DE MOTV OSCILLATORIO PENDVLORVM EX FILO TENSO DEPENDENTIVM.

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§. 1.

Concipiamus filum A E F B, in punctis A et B fixum, cui in punctis E et F appensa sint pendula E M et F N, quorum motum definiri oportet, postquam de statu aequilibrii vtcunque fuerint deturbata, cuiusmodi autem motus tanquam minimos spectamus, simulque a gravitate ipsius fili mentem abstrahimus, ita vt tantum massae duorum ponderum M et N in calculum ingrediantur. Tab. I. Fig. 1.

§. 2. Primum igitur consideremus statum aequilibrii, ac ponamus fili portiones A E $= a$, E F $= b$, F B $= c$; anguli autem, quibus hae portiones ad horizontalem A C inclinantur, sint α, β et γ; tum vero longitudinem pendulorum vocemus E M $= m$ et F N $= n$, dum

dum eorum pondera literis M et N exprimuntur. Hinc igitur erit recta horizontalis

$$AC = a \cos.\alpha + b \cos.\beta + c \cos.\gamma$$

recta vero verticalis

$$CB = a \sin.\alpha + b \sin.\beta + c \sin.\gamma.$$

§. 3. Ponamus nunc ſtatum aequilibrii ita eſſe turbatum, vt anguli α, β, γ, incrementa ceperint infinite parva ω, ω' et ω'', et quia illa intervalla A C et B C manent immutata, habebimus has duas aequationes:

$$a\,\omega \sin.\alpha + b\,\omega' \sin.\beta + c\,\omega'' \sin.\gamma = 0, \text{ et}$$

$$a\,\omega \cos.\alpha + b\,\omega' \cos.\beta + c\,\omega'' \cos.\gamma = 0.$$

ſicque ſtatim atque vnicus horum angulorum ω, ω', ω'' fuerit datus, bini reliqui ſimul determinantur.

§. 4. Sumamus nunc ambo pendula E M et F N de ſitu verticali declinare angulis Φ et Φ', itidem quam minimis, ac ductis inde ad verticalem horizontalibus M m et N n, vocemus quaſi-coordinatas

$$Am = x \text{ et } mM = y;\ An = x' \text{ et } nN = y'$$

eritque

$$x = a \sin.(\alpha+\omega) + m \cos.\Phi = a \sin.\alpha + a\,\omega \cos.\alpha + m \text{ et}$$

$$y = a \cos.(\alpha+\omega) + m \sin.\Phi = a \cos.\alpha - a\,\omega \sin.\alpha + m\Phi$$

deinde

$$x' = a \sin.(\alpha+\omega) + b \sin.(\beta+\omega') + n \cos.\Phi'$$
$$= a \sin.\alpha + a\,\omega \cos.\alpha + b \sin.\beta + b\,\omega' \cos.\beta + n$$

$$y' = a \cos.(\alpha+\omega) + b \cos.(\beta+\omega') + n \sin.\Phi'$$
$$= a \cos.\alpha - a\,\omega \sin.\alpha + b \cos.\beta - b\,\omega' \sin.\beta + n\Phi'.$$

§. 5.

§. 5. Quia nunc pondera M et N primum verticaliter deorsum vrgentur, viribus M et N, fila autem EM et FN aequalibus viribus tenduntur, vires verticales se mutuo destruunt. Horizontaliter autem pondera ad axem AB pelluntur, viribus $M\Phi$ et $N\Phi'$, sicque ex motus principiis erit

$$\frac{ddx}{2gdt^2} = 0,\ \frac{ddy}{2gdt^2} = -\Phi,\ \frac{ddx'}{2gdt^2} = 0 \text{ et } \frac{ddy'}{2gdt^2} = -\Phi',$$

vnde restitutis valoribus habebimus has quatuor aequationes:

$$\text{I. } +\frac{a\,d\,d\,\omega\cos.\alpha}{2\,g\,dt^2} = 0.$$

$$\text{II. } -\frac{a\,d\,d\,\omega\sin.\alpha + m\,d\,d\,\Phi}{2\,g\,dt^2} = -\Phi.$$

$$\text{III. } +\frac{a\,d\,d\,\omega\cos.\alpha + b\,d\,d\,\omega'\cos.\beta}{2\,g\,dt^2} = 0.$$

$$\text{IV. } -\frac{a\,d\,d\,\omega\sin.\alpha - b\,d\,d\,\omega'\sin.\beta + n\,d\,d\,\Phi'}{2\,g\,dt^2} = -\Phi'.$$

