x^n e^{\alpha x} (1 - x)$$

§. 22. Hic e denotat numerum cuius logarithmus hyperbolicus est vnitas, ita vt $d. e^{\alpha x} = \alpha\, dx\, e^{\alpha x}$. Hinc ergo erit

$$ds = n x^{n-1} dx\, e^{\alpha x} + (\alpha - (n+1)) x^n dx\, e^{\alpha x} - \alpha x^{n+1} dx\, e^{\alpha x},$$

vnde viciſſim integrando fit

$$s = n \int x^{n-1} dx\, e^{\alpha x} + (\alpha - (n+1)) \int x^n dx\, e^{\alpha x} - \alpha \int x^{n+1} dx\, e^{\alpha x}.$$

Quod ſi ergo poſt integrationem ſtatuatur $x = 1$, erit

$$n \int x^{n-1} dx\, e^{\alpha x} = (n + 1 - \alpha) \int x^n dx\, e^{\alpha x} + \alpha \int x^{n+1} dx\, e^{\alpha x}.$$

§. 23. Quodſi iam loco n ſucceſſiue ſcribamus numeros 1, 2, 3, 4, ac faciamus

$$A \int e^{\alpha x} dx = \tfrac{1}{\alpha}(e^{\alpha} - 1) \text{ et } B = \int x\, dx\, e^{\alpha x} = \tfrac{\alpha - 1}{\alpha \alpha} e^{\alpha} + \tfrac{1}{\alpha \alpha}$$

$$f = 1,\ f' = 2,\ f'' = 3,\ f''' = 4, \text{ etc.}$$

$$g = 2 - \alpha,\ g' = 3 - \alpha,\ g'' = 4 - \alpha, \text{ etc.}$$

$$h = \alpha,\ h' = \alpha,\ h'' = \alpha,\ h''' = \alpha, \text{ etc.}$$

prodibit iſta fractio continua:

$$\frac{A}{B} = 2 - \alpha + \cfrac{2\alpha}{3 - \alpha + \cfrac{3\alpha}{4 - \alpha + \cfrac{4\alpha}{5 - \alpha + \text{etc.}}}}$$

Adiungamus adhuc ſuperne $1 - \alpha + \alpha$, erit eius valor

$$1 - \alpha + \frac{(\alpha - 1) e^{\alpha} + 1}{e^{\alpha} - 1} = \frac{\alpha}{e^{\alpha} - 1},$$

vnde

vnde habebitur haec fractio continua satis concinna:

$$\frac{\alpha}{e^{\alpha}-1} = 1 - \alpha + \cfrac{\alpha}{2 - \alpha + \cfrac{2\alpha}{3 - \alpha + \cfrac{3\alpha}{4 - \alpha + \text{etc.}}}}$$

vnde patet, si fuerit $\alpha = 0$, ob $e^{\alpha} - 1 = \alpha$, fore vtique $1 = 1$.

§. 24. Consideremus nonnullos casus speciales; ac primo, si sit $\alpha = 1$, erit

$$\frac{1}{e-1} = 0 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \text{etc.}}}}}$$

quae fractio facile transfunditur in hanc:

$$\frac{1}{e-1} = \cfrac{1}{1 + \cfrac{\frac{1}{1}}{1 + \cfrac{\frac{1}{2}}{1 + \cfrac{\frac{1}{3}}{1 + \cfrac{\frac{1}{4}}{1 + \text{etc.}}}}}}$$

vnde fit

$$e - 1 = 1 + \cfrac{\frac{1}{1}}{1 + \cfrac{\frac{1}{2}}{1 + \cfrac{\frac{1}{3}}{1 + \text{etc.}}}}$$

 Haec

Haec autem porro a fractionibus partialibus liberata dat

$$e - 1 = + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \text{etc.}}}}}}$$

vnde sequitur

$$\frac{1}{e - 2} = 1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{4 + \text{etc.}}}}}$$

quae formae ob simplicitatem maxime sunt notatu dignae. Ex penultima, qua fit

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \text{etc.}}}}}$$

sumendo successiue 1, 2, 3, pluraue membra, orientur sequentes approximationes:

$$e = 2,0000$$
$$e = 3.0000$$
$$e = 2,6666$$
$$e = 2,7272$$
$$e = 2,7169$$

qui valores, alternatim maiores et minores, satis prompte ad veritatem conuergunt.

§. 25.