§. 6. Ex harum aequationum prima et tertia neutiquam concludere licet fore tam $\frac{ddx}{2gdt^2}$ quam $\frac{ddx'}{2gdt^2} = 0$; plus enim inde non sequitur, quam tensionem filorum EM et FN non exacte ponderibus M et N esse aequalem. Si enim ponamus tensionem fili $EM = P$ et fili $FN = Q$, hae aequationes ita se habebunt:

$$\text{I. } \frac{a\,d\,d\,\omega\cos.\alpha}{2\,g\,dt^2} = 1 - \frac{P}{M} \text{ et}$$

$$\text{III. } \frac{a\,d\,d\,\omega\cos.\alpha + b\,d\,d\,\omega'\cos.\beta}{2\,g\,dt^2} = 1 - \frac{Q}{N},$$

quae formulae cum sint quasi infinite paruae, erit vtique proxime $P = M$ et $Q = N$; quibus valoribus in aequatione secunda et quarta sine errore vti licet, quia ibi per quantitates minimas Φ et Φ' sunt multiplicatae.

§. 7. Ad has ergo aequationes resoluendas vtamur methodo iam aliquoties adhibita, dum scilicet angulis variabilibus ω, ω', ω''; Φ et Φ', rationem constantem tribuimus, vt eorum mutationes ad motum penduli simplicis reuocentur, cuius longitudo sit $= r$; motus vero hac aequatione exprimatur: $\frac{dd\,z}{2g\,dt^2} = -\frac{z}{r}$, et iam ponamus

$$\omega = \mathfrak{A}z,\ \omega' = \mathfrak{B}z,\ \omega'' = \mathfrak{C}z,\ \Phi = \mathfrak{M}z \text{ et } \Phi' = \mathfrak{N}z,$$

hincque ergo fiet

$$\frac{dd\,\omega}{2g\,dt^2} = -\frac{\mathfrak{A}z}{r};\quad \frac{dd\,\omega'}{2g\,dt^2} = -\frac{\mathfrak{B}z}{r};$$
$$\frac{dd\,\Phi}{2g\,dt^2} = -\frac{\mathfrak{M}z}{r};\quad \frac{dd\,\Phi'}{2g\,dt^2} = -\frac{\mathfrak{N}z}{r}.$$

§. 8. His igitur valoribus substitutis nanciscimur sequentes aequationes simplices:

$$\text{I} - \frac{\mathfrak{A}.a\,z\cos.\alpha}{r} = 1 - \frac{P}{M},$$
$$\text{II} + \frac{\mathfrak{A}\,a\,z\sin.\alpha - \mathfrak{M}m\,z}{n} = -\mathfrak{M}z,$$
$$\text{III} - \frac{\mathfrak{A}\,a\,z\cos.\alpha - \mathfrak{B}\,b\,z\cos.\beta}{r} = 1 - \frac{Q}{N},$$
$$\text{IV} + \frac{\mathfrak{A}\,a\,z\sin.\alpha + \mathfrak{B}\,b\,z\sin.\beta - \mathfrak{N}\,n\,z}{n} = -\mathfrak{N}z,$$

ex quarum prima et tertia deducimus tensiones P et Q, quae sunt:

$$P = M + \frac{\mathfrak{A}\,a\,z\,M\cos.\alpha}{r};\quad Q = N + \frac{Nz}{r}(\mathfrak{A}\,a\cos.\alpha + \mathfrak{B}\,b\cos.\beta);$$

ex secunda vero et quarta deducimus

$$\mathfrak{M} = \frac{\mathfrak{A}\,a\sin.\beta}{m - n} \text{ et } \mathfrak{N} = \frac{\mathfrak{A}\,a\sin.\alpha + \mathfrak{B}\,b\sin.\beta}{n - r}.$$

§. 9. Hos eosdem valores etiam substituamus in binis formulis, quae supra, ob puncta A et B fixa, sunt datae; vnde duas sequentes nanciscimur aequationes:

$$\mathfrak{A}\, a \text{ sin. } \alpha + \mathfrak{B}\, b \text{ sin. } \beta + \mathfrak{C}\, c \text{ sin. } \gamma = 0 \text{ et}$$

$$\mathfrak{A}\, a \text{ cos. } \alpha + \mathfrak{B}\, b \text{ cos. } \beta + \mathfrak{C}\, c \text{ cos. } \gamma = 0$$

ex quibus, eliminando $\mathfrak{C}$, elicimus

$$\mathfrak{A}\, a \text{ sin. } (\alpha - \gamma) + \mathfrak{B}\, b \text{ sin. } (\beta - \gamma) = 0,$$

hincque porro

$$\mathfrak{B} = + \frac{\mathfrak{A}\, a \text{ sin. } (\alpha - \gamma)}{b \text{ sin. } (\gamma - \beta)} \text{ et } \mathfrak{C} = \frac{\mathfrak{A}\, a \text{ sin. } (\alpha - \beta)}{c \text{ sin. } (\beta - \gamma)},$$

vnde sequentes formamus valores concinniores:

$$\mathfrak{A}\, a = f \text{ sin. } (\beta - \gamma);\ \mathfrak{B}\, b = f \text{ sin. } (\gamma - \alpha) \text{ et } \mathfrak{C}\, c = f \text{ sin. } (\alpha - \beta).$$

§. 10. Quodsi hos valores in praecedentes introducamus, fiet

$$P = M + \frac{M f z}{r} \text{ sin. } (\beta - \gamma) \text{ cos. } \alpha \text{ et}$$

$$Q = N - \frac{N f z}{r} \text{ sin. } (\alpha - \beta) \text{ cos. } \gamma;$$

tum vero

$$\mathfrak{M} = \frac{f \text{ sin. } (\beta - \gamma) \text{ sin. } \alpha}{m - r} \text{ et } \mathfrak{N} = \frac{-f \text{ sin. } (\alpha - \beta) \text{ sin. } \gamma}{n - r}.$$

§. 11. Superest autem, vt etiam motus fili rationem in computum ducamus; in quo cum nullam inertiam admittamus, vires, quibus puncta E et F sollicitantur, se mutuo destruere debent; haec autem puncta, praeter tensionem filorum P et Q, potissimum a tensione fili sollicitantur. Hunc in finem ponamus tensionem fili A E = A, fili E F = B et fili F B = C, quas vires secundum directiones verticalem et horizontalem resoluere oportet. Hinc igitur punctum E horizontaliter sinistrorsum trahetur his viribus:

I. $A \text{ cos. } (\alpha + \omega) - B \text{ cos. } (\beta + \omega') + P \text{ sin. } \Phi = 0,$

verticaliter autem sursum trahitur his viribus:

II.

II. $A \sin.(\alpha + \omega) - B \sin.(\beta + \omega') - P \cos. \Phi = 0.$

Punctum autem F horizontaliter sinistrorsum trahitur his viribus:

III. $B \cos.(\beta + \omega') - C \cos.(\gamma + \omega'') + Q \sin. \Phi' = 0,$

verticaliter autem sursum, his:

IV. $B \sin.(\beta + \omega') - C \sin.(\gamma + \omega'') - Q \cos. \Phi' = 0.$

§. 12. Resoluamus iam istos angulos, simulque loco ω, ω', ω''; Φ et Φ' suos valores assumtos scribamus, et obtinebemus sequentes formas:

I. $A \cos \alpha - B \cos. \beta - A \mathfrak{A} z \sin. \alpha + B \mathfrak{B} z \sin. \beta + P \mathfrak{M} z = 0.$