§. 25. Sumamus $\alpha = 2$ erit

$$\frac{2}{ee-1} = -1 + \cfrac{2}{0 + \cfrac{4}{1 + \cfrac{6}{2 + \cfrac{8}{3 + \text{etc.}}}}}$$

Ex hac fractione porro deducitur ista:

$$\frac{2(ee-1)}{ee+1} = 0 + \cfrac{4}{1 + \cfrac{6}{2 + \cfrac{8}{3 + \text{etc.}}}}$$

similique modo, si pro α maiores numeri accipiantur, reductio fieri poterit.

§. 26. Possunt etiam pro α numeri negativi accipi. Ita si fuerit $\alpha = -1$ fiet

$$\frac{e}{e-1} = 2 - \cfrac{1}{3 - \cfrac{2}{4 - \cfrac{3}{5 - \cfrac{4}{6 - \text{etc.}}}}}$$

quae reducitur ad hanc formam:

$$\frac{e}{e-1} = 2 + \cfrac{1}{-3 + \cfrac{2}{4 + \cfrac{3}{-5 + \cfrac{4}{6 + \text{etc.}}}}}$$

similique modo maiores valores expediri possunt.

§. 27. Statuamus etiam $\alpha = \frac{1}{2}$, ac reperietur ista expressio:

$$\frac{1}{2(\sqrt{e}-1)} = \frac{1}{2} + \cfrac{\frac{1}{2}}{\frac{3}{2} + \cfrac{1}{\frac{5}{2} + \cfrac{\frac{3}{2}}{\frac{7}{2} + \cfrac{\frac{5}{2}}{\frac{9}{2} + \text{etc.}}}}}$$

quae liberata a fractionibus partialibus euadit

$$\frac{1}{-1+\sqrt{e}} = 1 + \cfrac{2}{3 + \cfrac{4}{5 + \cfrac{6}{7 + \cfrac{8}{9 + \text{etc.}}}}}$$

Simili modo si sumus $\alpha = \frac{1}{3}$ erit

$$\frac{1}{3(\sqrt[3]{e}-1)} = 2:3 + \cfrac{1:3}{5:3 + \cfrac{2:3}{8:3 + \cfrac{3:3}{11:3 + \cfrac{4:3}{14:3 + \text{etc.}}}}}$$

quae a fractionibus partialibus liberata dat

$$\frac{1}{-1+\sqrt[3]{e}} = 2 + \cfrac{3}{5 + \cfrac{6}{8 + \cfrac{9}{11 + \cfrac{12}{14 + \text{etc.}}}}}$$

At

At fi ponatur $\alpha = \frac{2}{3}$, prodit haec fractio continua:

$$\frac{2}{3(\sqrt[3]{(ee-1)})} = 1:3 + \cfrac{2:3}{4:3 + \cfrac{4:3}{7:3 + \cfrac{6:3}{10:3 + \cfrac{8:3}{13:3 + \text{etc.}}}}}$$

quae a fractionibus partialibus liberata fit

$$\frac{2}{\sqrt[3]{(ee-1)}} = 1 + \cfrac{6}{4 + \cfrac{12}{7 + \cfrac{18}{10 + \cfrac{24}{13 + \text{etc.}}}}}$$

§. 28. His formulis tanquam principalibus ac fimplicioribus euolutis, fimili modo alias multo generaliores tractare licebit, quae ad fractiones continuas multo magis abfconditas perducent, vti ex cafibus qui fequuntur patebit.

V. Euolutio formulae.

$$s = x^n (a - b x^\theta - c x^{2\theta})^\lambda.$$

§. 29. Hinc igitur erit

$$ds = (a - b x^\theta - c x^{2\theta})^{\lambda - 1} (n a x^{n-1} dx - b(n + \lambda\theta) x^{n+\theta-1} dx$$
$$- c(n + 2\lambda\theta) x^{n+2\theta-1} dx),$$

vnde per partes integrando, tum vero ftatuendo $a - b x^\theta - c x^{2\theta} = 0$, (quod fit fi fuerit $x^\theta = -\frac{b \pm \sqrt{(bb + 4ac)}}{2c}$) habebitur ifta reductio generalis:

$$n a \int x^{n-1} dx (a - b x^{\theta} - c x^{2\theta})^{\lambda-1}$$
$$= (n + \lambda \theta) b \int x^{n+\theta-1} dx (a - b x^{\theta} - c x^{2\theta})^{\lambda-1}$$
$$+ (n + 2 \lambda \theta) c \int x^{n+2\theta-1} dx (a - b x^{\theta} - c x^{2\theta})^{\lambda-1}.$$