II. $A \sin. \alpha - B \sin. \beta + A \mathfrak{A} z \cos. \alpha - B \mathfrak{B} z \cos. \beta - P = 0.$

III. $B \cos. \beta - C \cos. \gamma - B \mathfrak{B} z \sin. \beta + C \mathfrak{C} z \sin. \gamma + Q \mathfrak{N} z = 0.$

IV. $B \sin. \beta - C \sin. \gamma + B \mathfrak{B} z \cos. \beta - C \mathfrak{C} z \cos. \gamma - Q = 0.$

§. 13. Quia autem praecipuum negotium in hoc versatur, vt tensiones A, B, C ex calculo exterminemus, potius vtamur formis prioribus, vtpote simplicioribus; ac primo quidem ex prima et secunda literam A definiamus hoc modo:

$$A = \frac{B \cos.(\beta + \omega') - P \Phi}{\cos.(\alpha + \omega)} = \frac{B \sin.(\beta + \omega') - P}{\sin.(\alpha + \omega)},$$

ex quo duplici valore colligimus

$$B = \frac{P(\cos.(\alpha + \omega) + \Phi \sin.(\alpha + \omega))}{\sin.(\alpha - \beta + \omega - \omega')}.$$

Eodem modo ex tertia et quarta eliciamus valores literae C, qui sunt

$$C = \frac{B \cos.(\beta + \omega') + Q \Phi'}{\cos.(\gamma + \omega'')} = \frac{B \sin.(\beta + \omega') - Q}{\sin.(\gamma + \omega'')},$$

vnde deducimus sequentem valorem:

$$B = \frac{Q(\cos.(\gamma + \omega'') + \Phi' \sin.(\gamma + \omega''))}{\sin.(\beta - \gamma + \omega' - \omega'')}.$$

Quod si iam hi duo valores ipsius P inter se aequentur, obtine-

obtinebimus aequationem finalem, ex qua quantitatem r determinari oportebit.

§. 14. Quoniam vero initio angulos α, β, γ, ita assumsimus, vt, dum pondera M et N verticaliter dependent, status aequilibrii resultet, nostrum calculum ad istum statum aequilibrii facillime accommodabimus, si modo angulos ω, ω' et ω'', vna cum angulis Φ et Φ' nullos statuamus, tum vero loco P et Q ipsa pondera M et N scribamus, quo facto bini valores, qui pro B essent proditurri, erunt

$$B = \frac{M \cos. \alpha}{\sin. (\alpha - \beta)} \text{ et } B = \frac{N \cos. \gamma}{\sin. (\beta - \gamma)},$$

quibus aequatis habebimus hanc aequationem pro statu aequilibrii

$$M \cos. \alpha \sin. (\beta - \gamma) = N \cos. \gamma . \sin. (\alpha - \beta)$$

quae reducitur ad hanc formam:

$$M (\sin. (\alpha + \beta - \gamma) - \sin. (\alpha - \beta + \gamma)) = N (\sin. (\alpha - \beta + \gamma) + \sin. (\alpha - \beta - \gamma)).$$

§. 15. Euidens nunc est, ex hac aequatione euoluta eam, quae ex superiori nostra aequatione oriretur, immediate formari posse, si loco angulorum α, β, γ, scribantur $(\alpha + \omega)$, $(\beta + \omega')$, $(\gamma + \omega'')$; deinde vero loco M scribi debebit $P + P \Phi \tang. (\alpha + \omega)$; at vero loco N scribere debebimus $Q + Q \Phi' \tang. (\gamma + \omega'')$. Quia vero supra inuenimus

$$P = M + \frac{M f z}{r} \sin. (\beta - \gamma) \cos. \alpha \text{ et}$$
$$Q = N - \frac{N f z}{r} \sin. (\alpha - \beta) \cos. \gamma,$$

nunc autem est $\Phi = M z$, loco $\tang. (\alpha + \omega)$ scribere sufficit $\tang. \alpha$, ita vt nunc sufficiat loco M scribere formulam

$$M + \frac{M f z}{r} \text{ sin. } (\beta - \gamma) \text{ cos. } \alpha + M \mathfrak{M} z \text{ tang. } \alpha.$$

Eodem modo loco N ſcribi conueniet hanc formulam:

$$N - \frac{N f z}{r} \text{ sin. } (\alpha - \beta) \text{ cos. } \gamma + N \mathfrak{N} z \text{ tang. } \gamma.$$

§. 16. Facillime igitur noſtram aequationem finalem impetrabimus, ſi aequationem ex ſtatu aequilibrii deductam, quae erat

$$M \text{ cos. } \alpha \text{ sin. } (\beta - \gamma) = N \text{ cos. } \gamma \text{ sin. } (\alpha - \beta),$$

differentiemus, omnibus literis M, N, α, β, γ, variabilibus ſumtis, ita ſcilicet vt ſit

$$d\alpha = \omega = \mathfrak{A} z = \frac{f z \text{ sin. } (\beta - \gamma)}{a},$$
$$d\beta = \omega' = \mathfrak{B} z = \frac{f z \text{ sin. } (\gamma - \alpha)}{b},$$
$$d\gamma = \omega'' = \mathfrak{C} z = \frac{f z \text{ sin. } (\alpha - \beta)}{c},$$
$$dM = \frac{M f z}{r} \text{ sin. } (\beta - \gamma) \text{ cos. } \alpha + M \mathfrak{M} z \text{ tang. } \alpha \text{ et}$$
$$dN = -\frac{N f z}{r} \text{ sin. } (\alpha - \beta) \text{ cos. } \gamma + N \mathfrak{N} z \text{ tang. } \gamma.$$

Supra vero iam inuenimus

$$\mathfrak{M} = \frac{f \text{ sin. } (\beta - \gamma) \text{ sin. } \alpha}{m - r} \text{ et } \mathfrak{N} = -\frac{f \text{ sin. } (\alpha - \beta) \text{ sin. } \gamma}{n - r}.$$

Hi igitur valores in aequatione illa differentiata ſubſtitui debent, quae ita ſe habet

$$d M \text{ cos. } \alpha \text{ sin. } (\beta - \gamma) - M d\alpha \text{ sin. } \alpha \text{ sin. } (\beta - \gamma)$$
$$+ M (d\beta - d\gamma) \text{ cos. } \alpha \text{ cos. } (\beta - \gamma) =$$
$$d N \text{ cos. } \gamma \text{ sin. } (\alpha - \beta) - N d\gamma \text{ sin. } \gamma \text{ sin. } (\alpha - \beta)$$
$$+ N (d\alpha - d\beta) \text{ cos. } \gamma \text{ cos. } (\alpha - \beta).$$

Calculus autem nimis fieret prolixus, ſi hos valores actu ſubſtituere vellemus. Sufficiat autem hic obſeruaſſe, quantitatem r ex iſta aequatione determinari poſſe, quandoquidem, facta euolutione, totum negotium ad aequationem cubicam reducetur.

Appli-

Applicatio ad exemplum.

Tab. I. Fig. 2.

§. 17. Sumamus primo puncta A et B, in quibus filum est fixum, in eadem recta horizontali esse sita; deinde sint portiones fili A E, E F, F B, inter se aequales, ita vt sit $b = a$ et $c = a$; tertio sit in statu aequilibrii portio media E F horizontalis, ita vt sit $\beta = 0$; hinc ergo manente angulo $BAE = \alpha$, fiet angulus $\gamma = -\alpha$, vnde sequitur intervallum $AB = 2a \cos.\alpha + a$. His positis aequatio pro statu aequilibrii dabit

$$M \cos.\alpha \sin.\alpha = N \cos.\alpha \sin.\alpha$$

vnde patet, pondera M et N inter se esse debere aequalia; pendulorum autem longitudines m et n maneant adhuc indeterminatae.