§. 30. Quodſi iam hanc formam cum noſtra generali initio tradita comparare velimus, valores pro litera n ſucceſſiue aſſumendi per differentiam θ augeri debent. Deinde non neceſſe eſt vt primus valor ipſius n, vt hactenus fecimus, ſumatur $= 1$; ſtatuamus igitur eius primum valorem $= \alpha$, et quaeramus valores binarum ſequentium formularum integralium, ſcilicet:

$$A = \int x^{\alpha-1} dx (a - b x^{\theta} - c x^{2\theta})^{\lambda-1} \text{ et}$$
$$B = \int x^{\alpha+\theta-1} dx (a - b x^{\theta} - c x^{2\theta})^{\lambda-1},$$

quae integralia ita ſunt capienda, vt euaneſcant poſito $x = 0$, quo facto ipſi x ille valor tribui debet, qui reddat formulam $a - b x^{\theta} - c x^{2\theta} = 0$. Quoniam autem hoc in genere exſequi non licet, iſtos valores per literas A et B indicare contenti ſimus, quos ergo tanquam cognitos ſpectemus.

§. 31. Praeterea vero literae f, g, h, cum ſuis deriuatis ſequentes induent valores:

$$f = \alpha a,\ f' = (\alpha + \theta) a,\ f'' = (\alpha + 2\theta) a,\ f''' = (\alpha + 3\theta) a, \text{ etc.}$$
$$g = (\alpha + \lambda\theta) b,\ g' = (\alpha + \theta + \lambda\theta) b,\ g'' = (\alpha + 2\theta + \lambda\theta) b, \text{ etc.}$$
$$h = (\alpha + 2\lambda\theta) c,\ h' = (\alpha + \theta + 2\lambda\theta) c,\ h'' = (\alpha + 2\theta + 2\lambda\theta) c; \text{ etc.}$$

Ex his igitur formabitur ſequens fractio continua:

$$\frac{\alpha a A}{B} = (\alpha + \lambda\theta) b + \cfrac{(\alpha + \theta)(\alpha + \lambda\theta) a c}{(\alpha + \theta + \lambda\theta) b + \cfrac{(\alpha + 2\theta)(\alpha + \theta + 2\lambda\theta) a c}{(\alpha + 2\theta + \lambda\theta) b + \cfrac{(\alpha + 3\theta)(\alpha + 2\theta + 2\lambda\theta) a c}{(\alpha + 3\theta + \lambda\theta) b \text{ etc.}}}}$$

quae

quae forma vtique est maxime generalis, cuius autem vlteriori euolutioni non immoramur.

VI. Euolutio formulae.

$$s = x^n (1 - x^\theta)^\lambda$$

§. 32. Hinc ergo fit

$$ds = n x^{n-1} dx (1-x^\theta)^\lambda - \lambda \theta x^{n+\theta-1} dx (1-x^\theta)^{\lambda-1},$$

vnde tantum duae formulae integrales orirentur; quam obrem huic differentiali denominatorem arbitrarium tribuamus $a + b x^\theta$, vt habeamus:

$$ds = \frac{(1-x^\theta)^{\lambda-1}}{a+bx^\theta} (n a x^{n-1} dx - (a(n+\lambda\theta) - bn) x^{n+\theta-1} dx$$
$$- b(n+\lambda\theta) x^{n+2\theta-1} dx.$$

Nunc igitur, ponendo post integrationem $x = 1$, deducimus hanc reductionem:

$$na\int \frac{x^{n-1} dx (1-x^\theta)^{\lambda-1}}{a+bx^\theta} = (a(n+\lambda\theta) - bn) \int \frac{x^{n+\theta-1} dx (1-x^\theta)^{\lambda-1}}{a+bx^\theta}$$
$$+ b(n+\lambda\theta) \int \frac{x^{n+2\theta-1} dx (1-x^\theta)^{\lambda-1}}{a+bx^\theta}.$$

§. 33. Hic iterum euidens est valores ipsius n per differentiam θ crescere debere. Statuatur autem primus valor ipsius $n = \alpha$, et quaerantur pro quouis casu oblato binae sequentes formulae integrales:

$$A = \int \frac{x^{\alpha-1} dx (1-x^\theta)^{\lambda-1}}{a+bx^\theta} \quad \text{et} \quad B = \int \frac{x^{\alpha+\theta-1} dx (1-x^\theta)^{\lambda-1}}{a+bx^\theta},$$

vbi scilicet post integrationem positum sit $x = 1$. Quibus

bus inuentis, cum hinc fiat

$$f = \alpha a,\ f' = (\alpha + \theta) a,\ f'' = (\alpha + 2\theta) a,\ f''' = (\alpha + 3\theta) a, \text{ etc.}$$