§. 18. His ergo constitutis, valores differentialium sequenti modo definientur:

$$d\alpha = \frac{f z \sin.\alpha}{a};\ d\beta = -\frac{f z \sin.\alpha}{a};\ d\gamma = \frac{f z \sin.\alpha}{a},$$

$$dM = \frac{M f z}{r} \sin.\alpha \cos.\alpha + \frac{M f z \sin.\alpha^2 \tang.\alpha}{m - r},$$

$$dN = -\frac{M f z}{r} \sin.\alpha \cos.\alpha - \frac{M f z \sin.\alpha^2 \tang.\alpha}{n - r}.$$

Quod si igitur hi valores in aequatione differentiali supra exhibita substituantur, in singulis terminis occurret factor $M f z$, quem ergo omittamus, vnde ista aequatio sequentem induet formam:

$$\left(\frac{\sin.\alpha \cos.\alpha}{m} + \frac{\sin.\alpha^2 \tang.\alpha}{m - r}\right) \sin.\alpha \cos.\alpha - \frac{\sin.\alpha^3}{a} - \frac{\cos.\alpha^2(\sin.2\alpha + \sin.\alpha)}{a}$$
$$= -\left(\frac{\sin.\alpha \cos.\alpha}{r} + \frac{\sin.\alpha^2 \tang.\alpha}{n - r}\right) \sin.\alpha \cos.\alpha + \frac{\sin.\alpha^3}{a}$$
$$+ \frac{\cos.\alpha^2(\sin.2\alpha + \sin.\alpha)}{a},$$

quae euoluta abit in formam sequentem:

$$\frac{2\,\sin.\,\alpha^2\cos.\,\alpha^2}{r} + \frac{\sin.\,\alpha^4}{m-r} + \frac{\sin.\,\alpha^4}{n-r} - \frac{2\sin.\,2\,\alpha\cos.\,\alpha^2}{a} - \frac{2\sin.\,\alpha\cos.\,\alpha^2}{a} = 0,$$

ex qua aequatione noſtram incognitam r quaeri oportet.

§. 19. Quod ſi iam vtrique pendulo eandem tribuamus longitudinem, vt ſit $n = m$, aequatio noſtra fiet

$$\frac{\sin.\,\alpha^2\cos.\,\alpha^2}{r} + \frac{\sin.\,\alpha^4}{m-r} - \frac{\sin.\,2\,\alpha\cos.\,\alpha^2}{a} - \frac{\sin.\,\alpha\cos.\,\alpha^2}{a} = 0,$$

quae reducitur ad hanc formam:

$$\frac{m\sin.\,\alpha^2\cos.\,\alpha^2 - r\sin.\,\alpha^2\cos.\,2\,\alpha}{r\,(m-r)} = \frac{\sin.\,2\,\alpha\cos.\,\alpha^2 + \sin.\,\alpha\cos.\,\alpha^2}{a}$$
$$= \frac{\cos.\,\alpha^2\,(\sin.\,2\,\alpha + \sin.\,\alpha)}{a},$$

hincque porro fit

$$a\,m\sin.\,\alpha^2\cos.\,\alpha^2 - a\,r\sin.\,\alpha^2\cos.\,2\,\alpha$$
$$= r\,(m-r)\cos.\,\alpha^2\,(\sin.\,2\,\alpha + \sin.\,\alpha),$$

quae aequatio iam tantum eſt quadratica, cum praecedens ad tertium gradum aſcendiſſet.

§. 20. Cum igitur caſu, quo ambo pendula longitudinem habent inaequalem, perueniatur ad aequationem cubicam, ponamus eius tres radices eſſe 1°) $r = k$; 2°) $r = k'$; 3°) $r = k''$; ex quibus triplex motus regularis oritur, quorumque ſinguli ſolutionem ſpecialem noſtri problematis exhibent, quarum vnam euoluiſſe ſufficiet, ex radice $r = k$ oriundam: Pro binis reliquis enim tantum opus eſt loco k vel k' vel k'' ſcribere. Cum igitur ex aequatione differentio-differentiali aſſumta ſit $z = \sin.\,(\zeta + t\sqrt{\frac{2g}{r}})$, denotante ζ angulum conſtantem quemcunque, erit ad quoduis tempus, tam pro ſitu fili quam amborum corporum M et N, vt ſiquitur:

$$\omega = \frac{f\,z\sin.\,\alpha}{a} = \frac{f}{a}\sin.\,\alpha\sin.\,(\zeta + t\sqrt{\tfrac{2g}{k}});$$

$\omega' =$

$$\omega' = -\frac{f}{a} \text{ sin. } 2\alpha \text{ sin. } (\zeta + t\sqrt{\tfrac{2g}{k}});$$
$$\omega'' = \frac{f}{a} \text{ sin. } \alpha \text{ sin. } (\zeta + t\sqrt{\tfrac{2g}{k}};$$
$$\Phi = \frac{f}{m-k} \text{ sin. } \alpha^2 \text{ sin. } (\zeta + t\sqrt{\tfrac{2g}{k}}) \text{ et}$$
$$\Phi' = \frac{f}{n-k} \text{ sin. } \alpha^2 \text{ sin. } (\zeta + t\sqrt{\tfrac{2g}{k}}),$$

quibus formulis totus motus, tam fili quam pendulorum appensorum, pro radice $r = k$ definitur.

§. 21. Similes igitur formulae etiam pro reliquis radicibus $r = k'$ et $r = k''$ prodibunt, in quibus quidem loco anguli constantis ζ scribi conueniet ζ' et ζ''; similique modo loco coefficientis f scribatur f' et f'', quo facto solutio generalis et completa, quae omnes plane motus in se complectatur, sequenti modo exhiberi poterit:

I. $$\omega = \frac{f}{a} \text{ sin. } \alpha \text{ sin. } (\zeta + t\sqrt{\tfrac{2g}{k}}) + \frac{f'}{a} \text{ sin. } \alpha \text{ sin. } (\zeta' + t\sqrt{\tfrac{2g}{k'}}) + \frac{f''}{a} \text{ sin. } \alpha \text{ sin. } (\zeta'' + t\sqrt{\tfrac{2g}{k''}}),$$

II. $$\omega' = -\frac{f}{a} \text{ sin. } 2\alpha \text{ sin. } (\zeta + t\sqrt{\tfrac{2g}{k}}) - \frac{f'}{a} \text{ sin. } 2\alpha \text{ sin. } (\zeta' + t\sqrt{\tfrac{2g}{k'}}) - \frac{f''}{a} \text{ sin. } 2\alpha \text{ sin. } (\zeta'' + t\sqrt{\tfrac{2g}{k''}}),$$

III. $$\omega'' = \frac{f}{a} \text{ sin. } \alpha \text{ sin. } (\zeta + t\sqrt{\tfrac{2g}{k}}) + \frac{f'}{a} \text{ sin. } \alpha \text{ sin. } (\zeta' + t\sqrt{\tfrac{2g}{k'}}) + \frac{f''}{a} \text{ sin. } \alpha \text{ sin. } (\zeta'' + t\sqrt{\tfrac{2g}{k''}}),$$

IV. $$\Phi = \frac{f}{m-k} \text{ sin. } \alpha^2 \text{ sin. } (\zeta + t\sqrt{\tfrac{2g}{k}}) + \frac{f'}{m-k'} \text{ sin. } \alpha^2 \text{ sin. } (\zeta' + t\sqrt{\tfrac{2g}{k'}}) + \frac{f''}{m-k''} \text{ sin. } \alpha^2 \text{ sin. } (\zeta'' + t\sqrt{\tfrac{2g}{k''}}),$$

V. $$\Phi' = \frac{f}{n-k} \text{ sin. } \alpha^2 \text{ sin. } (\zeta + t\sqrt{\tfrac{2g}{k}}) + \frac{f'}{n-k'} \text{ sin. } \alpha^2 \text{ sin. } (\zeta' + t\sqrt{\tfrac{2g}{k'}}) + \frac{f''}{n-k''} \text{ sin. } \alpha^2 \text{ sin. } (\zeta'' + t\sqrt{\tfrac{2g}{k''}}),$$

in quas formulas ingrediuntur sex constantes arbitrariae, scilicet f, f', f'' et ζ, ζ', ζ'', quemadmodum ratio omnium motuum possibilium postulat.