$$g = (\alpha + \lambda\theta) a - \alpha b,\ g' - (\alpha + \theta + \lambda\theta) a - (\alpha + \theta) b,$$

$$g'' = (\alpha + 2\theta + \lambda\theta) a - (\alpha + 2\theta) b, \text{ etc.}$$

$$h = (\alpha + \lambda\theta) b,\ h = (\alpha + \theta + \lambda\theta) b,\ h'' = (\alpha + \theta + 2\lambda\theta) b, \text{ etc.}$$

inde formabitur ſequens fractio continua:

$$\frac{\alpha a A}{B} = (\alpha+\lambda\theta)a - \alpha b + \cfrac{(\alpha+\theta)(\alpha+\lambda\theta)ab}{(\alpha+\theta+\lambda\theta)a - (\alpha+\theta)b + \cfrac{(\alpha+2\theta)(\alpha+\theta+\lambda\theta)ab}{(\alpha+2\theta+\lambda\theta)a - (\alpha+2\theta)b + (\alpha+3\theta)(\alpha+2\theta+\lambda\theta)ab \text{ etc.}}}$$

cuius formae vberiore euolutione ſuperſedemus.

VII. Euolutio formulae.

$$s = x^n (e^{\alpha x} (1 - x)^\lambda)$$

§. 34. Hinc ergo fit

$$ds = (1-x)^{\lambda-1} (n x^{n-1} dx - (n+\lambda-\alpha) x^n dx - \alpha x^n dx),$$

hinc igitur ſi poſt integrationem vbique ſtatuatur $x = 1$, quippe quo caſu fit $s = 0$, habebimus hanc reductionem:

$$n \int x^{n-1} dx\, e^{\alpha x} (1-x)^{\lambda-1} = (n+\lambda-\alpha) \int x^n dx\, e^{\alpha x} (1-x)^{\lambda-1}$$
$$+ \alpha \int x^{n+1} dx\, e^{\alpha x} (1-x)^{\lambda-1}.$$

§. 35. In his ergo formulis exponenti n valores vnitate creſcentes tribui debebunt, tum vero hic minimum eius valorem ſumamus $n = \delta$, atque valores literarum A et B ex his formulis erui oportebit, ponendo poſt integrationem $x = 1$,

$$A = \int x^{\delta-1} dx\, e^{\alpha x} (1-x)^{\lambda-1},\quad B = \int x^\delta dx\, e^{\alpha x} (1-x)^{\lambda-1}$$

deinde vero ob hos valores:

$$f = \delta,$$

$f = \delta$, $f' = \delta + 1$, $f'' = \delta + 2$, $f''' = \delta + 3$, etc.
$g = \delta + \lambda - \alpha$, $g' = \delta + 1 + \lambda - \alpha$, $g'' = \delta + 2 + \lambda - \alpha$, etc.
$h = \alpha$, $h' = \alpha$, $h'' = \alpha$, etc.

ſequitur iſta fractio continua:

$$\frac{\delta A}{B} = \delta + \lambda - \alpha + \cfrac{(\delta+1)\alpha}{\delta+1+\lambda-\alpha+\cfrac{(\delta+2)\alpha}{\delta+2+\lambda-\alpha+\cfrac{(\delta+3)\alpha}{\delta+3+\lambda-\alpha+\text{etc.}}}}$$

Vbi imprimis notari oportet, exponentes λ et δ neceſſario nihilo maiores accipi debere, quia alioquin formula principalis $x^n e^{\alpha x}(1-x)^\lambda$ caſibus $x = 1$ non euaneſceret.

§. 36. Si literis δ et λ tribuatur valor $= 1$, prodibit caſus iam ſupra tractatus; ac ſi his literis numeri integri aſſignentur, eiusmodi fractiones continuae orientur, quas per certas operationes ad priores reducere licebit. Verum ſi his literis δ et λ, vel alterutri, vel vtrique, fractiones aſſignemus, tum formae orientur ad priores prorſus irreductibiles, quarumque valor haud aliter quam per quantitates maxime tranſcendentes exprimere liceat. Velvti ſi fuerit $\delta = \frac{1}{2}$ et $\lambda = \frac{1}{2}$, valor literae A quaeri debebit ex hac formula integrali: $A = \frac{e^{\alpha x}\,dx}{\sqrt{(x - xx)}}$, cuius integratio ad quantitates maxime tranſcendentes perducit, ita vt valor talium fractionum continuarum prodeat maxime abſtruſus.